

Parte 2. Matemáticas presentes en múltiples aspectos de Arquitectura

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MAH

MASTER EN ARQUITECTURA,
ORGANIZACIÓN Y GESTIÓN DE
INFRAESTRUCTURAS HOSPITALARIAS

Madrid, 9 de febrero de 2008



1. Introducción

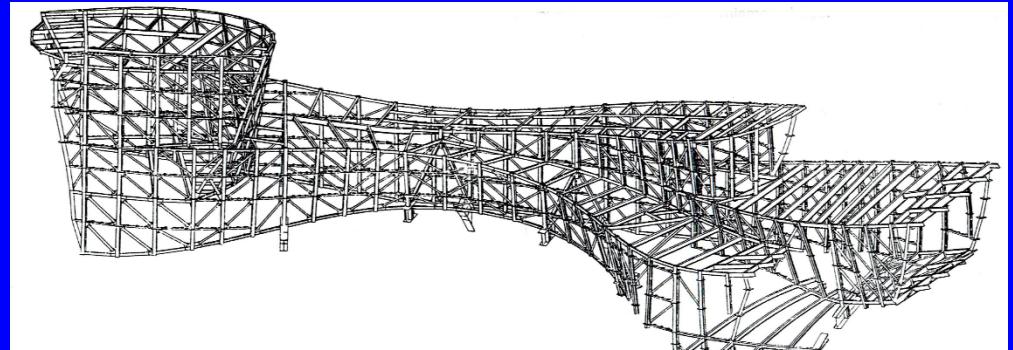
Detrás de las más atrevidas estructuras, desafiando la gravedad, retando a la indispensable robustez, aparentando una falsa inestabilidad, hay matemáticas ocultas cuya misión es pasar inadvertidas, ocultar el mérito técnico para resaltar la componente artística y creativa.

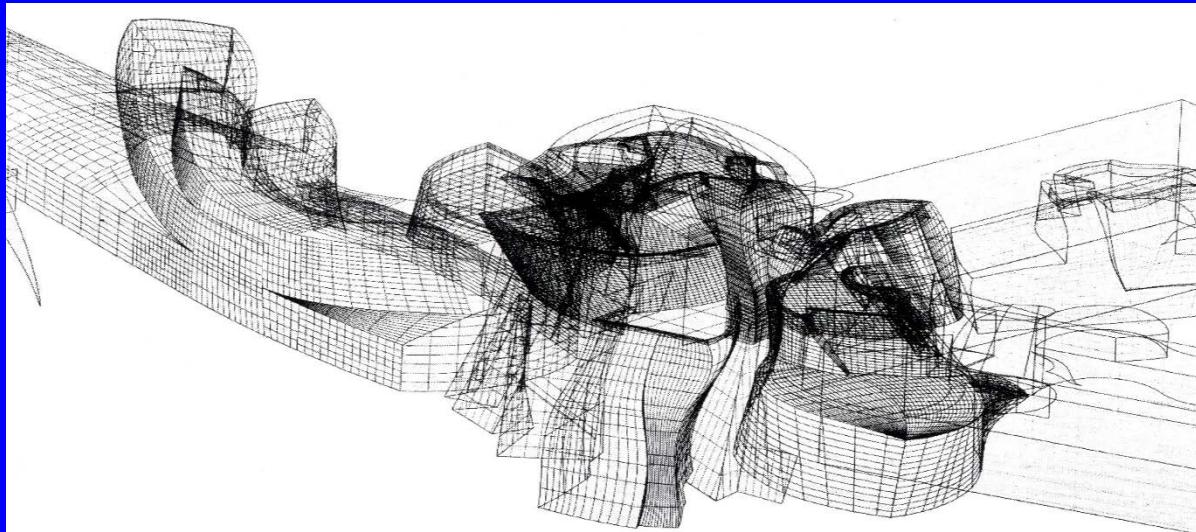


Museo Guggenheim, Bilbao, 1997

Arquitecto: Frank O. Gehry & Associates

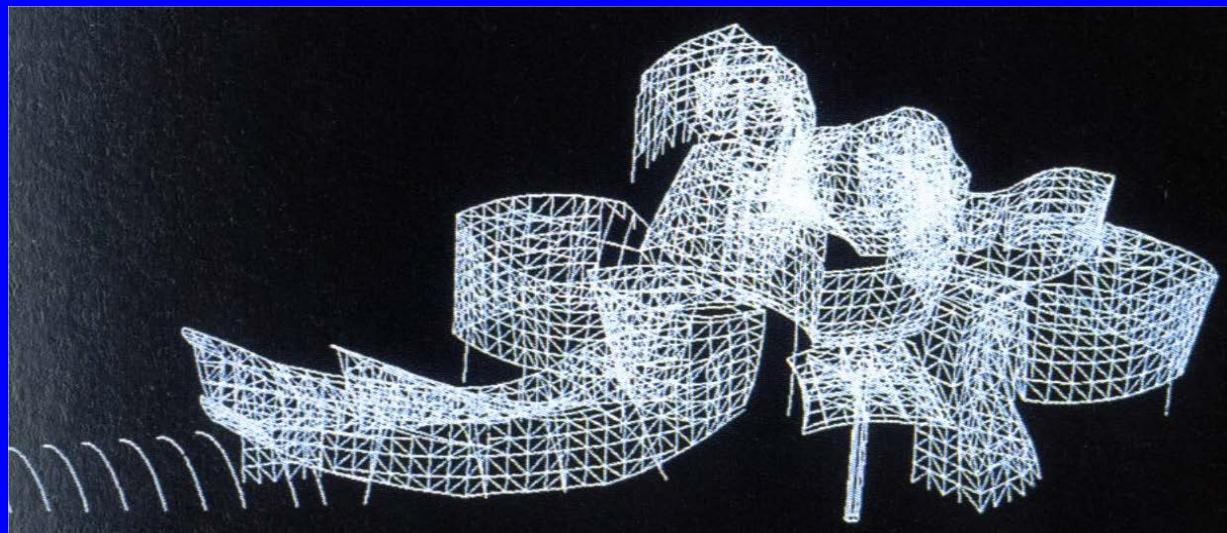
Ingeniero: Skidmore Owings & Merrill





Programa CATIA del
INRIA (Francia), análisis
estructural en Chicago

Mirage,
Boeing 777



Diseño y cálculo de
la estructura 3d:
diseño para corte y
ensamblaje de
elementos de titanio



Gustave Eiffel (de 1881 a 1882)

L. Euler y los Bernoulli como nexos de unión



<http://www.euler-2007.ch>

Leonhard Euler (1707-1783)

<http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Euler.html>

Leonhardi Euleri Opera Omnia: a centenary project: *Andreas Kleinert and Martin Mattmüller* in *Newsletter of the European Mathematical Society*, September 2007,

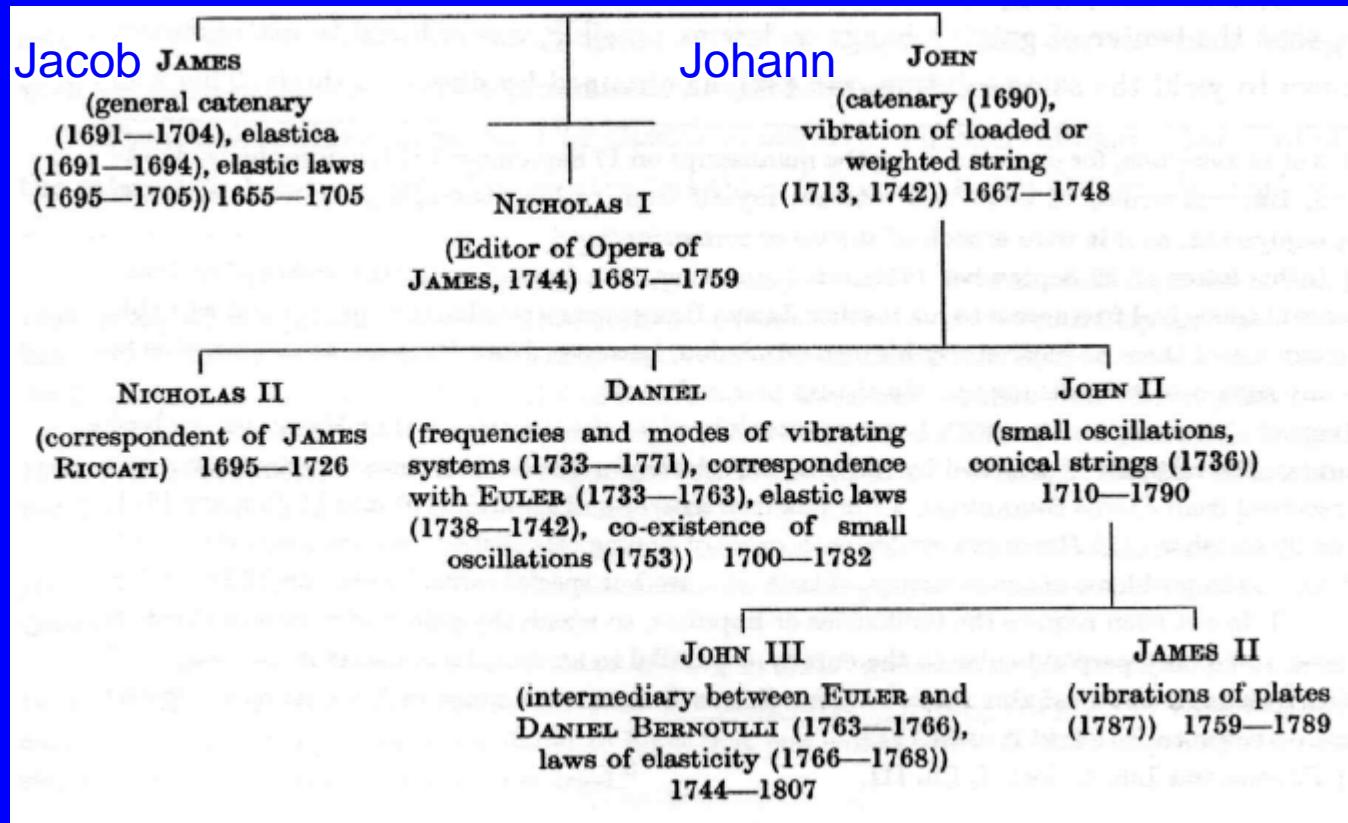
Facultad de Matemáticas,

Univ. Complutense de Madrid
20/08/2023



Shelf with *Opera Omnia* at Euler-Archiv Basel

Por extraño que parezca, entre 1691 y 1748, el estudio matemático de la vibración, deformación y elasticidad fue dominado (si no monopolizado) por los llamados “geometras de Bassel” (en particular, la familia Bernoulli y Euler).

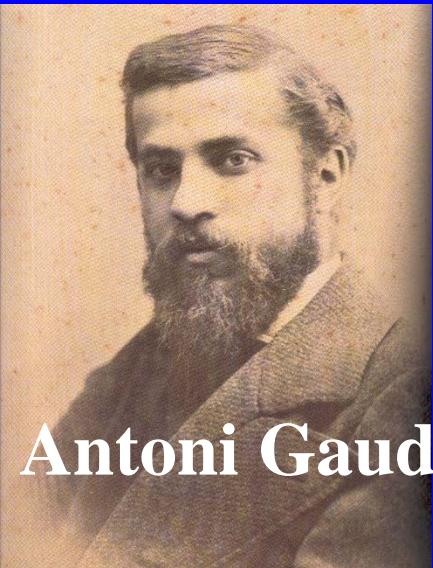


Plan del resto de la 2^a parte:

- 2. Gaudi: catenarias y transición de formas.
- 3. La más alta columna de Euler.
- 4. Columnas reforzadas y heterogéneas: rascacielos.
- 5. Matemáticas en la *Tour Eiffel*.
- 6. Puentes: resonancia.
- 7. Tensegridades.
- 8. Complejidad en matemáticas y arquitectura



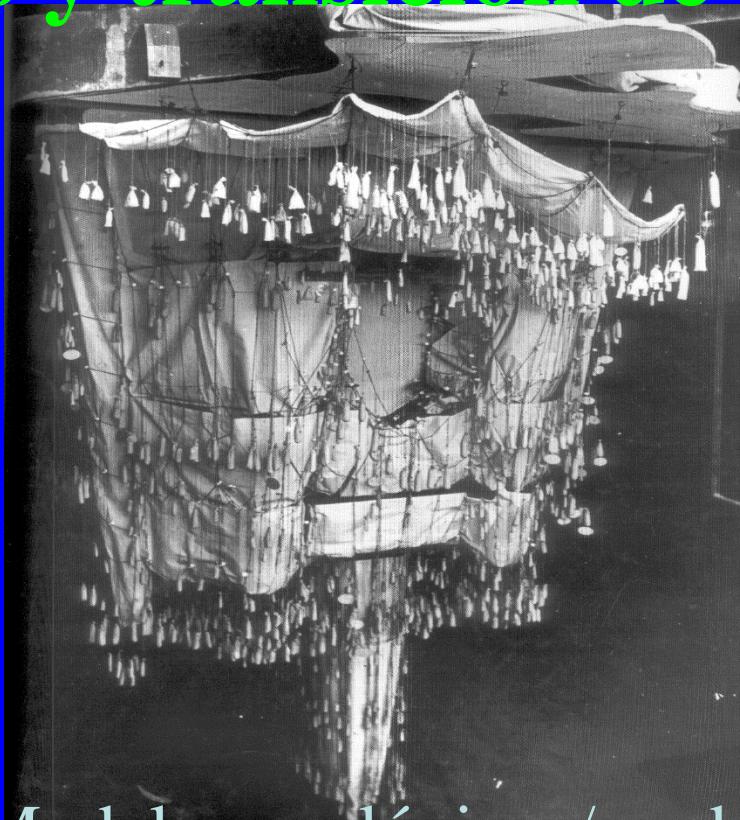
2. Gaudí: catenarias y transición de formas



Antoni Gaudí (1825-1926)



Iglesia de la
Colonia Güell



Modelos analógicos/modelos matemáticos
Inversión del camino de las técnicas de construcción: de la maqueta al cálculo, al dibujo y a la construcción.

La belleza de lo económico:

Guillermo de Occam (1285-1349) “Principio de la navaja (o de parsimonia)”: *Entia non sunt multiplicanda sine necessitate*



Gaudi:

“Para que un objeto sea extraordinariamente bello es necesario que su forma no contenga nada de superfluo”

“Soy geómetra, es decir, sintético”



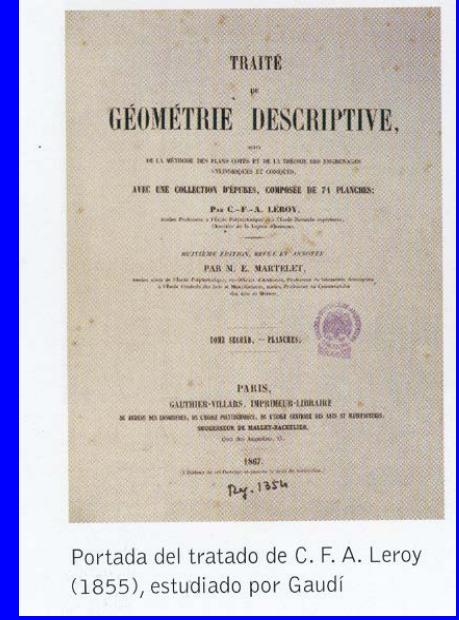
Gaudi:

“Yo lo calculo todo”

“Mis ideas estructurales y estéticas son una lógica *indisputable*. Convencido del perfeccionamiento que representan tengo el deber de aplicarlas”

“Este árbol cercano a mi obrador: éste es mi maestro”

“La geometría en la ejecución de las superficies no complica, sino que simplifica la construcción”



Portada del tratado de C. F. A. Leroy (1855), estudiado por Gaudí

C.F. A. Leroy (1855)



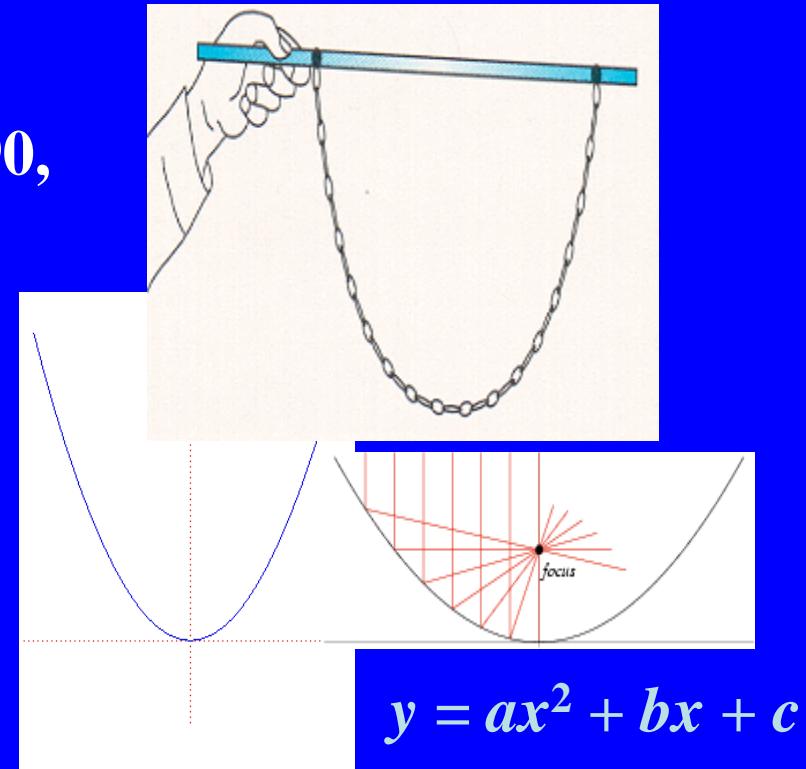
Gaudi : modelos antifuniculares

“La catenaria”:

Leonardo da Vinci (1452-1519) 1490,

En 1669, Jungius puso en tela de juicio la afirmación de Galileo Galilei (1562-1642): parábola

Huygens primero en utilizar el término de catenaria (carta a Leibniz en 1690)

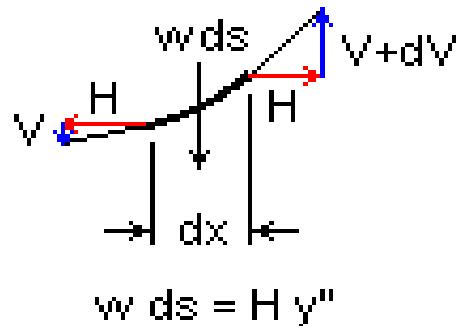


La ecuación fue obtenida por Leibniz, Huygens, y Johann Bernoulli en 1691 en respuesta a un problema de Jakob Bernoulli.

$$y = a \cosh(x/a)$$



Deducción:



$$w ds = H dy$$

$$ds = \sqrt{1 + y'^2} dx$$

$$\int_0^x \frac{dy'}{\sqrt{1 + y'^2}} = \int_0^x \frac{w}{H} dx$$

$$\sinh^{-1} y' = \frac{w}{H} x$$

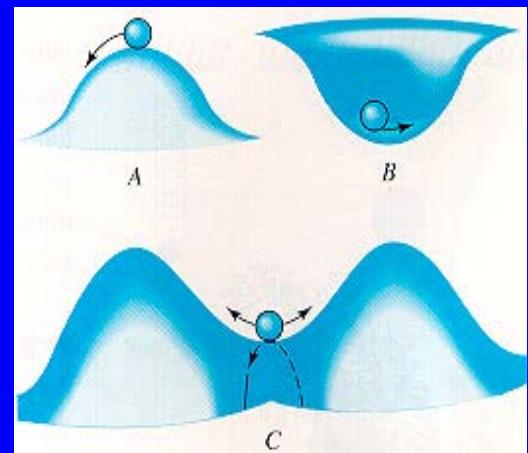
$$y' = \sinh \frac{w}{H} x$$

$$y = \frac{H}{w} \left(\cosh \frac{wx}{H} - 1 \right)$$

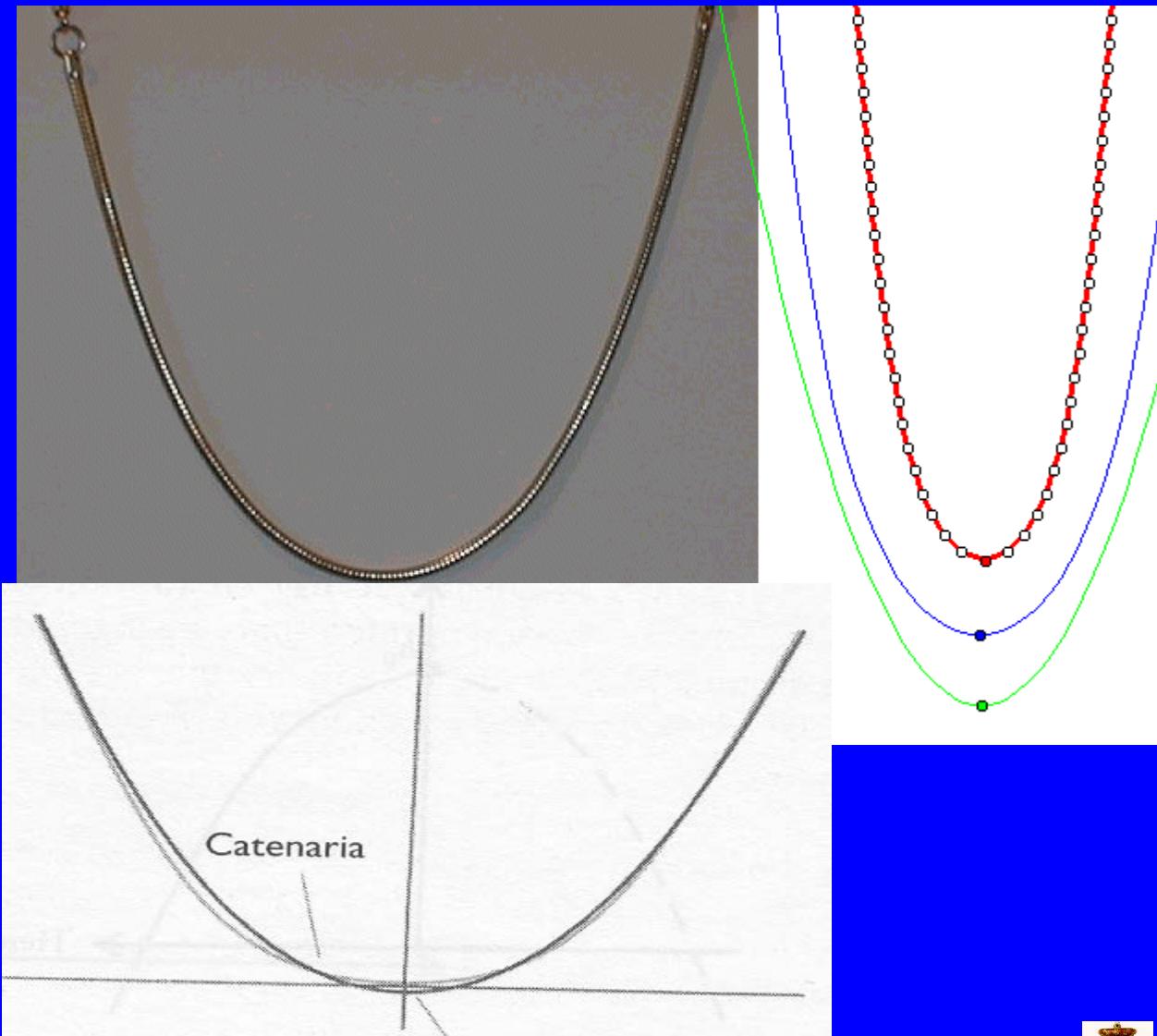
Finding the Catenary



Estado de equilibrio
de la cuerda vibrante
bajo la acción de la
gravedad



La sutileza de las matemáticas



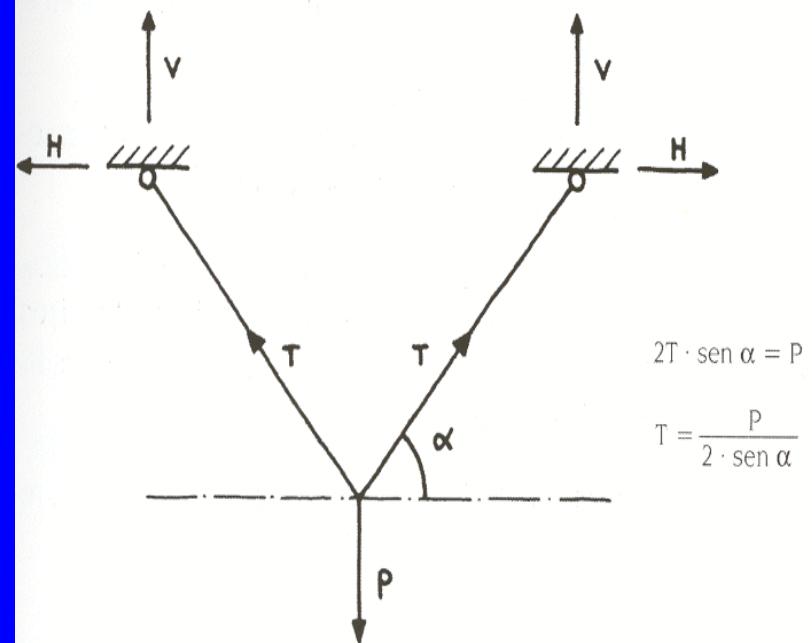
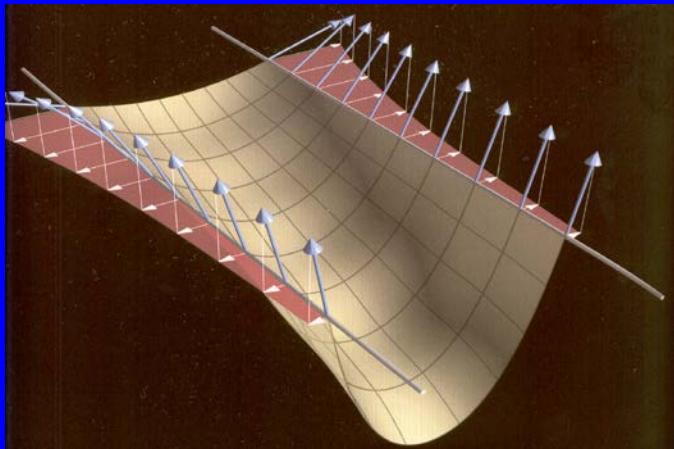
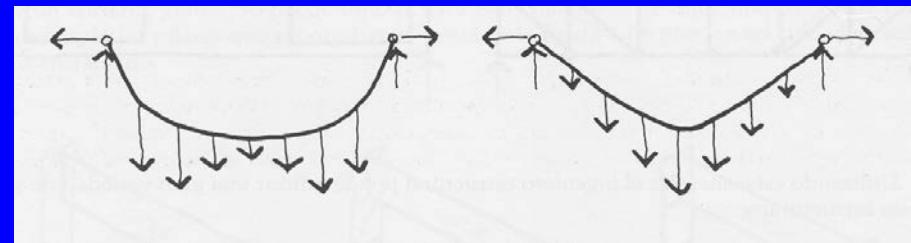
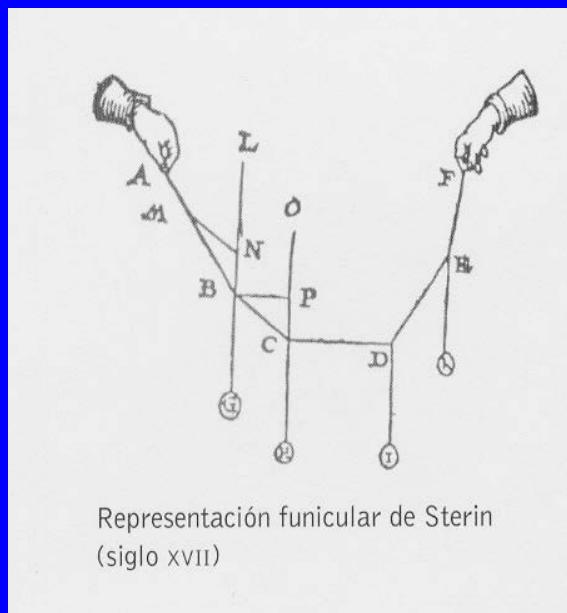
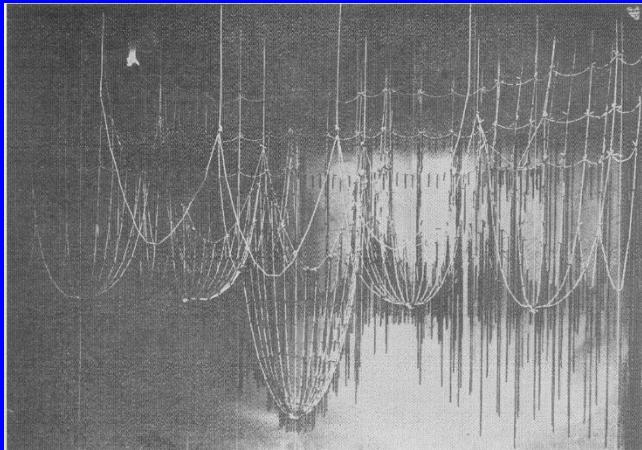


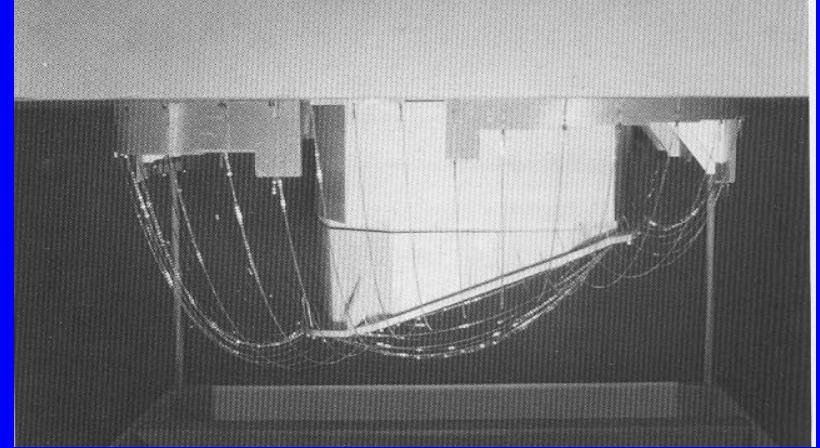
Fig. 1. Mecanismo resistente básico.



Algunas otras maquetas anti-funiculares



Primer estereofuncicular
de Gaudi



Suntory Pavilion,
Expo'85, Tsukuba, Japan

Arquitecto: Keizo Sataka

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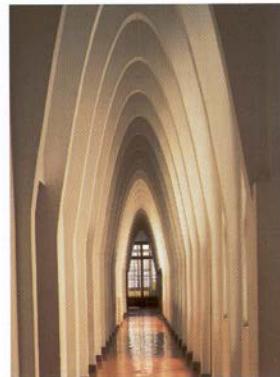
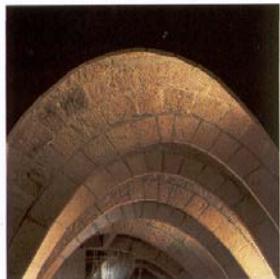


J.I. Diaz



16

La Catenaria como Arco

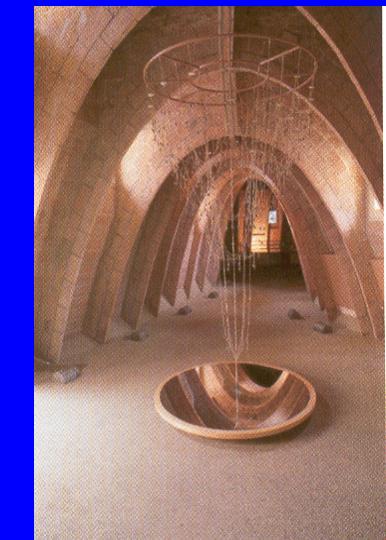
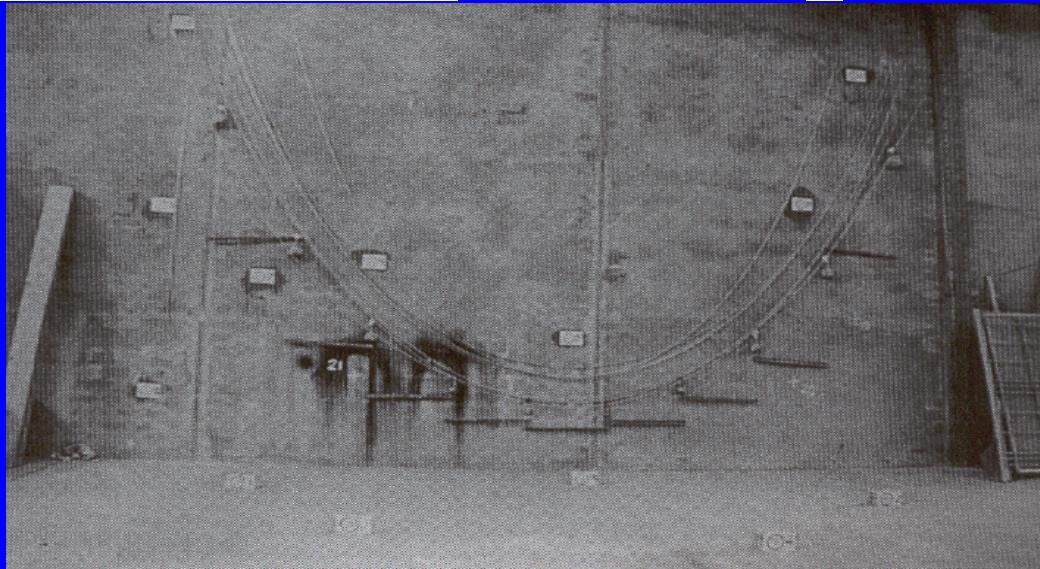
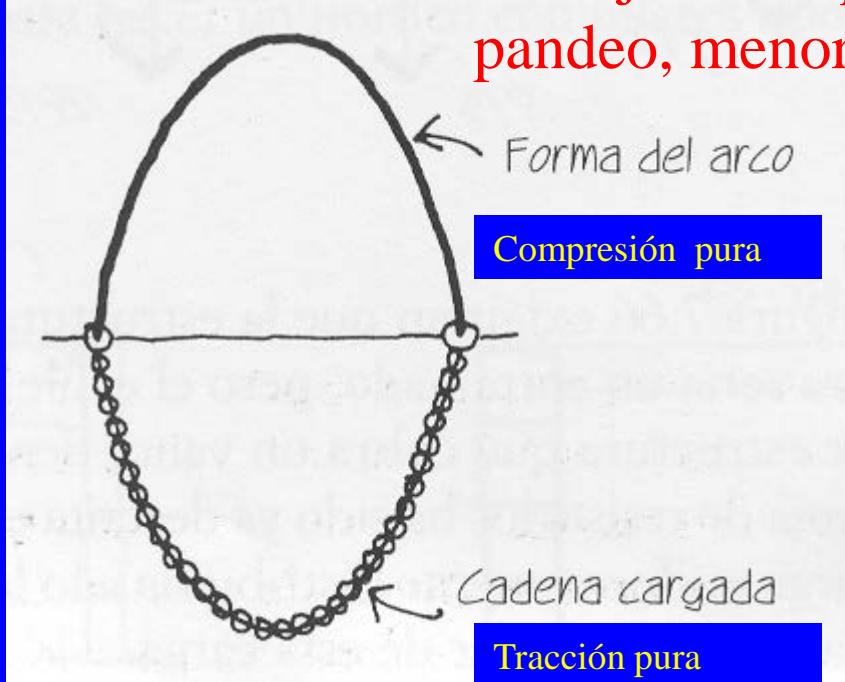


Arcos catenarios de las casas Milà y

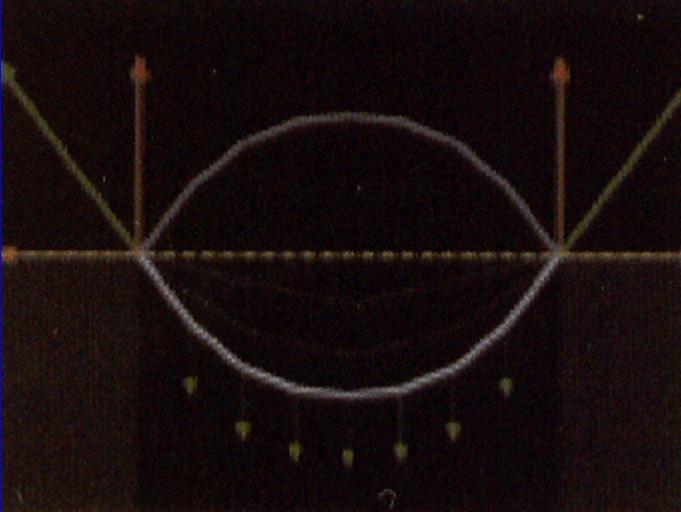
Casa Milà y

C₂Teresianas
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Trabajo a compresión, menor
pandeo, menor sección



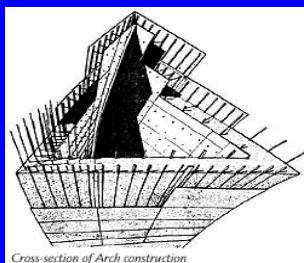
Traslación de arcos catenarios
de la Casa Milà. Modelo catenario
del Espai Gaudí



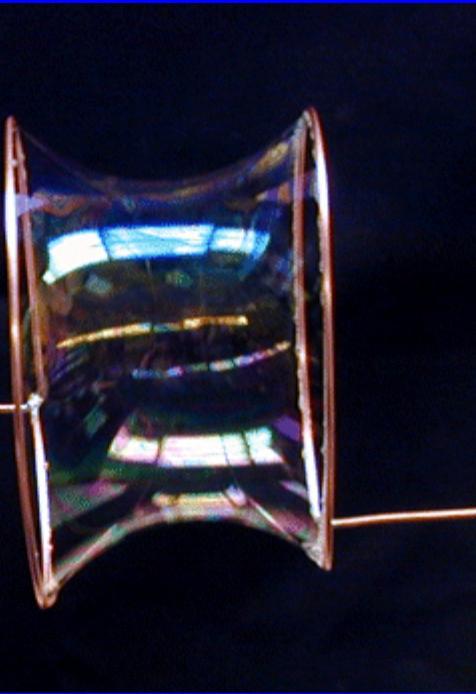
Arco Gateway de Saint Louis (EE.UU.)

Arquitecto Eero Saarinen (1963-67)

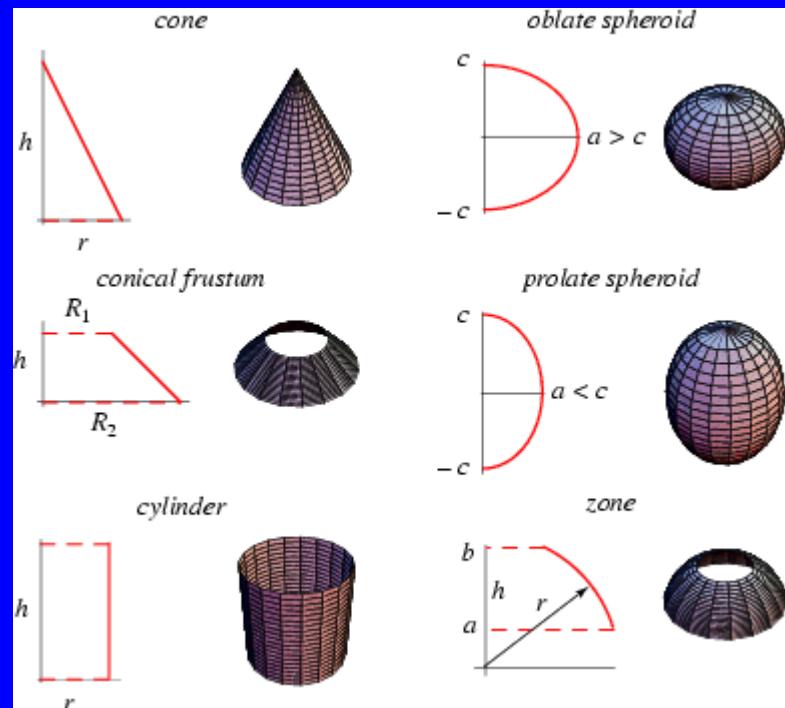
En conmemoración de la expansión al oeste de los EE.UU. en el sigloXIX



Catenoide:

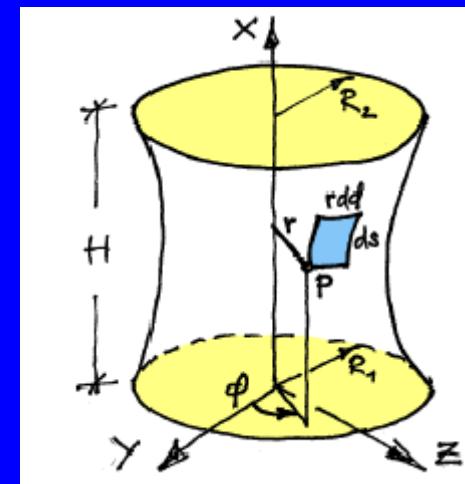


Superficies de revolución



Superficies de área mínima:

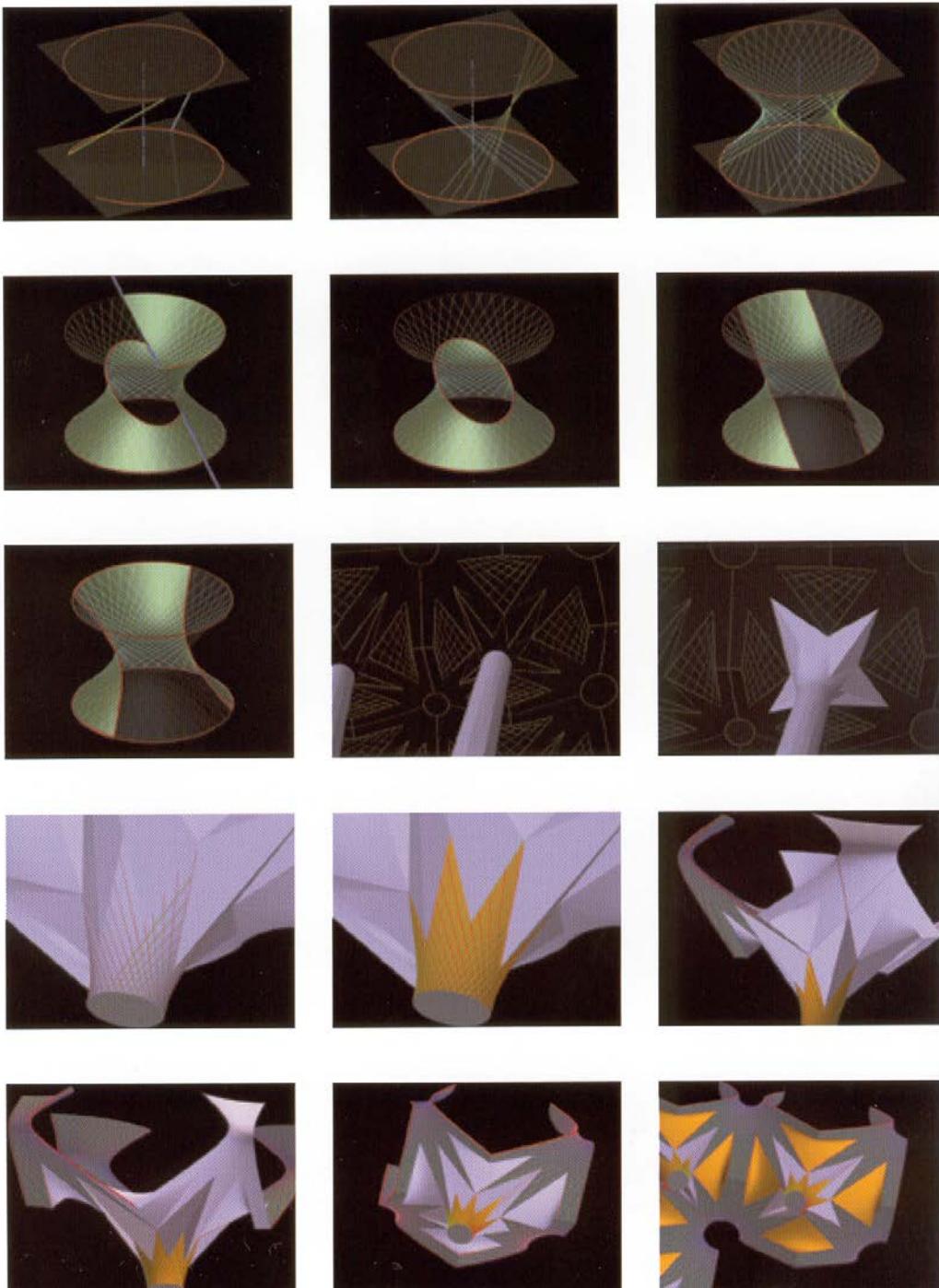
Leonhard Euler (1707-1783) 1743..



Hiperboloide de una hoja Superficies regladas

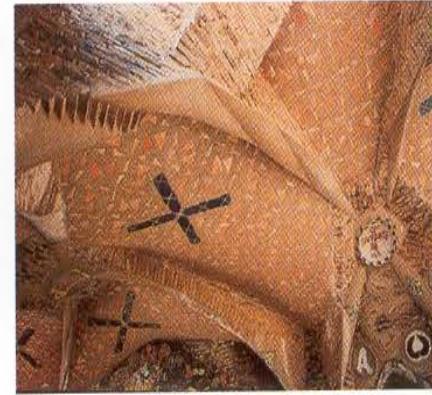
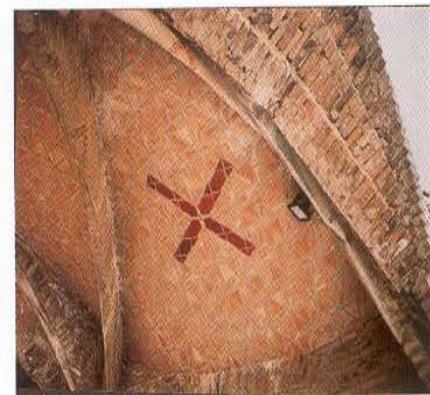
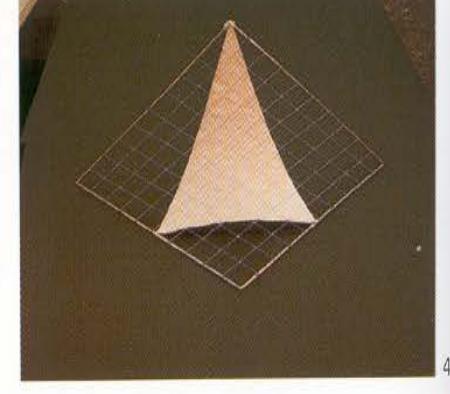
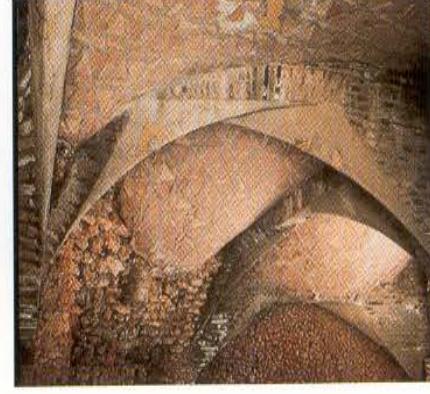
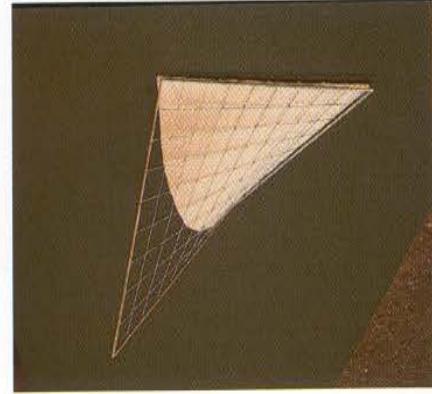
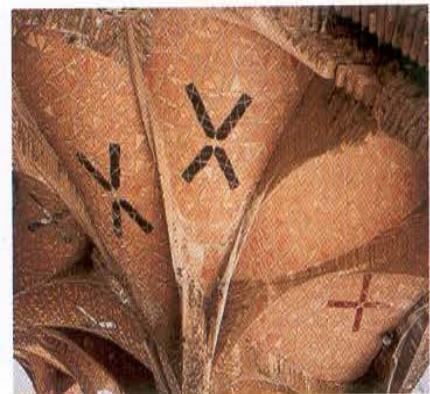
Gaudí:“El uso de las superficies regladas es lógico por su superioridad plástica y su facilidad constructiva”

Techo de la nave central de la Sagrada Familia



Superficies regladas

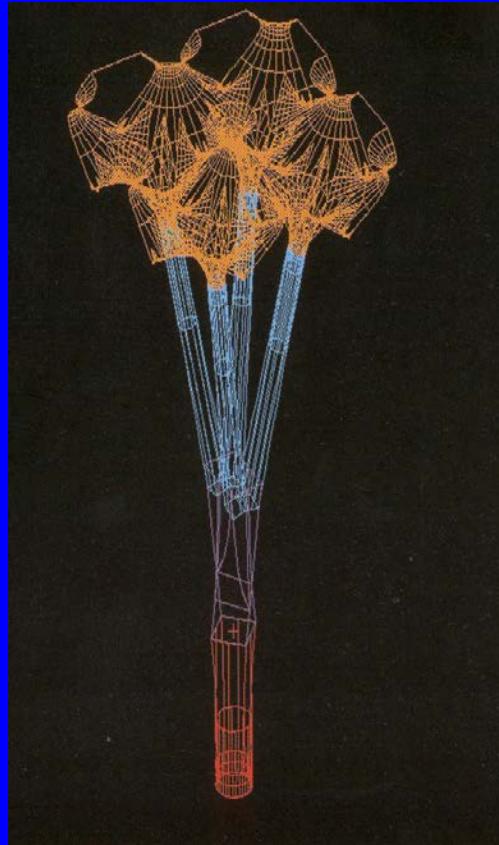
Parabolide hiperbólico



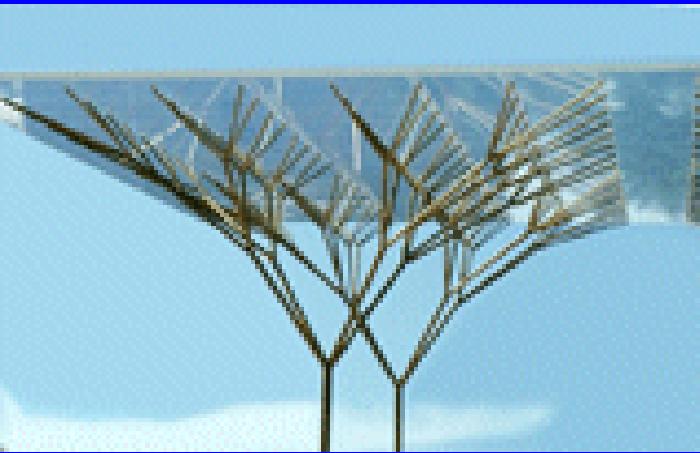
Soportal de la iglesia de la Colonia Güell

Félix Candela, Eduardo Torroja, ...
J.I. Díaz

Gaudí: fractalidad en el crecimiento de las ramas de los árboles

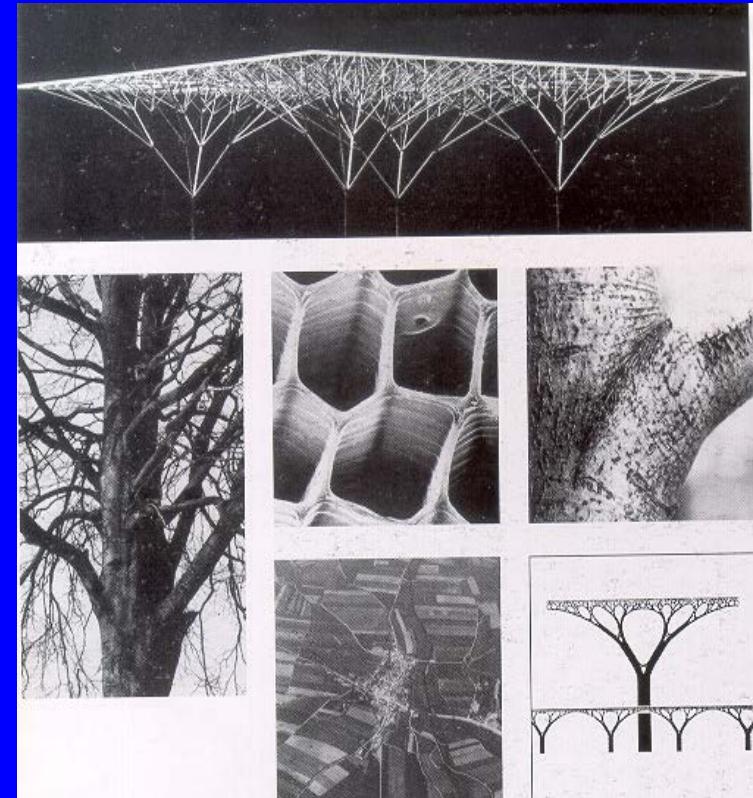
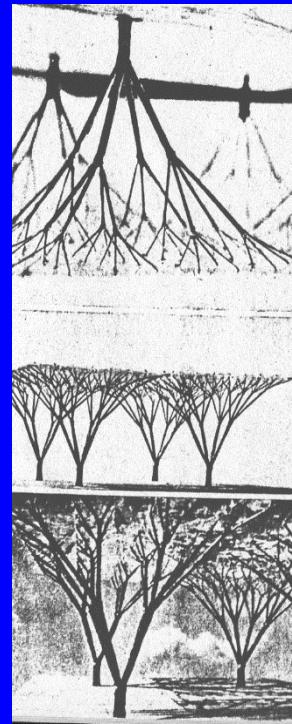


Estructuras arboriformes (fractales) Frei Otto, ...

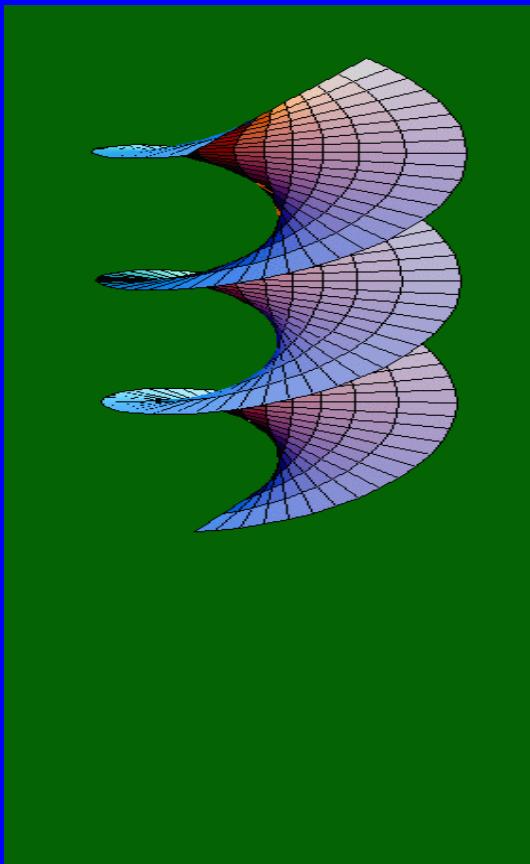


Trabajo a compresión, menor pandeo, menor sección

Yale University (1960)



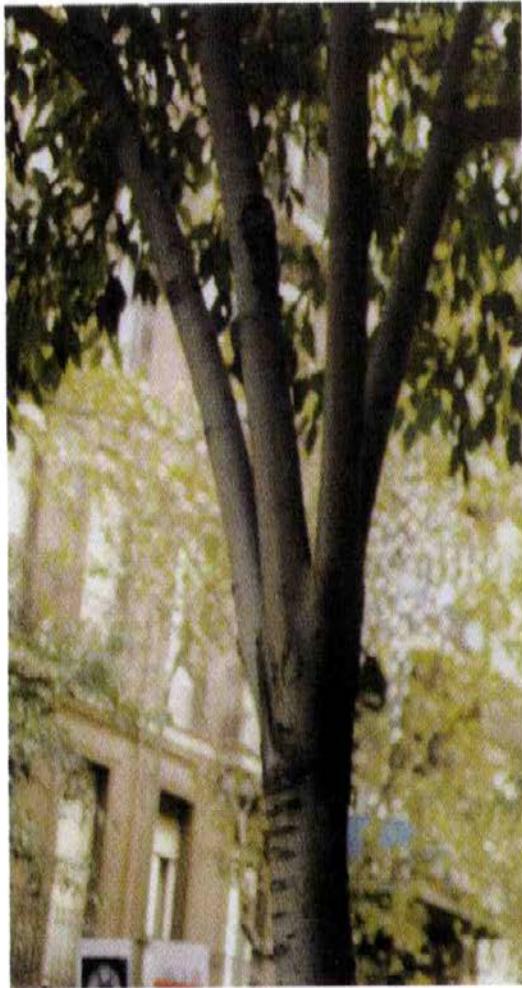
Deformación de una catenoide a una helicoide



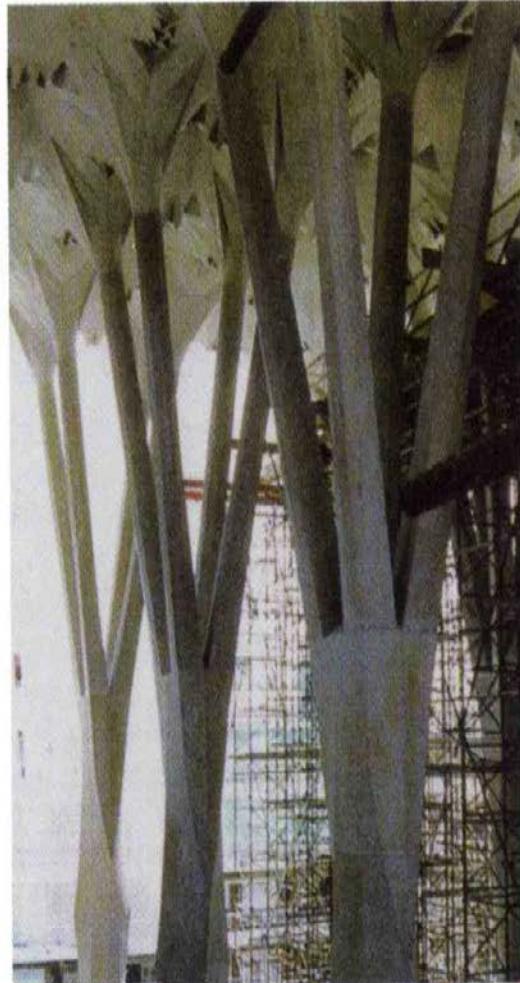
Escalera de una torre de la Sagrada Familia



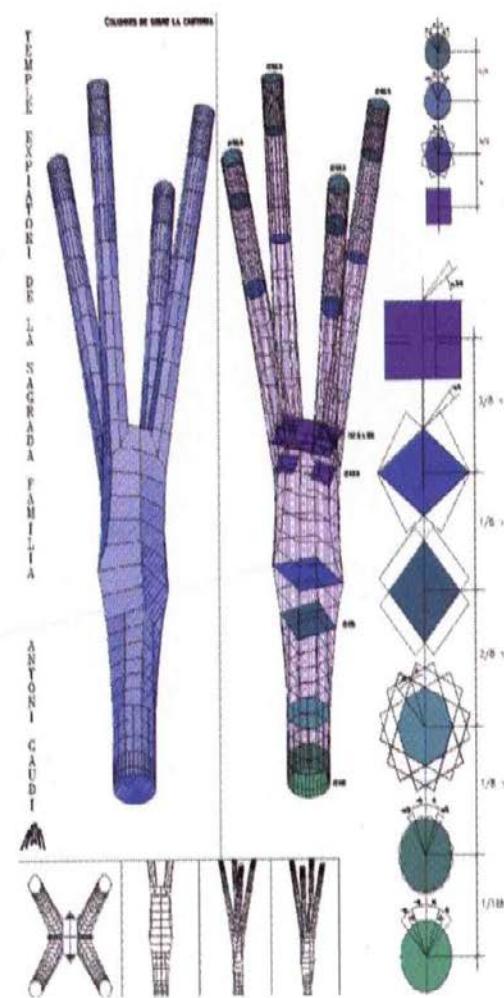
Columna de doble giro: Transición continua de formas y perfiles



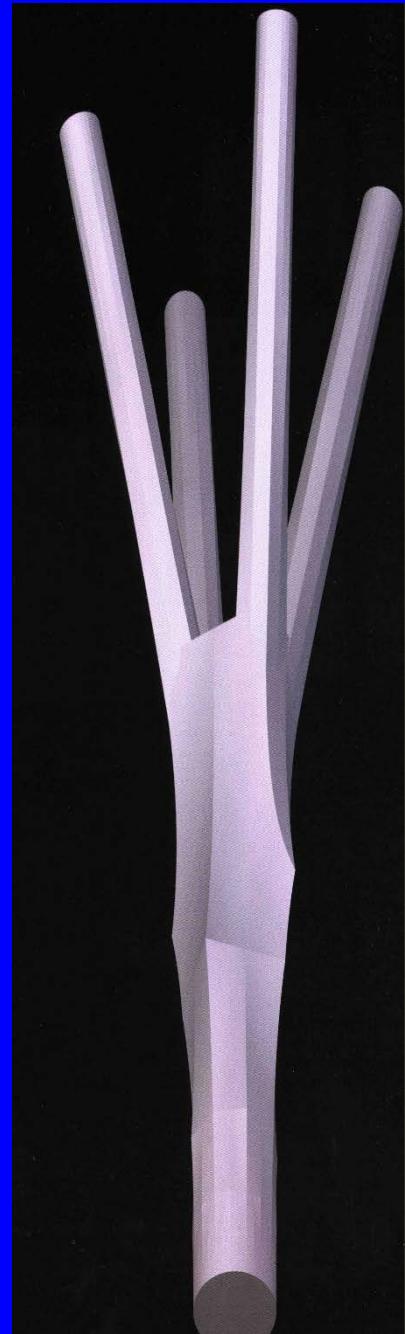
Tronco y ramas de un árbol



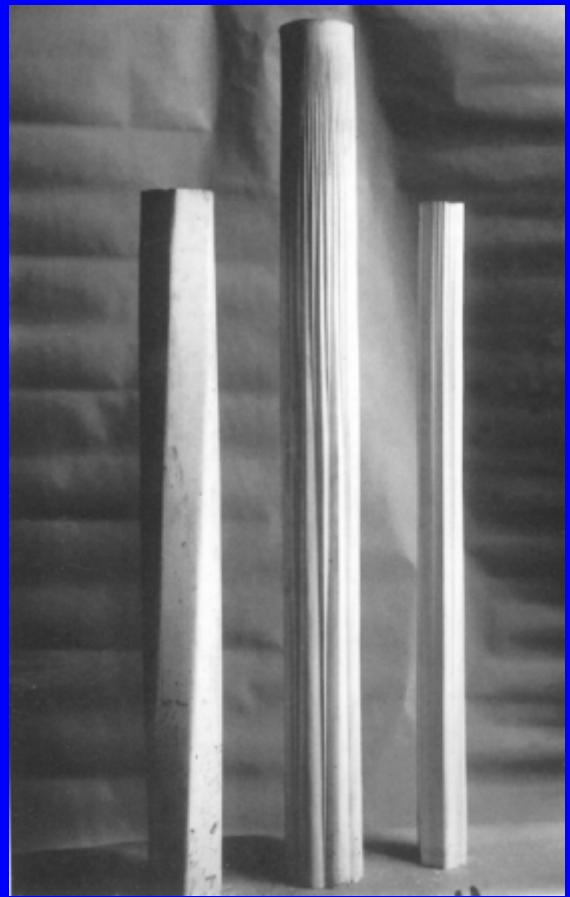
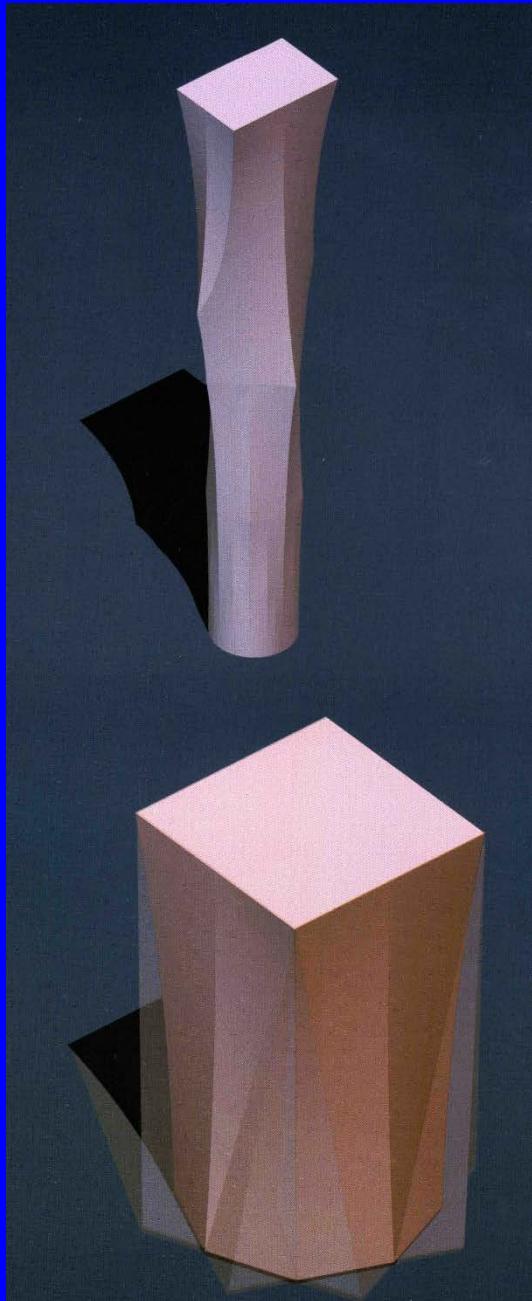
Columnas arboladas del templo
de la Sagrada Família



Esquemas geométricos de la columna
rectangular de la nave del templo
de la Sagrada Família



20/08/2023

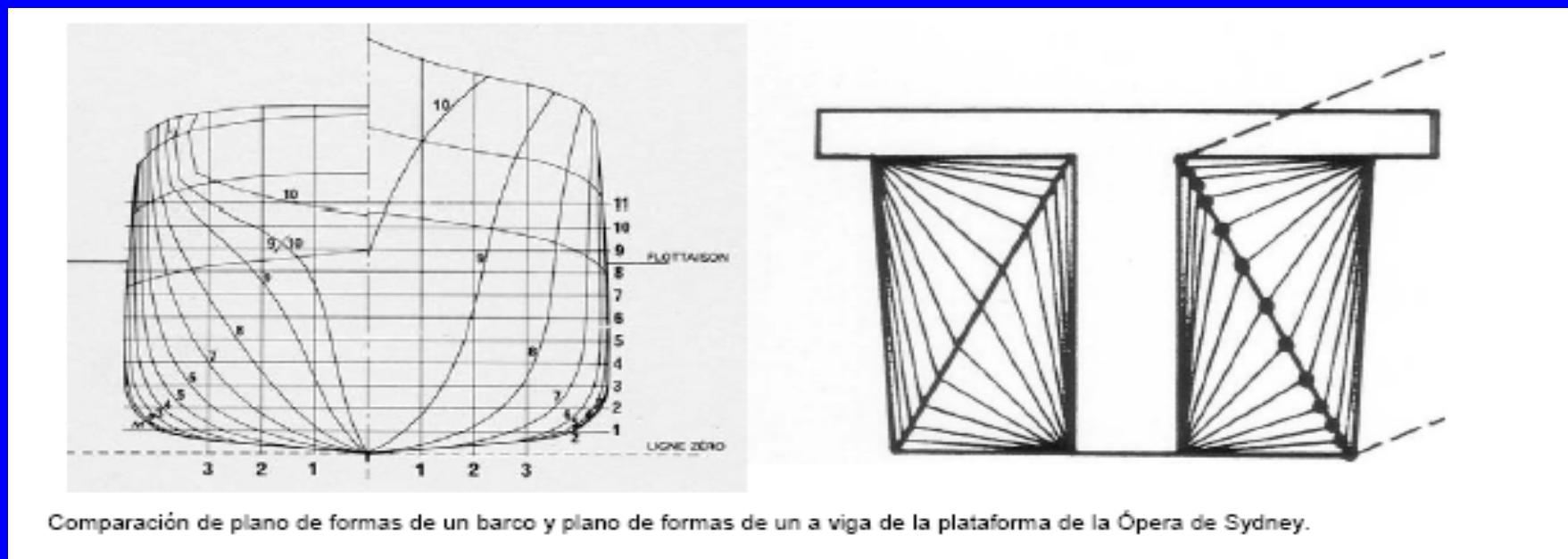
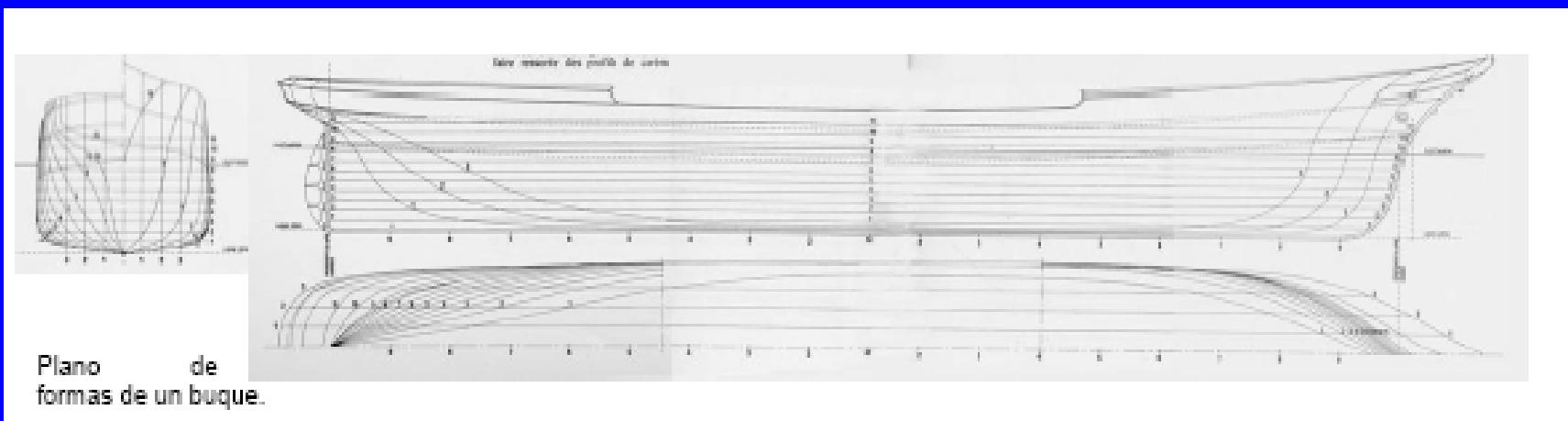


J.I. Díaz



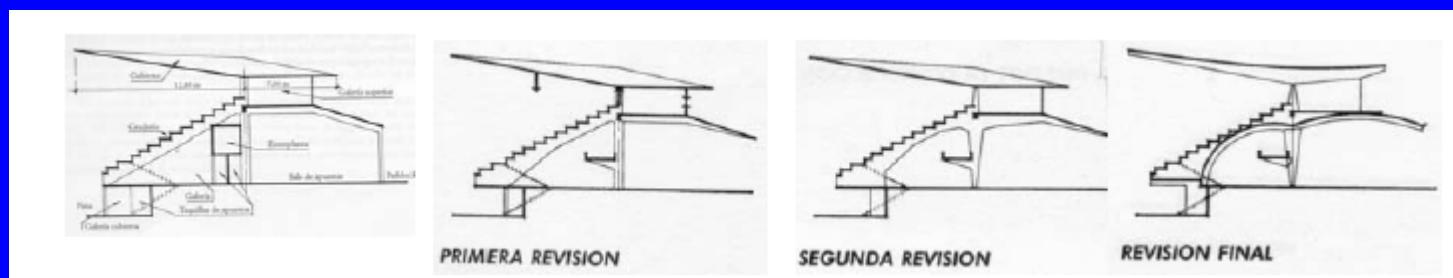
26

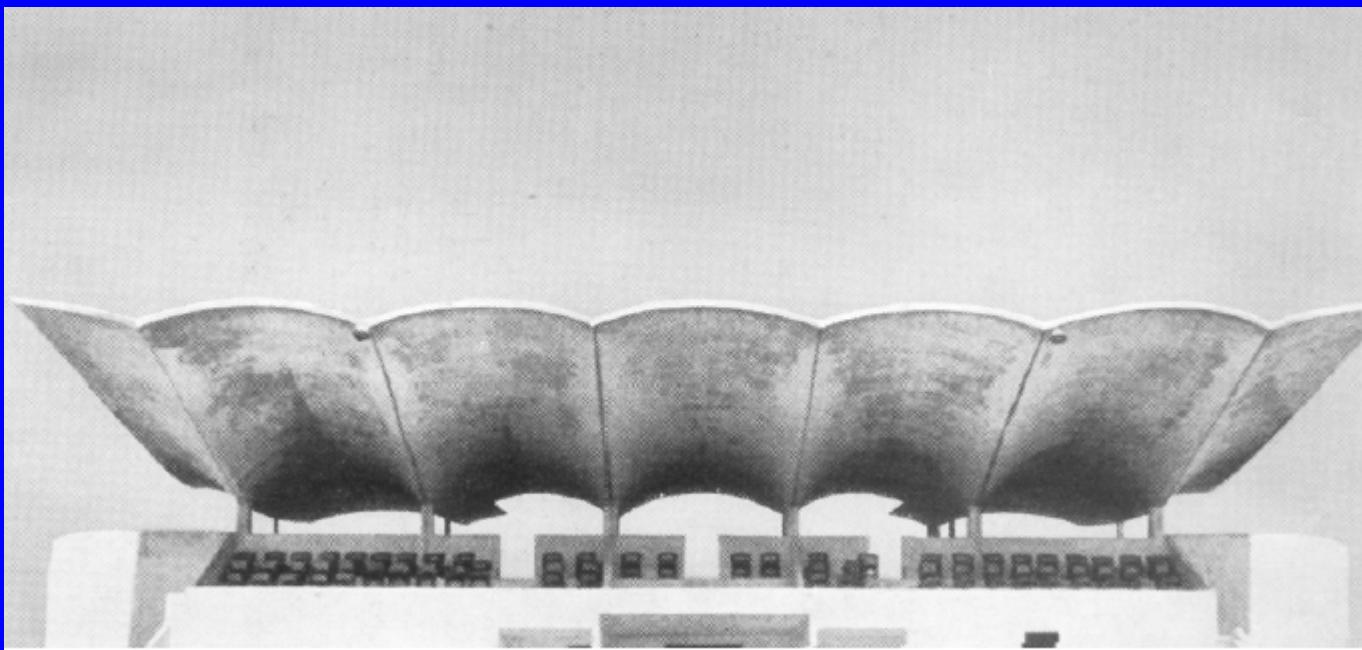
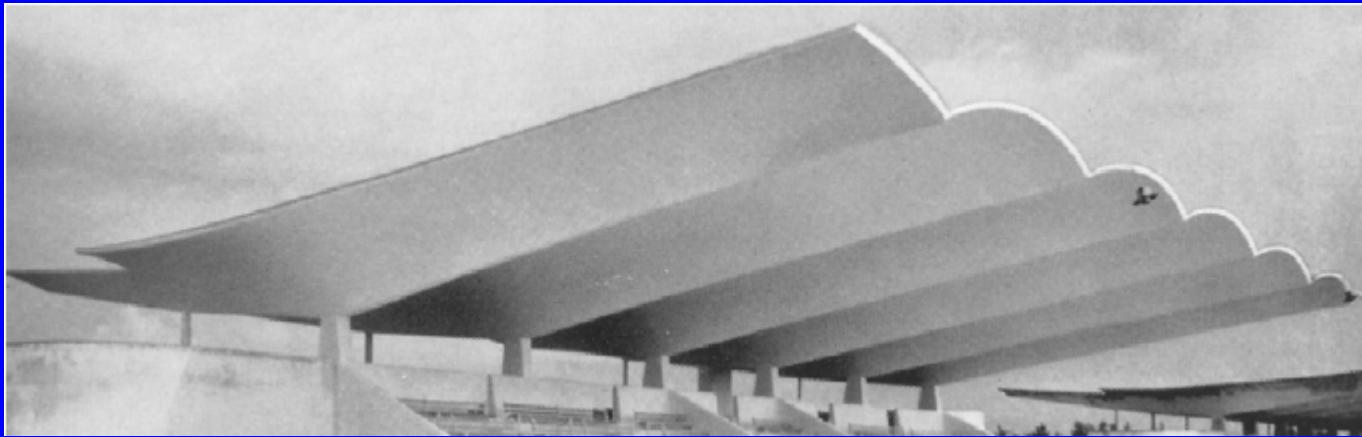
Transición continua de formas y perfiles en barcos y en otros arquitectos e ingenieros

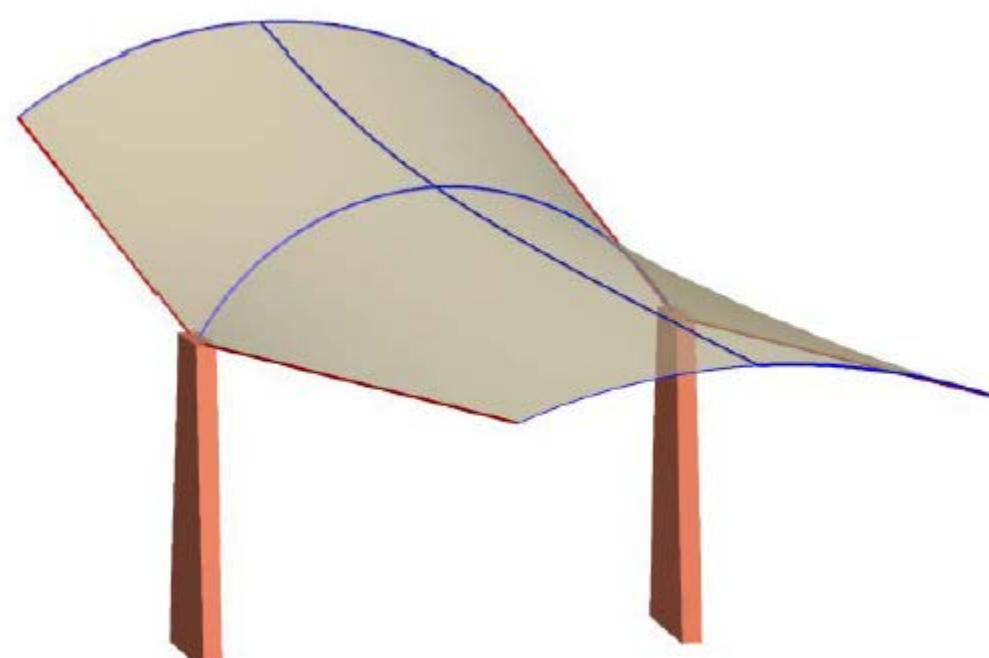
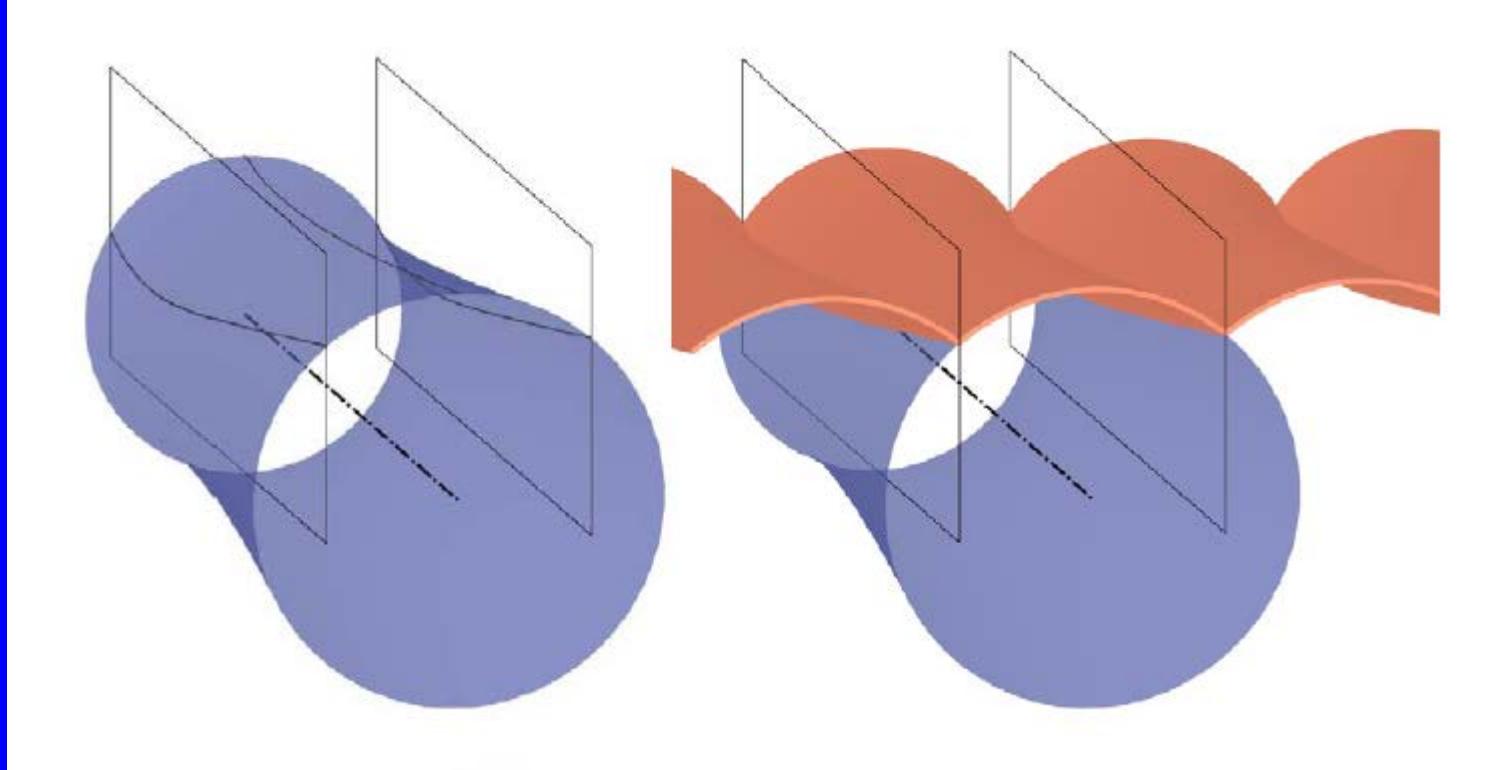


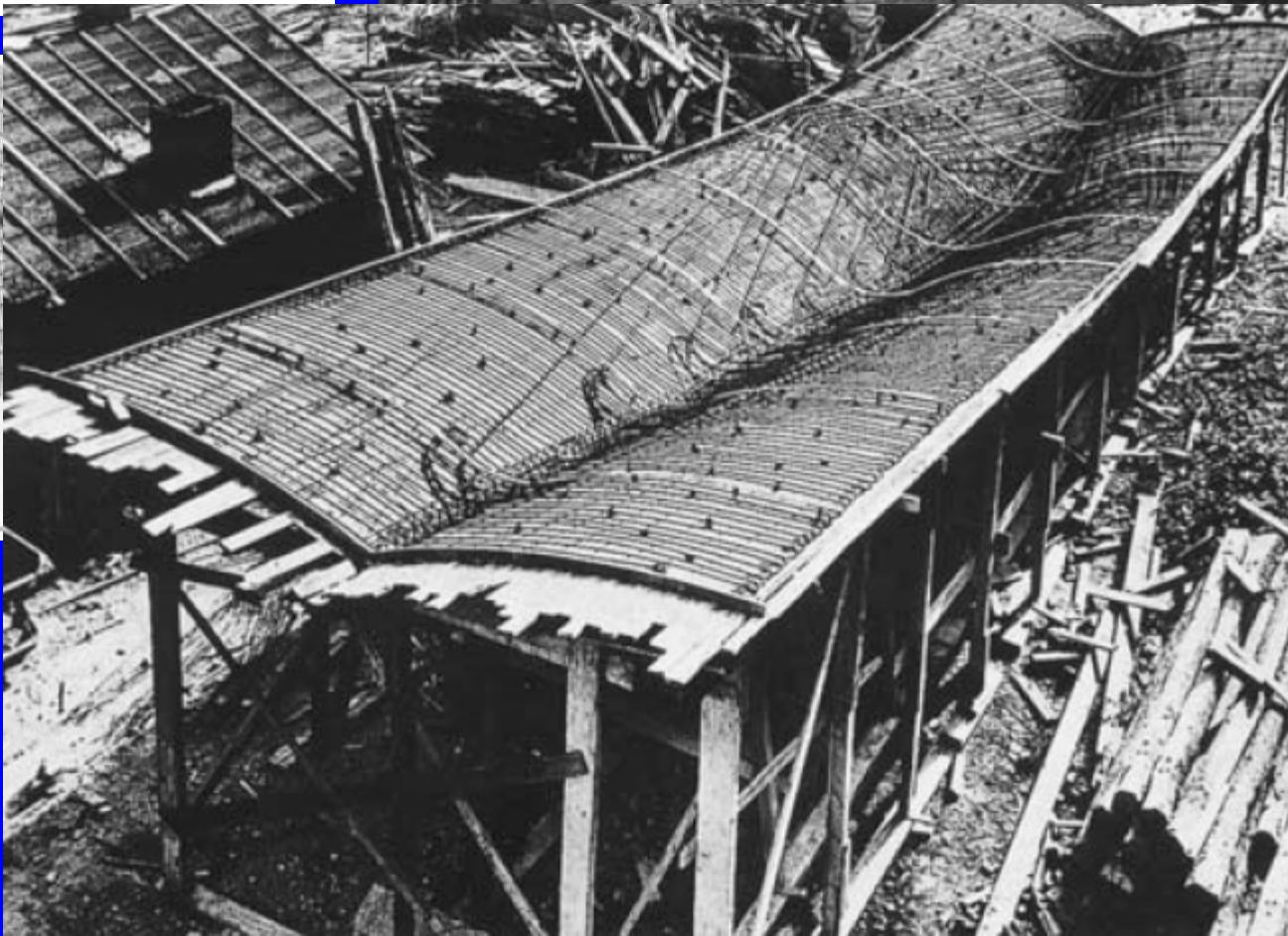
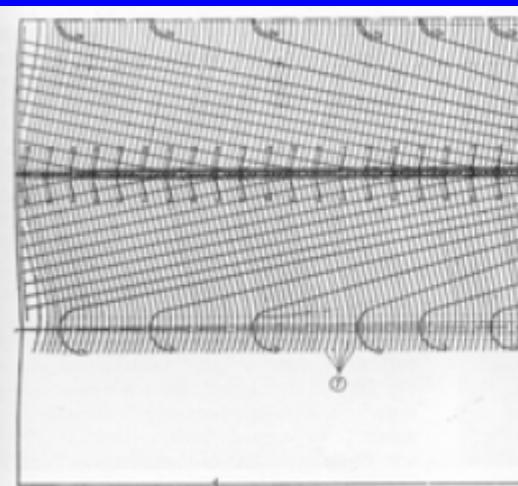
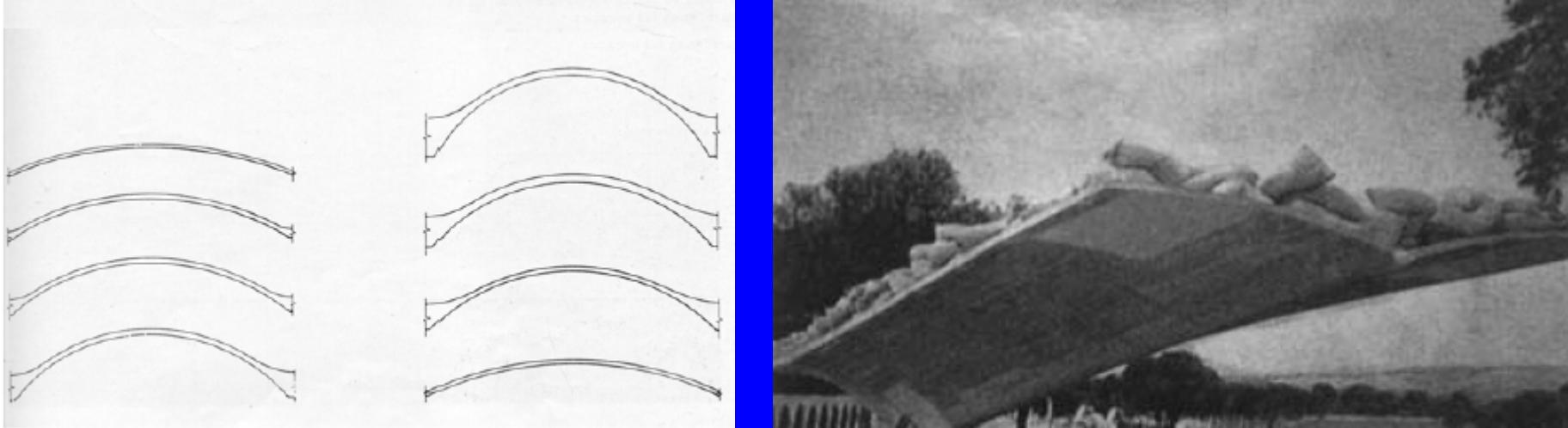
HIPÓDROMO DE LA ZARZUELA. MADRID, 1935.

C. Arniches, L. Domínguez y Eduardo Torroja. Con la empresa constructora Agroman E.C.









E. Torroja, The Structures of Eduardo Torroja, F. W. Dodge Corporation, New York, 1958.

“Las posibilidades de encontrar un modelo matemático son nulas en las presentes fechas”

On slender shells and related problems suggested by Torroja's structures

J. I. Díaz *and E. Sanchez-Palencia †



3. La más alta columna de Euler

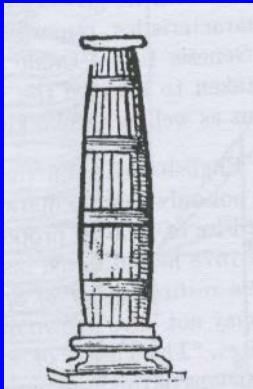
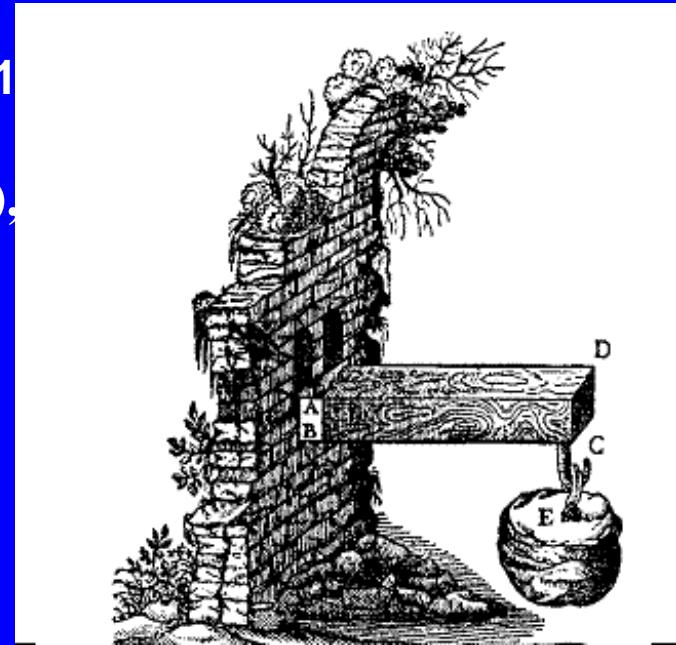
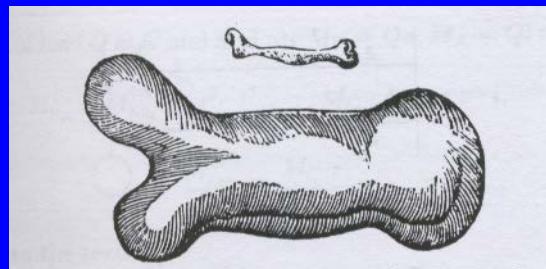


Estudios pioneros

Leonardo da Vinci (1452-1519)

Galileo Galilei (1562-1642),

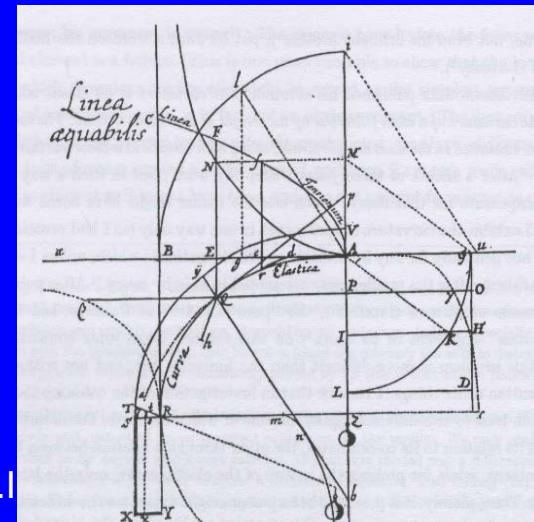
1638,



Ignace-Gaston Pardies

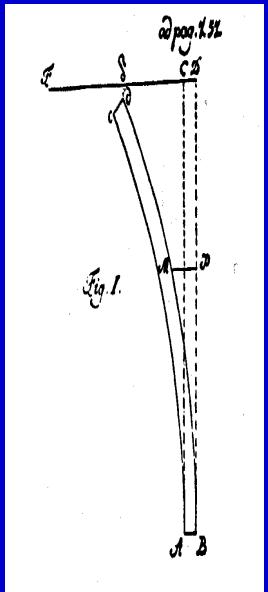
1673

Jacques Bernoulli (1654-1705),
1690



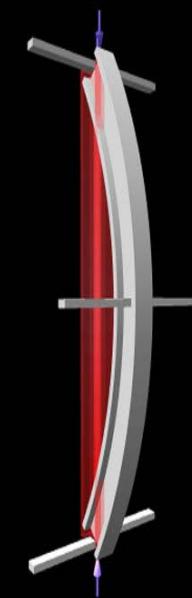
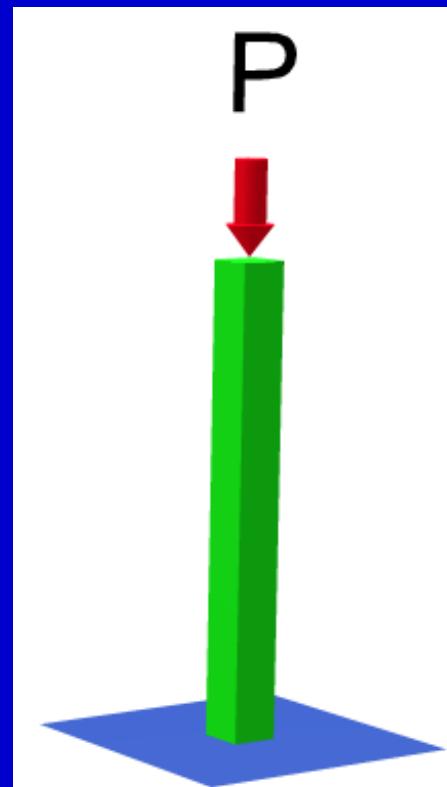
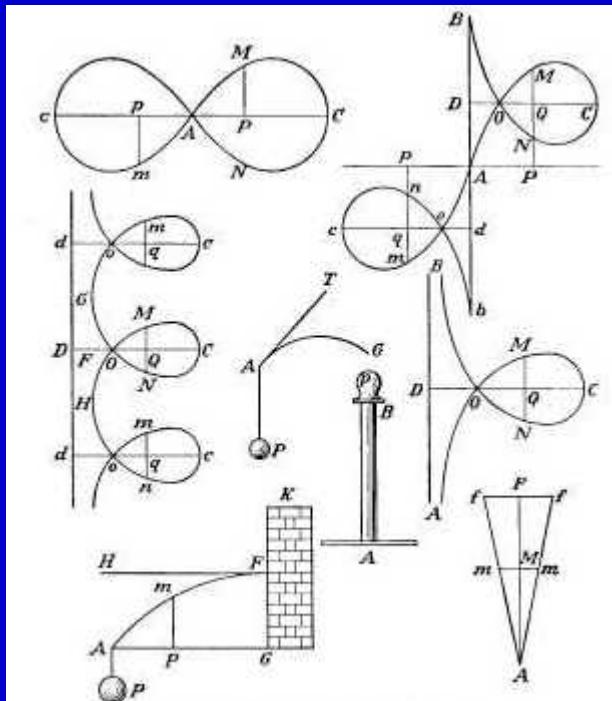
Jean Bernoulli (1667-1748):

On the curvature of extensible strings
(1691, ..., 1742)



Euler, 1727

Modulo de extensión
(modulo de Young)

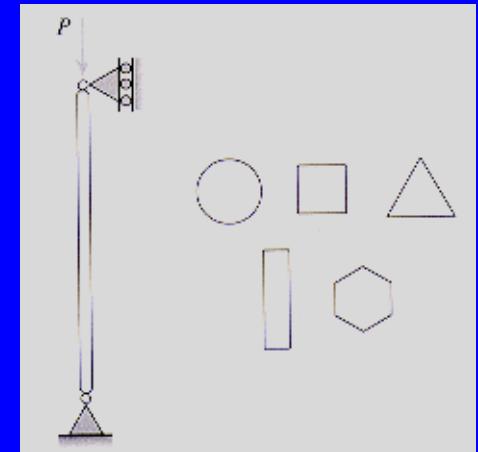
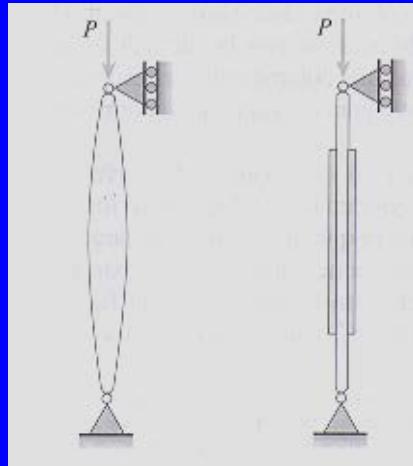


Optimización y Control

La mejor columna “elastica”: M. Vitruvio (I b. C.) *De Architecura*
25b. C., L.B. Alberti (1404-1472) 1450,

Euler 1744,
J. Lagrange (1736-1813) 1773,...

La más alta columna



L. Euler, *Leonhardi Euleri Opera Omnia*, Scientiarum Naturalium Helveticae edenda curvaverunt F. Rudio, A. Krazer, P. Stackel. Lipsiae et Berolini, Typis et in aedibus B. G. Teubneri, 1911—.

$$H_c = \left(\frac{9EI}{4\rho A} j_{-1/3}^2 \right)^{1/3}$$

$$H_c = \left(\frac{9EI}{16\pi\rho} j_{-1/3}^2 \right)^{1/4}$$

B. Keller and F. I. Niordson, “The Tallest Column”, Journal of Mathematics and Mechanics, 1986

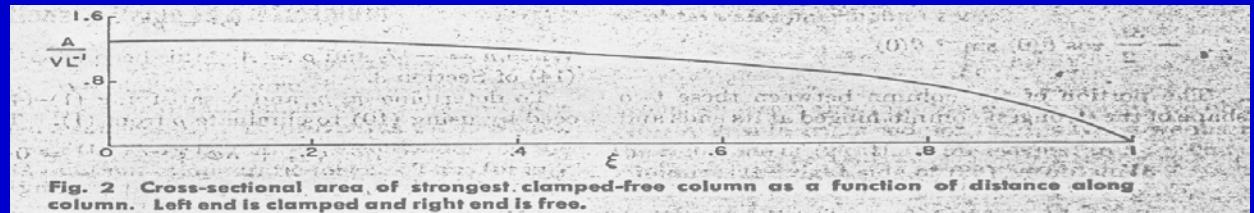
S. J. Cox and M. L. Overton, “On the optimal design of columns against buckling,” SIAM J. Math. Anal. 1992

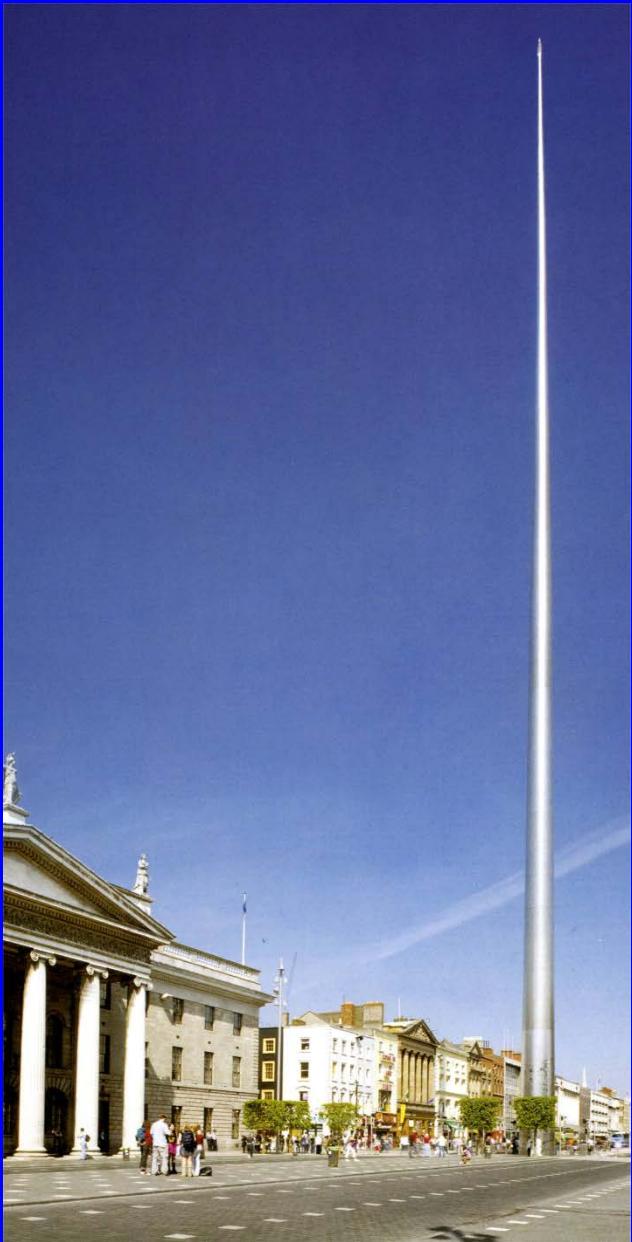
ON THE EULER BEST COLUMN: A SINGULAR NON LOCAL QUASILINEAR EQUATION WITH A BOUNDARY BLOWING UP FLUX CONDITION

J.I. DÍAZ AND M. SAUVAGEOT

$$\begin{cases} \left[\frac{A(u)}{u_x^3} \right]_x + (B(u) + \Lambda)u = 0 & \text{in } (0, 1), \\ u(0) = 0, \quad u'(1) = +\infty, \end{cases}$$

$$A(u)(x) = \left(\mu + \int_0^x u(t)^2 dt \right)^2 \quad B(u)(x) = \int_x^1 \frac{1}{u_x(t)^2} \left(\mu + \int_0^x u(t)^2 dt \right) dt.$$



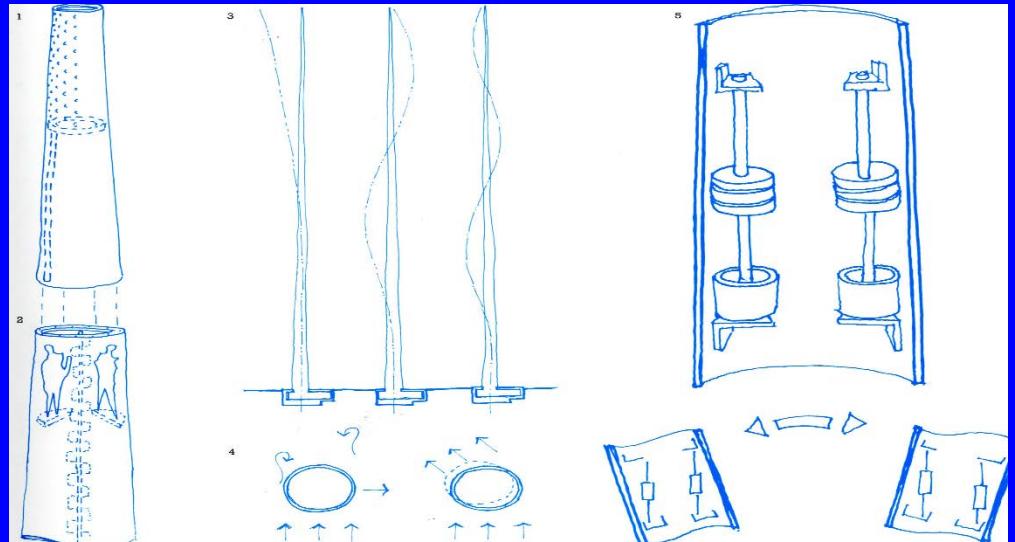


Spire. Dublin 2003

Arquitecto: Ian Ritchie Architects

Ingeniería: Arup

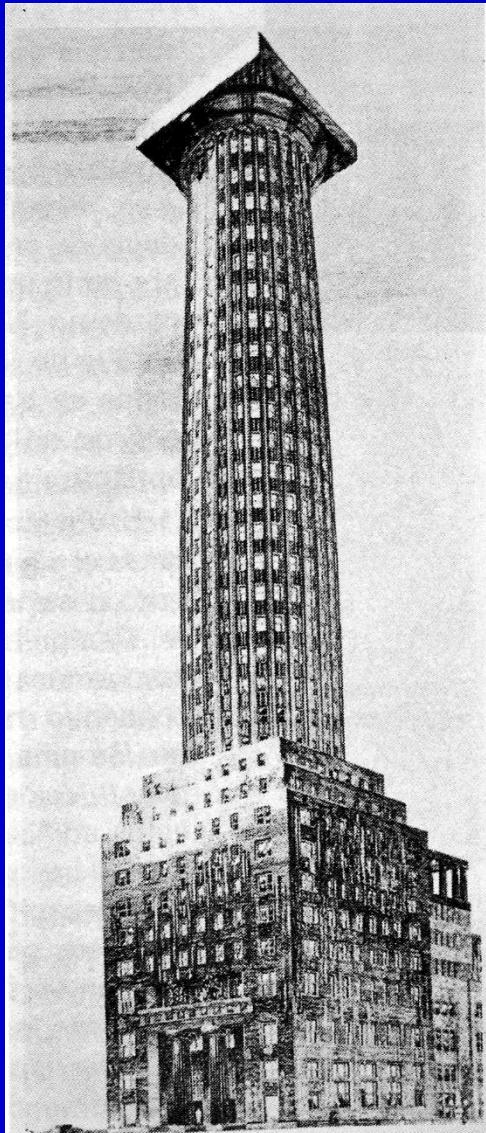
120 Metros, 3 mts de diámetro en la base



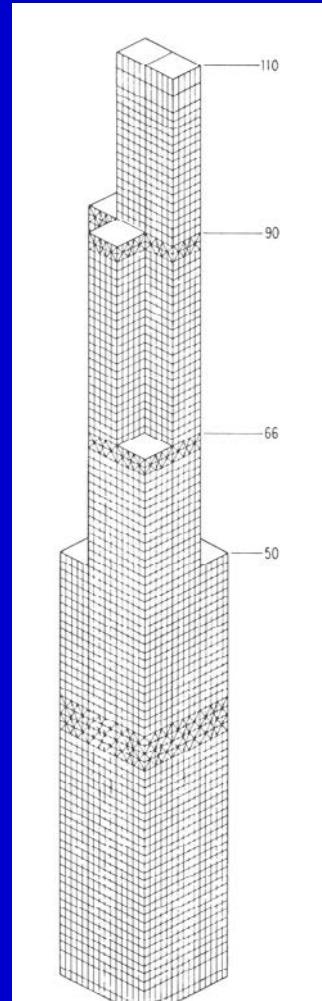
Nuevos materiales, nuevas
técnicas,...



4. Columnas reforzadas y heterogéneas: rascacielos

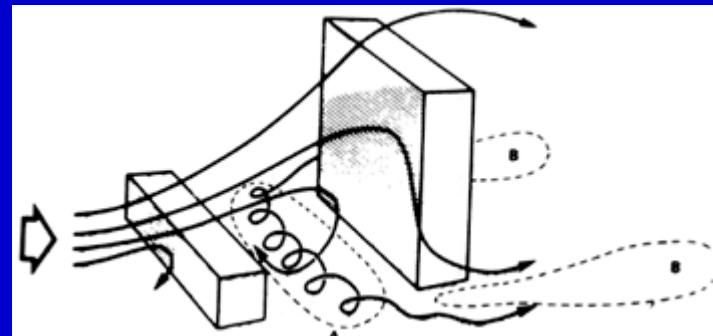
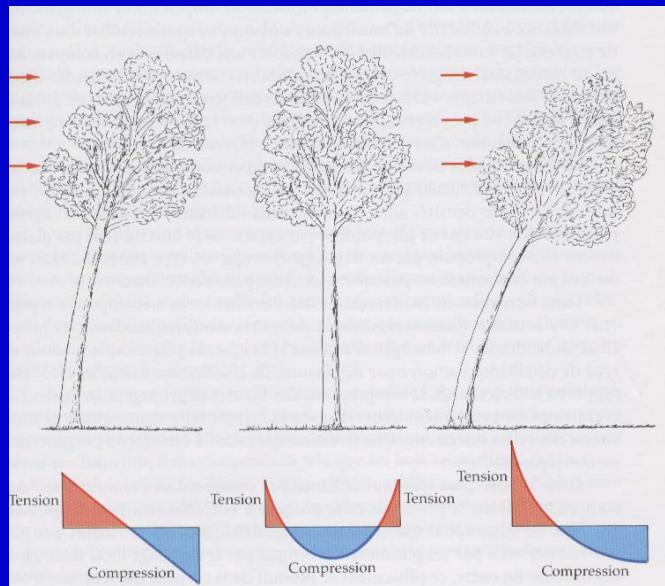
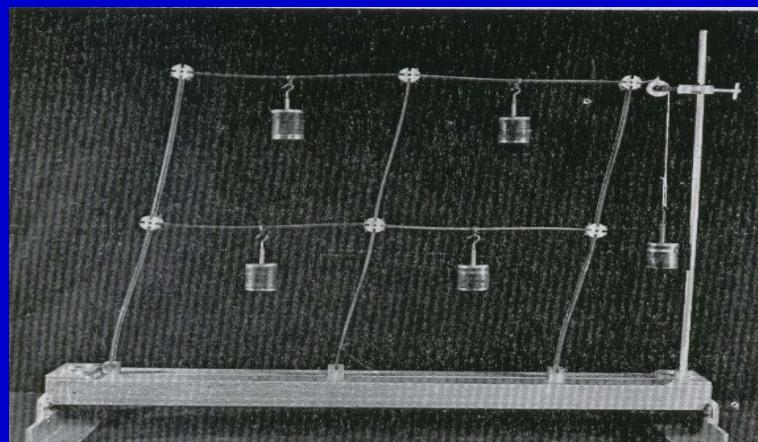
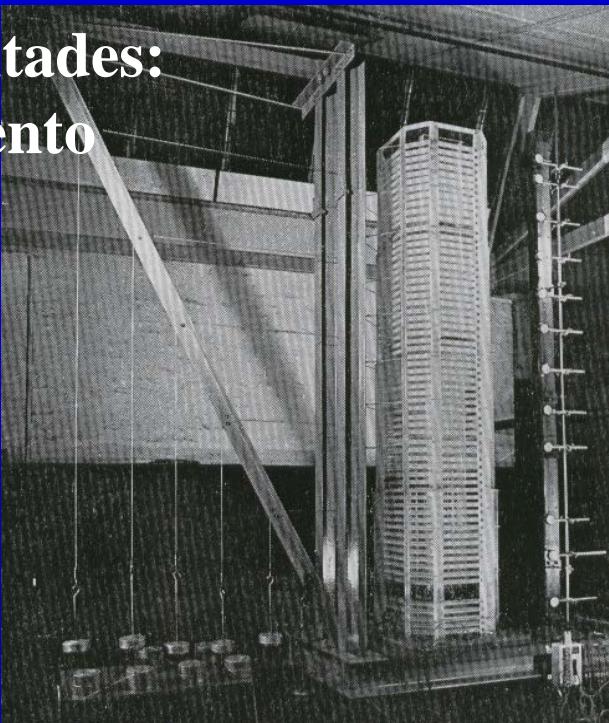


Lost, Chicago, 1920

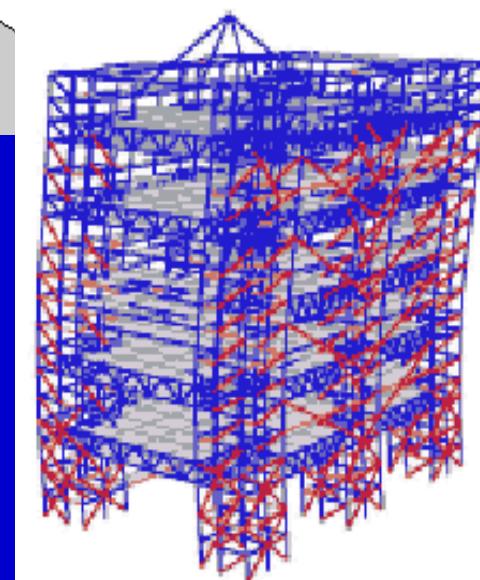
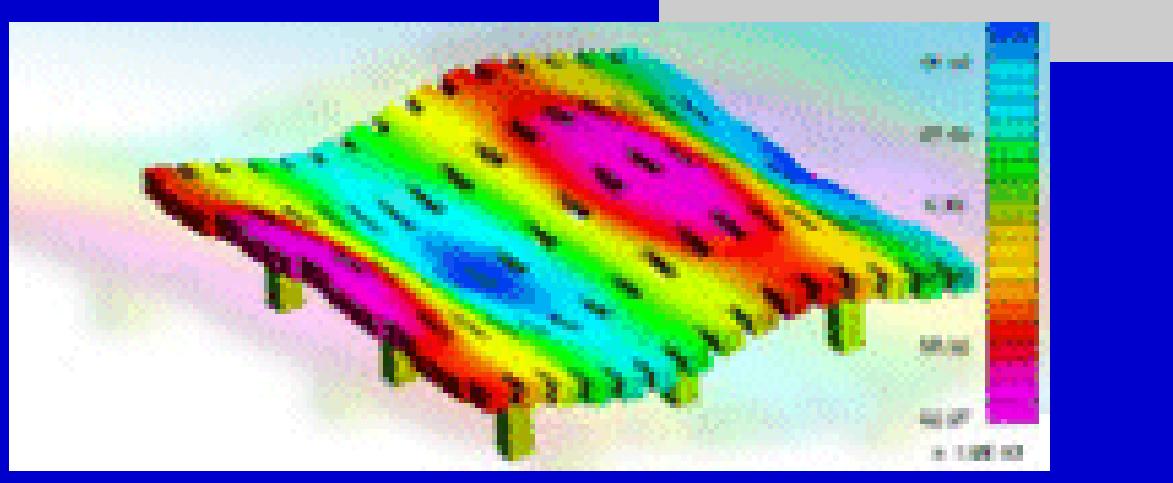
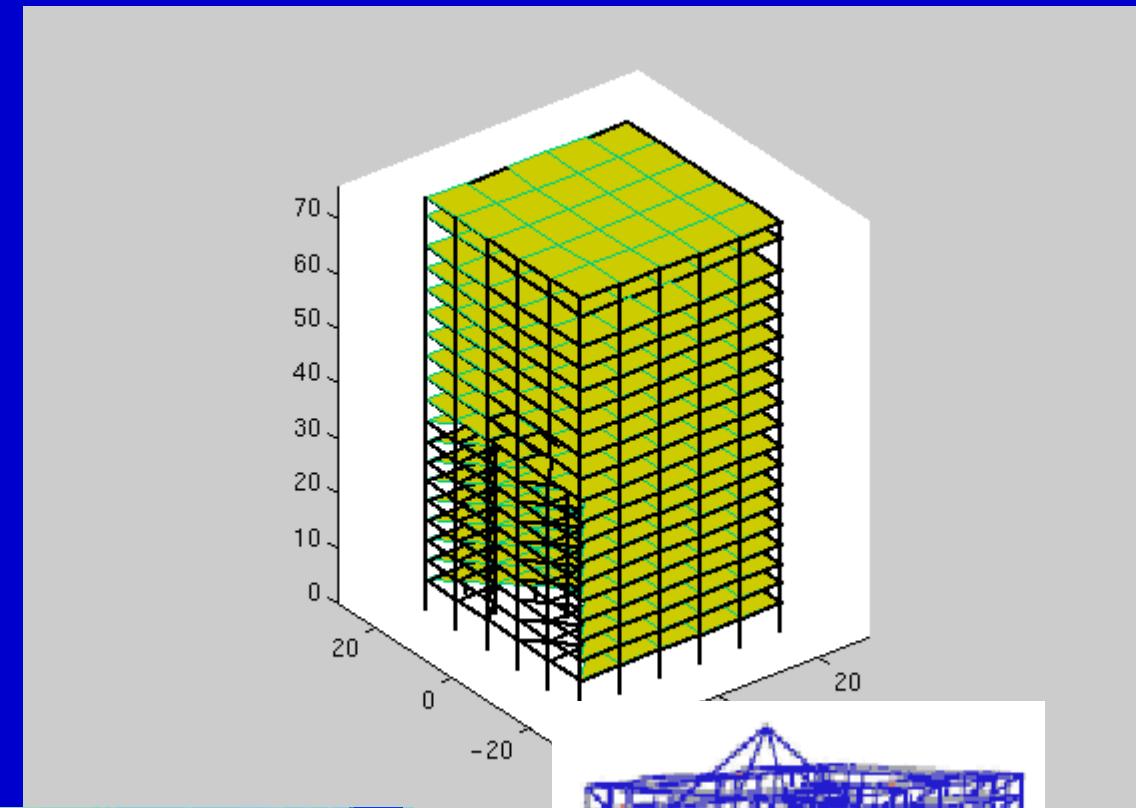
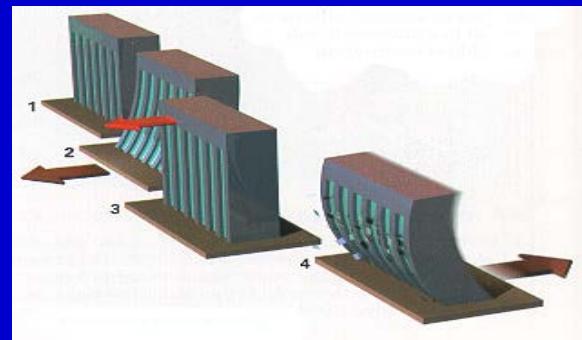


La Torre Sears de Chicago, 1974

Nuevas dificultades: estudios de viento

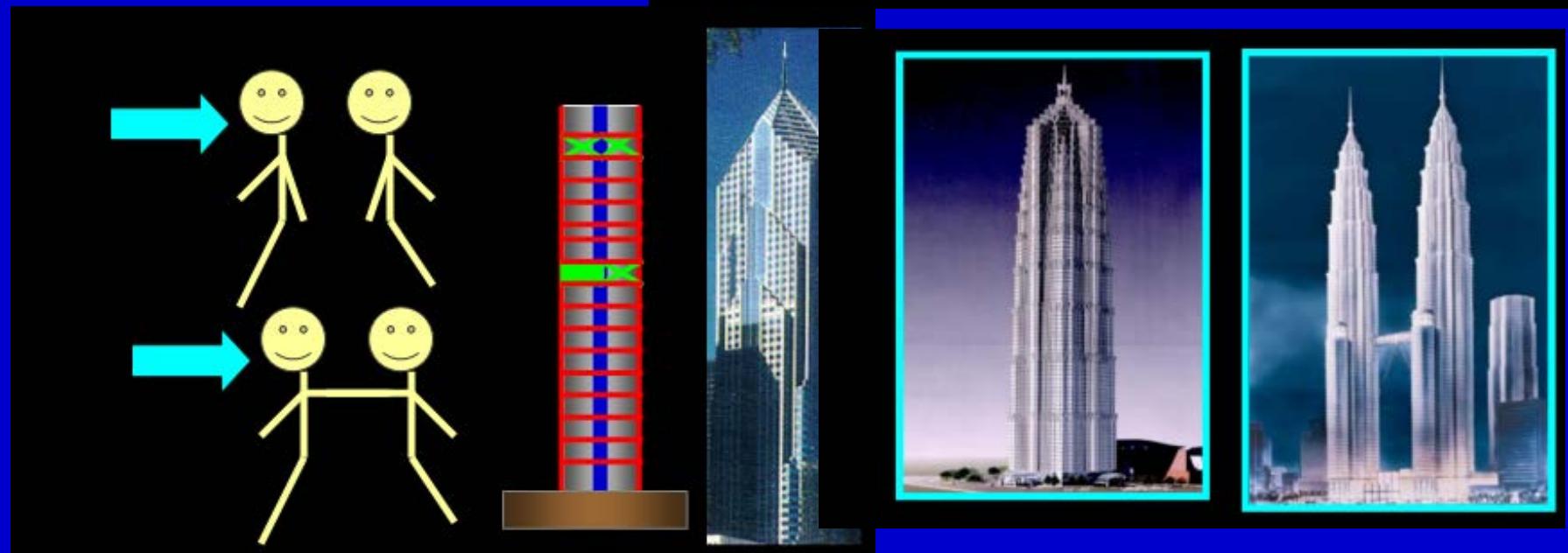
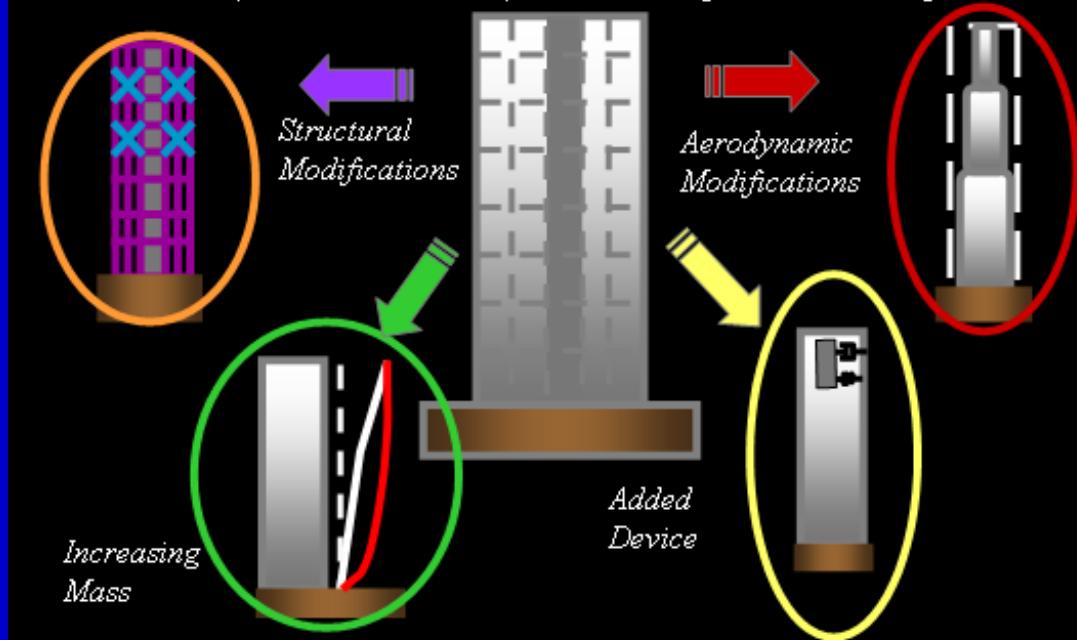


Dinámica: riesgos sísmicos

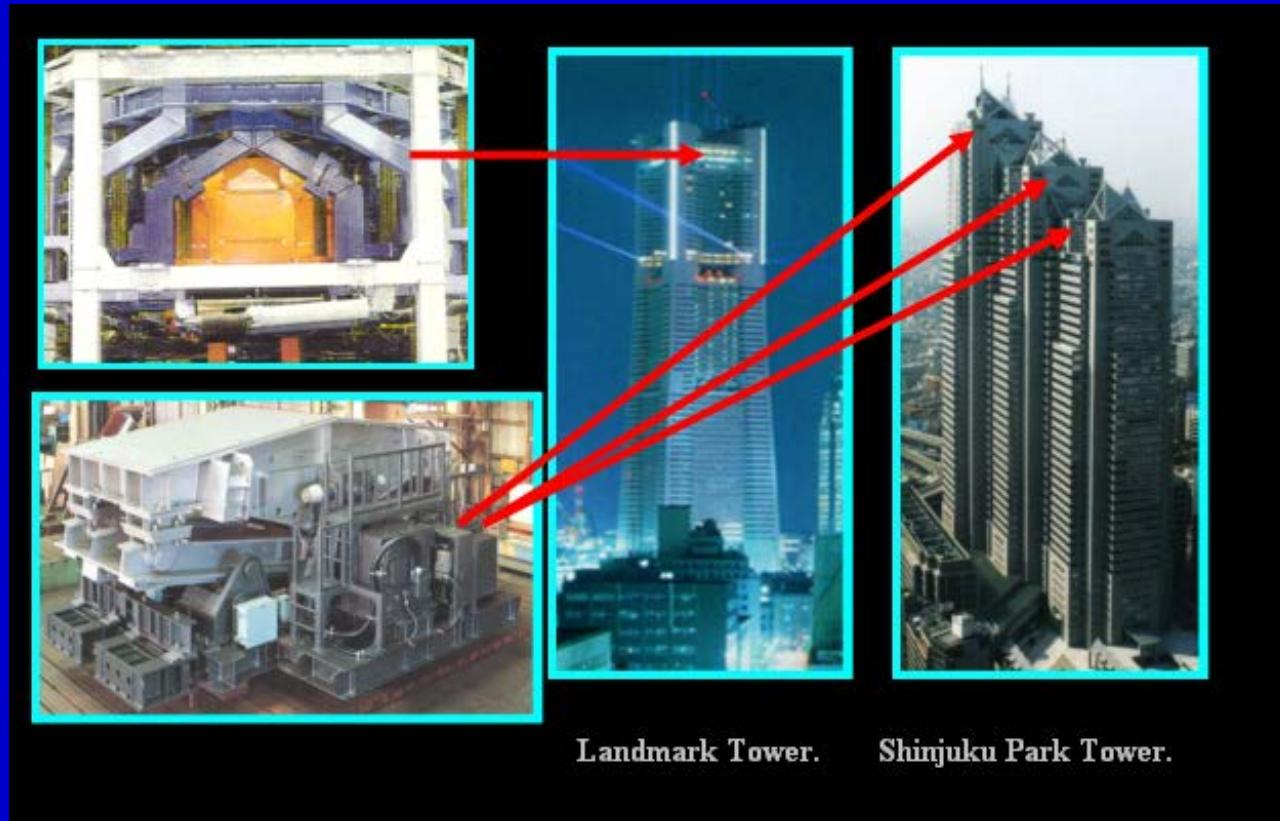
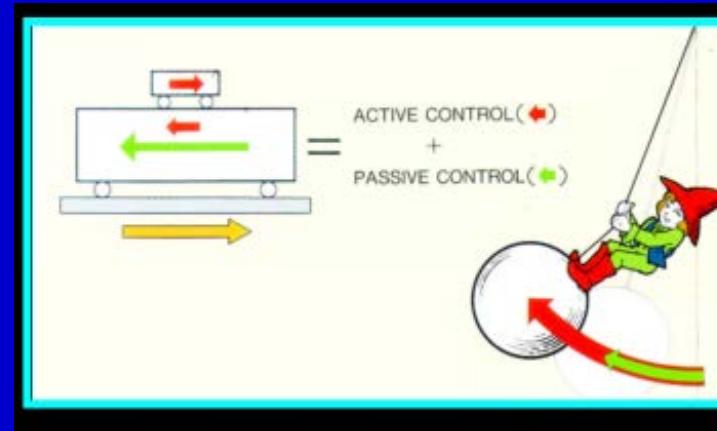


Optimización en el diseño

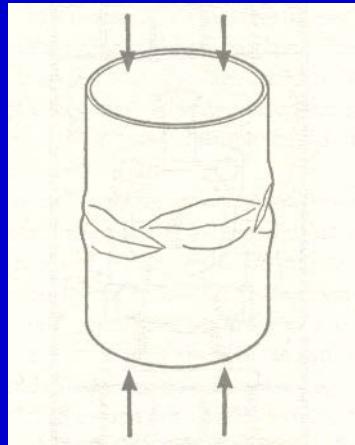
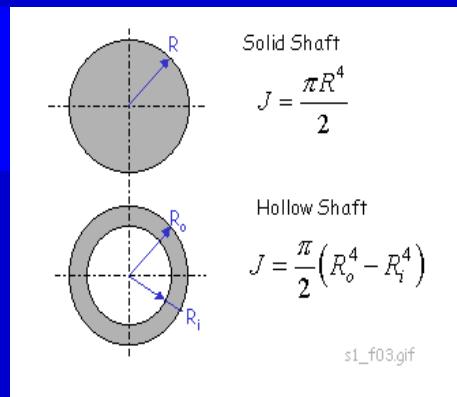
There are many solutions but they can be thought of in a few general ways.



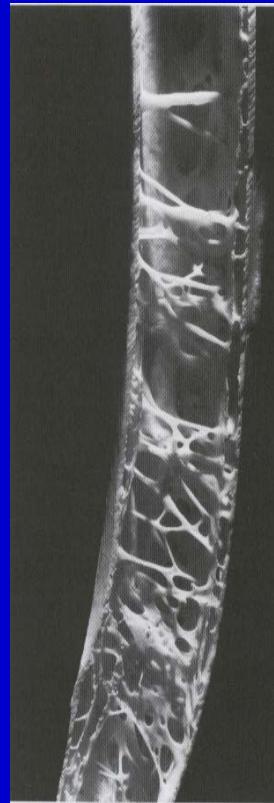
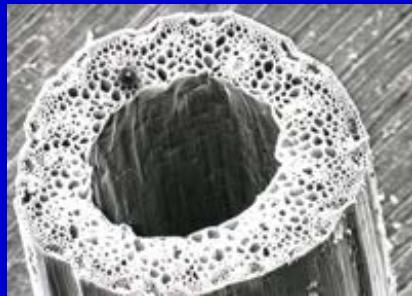
Control de movimientos sísmicos



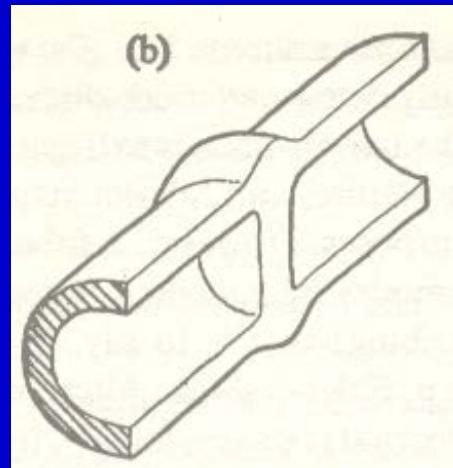
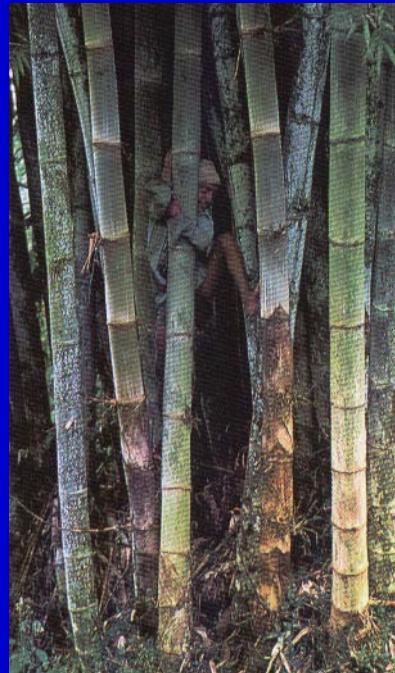
Regreso al estudio de la columna: consideraciones sobre la distribución de masa



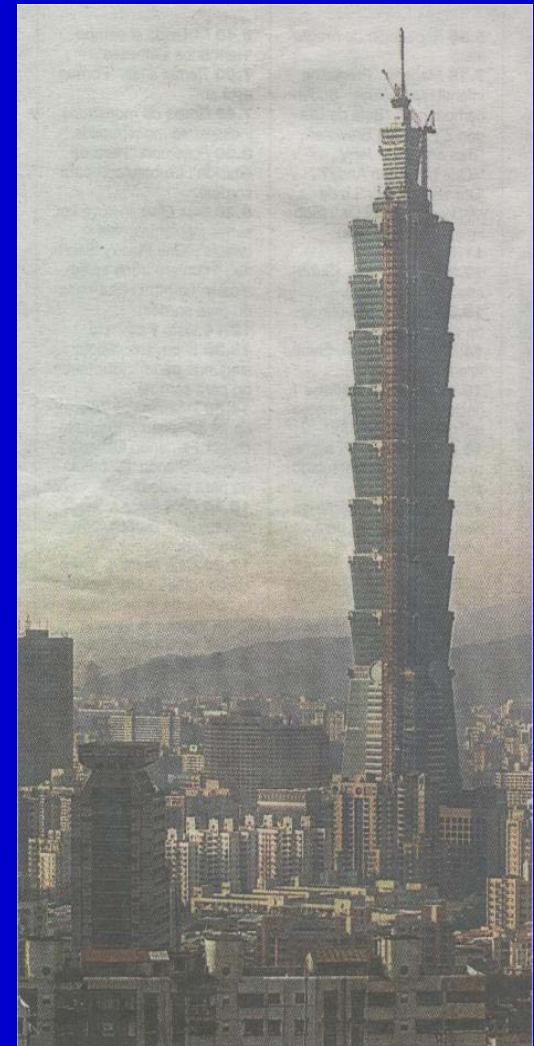
Fractura de Brazier



Una primera solución de reforzamiento



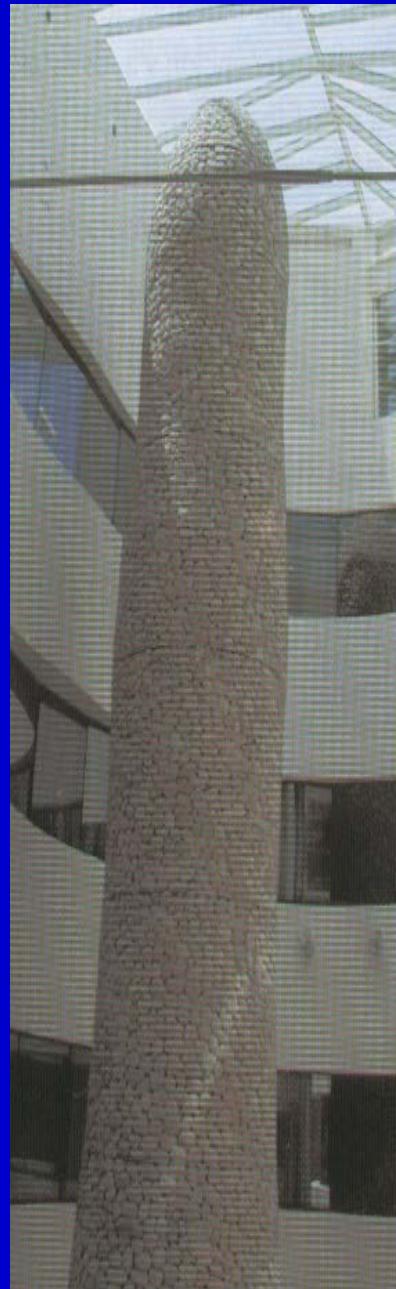
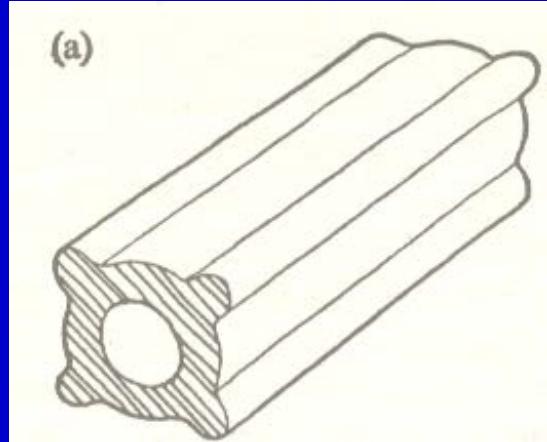
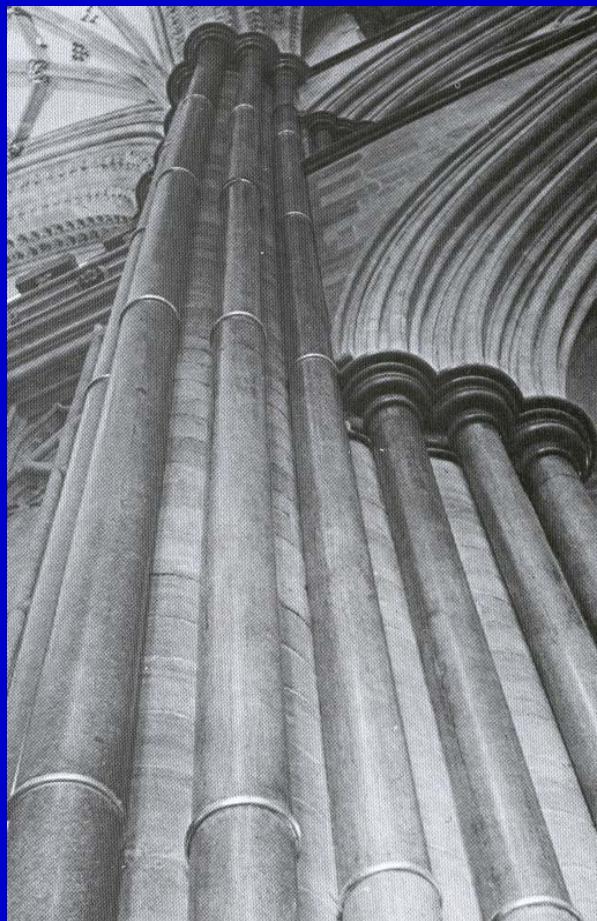
Torre Jinmao de Shanghai

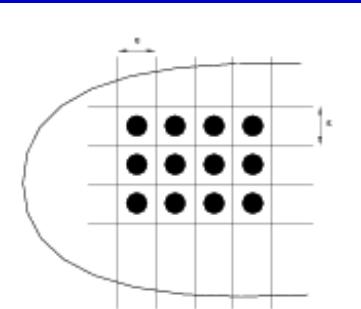
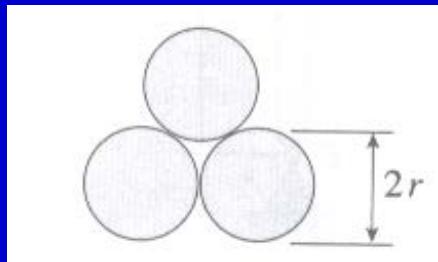


Taipei 101 C.Y. Lee & Partners
(amortiguamiento en planta 88),

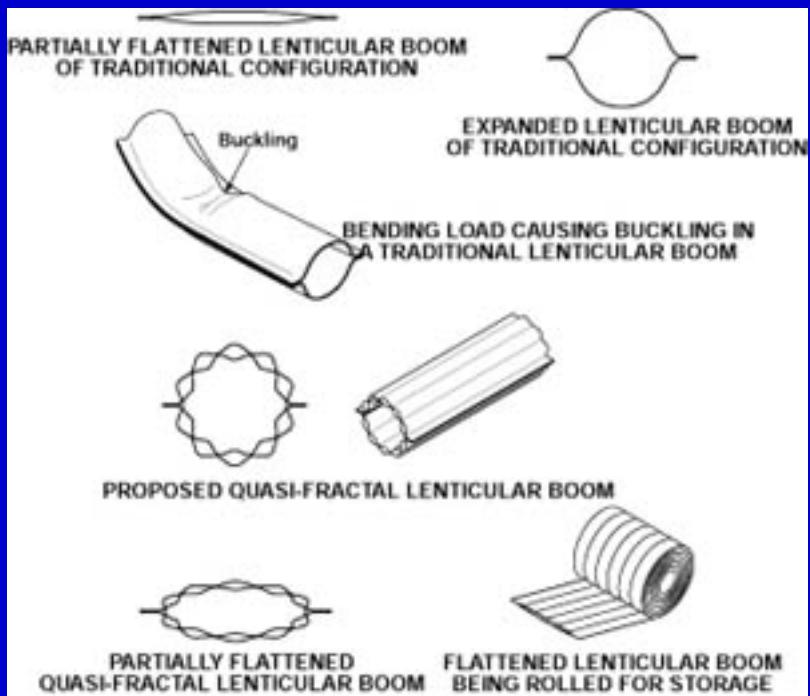
Una segunda solución de reforzamiento

Medios compuestos





Teoría de la Homogeneización

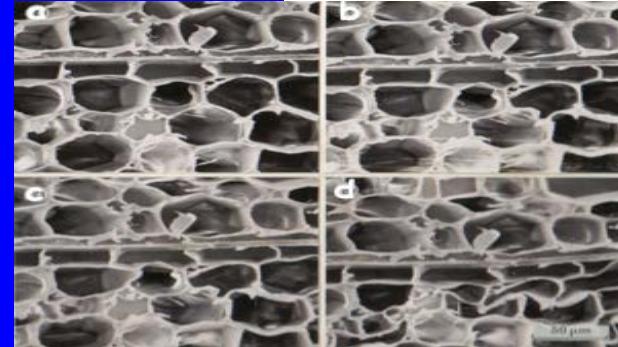
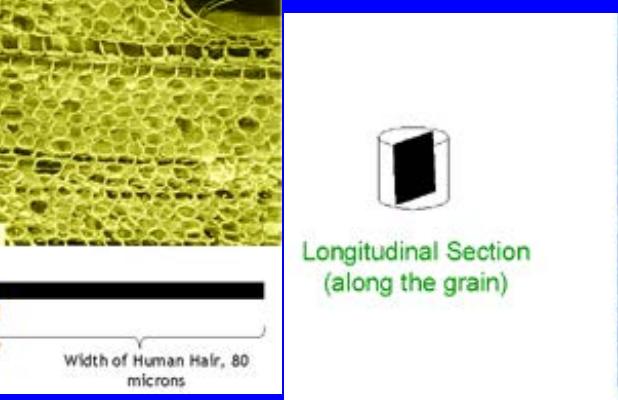
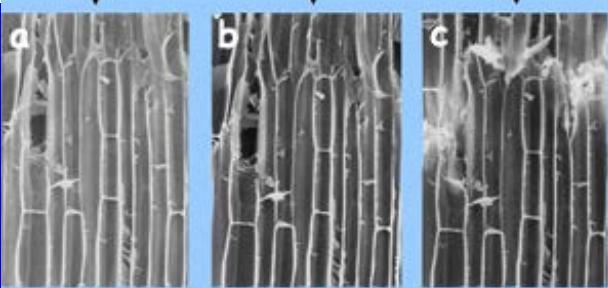
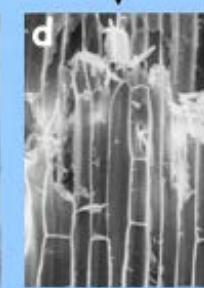
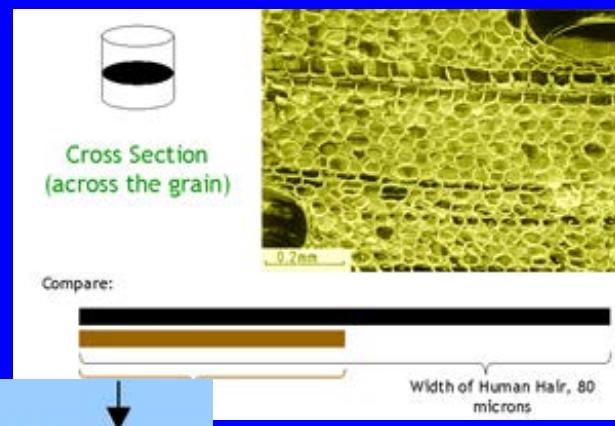
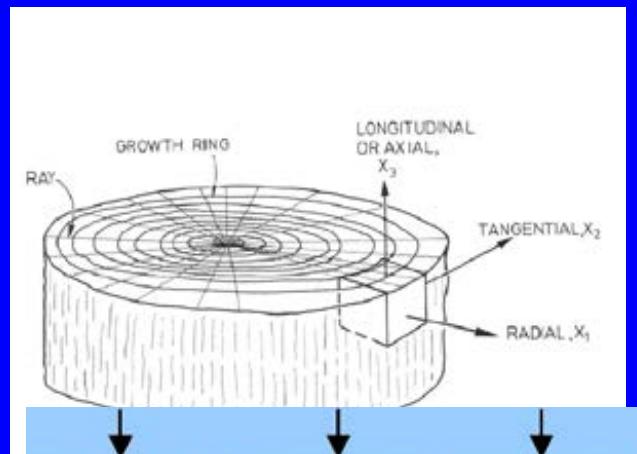


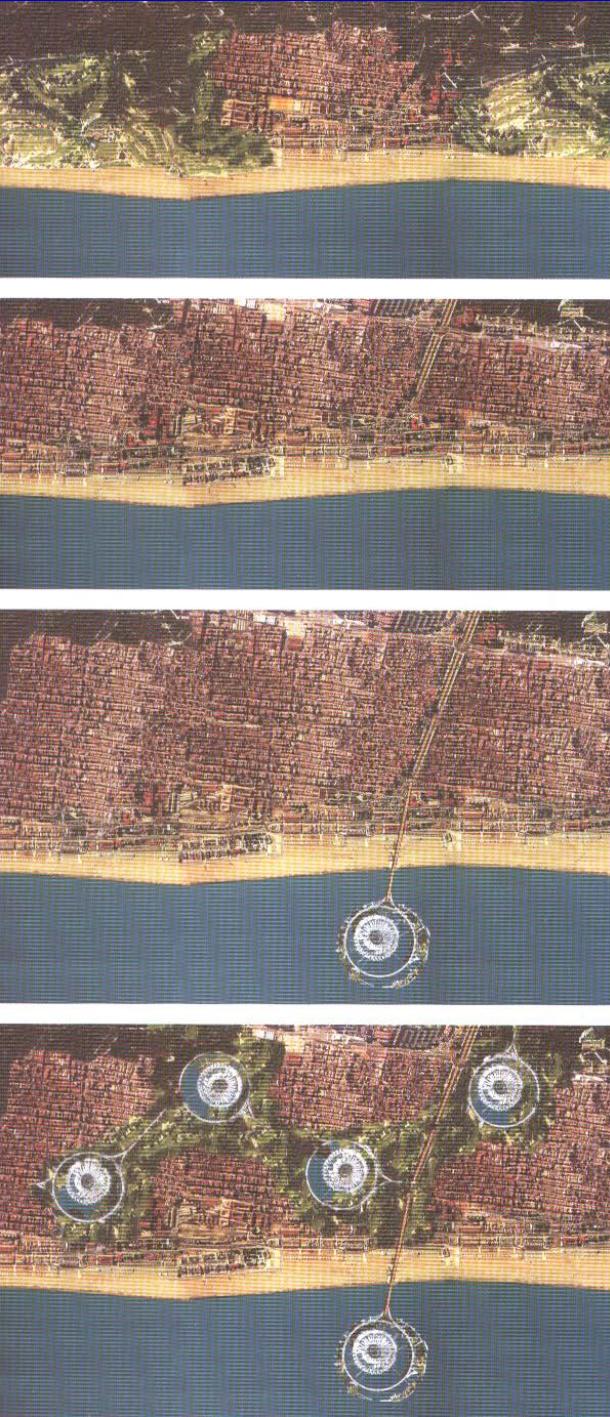
Sección Quasifractal
(NASA)



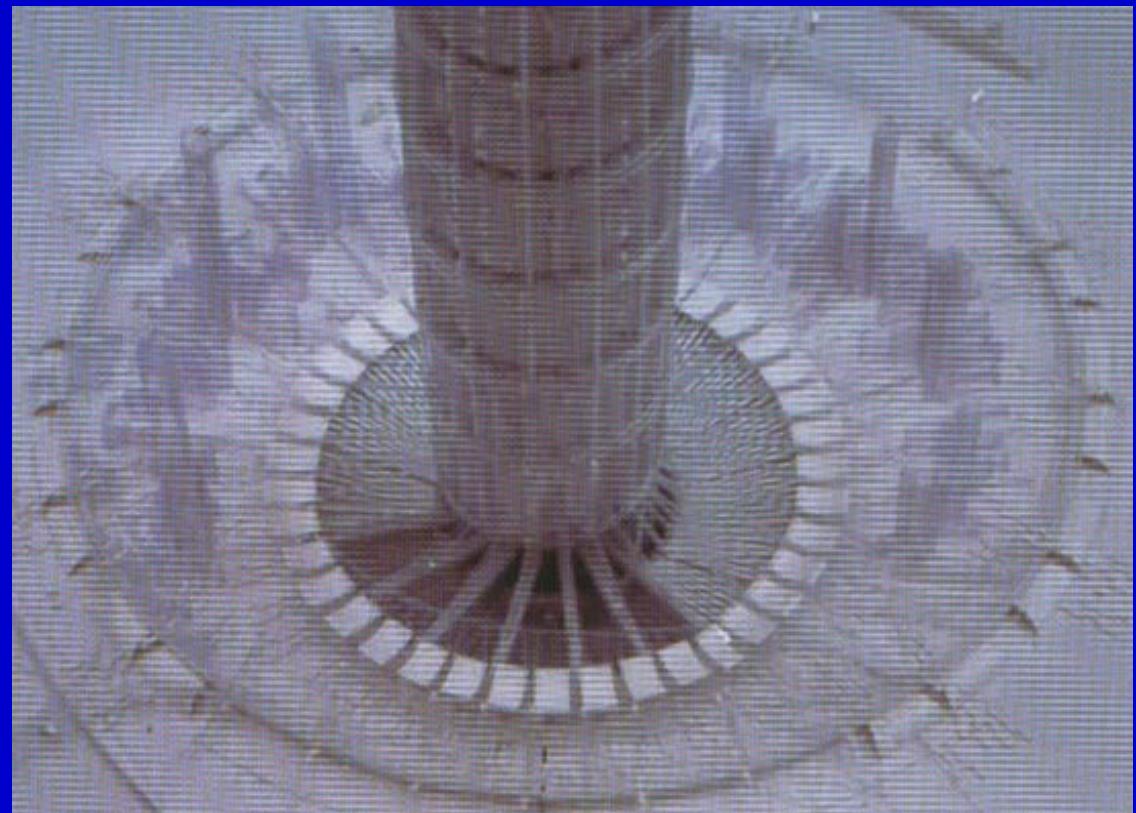
Sobre el proyecto Shangai bionic tower Cervera & Pioz

Inspiraciones naturales



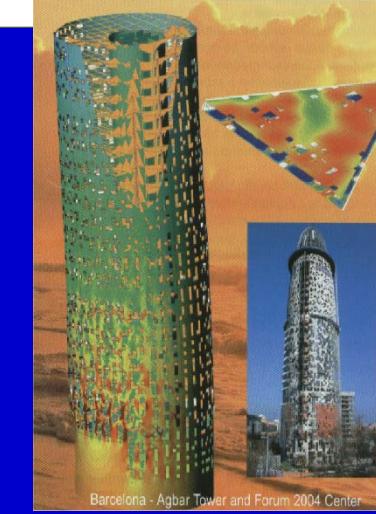
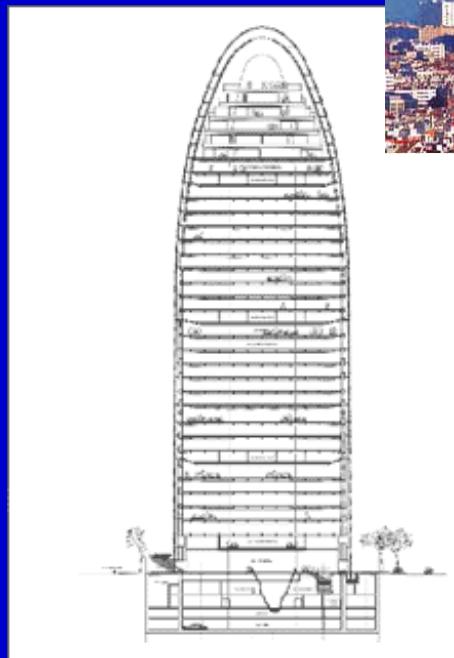


Metas ecológicas y sociales



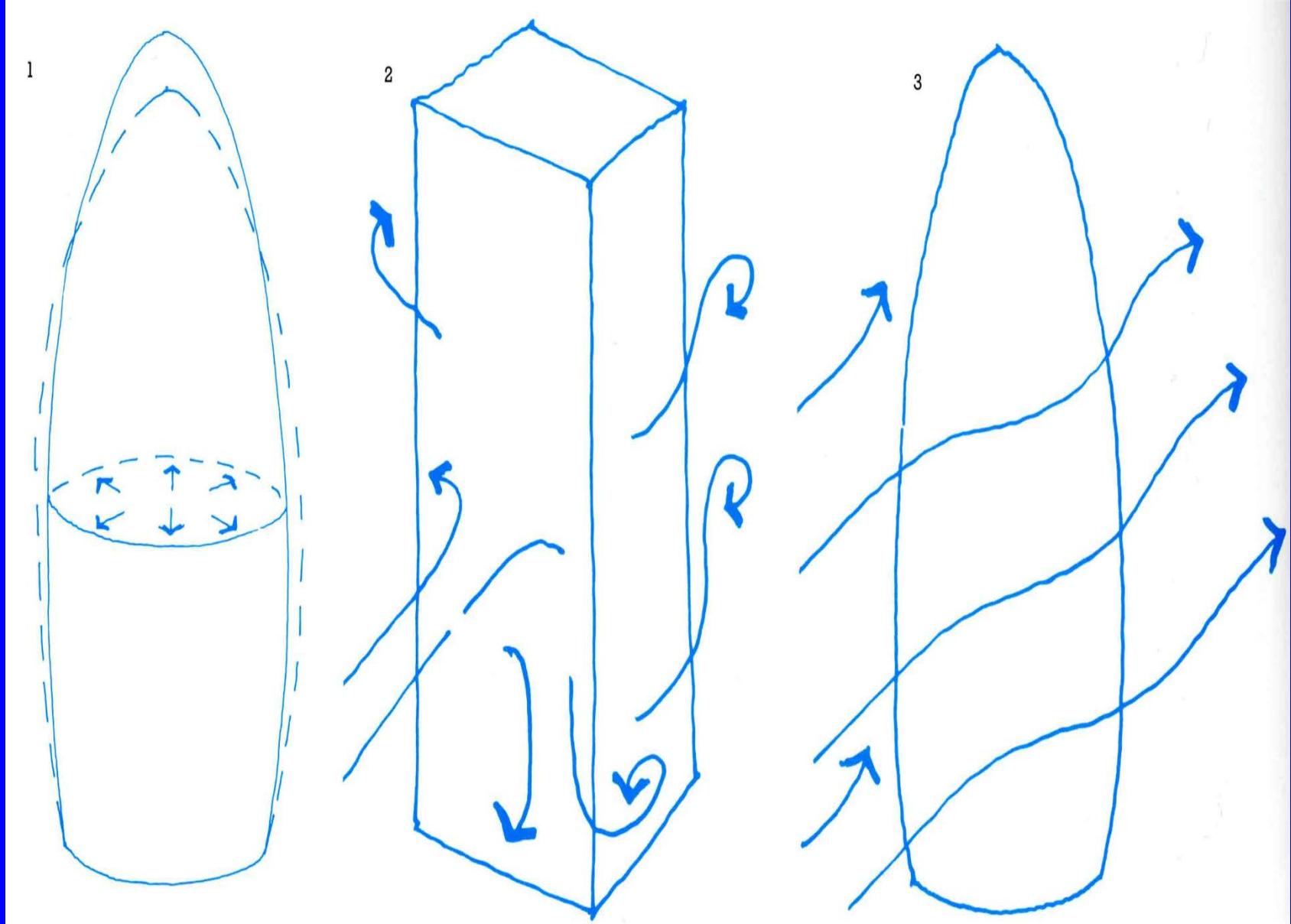
Torre Agbar, Barcelona 2004

Jean Nouvel

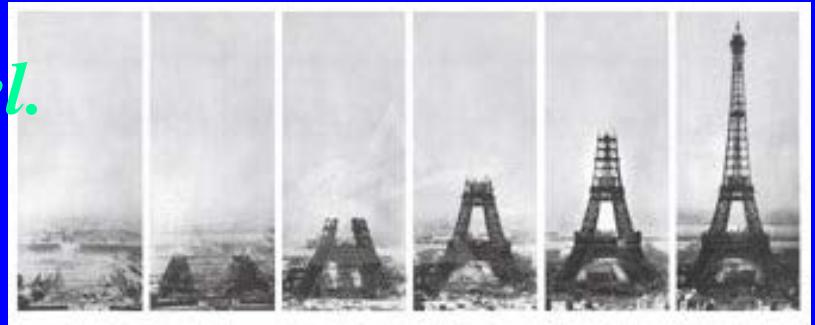


**Swiss Re Tower,
Norman Foster**





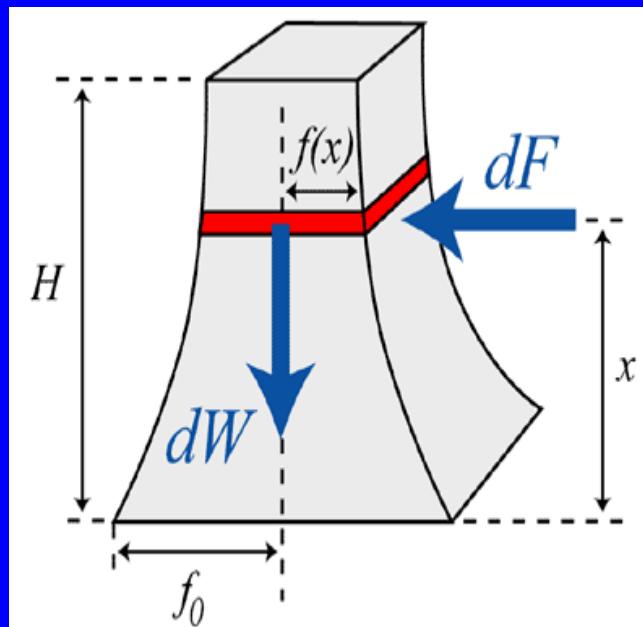
5. Matemáticas en la *Tour Eiffel*.



“Considero que la curvatura de las cuatro aristas, consecuencia de los cálculos matemáticos, dará una gran impresión de resistencia y belleza”

G. Eiffel /Le Tepms, 14 de febrero de 1887)

J. Gallant, Am. J. Phys. 2002



$$dW = 4\rho g f^2(x)dx$$

Elemento de volumen

$$dF = 2Pf(x)dx,$$

Elemento de fuerza del viento

Balance de momentos

$$\frac{1}{2} \int_x^H f^2(x)dx - \text{constant} \times (H-x) = \int_x^H xw(x)f(x)dx,$$

$$w(x) = \frac{P(x)}{4\rho(x)g f_0}$$



$$\frac{1}{2}f^2(x) - \text{constant} = xw(x)f(x).$$

$$\frac{1}{2}f^2(x) - \frac{1}{2}f_0^2 = xw(x)f(x).$$

$$f(x) = xw(x) - \sqrt{x^2 w^2(x) + f_0^2},$$

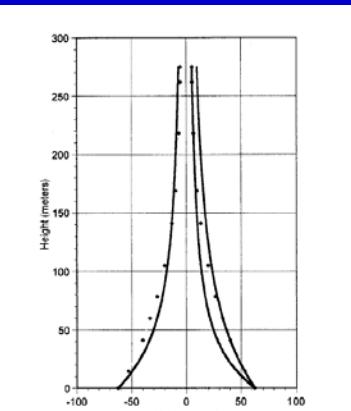


Fig. 2. The Eiffel function for three constant wind pressures ($w_0=0.700$ and $w_0=1.33$ on the right-hand side and $w_0=1.00$ on the left-hand side). The actual shape of the Tower are the data points.

$$w(x) = 0.690 - 1.53 \times 10^{-3}x + 3.96 \times 10^{-5}x^2 - 9.22 \times 10^{-8}x^3$$

$w(x)$ constante

$w(x)$ con presión variando con la altura

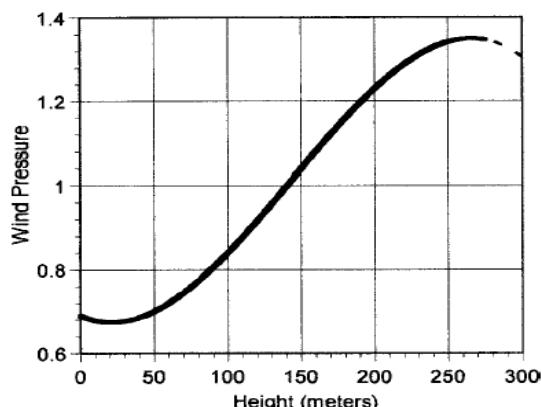


Fig. 3. The dimensionless maximum wind pressure $w(x)$ of Eq. (5) as a function of height.

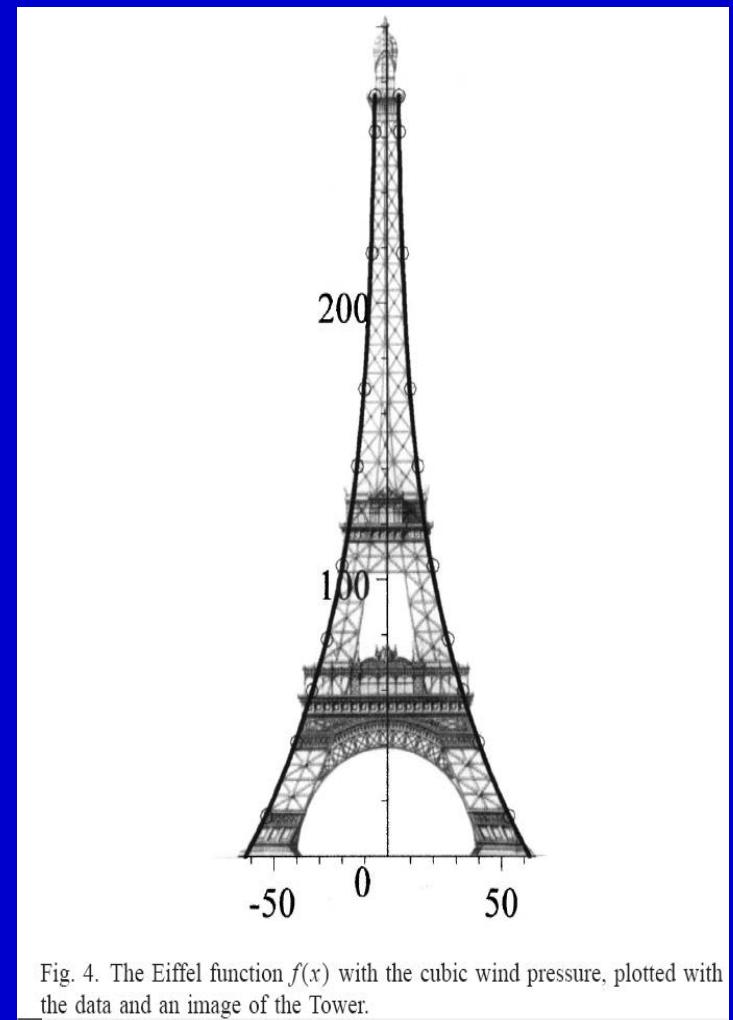
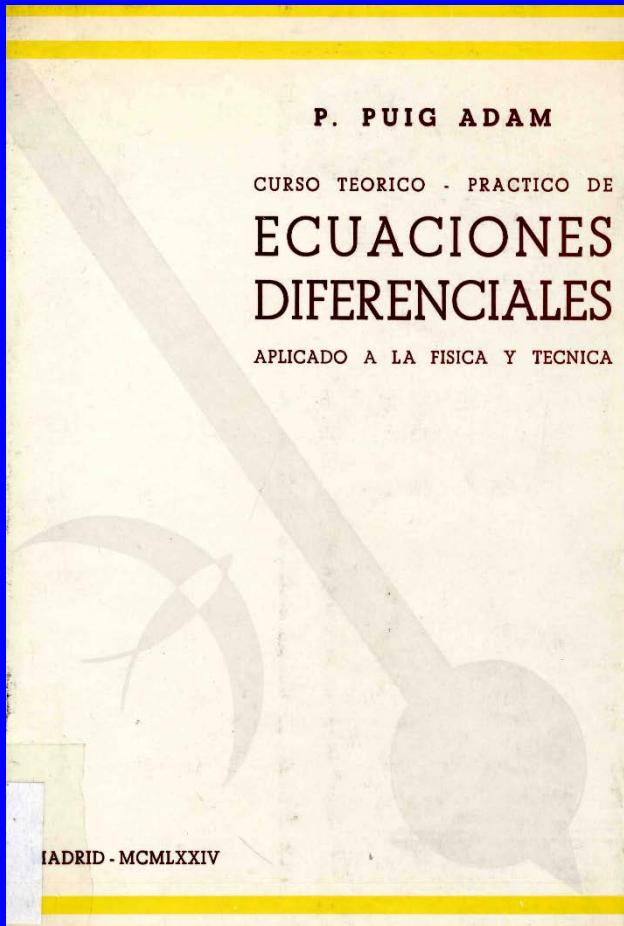


Fig. 4. The Eiffel function $f(x)$ with the cubic wind pressure, plotted with the data and an image of the Tower.

P. Weidman and I.Pinelis: Model equations for the Eiffel Tower profile, CR Mecanique AS 332 (2004)



14

LECCION 2.*—§ 4

El haz integral está, pues, formado por *espirales logarítmicas*. Las dos formas distintas en que se ha presentado la constante nos indica que de una espiral se puede pasar a otra en dos formas equivalentes

- 1.* Agregando una constante C al argumento, Dos espirales son congruentes, pudiendo obtenerse una de otra por un simple giro.
- 2.* Multiplicando el radio vector por una constante K . Dos espirales son homotéticas respecto del origen (*). Obsérvese además que cada espiral es homotética de sí misma en infinitas homotecias de razón $e^{2m\pi}$ siendo n entero, pues al aumentar el argumento en 2π el módulo queda multiplicado por $e^{2m\pi}$.

4. Perfil de igual resistencia.—Supongamos una columna sosteniendo un peso P . Este peso produce una presión en la base superior de P/S_0 , siendo S_0 la superficie de dicha base. A la distancia x de ésta la sección, de área $S(x)$, soporta el peso P más el de la porción de columna que tiene encima $\delta \int_0^x S(x) dx$ (δ peso específico), sufriendo una presión $= (P + \delta \int_0^x S(x) dx)/S(x)$

Si nos dan la forma de la columna podremos averiguar $S(x)$ (constante en el caso de columna cilíndrica) y, por tanto, la variación de la presión a lo largo de la columna. Si, reciprocamente, queremos que esta presión obedezca a una ley prefijada, por ejemplo que sea *uniforme* $= k$, se habrá de tener



$$kS(x) = P + \delta \int_0^x S(x) dx$$

y derivando

$$\delta S'(x) = \delta S(x) \quad \frac{S'(x)}{S(x)} = \frac{\delta}{k}$$

de donde

$$S = C e^{\frac{\delta}{k} x}$$

y como para $x=0$, $S_0 = C$ resulta $S = S_0 e^{\frac{\delta}{k} x}$

que da la variación de la sección para que la columna tenga igual resistencia a la compresión en toda su longitud. Si la columna es de revolución, de la variación de S deducimos la del radio, pues se tendrá

$$\pi R^2 = \pi R_0^2 e^{\frac{\delta}{k} x} \quad \text{de donde} \quad R = R_0 e^{\frac{\delta}{k} x}$$

Más general, si queremos que la presión varie a lo largo de la columna según una ley prefijada $f(x)$ (con $f(0)=P : S_0$) se tendrá

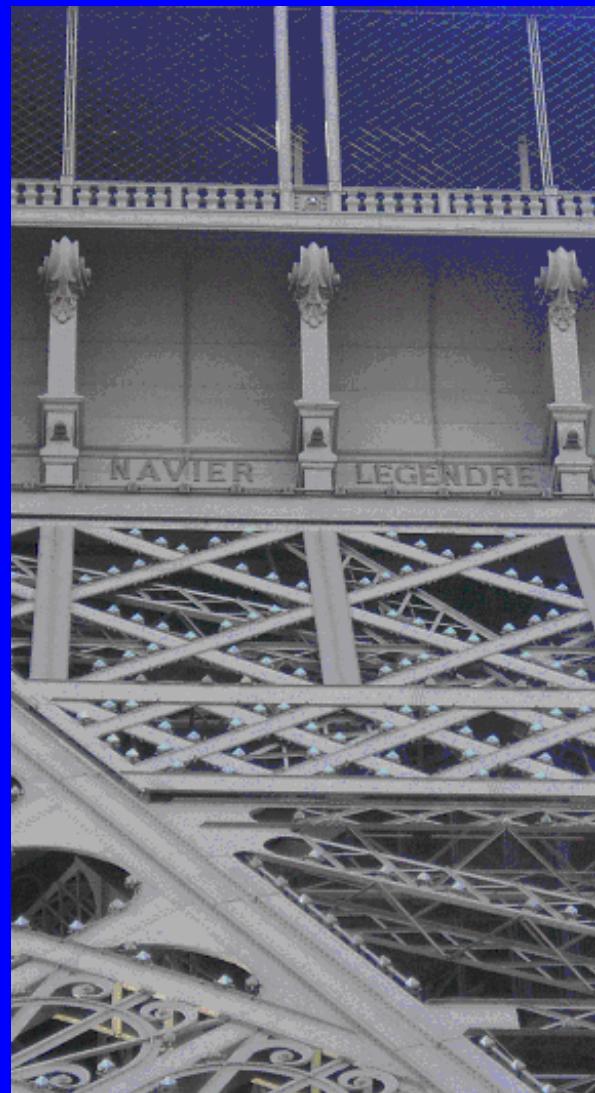
$$f(x) S(x) = P + \delta \int_0^x S(x) dx$$

Derivando, $fS' + f'S = \delta S$, de donde

$$\frac{S'}{S} = \frac{\delta}{f(x)} - \frac{f'(x)}{f(x)}$$

(*) No debe causar extrañeza al principiante el hecho de que dos curvas se puedan relacionar a un tiempo por dos transformaciones distintas, pues todo depende de la elección de puntos homólogos.

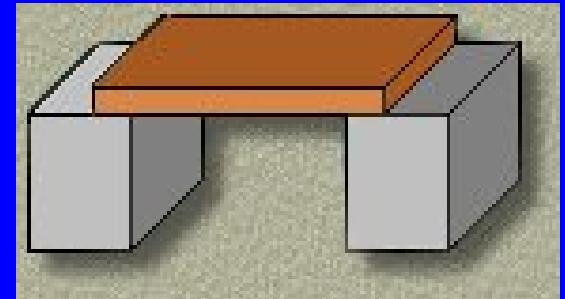




6. Puentes: resonancia.

Some bridge structures

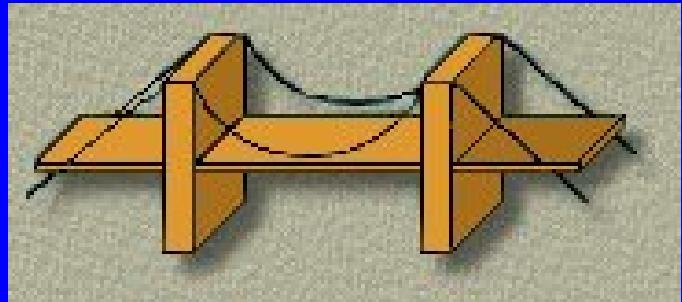
Beam bridge – it is a very strong structure over short distances, but it is not strong enough for lengthy ones.



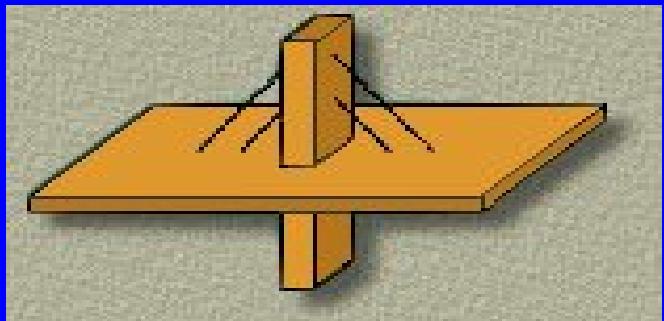
Arch bridge – the compression pushes the weight away from the arch and against the side walls and the material of the arch itself.



Suspension bridge – the structure hangs from cables anchored at each of its ends and also the towers support it. Suspension bridges are good for long distances.



Cablestay bridge – the structure is supported by cables from a central tower.



What are the main forces acting on a bridge structure?

In all bridge structures, the designers try to balance the various forces which act on the bridge structure so that they are even. If they are not, the structure can collapse.

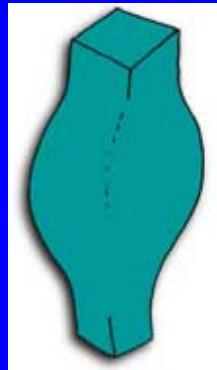
Load – is the amount of push or pull.

- ° load from its own weight
- ° load from things crossing it
- ° load from winds pushing against it

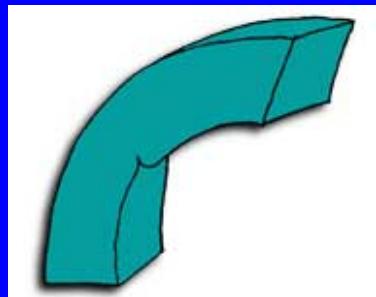
Gravity – pull downward the structure.



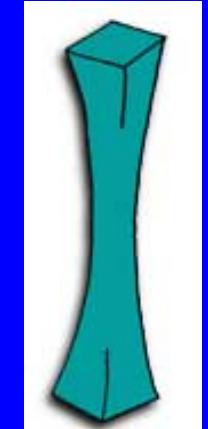
Compression – the effect of pushing forces on parts of a structure.



Bending – happens when a load is imposed on the side of a structure.



Torsion – happens when the ends of a part of a structure are turned in opposite directions.



The Tacoma Narrows suspension bridge

Bridge's designer

Leon Moisseiff (consulting engineer for the Golden Gate, Bronx-Whitestone, San Francisco-Oakland Bay bridges)

Characteristics

- was the third longest bridge in the world,
- was much narrower, lighter and more flexible than any other bridge of its time.

Nickname

« **Galloping Gertie** » - the unique characteristics of the bridge coupled with its low dampening ability caused large vertical oscillations even at the most moderate of winds.



November 7, 1940



Oscillations – while the wind was not extraordinary, the bridge was undulating noticeably (wind speed: 42mph, amplitude: 1m, 38 oscillations/min).



Torsional movement – when the twisting motion was at its maximum, the elevation of the sidewalk at the right was 9m higher than the sidewalk at the left.





Collapse – occurred at about 11:00 a.m.



The image shows the fallen part of the roadbed (approx. 15m) after the failure.



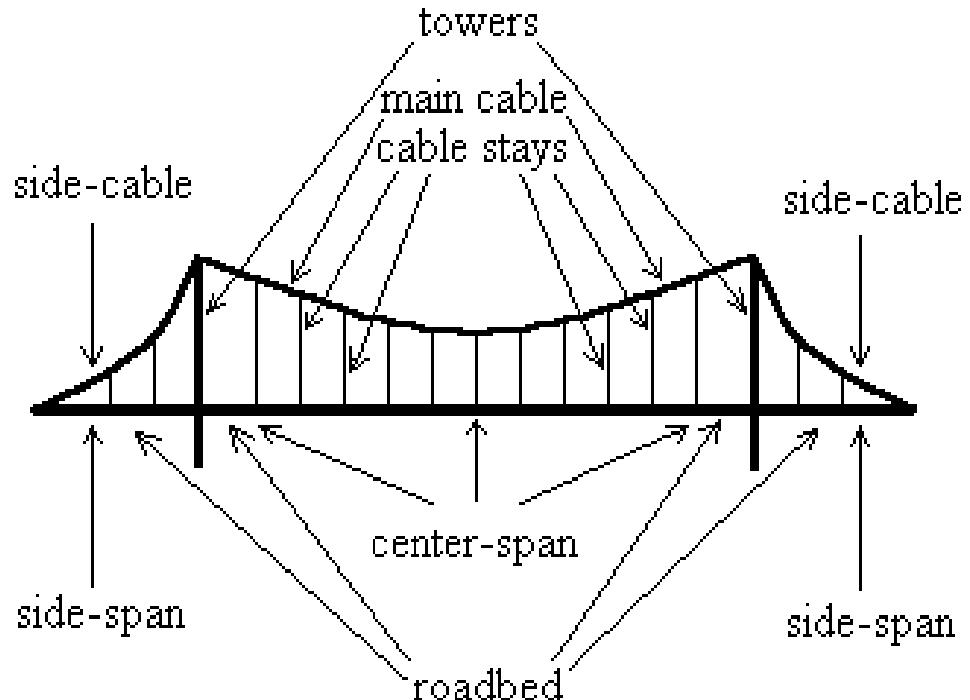
Possible causes of the failure (FWA –Federal Works Agency)

- Vertical oscillations were caused by the force of the wind and caused no structural damage.
- The torsional motion caused high stress throughout the bridge, which lead to the failure of the suspended structure.
- The Tacoma's exceptional flexibility coupled with its inability to absorb dynamic forces made the wild oscillations which destroyed it possible (The FWA believed that wind induced oscillations approached to the natural frequencies of the structure causing resonance – the process by which the frequency on an object matches its natural frequency causing a dramatic increase in amplitude – and this explains why the relatively low speed wind (42 mph) caused the spectacular oscillations and the destruction of the brigde).



Some mathematical models:

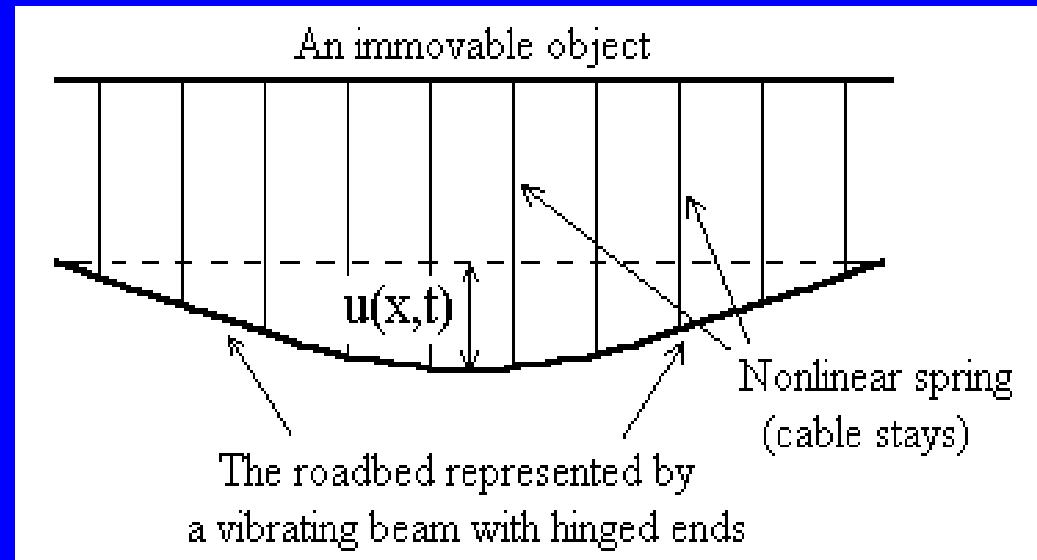
The main components
in a model of a
one-dimensional
suspension bridge



- take into account only the length of the bridge;
- neglect the influence of the towers and side parts.

The first class of models – a single beam

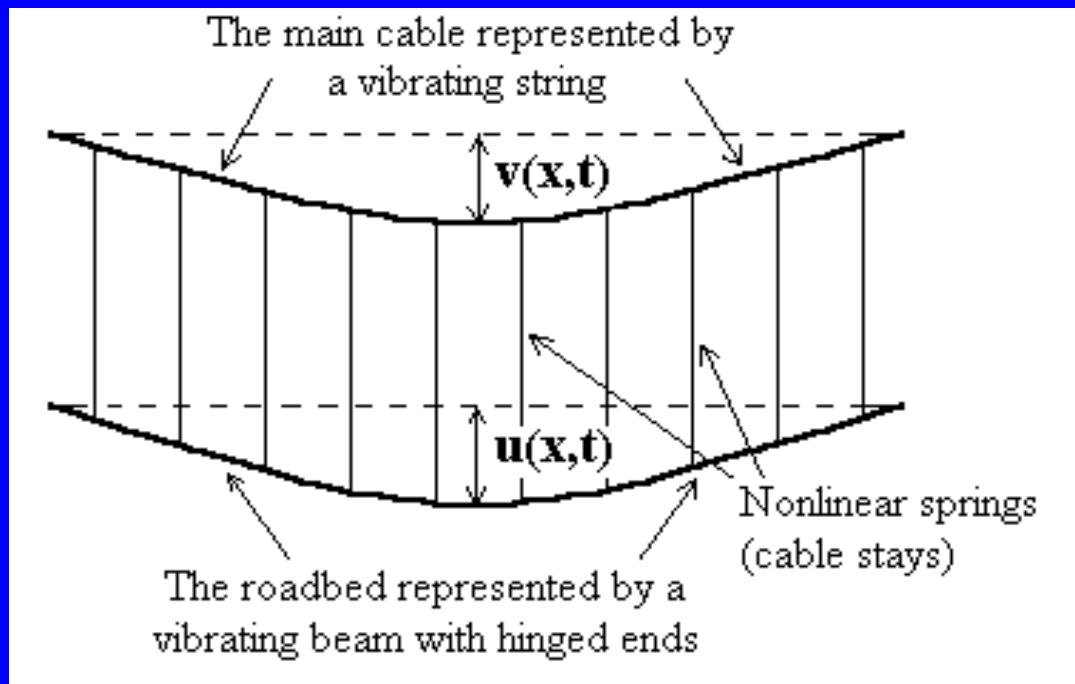
The suspension bridge is seen as a beam of length L , with hinged ends, subject to three separate forces:



- the stays, holding it up as nonlinear springs with spring constant k ,
- the weight per unit length of the bridge, $W(x)$ pushing it down,
- the external forcing term, which it is assumed to be periodic.

A second class of models – beam & vibrating string

The suspension bridge is seen as a beam of length L , like in the previous model.



The main cable is treated like a vibrating string, coupled with the beam of the roadbed by nonlinear cable stays.

L. Euler: Solución general de problemas de vibración lineales (1739).

De novo genere oscillationum (1739)

$$M\ddot{x} + Kx = F \sin \omega_d t \quad \omega = \sqrt{K/M}, \quad n \equiv \frac{\omega_d}{\omega}$$

Resonancia

Carta a John Bernoulli (1739) y *De integratione equationum differentialium altiorum graduum* (1743)

$$\sum_{k=0}^n A_k \frac{d^k y}{dx^k} = 0, \quad A_k = \text{const.}$$

$$p = \alpha \pm i\beta$$

$$\sum_{k=0}^n A_k p^k = 0.$$

$$y = C e^{px}, \quad e^{\alpha x} (C \cos \beta x + D \sin \beta x)$$

No considera el caso de raíces múltiples

Da como ejemplo

$$K^4 \frac{d^4 y}{dx^4} = y,$$

$$y = C e^{\frac{x}{K}} + D e^{-\frac{x}{K}} + E \sin \frac{x}{K} + F \cos \frac{x}{K}$$

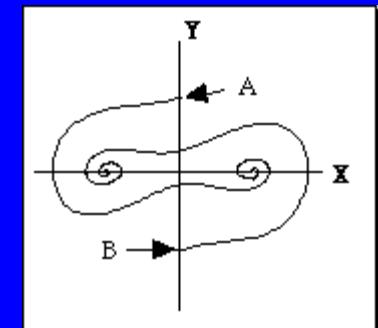
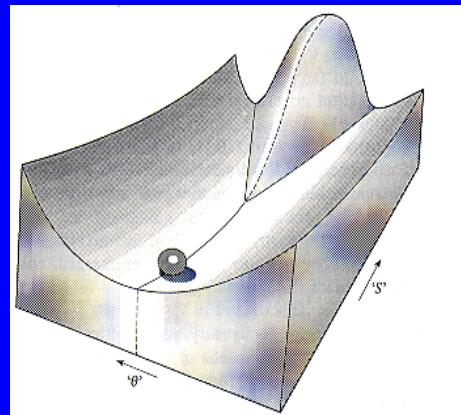
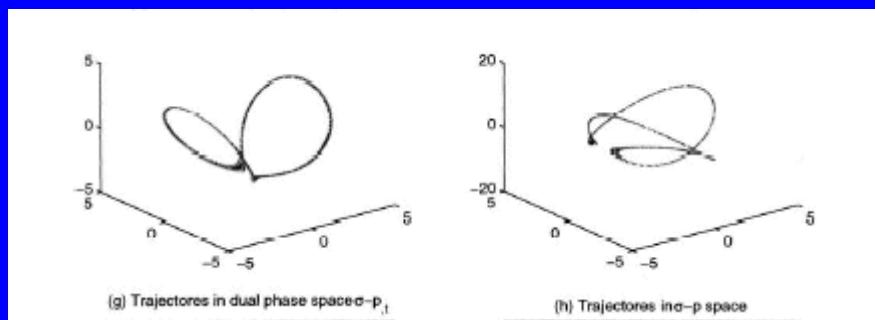
Primera noción de **solución general de una EDO lineal**, objeciones de John Bernoulli con el caso de “raíces imposibles” (numeros complejos)



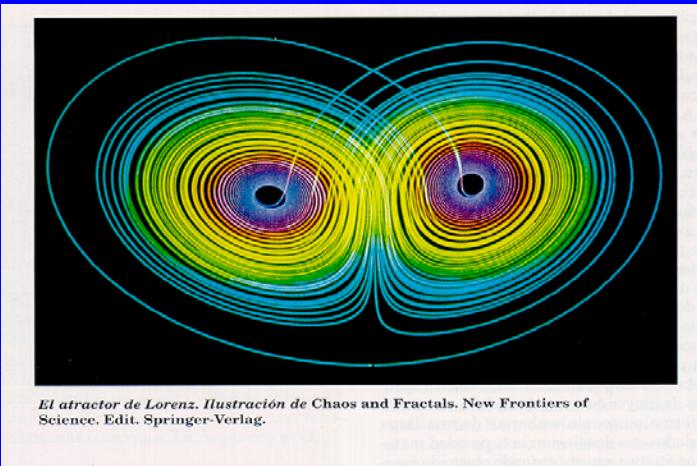
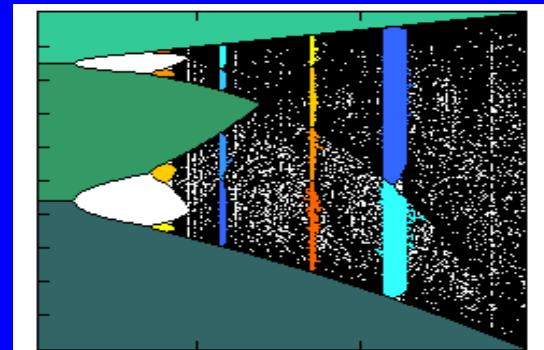
$$\frac{d^2x}{dt^2} + b \frac{dx}{dt} - x + x^3 = F \cos(\omega t)$$

Ecuación de Duffing

Con peso: dos equilibrios

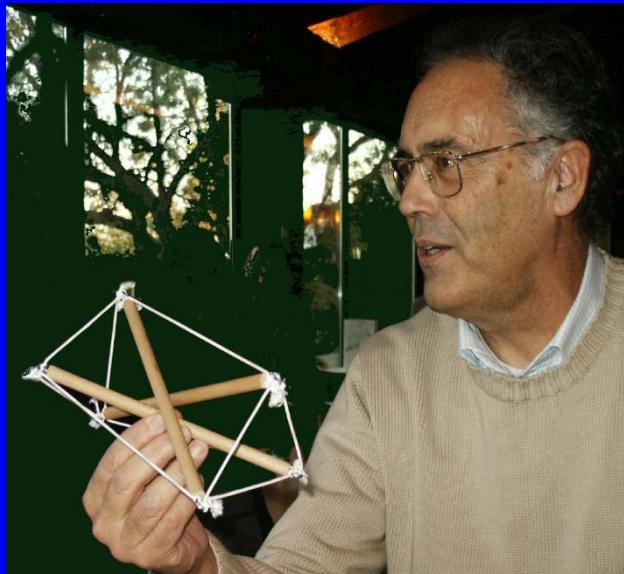


Cascada de bifurcaciones de soluciones periódicas



7. Tensegridades

Esas matemáticas nos pueden dar pistas de cómo combatir movimientos sísmicos en desafíos de la magnitud de rascacielos, o porqué son estables formaciones ingeniosas de gomas y varillas (*tensegridades*) que tanto atrajeron a uno de los más brillantes matemáticos de la historia de nuestro país: nuestro añorado Miguel de Guzmán.



- La tensión nació en la escultura de K. Snelson, 1949

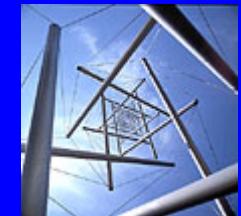
- alcanzó cierto renombre a la sombra de Buckminster Fuller (<http://www.bfi.org/>) y va encontrando diversas aplicaciones recientemente, por ejemplo

- en medicina: arquitectura de la célula, D. Ingber, Harvard

- estructuras desplegables: S. Pellegrino, Cambridge U.K.

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Un libro con amplia información es:
R. Motro, Tensegrity. Structural Systems for the Future (Kogan Page Science, London, 2003)



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