Free boundary and large solutions for linear equations with absorption Hardy type potentials

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Topics in Elliptic and Parabolic EDPs

Naples, September 11th,2014

Congratulations Guido !!!

Souvenirs of my first visit to Naples (September 1990)

C. R. Acad. Sci. Paris, t. 314, Série I, p. 1015-1020, 1992 1015

Équations aux dérivées partielles/*Partial Differential Equations*

Équations elliptiques et symétrisation de Steiner

Angelo Alvino, Jesus Ildefonso Diaz, Pierre-Louis Lions et Guido Trombetti

Elliptic Equations and Steiner Symmetrization

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My initial subject for this lecture

Steiner symmetrization for concave semilinear elliptic and parabolic equations and the obstacle problem

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but ....
Cambridge 2014....
Schrödinger equations ....
last minutes ...

1. Introduction

Given Ω bounded regular open set of $\mathbb{R}^N$, $N \geq 1$, we shall consider different LINEAR problems associated to a "Hardy type potential" $V(x)$, $V \in L^1_{loc}(\Omega)$, depending on the distance to the boundary and satisfying

$$\frac{\underline{C}}{d(x,\partial\Omega)^2} \leq V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^2} \quad \text{a.e. } x \in \Omega, \tag{1}$$

We shall start with the Dirichlet problem

$$LP(V,f)\begin{cases} -\Delta u + V(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f \in L^1(\Omega, \delta)$, $f \geq 0$ (for simplicity), with $\delta(x) := d(x, \partial\Omega)$ and we shall prove the existence of "free boundary solutions" (*i.e.* such that $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$).

We shall also mention also how to extend this result to the non-monotone spectral type problem

$$LP(V,\lambda,\Omega)\begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Absorption Hardy potential type (in contrast to many other results with $V(x)u$ ath the right hand side of the equation).

We shall also consider "large solutions" ($u = \infty$ on $\partial\Omega$) to this LINEAR equation, i.e. solutions of

$$LSP(V,f)\begin{cases} -\Delta u + V(x)u = f(x) & \text{in } \Omega, \\ u = +\infty & \text{on } \partial\Omega. \end{cases}$$

The results were motivated by a pioneering article of 1928 by G. Gamow dealing with "bound-state solutions" of the Schrödinger equation for "infinite well type potentials" which we shall recall in Section 2.

Curiously enough, our proof uses some results for semilinear equations

$$-\Delta u + |u|^{m-1}u = f(x) \text{ on } \Omega,$$

under radically different conditions ($m < 1$ for free boundary solutions and $m > 1$ for large solutions).

We emphasize that both phenomena are possible, in this linear framework, for such type of potentials $V(x)$.

In a last part, some applications of the rearrangement techniques will be also presented.

2. The infinite well potential versus the tunneling effect: Gamow (1927)

In his 1928 pioneering article Gamow proved, for the first time, the *tunneling effect* which, among many other applications lead to the construction of the electronic microscope and the correct study of the alpha radioactivity. Most of his study was concerning with *bound states* $\psi(x,t) = e^{-\mathrm{i}Et}u(x)$ of the Schrödinger equation in $\mathbb{R}^N$, $N \geq 1$,

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \text{ in } (0,\infty)\times\mathbb{R}^N,$$

associated to the potential $V(x)$, for a single elementary particle of mass m and energy E which we shall denote also by λ). Here $\mathrm{i} = \sqrt{-1}$ and $\hbar$ is the renormalized Plank constant.

Gamow was specially interested in the Coulomb potential

$$V(x) = \frac{k}{|x|}, \quad x \in \mathbb{R}^N,$$

Zur Quantentheorie des Atomkernes.

Von G. Gamow, z. Zt. in Göttingen.

Mit 5 Abbildungen. (Eingegangen am 2. August 1928.)

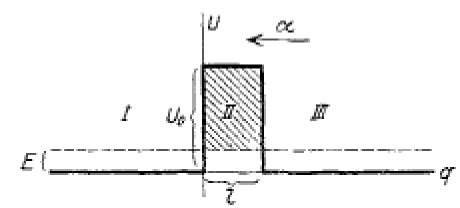


Fig. 2.

finite well potential

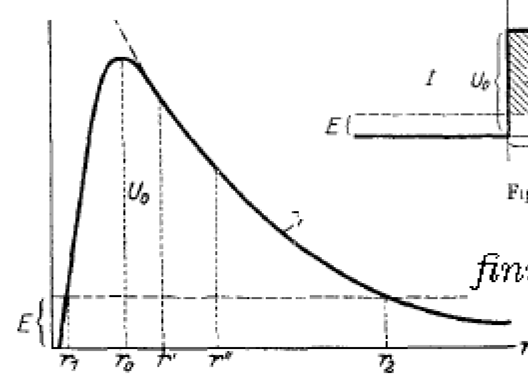


Fig. 1.

We recall that in Quantuum Mechanics,

$\psi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}$ the matter wave function (L. de Broglie 1924: wave-particle duality)

$\hbar > 0$ renormalized Plank constant, m mass of the elementary particle,

$V(x) \in \mathbb{R}$ the external potential

Crucial fact: $|\psi(x,t)|^2$ represents the probability density (Max Born 1926) to find the particle at point x and time t :

$$\frac{\partial}{\partial t} |\psi|^2 + \operatorname{div} \mathbf{J} = 0$$

$$\mathbf{J} := \frac{i\hbar}{2m}(\psi \nabla \psi^* - \psi^* \nabla \psi) = \frac{\hbar}{2m} \operatorname{Re}(\frac{1}{i} \psi^* \nabla \psi)$$

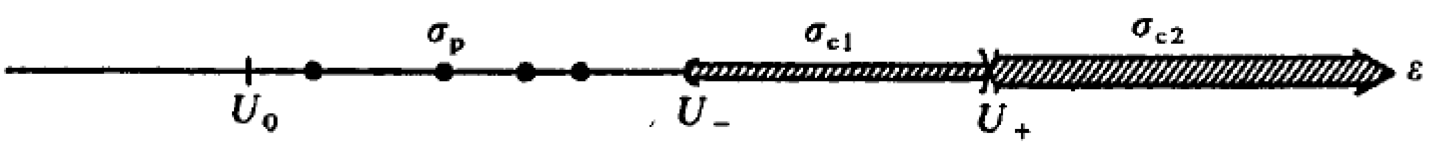
(ψ^* =the complex conjugate of ψ^*).

Question: how to "confine" (or "localize") the particle (and how to measure its linear momentum $\mathbf{p}$) ??

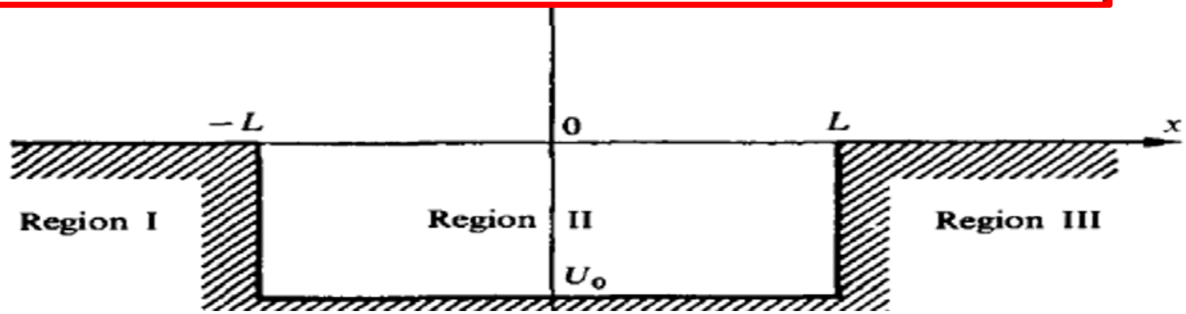
Simplifications for the linear Schrödinger equation (attributed by him, in 1935, to George Gamow (1904-1968) and repeated in any text book in Quantum Mechanics):

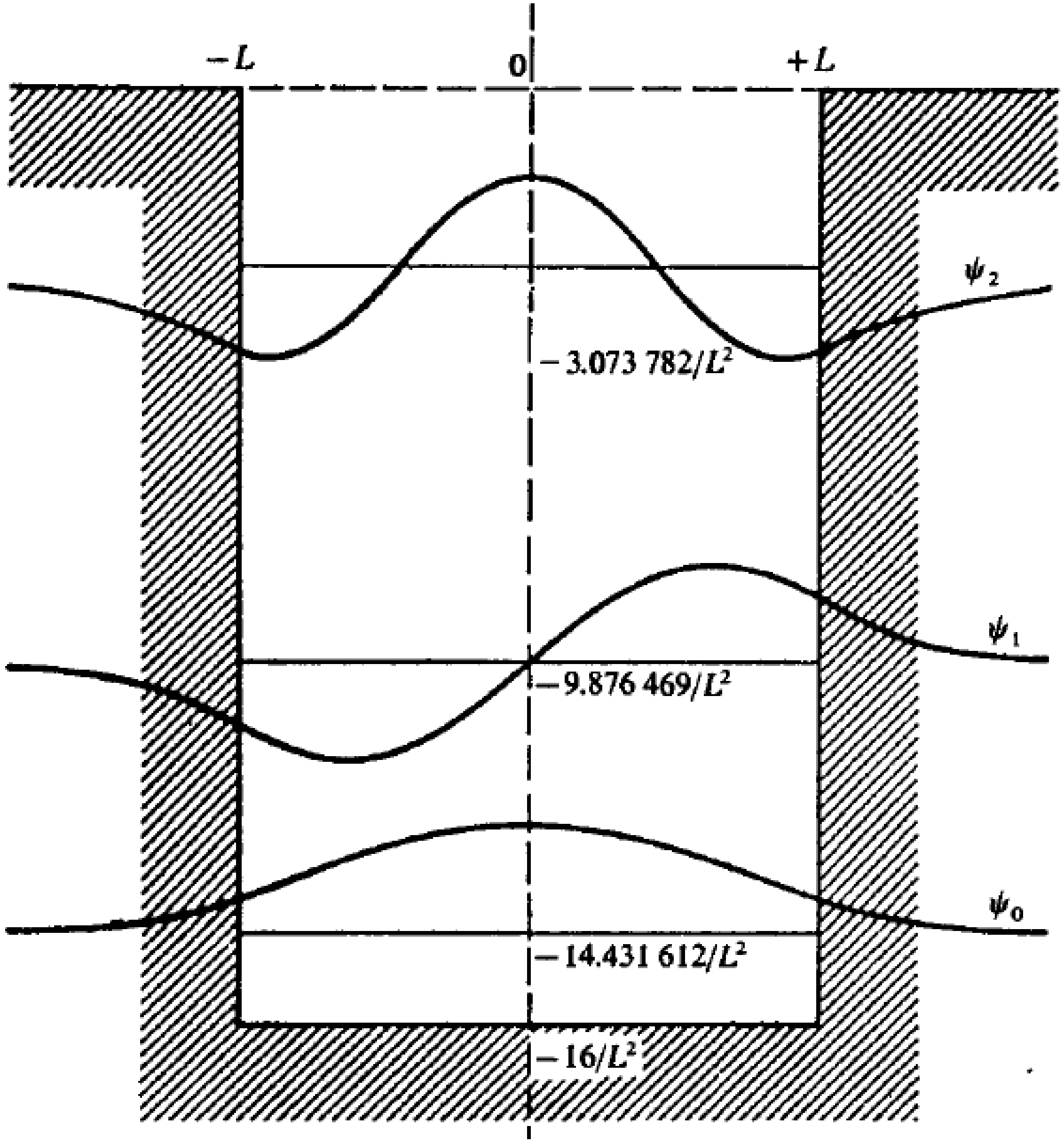
$$\frac{d^2\psi(x)}{dx^2} + [\varepsilon - U(x)]\psi(x) = 0 \,,$$

$$\varepsilon \equiv \frac{2M}{\hbar^2}E, \quad U(x) \equiv \frac{2M}{\hbar^2}V(x)\,, \qquad U_0 \equiv \inf U(x)\,, \quad U_\pm \equiv \lim_{x\to\pm\infty} U(x)\,.$$



"the square well potential": tunneling effect

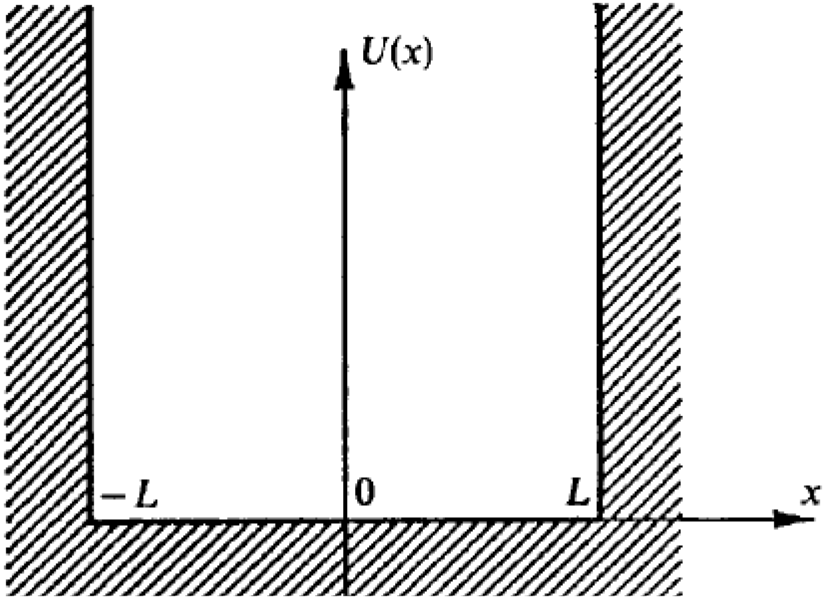




Wave functions and potential for a square potential well with $L|U_0|^{1/2} = 4$

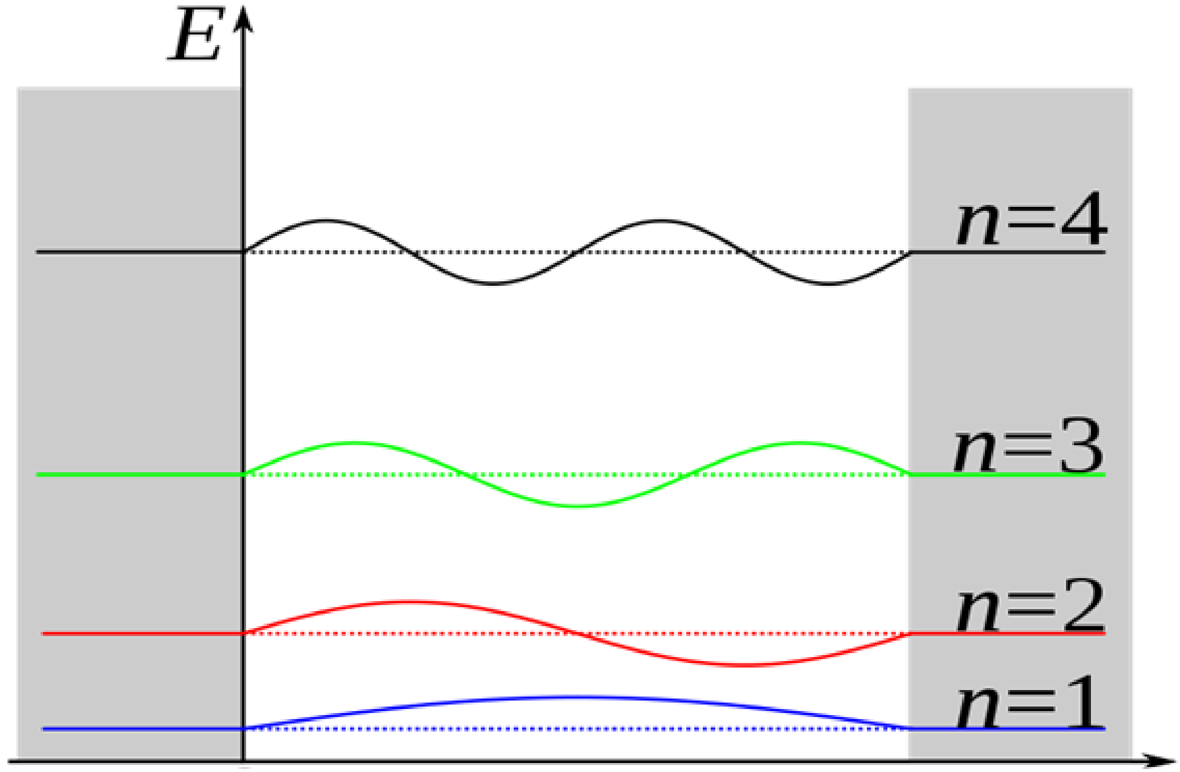
"the infinite well potential"

$$U(x) = \begin{cases} U_0 & \text{if } x \in [-L, L], \\ +\infty & \text{if } x \notin [-L, L]. \end{cases} \quad U(x) \equiv \frac{2m}{\hbar^2} V(x)$$



$$\psi_n(x) = \begin{cases} C \sin \frac{n\pi}{2L}(x + L) & \text{if } x \in [-L, L] \\ 0 & \text{if } x \notin [-L, L], \end{cases}$$

$$\varepsilon_n - U_0 = \left(\frac{\pi}{2L}\right)^2 n^2 := \lambda_n, \; n = 1, 2, ... (\text{i.e. } E_n := \frac{\hbar^2}{2m} \varepsilon_n)$$



Generalizations to

$$U(x) = \begin{cases} U_0(x) & \text{if } x \in [-L, L], \\ +\infty & \text{if } x \notin [-L, L]. \end{cases}$$

$U_0 \in L^1(-L, L)$ M.A. Naimark's book 1968

$U_0(x) = \delta_0(x)$ Quantum Dots: Y. N. Joglekar (2009)

But,....., such ψ_n are not solutions of

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial x^2}+V(x)\psi_n=E_n\psi, \text{ in } \mathbb{R}$$

since the second derivative develops two Dirac mass:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi_n}{\partial x^2}+V(x)\psi_n=E_n\psi+k_n(L)\delta_{\{L\}}-k_n(-L)\delta_{\{-L\}}, \text{ in } \mathbb{R}$$

with

$$k_n(-L)=\frac{\hbar^2}{2m}\frac{\sqrt{2}}{L^{3/2}}n\pi \text{ and } k_n(L)=\frac{\hbar^2}{2m}\frac{\sqrt{2}}{L^{3/2}}n\pi(-1)^n$$

The argument to confine (to localize) the particle on some interval $[-L,L]$ must require much more complicated arguments (i.e. more complicated **"infinite well potentials"**) preventing the formation of the two Dirac Deltas and bringing a correct justification of the product $V(x)\psi_n(x)=\infty\cdot 0=0$.

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via free boundary solutions: the one-dimensional case. Submitted

In contrast with the case of the tunneling effect (corresponding to the treatment of the finite well potential (2)) the usual study of the *infinite well potential included,* such as it is presented in most of the textbooks, contains an important mistake which, curiously enough, it seems unadvertised before: it said there that the only way to solve the equation in $\mathbb{R}^N$ outside Ω is by imposing that the solution $u(x)$ of (3) let $u(x) \equiv 0$ if $x \notin \Omega$ and thus problem (3) was reduced to solve the associated Dirichlet problem on Ω

$$(DP) \begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The main goal today is to present a set of results offering an alternative (in a weaker sense). Here we shall deal merely with nonnegative solutions $u \geq 0$ of (DP). The case of nodal solutions (changing sign on Ω) will be developed in a different work (Díaz and Hernández [2014] trough a suitable application of the "bifurcation from the infinity", Rabinowitz 1973, and additional arguments).

We shall work with the notion of *free boundary solutions, i.e. such that* $\frac{du}{dx}(-L) = \frac{du}{dx}(L) = 0$.

Connections with an auxiliary *inverse free boundary problem*: given an open bounded set Ω of $\mathbb{R}^N$ find a potential $V(x)$ such that the solution of $-\Delta u + V(x)u = \lambda u$ in $\mathbb{R}^N$ gives rise to a free boundary given by $\partial\Omega$)

Our main result is the following

Theorem 1. *Let* $\Omega = (-L, L)$ *and let* $V \in L^1_{loc}(\Omega)$ *be such that*

$$\frac{\underline{C}}{d(x,\partial\Omega)^\alpha} \leq V(x) \leq \frac{\overline{C}}{d(x,\partial\Omega)^\alpha} \text{ a.e. } x \in \Omega, \tag{5}$$

for some $\alpha \in [0,2]$ *and some* $\overline{C} > \underline{C} \geq 0$.
Then there exist a $\lambda > 0$ *for which problem* (DP) *has a free boundary solution if and only if* $\alpha = 2$. *Moreover, in this case there exists* $\underline{m}, \overline{m} \in (0,1)$ *such that* $u \in C^{2/(1-\overline{m})}(\overline{\Omega})$ *and*

$$\underline{K} d(x,\partial\Omega)^{2/(1-\underline{m})} \leq u(x) \leq \overline{K} d(x,\partial\Omega)^{2/(1-\overline{m})} \text{ for any } x \in \overline{\Omega}, \tag{6}$$

for some constants $\overline{K} > \underline{K} > 0$.

Corollary 1. *Let* $V_\infty(x : L, U_0(.))$ *with* $U_0(.)$ *satisfying (5) with* $\alpha = 2$. *Then there exists a* $\lambda > 0$ *for which the Shrödinger equation (3), with* $N = 1$, *admits a solution* $u \in C^1(\mathbb{R})$ *satisfying (6), for suitable* $\underline{m}, \overline{m} \in (0,1)$, $\overline{K} > \underline{K} > 0$, *and such that* $u \equiv 0$ *on* $\mathbb{R} - \Omega$.

For nodal free boundary solutions see:

J.I.Díaz and J. Hernández. On branches of nodal solutions bifurcating from the infinity for a semilinear equation related to the linear Schrödinger equation: free boundary solutions and solutions with compact support. Submitted

3. On *free boundary solutions* for the linear problem

Let $V \in L^1_{loc}(\Omega)$, $V \geq 0$ and let $f \in L^1(\Omega, \delta)$. We recall that a function $u \in L^1(\Omega)$ is a *very weak solution* of

$$LP(V,f) \begin{cases} -\Delta u + V(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

if $V(x)u \in L^1(\Omega, \delta)$ and for any $\varphi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$

$$-\int_\Omega u\Delta\varphi\, dx + \int_\Omega V(x)u\varphi\, dx = \int_\Omega f\varphi\, dx.$$

The existence and uniqueness (in fact comparison principle) of a very weak solution to $DP(V,f)$ was given (under a more general statement) in

J.I. Díaz, J.M. Rakotoson, On very weak solutions of semilinear elliptic equations with right hand side data integrable with respect to the distance to the boundary, Discrete and Continuum Dynamical Systems, Vol.27, 3, 2010, 1037-1058.

Previous references: unpublished paper by Haim Brezis (1972), Brezis, Cazenave, Martel and Ramiandrisoa (1996), series of papers by L. Veron,...).

Additional regularity: Díaz and Rakotoson (2009)

- $u \in W^{1,q}(\Omega, \delta)$ for any $1 \leqslant q < \frac{2N}{2N-1}$
- if additionally $f \in L^1(\Omega, \delta^\alpha)$, for some $\alpha \in [0,1)$, then $u \in W^1(\Omega, |\cdot|_{\frac{N}{N-1},\infty})$, where the Sobolev-Lorentz spaces are defineed by $W^1(\Omega, |\cdot|_{p,q}) = \{v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega)\}$, $1 \leq p, q \leq +\infty$, with

$$L^{p,q}(\Omega) = \{v : \Omega \to \mathbb{R} \text{ measurable: } |v|^q_{L^{p,q}} = \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty\},$$

$$|v|_{**}(t) = \frac{1}{t} \int_0^t |v|_*(s) ds, \text{ for } t \in \Omega_* =]0, |\Omega|[,$$

and where v_* denotes the decreasing rearrangement of v, defined by

$$v_* : \Omega_* =]0, |\Omega|[\to \mathbb{R}, \quad v_*(s) = \inf\{t \in \mathbb{R} : |v > t| \leqslant s\}.$$

A curious fact, which seems to have been quite very few advertized in the literature, is that, in spite of the large gererality of the application of the L^1-theory, there are very simple cases of Hölder continuous data $f \in C^\gamma(\Omega)$, which are not globally integrable $f \notin L^1(\Omega)$, for which the basic problem $DP(0, f)$ admits a unique solution *classical solution* $u \in C^0(\overline{\Omega}) \cap C^2(\Omega))$ [see, e.g. Lemma 4.9 and Problem 4.6 in the book by Gilbarg and Trudinger].

Note that if $V \neq 0$, in contrast with some nonexistence results (Brezis-Cabré 1998 for $V \leq 0$), the above result on the existence and uniqueness of solution (for $V \geq 0$) does not require any global integrability asumption on V.

Theorem A. *Let Ω be a C^2 domain and let $V \in L^1_{loc}(\Omega)$, such that*

$$\frac{\underline{C}}{d(x,\partial\Omega)^2} \leq V(x) \quad a.e.\ x \in \Omega, \tag{2}$$

for some $\underline{C} > 0$. Let $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ a.e. $x \in \Omega$ such that

$$f(x) \leq \overline{K} d(x,\partial\Omega)^{(1+m)/(1-m)} \quad a.e.\ x \in \Omega, \tag{3}$$

for some suitable $m \in (0,1)$ and $\overline{K} > 0$. Then the very weak solution u of $LP(V,f)$ satisfies

$$0 \leq u(x) \leq K d(x,\partial\Omega)^{2/(1-m)} \ a.e.\ x \in \Omega, \tag{4}$$

for some constant K. In particular $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ (i.e. u is a free boundary solution). Moreover if

$$V(x) \leq \frac{C}{d(x,\partial\Omega)^2} \quad a.e.\ x \in \Omega, \tag{5}$$

there exists $\underline{m} \in (0,1)$ and $\underline{K} > 0$ such that for any $f \in L^1(\Omega : \delta)$, $f \geq 0$ the very weak solution u of $LP(V,f)$ satisfies

$$\underline{K} d(x,\partial\Omega)^{2/(1-\underline{m})} \leq u(x) \ a.e.\ x \in \Omega. \tag{6}$$

Lemma 1. *Let $V_0 > 0$ and $m \in (0,1)$ be given and let $v \geq 0$ the very weak solution of the semilinear problem*

$$SP(m,V_0) \equiv \begin{cases} -\Delta v + V_0 v^m = f(x), & in\ \Omega, \\ v = 0, & on\ \partial\Omega, \end{cases}$$

for some. Assume that

$$v(x) \leq K d(x,\partial\Omega)^{2/(1-m)} \ a.e.\ x \in \Omega,$$

for some constant $K = K(\Omega, m, V_0) > 0$. Then, if

$$\frac{V_0 K^{-(1-m)}}{d(x,\partial\Omega)^2} \leq V(x) \quad a.e.\ x \in \Omega, \tag{5}$$

we have that $0 \leq u \leq v$ a.e. $x \in \Omega$. In particular u is a free boundary solution of $LP(V, f)$.

Proof. From the assumptions

$$\frac{V_0}{v(x)^{1-m}} \leq \frac{V_0}{K^{1-m}} \frac{1}{d(x,\partial\Omega)^2} \leq V(x) \text{ a.e. } x \in \Omega.$$

Then if we define

$$\underline{V}(x) := \frac{V_0}{|v(x)|^{1-m}}$$

then

$$-\Delta u + V(x)u = f(x) = -\Delta v + \underline{V}(x)v \geq -\Delta v + V(x)v$$

and we get the result by the comparison principle.

Lemma 2. *Let $V_0 > 0$ and $m \in (0,1)$. Let $v \geq 0$ be the very weak solution of the semilinear problem $SP(m, V_0)$. Let $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ a.e. $x \in \Omega$ be such that*

$$f(x) \leq K_f d(x,\partial\Omega)^{(1+m)/(1-m)} \quad a.e.\ x \in \Omega, \tag{6}$$

for some $K_f > 0$. Then, there exists $K = K(\Omega, m, V_0, K_f) > 0$ such that

$$v(x) \leq K d(x, \partial\Omega)^{2/(1-m)} \ a.e. \ x \in \Omega.$$

Moreover,

$$\begin{cases} \frac{V_0}{K(\Omega,m,V_0,K_f)^{1-m}} \searrow 0 \ \text{ if } \ V_0 \searrow 0 \text{ and } m \searrow 0. \\ \frac{V_0}{K(\Omega,m,V_0,K_f)^{1-m}} \nearrow \infty \ \text{ if } \ V_0 \nearrow \infty \text{ and } m \nearrow 1. \end{cases} \tag{9}$$

Proof. It is a special case of the family of local supersolutions constructed in Theorem 1.15 of Diaz (Pitman, 1985) [see also Alvarez-Díaz,*Discrete and Continuum Dynamical Systems*, **25**, (2009), 1-17].

Proof of Theorem A. Given $\underline{C} > 0$ take $V_0 > 0$ *and* $m \in (0,1)$ such that

$$\underline{C} \geq \frac{V_0}{K(\Omega, m, V_0, K_f)^{1-m}}$$

and apply the above Lemma 1 and 2. The proof of the estimate from above (6) is similar (the local subsolution for the semilinear equation was given also in Theorem 1.15 of Diaz (Pitman, 1985)).

Remark 1. For not so singular potentials the result fails. Indeed, if we assume

$$\frac{\underline{C}}{d(x, \partial\Omega)^{\alpha}} \geq V(x) \quad a.e. \ x \in \Omega,$$

for some $\alpha \in [0,2)$ then, for any $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ *a.e.* $x \in \Omega$ the solution u of $LP(V,f)$ verifies $u(x) \geq Kd(x,\partial\Omega) > 0$ on Ω and for some $K > 0$ and $\frac{\partial u}{\partial n} > 0$ *on* $\partial\Omega$ (easy adaptation of the Hopf strong maximum principle: Protter- Weimberger (1984), Bertsch and Rostamian, *J. Diff. Eq.*(1985).

Remark 2. By choosing $m = 0$ (i.e. "the obstacle problem) in the above proof it is possible to show that the conclusion of Theorem A remains true if, for instance, $f(x)$ is bounded and changes sign (still $u \geq 0$ on Ω !).

Let us consider now the spectral problem for the linear Schroedinger equation

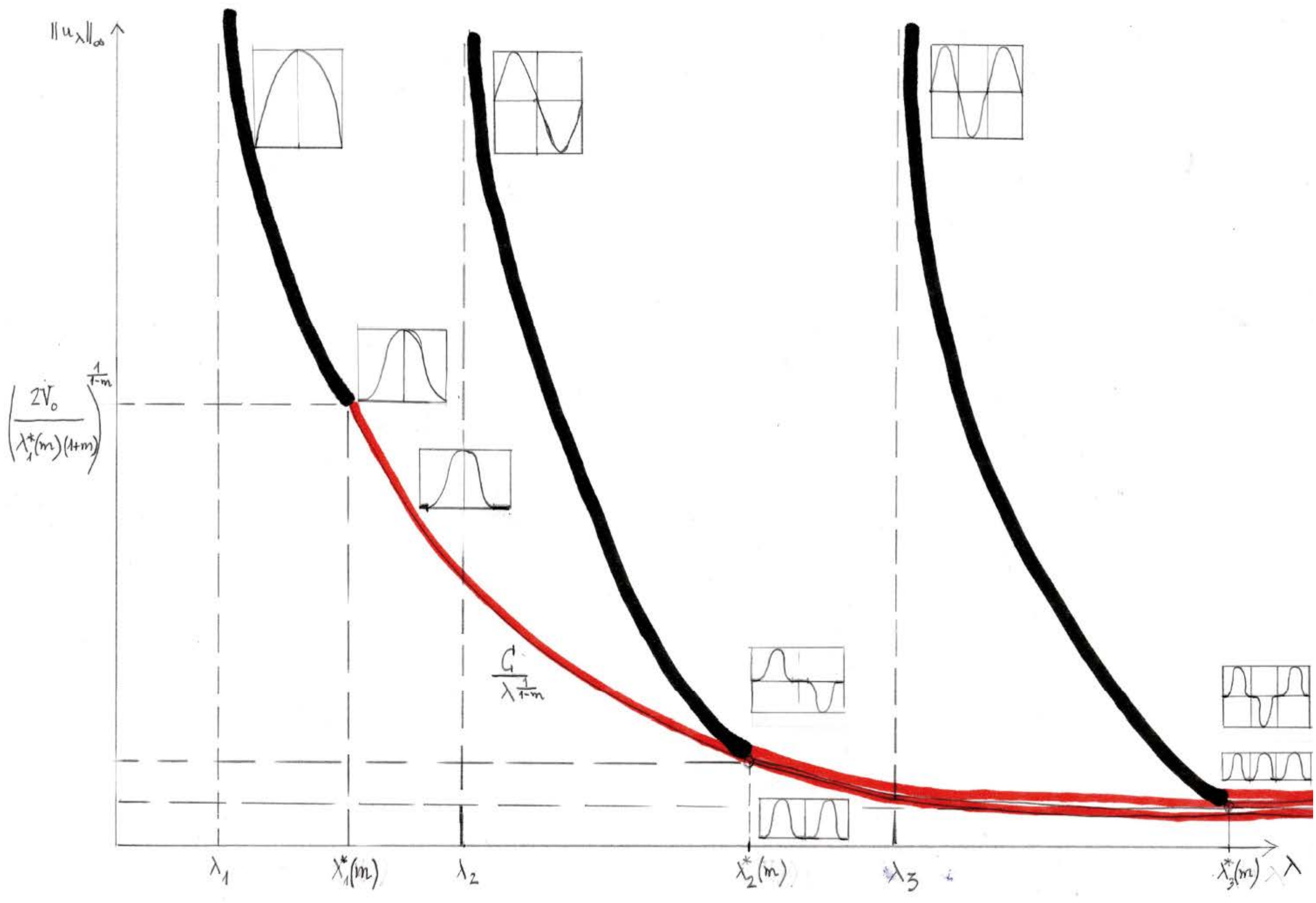
$$LP(V,\lambda,\Omega)\begin{cases} -\Delta u + V(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Now the comparison principle cannot be apply and the results involves more technical details and difficulties.

As already said, the case $N = 1$ was considered in

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via free boundary solutions: the one-dimensional case. Submitted

J.I.Díaz and J. Hernández. On branches of nodal solutions bifurcating from the infinity for a semilinear equation related to the linear Schrödinger equation: free boundary solutions and solutions with compact support. Submitted



$\|u_\lambda\|_\infty$
$\left(\frac{2V_0}{\lambda_1^*(m)(1+m)}\right)^{\frac{1}{1-m}}$
$\frac{C}{\lambda^{\frac{1}{1-m}}}$
λ_1
$\lambda_1^*(m)$
λ_2
$\lambda_2^*(m)$
λ_3
$\lambda_3^*(m)$
λ

The proof of Theorem 1 of Section 2 (for the linear problem) requires the estimate $v(x) \leq Kd(x, \partial\Omega)^{2/(1-m)}$ *a.e.* $x \in \Omega$ for the solution of the sublinear problem.

This is obtained trough the study of the ODE in the line of the paper

Díaz- Hernández (1999): Global bifurcation and continua of nonnegative solutions for a quasi-linear elliptic problem, CRAS.

The iterative method of super and subsolutions is used replacing the direct application of the maximum principle.

Remark 3. The conclusion of Theorem 1 holds for the Ndimensional case (and $\partial\Omega$ of class C^2): the existence of solutions with compact support, for λ large enough, was shown in the recent paper:

Díaz- Hernández-Il'yasov: On the existence of positive and nonnegative solutions for a spectral nonlinear elliptic problem with strong absorption (submitted).

The estimate $v(x) \leq Kd(x, \partial\Omega)^{2/(1-m)}$ *a.e.* $x \in \Omega$ near the boundary uses local barrier functions like in Theorem 1.15 of Diaz (Pitman, 1985)) since on the level sets $\{x \in \Omega : 0 \leq v(x) \leq \Lambda\}$ we can apply the comparison principle (if Λ is small enough).

Details will be given in

D: On the ambiguous treatment of the Schrödinger equation for infinite potential well and an alternative via free boundary solutions: the multi-dimensional case.

4. On *large solutions* for the linear problem

Given $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ *a.e.* $x \in \Omega$, the existence of a large solution of the semilinear

$$LSSP(m, V_0) \begin{cases} -\Delta v + V_0 v^m = f(x) & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases}$$

was shown, now for $m > 1$, in Theorem 2.10 of Díaz- Rakotoson (2010).

Concerning the linear problem we have:

Theorem B. *Let Ω be a C^2 domain and let $V \in L^1_{loc}(\Omega)$, such that*

$$\frac{\underline{C}}{d(x, \partial\Omega)^2} \leq V(x) \quad a.e.\ x \in \Omega, \tag{10}$$

for some $\underline{C} > 0$. Let $f \in L^1(\Omega : \delta)$, $f(x) \geq 0$ a.e. $x \in \Omega$ such that

$$f(x) \leq \overline{K} d(x, \partial\Omega)^{-(1+m)/(m-1)} \quad a.e.\ x \in \Omega, \tag{11}$$

for some suitable $m > 1$ and $\overline{K} > 0$. Then there exists a solution to the linear problem

$$LSLP(V, f) \begin{cases} -\Delta u + V(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover

$$u(x) \leq Kd(x, \partial\Omega)^{-2/(m-1)} \ a.e. \ x \in \Omega, \tag{12}$$

for some constant K. Moreover, the reverse estimate holds if

$$\frac{\overline{C}}{d(x, \partial\Omega)^2} \geq V(x) \quad a.e. \ x \in \Omega, \tag{13}$$

and

$$f(x) \geq Kd(x, \partial\Omega)^{-(1+m)/(m-1)} \ \ a.e. \ x \in \Omega, \tag{14}$$

for some suitable $\underline{m} > 1$ and $\underline{K} > 0$.

Remark 4. The proof is entirely similar once we know the blow-up estimate (for the semilinear problem) $0 \leq v(x) \leq Kd(x, \partial\Omega)^{-2/(m-1)}$ *a.e.* $x \in \Omega$. This (and the reverse estimate) was proved in

Alarcón, S., Díaz, G., Letelier, R., Rey, J.M.: Expanding the asymptotic explosive boundary behavior of large solutions to a semilinear elliptic equation. Nonlinear Anal. (2010),

with the universal constant

$$K = \left(\frac{2(m+1)}{V_0(m-1)^2}\right)^{\frac{1}{m-1}} \pm \epsilon$$

respectively, for any $\epsilon > 0$.

Notice that, again,

$$V_0 v^m \geq \frac{V_0 K^{(m-1)}}{d(x, \partial\Omega)^2} v$$

(and analogously for the reverse estimate).

Remark 5. I conjecture that for not so singular potentials the result fails: i.e. if we assume

$$\frac{C}{d(x, \partial\Omega)^\alpha} \geq V(x) \quad a.e. \ x \in \Omega,$$

for some $\alpha \in [0, 2)$ then there is no large solutions to the linear problem $LSLP(V, f)$.

Remark 6. The estimate $v(x) \leq K d(x, \partial\Omega)^{-2/(m-1)}$ *a.e.* $x \in \Omega$ was also proved for the large solution of the higher order semilinear equations

$$LSSP(m, V_0) \begin{cases} (-\Delta)^\alpha v + V_0 v^m = f_0 & \text{in } \Omega, \\ v = +\infty & \text{on } \partial\Omega, \end{cases}$$

for a given $\alpha \in \mathbb{N}$, when Ω is a ball and $f_0 \geq 0$:

J. I. Díaz, M. Lazzo, P.G. Schmidt, Asymptotic Behavior of Large Radial Solutions of a Polyharmonic Equation with Superlinear Growth. Journal Differential Equations (August 2014).

The extension to the associate **linear** problem is in progress.

5. Rearrangement results

The main goal of this section is to present here an extension of several results in the literature in the framework of the space $L^1(\Omega, \delta)$.
Given $V \in L^1_{loc}(\Omega)$, $V(x) \geqslant 0$ *a.e.* $x \in \Omega$, we consider the radially symmetric problem

$$SP(\Omega^\# : V^\#, F) \equiv \begin{cases} -\Delta U + V^\#(|x|)U = F(|x|) \text{ in } \Omega^\#, \\ U = 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega^\#. \end{cases}$$

Here we are assuming that $F \in L^1_{loc}(\Omega^\#)$ is given function and that we have

$$F(x) \geqslant f_\#(x) \geqslant 0 \text{ a.e. } x \in \Omega^\#, \tag{15}$$

where $\Omega^\# = B_R(0)$ is the N-dimensional ball centered at the origin and with measure $|\Omega|$ and $f_\#(x)$ the decreasing rearrangement of f. Notice that the assumption $F \in L^1(\Omega^\#, \delta_{\Omega^\#})$ and the condition that $F_\# \in L^1(\Omega^\#, \delta_{\Omega^\#})$ implies that necessarely $F_\# \in L^1(\Omega^\#)$ (see Remark 2 of Diaz-Rakotoson (2009)) and so, the only relevant weighted integrability concerns the term U^V, when U^V is the unique solution of $SP(\Omega^\# : V^\#, F)$.

Theorem C. *Assume $F \in L^1(\Omega^{\#})$. Then, if u is the (unique) very weak solution of $LSLP(V,f)$ we have*

$$\int_{B_r(0)} V^{\#}(|x|)u_{\#}(|x|)\delta_{\Omega^{\#}}(x)\,\mathrm{d}x \leqslant \int_{B_r(0)} V^{\#}(|x|)U^V(|x|)\delta_{\Omega^{\#}}(x)\,\mathrm{d}x \text{ for any } r \in (0,R). \tag{16}$$

In particular,

$$||V(.)u||_{L^1(\Omega,\delta)} \leqslant ||V^{\#}(|x|)U^V)||_{L^1(\Omega^{\#},\delta_{\Omega^{\#}})}. \tag{17}$$

Proof. We approximate functions f, V, and F_n and denote by u_n and U_n^V to the solutions of the associated problems. By applying the rearrangement's comparison (see, e.g. Theorem 1.26 of Diaz (Pitman, 1985) we get

$$\int_{B_r(0)} V^{\#}(|x|)u_{n,\#}(|x|)\mathrm{d}x \leqslant \int_{B_r(0)} V^{\#}(|x|)U_n^V(|x|)\,\mathrm{d}x \text{ for any } r \in (0,R). \tag{18}$$

Note the absence of the weight $\delta_{\Omega^{\#}}$ in (18). But from the fact that functions

$$u_{n,\#}(|x|)),\ U_n^V(|x|) \text{ and } \delta_{\Omega^{\#}}(|x|)$$

are decreasing with the radii $|x|$ we get that (18) implies the same inequality with the weight $\delta_{\Omega^{\#}}$ for any $r \in (0,R]$,

$$\int_{B_r(0)} V^{\#}(|x|)u_{n,\#}(|x|)\delta_{\Omega^{\#}}(x)\,\mathrm{d}x \leqslant \int_{B_r(0)} V^{\#}(|x|)U_n^V(|x|))\delta_{\Omega^{\#}}(x)\,\mathrm{d}x \text{ for any } r \in (0,R). \tag{19}$$

Since we have convergence in $L^1(\Omega, \delta)$ and $L^1(\Omega^\#, \delta_{\Omega^\#})$ of both terms (D-R (2009)), we arrive to the inequality (16) by passing to the limit in (19). Estimate (17) follows from equiintegrability properties (see e.g. Theorem 1.25 in Diaz (Pitman, 1985)) and the inequality $(\delta_\Omega)_\#(x) \leqslant \delta_{\Omega^\#}(x)$ for any $x \in \Omega^\#$ (see Betta and Mercaldo, Differential Integral Equations (1997)).

Remark 7. Many variants are possible. For instance, if we assume

$$\underline{C}((\delta_\Omega)_\#(x))^{-\alpha} \leqslant V^\#(x) \leqslant \overline{C}(\delta_{\Omega^\#}(x))^{-\gamma} \text{ a.e. } x \in \Omega^\#, \tag{20}$$

for some $0 < \underline{C} \leqslant \overline{C}$ and $\alpha, \gamma \in$. Then estimate (16) must be replaced by

$$\int_\Omega |u| \delta_\Omega^{(1-\alpha)}(x)\,\mathrm{d}x \leqslant \frac{\overline{C}}{\underline{C}} \int_{\Omega^\#} |U^V| \delta_{\Omega^\#}^{(1-\gamma)}(x)\,\mathrm{d}x. \tag{21}$$

We end this section with a comparison result for large solutions which this time involves the symmetric increasing rearrangement of functions. To consider several further applications, it is useful to shift the formulation of the semilinear problem (SP) to a general term $\beta(\widetilde{u}(|x|))$. We easily get

$$\int_{B_r(0)} \beta(\widetilde{v}(|x|))\delta_{\Omega^\#}(x)\,\mathrm{d}x \geqslant \beta(\widetilde{W}(|x|))\delta_{\Omega^\#}(x)\,\mathrm{d}x \text{ for any } r \in (0, R), \tag{22}$$

anf from here, if β is strictly increasing we get

$$\operatorname*{ess\,inf}_{\Omega^\#} W \leqslant \operatorname*{ess\,inf}_{\Omega} v.$$

Corollary 2. *Assume* $\frac{\overline{C}}{d(x,\partial\Omega)^2} \geq V(x)$ *a.e.* $x \in \Omega$, *Then if* $m > 1$ *and* V_0 *are taken as in the proof of Theorem C then*

$$\operatorname*{ess\,inf}_{\Omega^{\#}} W \leqslant \operatorname*{ess\,inf}_{\Omega} u.$$

Remark 8. Estimates of this type were proved previously by M. R. Posteraro CRAS (1996) for the semilinear case $\beta(u) = e^u$.

Many problems remain open for the case of linear equations with absorption Hardy potentials

Thanks for your attention