On the numerical analysis of very weak solutions of some elasticity problems

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Numerical methods for ordinary
and partial differential equations
and applications
(dedicated to the 65th birthday of Professor Francisco J. Lisbona)

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1. Introduction

Given a linear boundary value problem on a bounded regular open set Ω of $\mathbb{R}^N$

$$(P_L)\begin{cases} Lu = f(x) & \text{in } \Omega, \\ + \text{ boundary conditions} \equiv (BC) & \text{on } \partial\Omega, \end{cases}$$

where Lu denotes an elliptic differential operator (of order $2m$, $m \in \mathbb{N}$) in divergence form, the usual notion of **weak solution** consists in to introduce the associated "energy space" $V \subset H^m(\Omega)$ (the Sobolev space or order m, i.e. $D^\alpha u \in L^2(\Omega)$ for any $\alpha \in \mathbb{N}^N$, $|\alpha| \leq m$) and then, if we assume that

$$f \in V' \tag{0.1}$$

u is characterized trough the introduction of the associated bilinear form $a : V \times V \to \mathbb{R}$ and the condition

$$a(u, \zeta) = \langle f, \zeta \rangle_{V'V}\ , \text{ for any } \zeta \in V.$$

A weaker notion of solution can be introduced leading to a correct mathematical treatment for a more general class of data f (i.e. for f not necessarily in V'). For instance, for $f \in L^1_{Loc}(\Omega)$ the notion of **very weak solution of problem** (P_L) can be introducing by integrating $2m-$times by parts (and not merely $m-$times as before) and by requiring, merely that $u \in L^1(\Omega)$ and that

$$\int_\Omega u(x)L^*\zeta(x)dx = \int_\Omega f(x)\zeta(x)dx.$$

for any $\zeta \in W := \overline{\{\zeta \in C^{2m}(\overline{\Omega})\colon \zeta \text{ satisfies } (BC)\}}^{W^{2m,\infty}(\Omega)}$, once we assume that

$$\int_\Omega |f(x)\zeta(x)|\, dx < \infty, \text{ for any } \zeta \in W.$$

Here L^* denotes the adjoint operator of L.

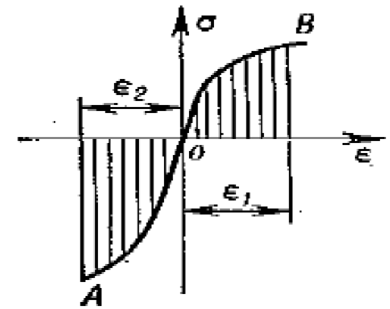
Most of the theory on very weak solutions available in the literature deals with second order equations. Recently, sharper results have been obtained, to this case, when $f \in L^1(\Omega, \delta)$, with $\delta = dist\,(x, \partial\Omega)$. That was originally proved by Haim Brezis, at the seventies, in a famous unpublished manuscript concerning Dirichlet boundary conditions (see also his 1996 paper with Cazenave, Martel, and Ramiandrisoa). For more recent references see J.I. D., J.M. Rakotoson (2009), (2010).

The main goal of my past lecture at Jaca 2011 (see J.I. D. On the very weak solvability of the beam equation. *Rev. R. Acad. Cien. Serie A. Mat* (**RACSAM**) 105 (2011), 167–172) was to present some new results proving that in the case of **higher order** equations the class of $L^1_{Loc}(\Omega)$ data for which the existence and uniqueness of a very weak solution can be obtained is, in general, larger than $L^1(\Omega, \delta)$ (the optimal class for the case of second order equations). For instance, for the case of the beam equation with Dirichlet boundary ($u = u' = 0$ on the boundary) I proved that **the optimal class of data** is the space $L^1(\Omega, \delta^2)$ but, for instance, for the simply supported beam ($u = u'' = 0$ on the boundary) **the optimal class of data** is again $L^1(\Omega, \delta)$. One of my main arguments was the use of the Green function $G(x, y)$ associated to the corresponding boundary value problem.

An important open problem in our days is the searching of solutions (beyond the class of weak solutions) for the case in which the operator L is **nonlinear.** Obviously, we can not integrate $2m-$times by parts and, which seem to be more important, we do not have any kind of Green function associated to the problem.

The main goal of this lecture is to present some new results concerning very weak solutions for **nonlinear problems** and now with indications about its numerical approximation. We point out that, without lost of generality we can assume that the beam is represented by the interval $(0, L)$ with $L = 1$ (which we shall do in the rest of the lecture). To fix ideas I will concentrate my attention in the **nonlinear beam equation with simply supported boundaries**

$$(B_{SS}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1 \\ u(0) = \phi(u'')(0) = 0, & \\ u(1) = \phi(u'')(1) = 0, & \end{cases}$$



where $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous strictly increasing function such that $\phi(0) = 0$. A standard example corresponds to the linear case $\phi(s) = EIs$ for any $s \in \mathbb{R}$ (E, I positive constants) but many other cases arise in the more diverse applications (case of non Hookean material such as cat iron, stone, "caucho", many bioelastic material and most of the composite materials such as concrete). Again, by dimensional analysis we can assume equal to one any constant arising in the constitutive law of the material. So, for instance, a case very treated in the literature is $\phi(s) = |s|^{\alpha-1} s$ for some $\alpha > 0$ (notice that $\alpha = 1$ reproduces, the linear case).

We shall also make some comments on the case of a **nonlinear cantilever beam**

$$(B_{Cant})\begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, & \\ \phi(u'')(1) = \phi(u'')'(1) = 0, & \end{cases}$$

as well as the N-dimensional problem

$$(P_{Nd})\begin{cases} -\Delta\phi(-\Delta u(x)) = f(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = \phi(-\Delta u) = 0, & \text{on } \partial\Omega. \end{cases}$$

The plan of the rest of the lecture is the following:

2. Very weak solutions and the optimal class of data for the nonlinear beam equation with simply supported boundaries.

3. Main idea of the proofs: some *magic* representation formulas.

4. Remarks on the numerical approach of very weak solutions of the one-dimensional problem.

5. Further remarks.

6. On the N-dimensional formulation and its numerical approximation.

The sections 2, 3 and 4 form an earlier presentation of a joint work in collaboration with E. Castillo (RAC and Universidad de Cantabria).

Section 5 corresponds to a work in progress with I. Arregui and C. Vazquez (Universidade de A Coruña) .

2. Very weak solutions and the optimal class of data for the nonlinear beam equation with simply supported boundaries.

To fix ideas I will concentrate my attention in the **nonlinear beam equation with simply supported boundaries**

$$(B_{SS}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = \phi(u'')(0) = 0, \\ u(1) = \phi(u'')(1) = 0, \end{cases}$$

where

$$\phi : \mathbb{R} \to \mathbb{R} \text{ is a continuous strictly increasing function such that } \phi(0) = 0. \tag{1.2}$$

Definition 2.1. Given $f \in L^1_{Loc}(0,1)$ a function $u \in W^{2,1}_{loc}(0,1)$ is a distributional solution of the differential equation $\phi(u''(x))'' = f(x)$ in $D'(0,1)$ if $\phi(u''(x)) \in L^1_{Loc}(0,1)$ and

$$\left\langle \phi(u''(x)), \frac{d^2\zeta}{dx^2} \right\rangle_{D'D} = \langle f, \zeta \rangle_{D'D}$$

for any $\zeta \in D(0,1) = C^\infty_c(0,1)$.

Let us denote the boundary conditions by

$$(BC) \equiv \begin{cases} u(0) = \phi(u'')(0) = 0, \\ u(1) = \phi(u'')(1) = 0, \end{cases}$$

Definition 2.2. Given $f \in L^1(0,1:\delta)$, with $\delta = dist\,(x, \partial\Omega)$, a function $u \in W^{2,1}_{loc}(0,1)$ is a "very weak solution" of (B_{SS}) if $u \in W^{2,1}(0,1) \cap W^{1,1}_0(0,1)$, $\phi(u''(x)) \in L^1(0,1)$ and for any $\zeta \in W^{2,\infty}(0,1) \cap W^{1,\infty}_0(0,1)$ we have

$$\int_0^1 \phi(u''(x)) \frac{d^2\zeta}{dx^2}(x)dx = \int_0^1 f(x)\zeta(x)dx.$$

The main result of this section is the following:

Theorem 2.1. *a)* ***Sufficiency.*** *Assume (2.2). Then, for any $f \in L^1(0,1:\delta)$ there exists a unique very weak solution of (B_{SS}). Moreover, the (nonlocal) operator $D : L^1(\Omega, \delta) \to L^1(\Omega)$ defined by $D(f) = u$ satisfies that if $D(g) = v$ then the weak maximum principle holds:*

$$\begin{aligned} f(x) &\leq g(x) \text{ implies that} \\ -u''(x) &\leq -v''(x) \text{ and so that } u(x) \leq v(x) \quad a.e.\ x \in \Omega. \end{aligned}$$

Moreover, we have the estimate

$$\int_0^1 [u(x) - v(x)]_+ \, dx \tag{2.3}$$

$$\leq \int_0^1 \left[-\int_0^1 \phi^{-1} \left(\int_0^1 -[f(\sigma) - g(\sigma)]_+ G(s,\sigma) d\sigma \right) G(x,s) ds \right] dx \tag{2.4}$$

where, in general, $h_+ = \max(0, h)$, *and* $G(s,\sigma)$ *is the Green function for the operator* $-\frac{d^2}{dx^2}$ *with homogeneous Dirichlet boundary conditions on* $(0,1)$*: i.e.*

$$G(t,\sigma) = \begin{cases} t(1-\sigma) & 0 \leq t \leq \sigma \leq 1, \\ \sigma(1-t) & 0 \leq \sigma \leq t \leq 1. \end{cases}$$

Moreover u *is smoother than said at Definition 2.2 since, at least,* $u \in C^2([0,1])$ and $\phi(u''(x)) \in C^1((0,1)) \cap C^0([0,1])$.

b) ***Strong maximum principle.*** *Let* $f \in L^1(0,1:\delta)$ *with* $f \geq 0$ *a.e.* $x \in (0,1)$, $f \neq 0$. *Then the very weak solution satisfies that*

$$\phi(u'')(x) \leq -C\left(\int_0^1 \left[\int_0^1 f(\sigma)G(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(x) < 0, \qquad (2.5)$$

for any $x \in (0,1)$, *and*

$$u(x) \geq -C(\int_0^1 \phi^{-1}\left\{-C\left(\int_0^1 \left[\int_0^1 f(\sigma)G(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(t)dt\right\}\delta(x) > 0, \qquad (2.6)$$

for any $x \in (0,1)$ *and for some positive constant* C *independent of* f.

c) **Necessity.** *Assume that* $f \in L^1_{Loc}(0,1)$, *such that* $f \geq 0$ *a.e.* $x \in (0,1)$. *Then if* $\int_0^1 f(x)\delta(x)dx = +\infty$ *it can not exists any* $u \in C^2([0,1])$ *with* $\phi(u''(x)) \in C^0([0,1])$ *satisfying the boundary conditions* (BC) *and being also solution in* $D'(0,1)$ *of the differential equation.*

Remark. 2.1. Theorem 2.1 improves some previous results in the literature on the nonlinear formulation (see, e.g. G. Shi and J. Zhang: Positive solutions for higher order singular p-Laplacian boundary value problems, *Proc. Indian Acad. Sci. (Math. Sci.)*, **118 (2008)**, 295-305 and its references) and contains also some slight improvements with respect the mentioned results by Díaz (Racsam 2011) for the case in which ϕ is linear, for which there is a great amount of previous results in the literature: (see, e.g. Gupta (1988), Agarwall (1989), Bernis (1996), Yao (2008),...). We also point out that the integral $\int_0^1 f(x)\delta(x)dx$ is "equivalent " to the integral $\int_0^1 f(x)x(1-x)dx$ in the sense that there exists two positive constants C_1 and C_2 such that

$$C_1 \int_0^1 |f(x)|\, x(1-x)dx \leq \int_0^1 |f(x)|\, \delta(x)dx \leq C_2 \int_0^1 |f(x)|\, x(1-x)dx.$$

The optimal growth condition on the data can be easily understood in terms of the physical modelling. For instance the assumption $f \in L^1(0,1 : \delta)$ is equivalent to the global integrability of the stress function $m(x)$ of the beam ($m \in L^1(0,1)$): see Theorem 4.1, part ii).

Remark. 2.2. Estimate (2.3) is new in the literature. Notice this estimate implies not only the usual *weak maximum principle* (for instance, if $f(x) \geq 0$ *a.e.* $x \in (0,1)$ then, necessarily $u(x) \geq 0$ for any $x \in (0,1)$) but also a new stronger conclusion: the u and u'' have a constant sign once we merely know that

$$\text{the function } x \to \int_0^1 f(\sigma)G(x,\sigma)d\sigma \text{ is } \geq 0 \text{ a.e. on } \Omega. \tag{2.7}$$

This generalization with respect the usual version of the *weak maximum principle* is quite surprising since (it is not difficult to show that) there is a continuum of changing sign functions but such that $f(x)$ satisfies property (2.7). Indeed: we have

$$\int_0^1 f(\sigma)G(x,\sigma)d\sigma = (1-x)\int_0^x f(\sigma)\sigma d\sigma + x\int_x^1 f(\sigma)(1-\sigma)d\sigma. \qquad (2.8)$$

Now denote by

$$g(x) = \int_0^x f(\sigma)\sigma d\sigma.$$

Then $f \in L^1(0,1:\delta)$ implies that $g \in W^{1,1}(0,1)$ with $g(0)=0$ and $g'(s) = f(s)s$ for a.e. $s \in (0,1)$. So, since there is a continuum of functions $g \in$

$W^{1,1}(0,1)$, not neceserely being increasing, such that $g(0)=0$ and $g(s) \geq 0$ on $(0,1)$ we arrive to our conclusion by taking $f(s) := g'(s)/s$ for such a given function g (note that f can be discontinuous). A similar argument applies to the second term of (2.8).

Remark. 2.3. The estimates implying the strong maximum are new in the literature (even for the linear case). We also point out that the nonexistence result can be also obtained by showing that if $m \in C^0([0,1]) \cap C^1(0,1)$, $m'' \in L^1_{Loc}(0,1)$ and $m'' \geq 0$ *a.e.* $x \in (0,1)$, then necesarily $\int_0^1 m''(x)x(1-x)dx < +\infty$. Indeed, by Taylor formula applyied to $x = 1/2$ we have that

$$m(x) = m(\frac{1}{2}) - (\frac{1}{2} - x)m'(\frac{1}{2}) + \int_x^{1/2} (\sigma - x)m''(\sigma)d\sigma.$$

Since the integral has a constant sign , letting $x \to 0$ we get that

$$\int_0^{1/2} m''(\sigma)\sigma d\sigma < +\infty.$$

The convergence result

$$\int_{1/2}^1 m''(\sigma)(1-\sigma)d\sigma < +\infty$$

is similar.

Remark. 2.4. The mathematical results for the case of the cantilever beam (B_{Cant}) present some relevant differences since the optimal growth condition on $f(x)$ becomes

$$\phi^{-1}[\int_x^1 \int_s^1 f(\sigma)d\sigma ds] \in L^1(0,1),$$

which obviously depends of the constitutive law function ϕ (in contrast with the case of the simply supported beam problem (B_{SS}) for which, unexpectedly, the optimal class of data is independent of ϕ).
For instance, if $f(x) = cx^a$ and $\phi(s) = |s|^{\alpha-1}s$ for some $\alpha > 0$ then the optimal solvability condition for problem (B_{SS}) is $a > -2$ (for any value of α !!) but for problem (B_{Cant}) the optimal solvability condition is $a > -(\alpha+1)$ (which depends on α). Notice that, in both cases, the solvability is possible beyond the condition $f \in L^1(0,1)$ (which would require to assume $a > -1$).

Remark. 2.5. The case of a nonlinear beam equation with clamped boundaries

$$(B_{Clam}) \begin{cases} \phi(u''(x))'' = f(x) & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, \\ u(1) = u'(1) = 0, \end{cases}$$

seems to be more delicate. When ϕ is linear, it was shown in Díaz (Racsam

2011) that the optimal set of data is $L^1(\Omega, \delta^2)$, i.e.

$$\int_0^1 |f(x)|\, x^2(1-x)^2 dx < +\infty.$$

We conjecture that in the case of a nonlinear constitutive equation the solvability requires two kind of conditions: one independent of ϕ, $f \in L^1(0,1:\delta)$, and other one depending on ϕ,

$$\phi^{-1}[\int_0^x \int_s^1 f(\sigma) d\sigma ds] \in L^1(\Omega, \delta).$$

4. Main idea of the proofs: some *magic* reprentation formulas.

The proof of the part a) (sufficiency) of Theorem 2.1 will made use of the following representation formulas:

Theorem 3.1. i) *Assume that*

$$f \in L^1(\Omega),\ i.e.\ \int_0^1 |f(x)|\, dx < +\infty. \tag{3.9}$$

Then the unique very weak solution u of (B_{SS}) is given trough the (nonlocal) operator $D : L^1(\Omega) \to L^1(\Omega)$, $D(f) = u$, defined by the representation formula

$$\begin{aligned} u(x) = &\int_0^x \{ \int_0^\theta \phi^{-1}(\int_0^t (t-r)f(r)dr - \tfrac{t}{1}\int_0^1 (1-\sigma)f(\sigma)d\sigma)dt \\ &- \tfrac{1}{1}\int_0^1 \int_0^s \phi^{-1}\left(\int_0^t (t-r)f(r)dr - \tfrac{t}{1}\int_0^1 (1-\sigma)f(\sigma)d\sigma \right) dtds \} d\theta. \end{aligned} \tag{3.10}$$

ii) In the more general case of

$$f \in L^1(\Omega, \delta), \ i.e. \ \int_0^1 |f(x)|\, x(1-x)dx < +\infty \tag{3.11}$$

the unique very weak solution u of (B_{SS}) is given trough the (nonlocal) operator $D : L^1(\Omega, \delta) \to L^1(\Omega)$, $D(f) = u$, defined by the representation formula

$$u(x) = -\int_0^1 \phi^{-1}\left(\int_0^1 -f(\sigma)G(s,\sigma)d\sigma \right) G(x,s)ds \text{ for any } x \in [0,1]. \tag{3.12}$$

where

$$G(t,\sigma) = \begin{cases} t(1-\sigma) & 0 \le t \le \sigma \le 1, \\ \sigma(1-t) & 0 \le \sigma \le t \le 1. \end{cases}$$

Proof of Theorem 3.1. Part i). We shall adapt to our framework a method introduced in the book by E. Castillo, A. Iglesias and R. Ruiz-Cobo. *Functional Equations in Applied Sciences*. Elsevier, 2004 (see also the previous spanish version E. Castillo and R. Ruiz-Cobo. *Ecuaciones Funcionales en la Ciencia, la Economía y la Ingeniería.* Editorial Reverté, Barcelona, 1993).

If we denote by $u(x), w(x), m(x), q(x)$ and $f(x)$ to the deflection, the rotation of the beam, the curvature, the shear and the load density at point x, respectively, then, from the beam differential equation we get that the system of four first order differential equations

$$\begin{cases} q'(x) &= f(x), \\ m'(x) &= q(x), \\ w'(x) &= \phi^{-1}(m(x)), \\ u'(x) &= w(x), \end{cases} \tag{3.13}$$

which is the mathematical model in terms of differential equations, when we are interested in q, m, w and u. It is well known that the system (3.13) of four first order differential equations is equivalent to the fourth order differential equation .In the new approach, the equilibrium equations are stated for discrete pieces.

Integrating in the first equation we get the equilibrium of vertical forces in this piece, i.e.

$$q(x+h) = q(x) + A(x,h), \tag{3.14}$$

where

$$A(x,h) = \int_x^{x+h} f(t)dt. \tag{3.15}$$

From (3.14) by choosing adequate values of x and h we get

$$q(s) = q(0) + A(0,s).$$

Integrating in the second equation we obtain the equilibrium of moments which leads to

$$m(x+h) = m(x) + hq(x) + B(x,h), \tag{3.16}$$

where

$$B(x,h) = \int_x^{x+h} (x+h-t)f(t)dt. \tag{3.17}$$

Similarly, from (3.16) by choosing adequate values of x and h we get

$$m(s) = m(0) + sq(0) + B(0, s).$$

Now using the material constitutive equation we get

$$w(x+h) \quad = \quad w(x) + C(x, h), \tag{3.18}$$

where

$$C(x, h) = \int_{x}^{x+h} \phi^{-1}(m(s))ds, \tag{3.19}$$

from which we get

$$w(s) = w(0) + C(0, s).$$

In addition we have

$$u(x+h) \quad = \quad u(x) + D(x, h) \tag{3.20}$$

where

$$D(x, h) = \int_{x}^{x+h} w(s)ds, . \tag{3.21}$$

from which we get

$$u(s) = u(0) + D(0,s).$$

Thus, we get the system of four "first order" functional equations

$$\begin{aligned} q(x+h) &= q(x) + A(x,h), && (3.22)\\ m(x+h) &= m(x) + hq(x) + B(x,h), && (3.23)\\ w(x+h) &= w(x) + C(x,h), && (3.24)\\ u(x+h) &= u(x) + D(x,h). && (3.25) \end{aligned}$$

The system (3.22)-(3.25) is equivalent to the system of differential equations (3.13). Note that the functions $A(x,h)$, $B(x,h)$, $C(x,h)$ and $D(x,h)$ become known as soon as the load function $f(x)$ is known and that in some cases we can solve the problem with $1, 2, 3$ or all equations in (3.22)-(3.25), depending on the boundary conditions. Note also that the system (3.22)-(3.25) can be considered as a system of difference equations (simply make $h = \Delta x$) which gives the exact solution at the interpolating points for any value of $h = \Delta x$.

The system (3.22)-(3.25) can be written as

$$q(1) = q(0) + \int_0^1 f(t)dt, \tag{3.26}$$

$$m(1) = m(0) + 1q(0) + \int_0^1 (1-t)f(t)dt, \tag{3.27}$$

$$w(1) = w(0) + \int_0^1 \phi^{-1}\left(m(0) + tq(0) + \int_0^t (t-r)f(r)dr\right)dt, \tag{3.28}$$

$$u(1) = u(0) + \int_0^1 \Bigg[w(0) + \int_0^s \phi^{-1}\left(m(0) + tq(0) + \int_0^t (t-r)f(r)dr\right)dt\Bigg) ds, \tag{3.29}$$

which, for general boundary conditions, is a non-linear system of 4 equations in the eight unknowns

$$q(1), q(0), m(1), m(0), w(1), w(0), u(1), u(0).$$

Thus, an adequate selection of four of them leads to a system from which the other 4 can be obtained. Once we know all boundary values, we can use the following closed formula for the shear, moment, rotation and displacement of the beam, as follows[1]:

$$q(h) = q(0) + \int_0^h f(t)dt, \tag{3.30}$$

$$m(h) = m(0) + hq(0) + \int_0^h (h-t)f(t)dt, \tag{3.31}$$

$$w(h) = w(0) + \int_0^h \phi^{-1}\left(m(0) + tq(0) + \int_0^t (t-r)f(r)dr\right) dt, \tag{3.32}$$

$$u(h) = u(0) + \int_0^h \left[w(0) + \int_0^s \phi^{-1}\left(m(0) + tq(0) + \int_0^t (t-r)f(r)dr\right) dt\right) ds, \tag{3.33}$$

[1] Formulas (3.26)-(3.29) are valid for any arbitrary value h of L.

To obtain $q(1)$, $q(0)$, $w(1)$ y $w(0)$, from (3.34)-(3.37) we proceed as follows:

we obtain $q(0)$ from (3.35):

$$q(0) = -\frac{1}{1}\int_0^1 (1-t)f(t)dt,$$

Then we obtain $q(1)$ from (3.34)

$$q(1) = \int_0^1 f(t)dt - \frac{1}{1}\int_0^1 (1-t)f(t)dt.$$

We get $w(0)$ from (3.37):

$$w(0) = -\frac{\int_0^1\int_0^s \phi^{-1}\left(-\frac{t}{1}\int_0^1(1-\sigma)f(\sigma)d\sigma + \int_0^t(t-r)f(r)dr\right)dtds}{1},$$

and $w(1)$ from (3.36):

$$\begin{aligned} w(1) \;&=\; -\frac{1}{1}\int_0^1\int_0^s \phi^{-1}\left(-\frac{t}{1}\int_0^1(1-\sigma)f(\sigma)d\sigma + \int_0^t(t-r)f(r)dr\right)dtds \\ &\quad + \int_0^1 \phi^{-1}\left(-\frac{t}{1}\int_0^1(1-\sigma)f(\sigma)d\sigma + \int_0^t(t-r)f(r)dr\right)dt. \end{aligned}$$

Finally, we use the closed expressions (3.30)-(3.33) to get

$$m(h) = -\frac{h}{1}\int_0^1 (1-t)f(t)dt + \int_0^h (h-t)f(t)dt$$

and the representation formula (3.10). It is a very easy task to check that all the above integrals are well justified thanks to the assumption (3.9).

Part ii). It is clear that the general hypothesis (3.11) does not allow to justify the above process (for instance, since $q(1) - q(0) = \int_0^1 f(t)dt$ one (or both) of the boundary values of the shear $q(x) = \phi(u''(x))'$ could be infinite). It seems possible to adapt the above arguments to the case of the general hypothesis (3.11), nevertheless, this time we can get a shorter representation formula (the one given by (3.12)). Indeed, since

$$\begin{cases} m''(x) = f(x) & \text{in } \Omega = (0,1), \\ m(0) = m(1) = 0, \end{cases}$$

and f satisfies (3.11) we know that

$$m(x) = -\int_0^1 f(\sigma)G(x,\sigma)d\sigma \text{ for any } x \in [0,1]. \tag{3.38}$$

Thus

$$\begin{cases} u''(x) = \phi^{-1}(-\int_0^1 f(\sigma)G(x,\sigma)d\sigma) & \text{in } \Omega = (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{3.39}$$

and since $\phi^{-1}(\int_0^1 f(\sigma)G(x,\sigma)d\sigma) \in C([0,1]) \subset L^1(\Omega,\delta)$ we get formula (3.12).

Proof of the part a) (sufficiency) of Theorem 2.1. The existence of a very weak solution *of* (B_{SS}) and the definition of the (nonlocal) operator $D : L^1(\Omega,\delta) \to L^1(\Omega)$ defined by $D(f) = u$ is consequence of the representation formula (3.12) given in Theorem 3.1. In order to prove the uniqueness let $D(g) = v$. Then , since

$$\begin{cases} (m_f - m_g)''(x) = f(x) - g(x) & \text{in } \Omega = (0,1), \\ m_f - m_g(0) = m_f - m_g(1) = 0, \end{cases}$$

and f satisfy (3.11) we know that

$$(m_f - m_g)(x) = -\int_0^1 (f(\sigma) - g(\sigma))G(x,\sigma)d\sigma \text{ for any } x \in [0,1].$$

Thus

$$[-\phi(u'')(x) + \phi(v'')(x)]_+ = \left[\int_0^1 (f(\sigma) - g(\sigma))G(x,\sigma)d\sigma\right]_+ .$$

So that $f(x) \leq g(x)$ implies that $\phi(u'')(x) \geq \phi(v'')(x)$ and , as ϕ is strictly increasing, $-(u-v)''(x) \leq 0$ on $\Omega = (0,1)$. But, since $u - v = 0$ on $\partial\Omega$ we deduce the comparison $u(x) \leq v(x)$ on Ω. Obviously this implies the uniqueness of very weak solution.

To get the quantitative estimate we can adapt the argument used in Diaz (Racsam 2011). Indeed, from the representation formula we get that

$$\int_0^1 u(x)dx = \int_0^1 D(f)(x)dx, \tag{3.40}$$

here $D(f)$. Then can use the following result:

Lemma 3.1 (Crandall-Tartar 1980). *Let X, Y two vector lattices and λ_X, λ_Y be nonnegative linear functionals on X and Y respectively. Let $C \subseteq X$ and $f, g \in C$ imply $f \vee g \in C$. Let $T : C \to Y$ satisfy $\lambda_X(f) = \lambda_Y(T(f))$ for $f \in C$. Then $(a) \Rightarrow (b) \Rightarrow (c)$ where $(a), (b), (c)$ are the properties:(a) $f, g \in C$ and $f \leq g$ imply $T(f) \leq T(g)$,(b) $\lambda_Y((T(f) - T(g))_+) \leq \lambda_X((f - g)_+)$ for $f, g \in C$, (c) $\lambda_Y(|T(f) - T(g)|) \leq \lambda_X(|f - g|)$.Moreover, if $\lambda_Y(F) > 0$ for any $F > 0$, then $(a), (b), (c)$ are equivalent.*

Now, to prove the L^1-estimate we take $C = X = L^1(0, 1 : \delta)$, $Y = L^1(0, 1)$, $\lambda_Y(e) = \int_0^1 e(x)dx$, $T(f) = D(f)$ and

$$\lambda_X(f) = \int_0^1 \left[-\int_0^1 \phi^{-1} \left(\int_0^1 -f(\sigma)G(s, \sigma)d\sigma \right) G(x, s)ds \right] dx.$$

Then, thanks to (3.40) and the weak maximum principle we get (b) of Lemma 3.1 which is the wanted L^1-estimate (2.3).

With respect to the additional regularity of the very weak solution it is enough to use that, from (3.38) $m = \phi(u'') \in C^1((0, 1)) \cap C^0([0, 1])$ and from (3.39) and the above regularity we get that $u \in C^2([0, 1])$.

The proof of the strong maximum principle uses the following estimate: if

$$\begin{cases} -U''(x) = F(x) & \text{in } \Omega = (0,1), \\ U(0) = U(1) = 0, \end{cases}$$

with $F \in L^1(0,1:\delta)$, $F \geq 0$ then there exists a positive constant C such that

$$U(x) \geq C(\int_0^1 F(s)\delta(s)ds)\delta(x) > 0 \text{ for any } x \in (0,1)$$

(That was proved first by J.M. Morel and L. Oswald (unpublished 1985) and later by H. Brezis and X. Cabré (Some simple nonlinear PDE's without solutions, Bull UMI, 1(1998), 223–262: Lemma 3.2). Thus, applying it to function $m(x)$ we get the strong negativeness for $\phi(u'')$, estimate (2.5), and applying it again, now to (3.39), we conclude the strict positivity of u estimate (2.6).

To prove part c), and more specifically the complete blow up (in the whole interval $(0,1)$) when $f \notin L^1(0,1:\delta)$ we truncate f generating $f_n(x) = \min(f(x), n)$. Now, if u_n is the associated solution ($f_n \in L^\infty(0,1) \subset L^1(0,1:\delta)$) then $u_n(x) \geq \alpha(\|f_n\|_{L^1(0,1:\delta)})\delta(x)$, for a suitable increasing function α such that $\alpha(n) \nearrow +\infty$ as $n \nearrow +\infty$, which implies that $u_n(x) \nearrow +\infty$ for any $x \in (0,1)$. The proof of Theorem 2.1 is now completed.

3. Remarks on the numerical approach of very weak solutions.

Remark 4.1. It takes sense to search for some discrete approximation $D^N[f]$ of the nonlocal operator $D[f]$) given in Theorem 3.1 (but without approximating the function f) allowing to compute an approximation u^N of the solution u in a faster way. So $D^N : L^1(\Omega, \delta) \to S^N$, where S^N is a *specialized* (finite dimensional) subspace of $L^1(\Omega)$. In the linear case ($\phi(s) = s$) the nonlocal operator is given by the direct Green function G_L associated to the (linear) operator L on the open interval $\Omega = (0, 1)$ and with the corresponding boundary conditions

$$u(x) = D[f](x) = \int_0^1 f(s) G_L(x, s) ds$$

and so the approximation $D^N[f]$ can be now searched trough the notion of "Discrete Green Function" $G_L^N(x, s)$ leading to

$$u^N(x) = D^N[f](x) = \int_0^1 f(s) G_L^N(x, s) ds.$$

where we impose now that $u^N \in S^N$.

This point of view (which is very related with the study of the discrete maximum principle) was adopted by many authors:

*P.G. Ciarlet, Discrete variational Green's function. I. *Aequationes Math.* **4** (1970) 74–82.

*P.G. Ciarlet and R.S.Varga, Discrete variational Green's function. II. One dimensional problem. *Numer. Math.* **16** (1970) 115–128.

*C. R. Deeter and G. J. Gray, The discrete Green's function and the discrete kernel function, *Discrete Math.* **10** (1974), 29-42.

*D. H. Mugler, Green's functions for the finite difference heat, Laplace and wave equations. In *Anniversary volume on approximation theory and functional analysis* (Oberwolfach, 1983), Internat. Schriftenreihe Numer. Math. 65, Birkhauser, Basel-Boston, Mass., 1984, 543-554.

*F. Chung and S.-T. Yau: Discrete Green's functions, Journal of Combinatorial Theory (A), 91, (2000), 191-214.

* T. Vejchodský and P. Solín , Discrete Green's function and Maximum Principles, In *International Conference Programs and Algorithms of Numerical Mathematics 13 (in honor of Ivo Babuska's 80th birthday)*, Edited by J. Chleboun, K. Segeth and T. Vejchodsky, Mathematical Institute Academy of Sciences of the Czech Republic, Prague 2006, 247-252.

For instance, we can take as S^N the L-spline subspace $Sp_0(L, \Pi^N, z)$ satisfying the homogeneous boundary conditions (see the Ciarlet and Varga's paper), but many other finite dimensional subspaces can be considered as well.

The continuous dependence estimate (2.3) obtained in Theorem 2.1 allows to extend **to the nonlinear case** (corresponding to constitutive laws ϕ non necessarily linear) the theory on the Discrete Variational Green's Function (usually restricted to functions f in $L^2(\Omega)$) **and on the optimal solvability space** $L^1(\Omega, \delta)$**:**

Corollary 4.2 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $\phi(0) = 0$ and let $f \in L^1(\Omega, \delta)$. For any $k > 0$ let $T_k : L^1(\Omega, \delta) \to L^\infty(\Omega)$ be the truncation operator*

$$T_k(f)(x) = \begin{cases} \min(f(x), k) & \text{if } f(x) \geq 0, \\ \max(f(x), -k) & \text{if } f(x) \leq 0. \end{cases}$$

Let S^N be a finite dimensional subspace of $H_0^1(\Omega)$ and let G_Δ^N be the discrete variational Green function (in the Ciarlet-Varga sense) associated to the problem

$$\begin{cases} -U''(x) = F(x) & \text{in } \Omega = (0, 1), \\ U(0) = U(1) = 0. \end{cases}$$

For any $k > 0$ and $N \in N$ consider the discrete version of the nonlocal operator $D^N : L^\infty(\Omega) \to S^N$ given by

$$D^N[f](x) = -\int_0^1 \phi^{-1}\left(\int_0^1 -f(\sigma)G_\Delta^N(s,\sigma)d\sigma\right)G_\Delta^N(x,s)ds.$$

Let u_k^N and u be the discrete and continuous very weak solutions corresponding to $T_k(f)$ and f respectively (i.e. $u_k^N = D^N[T_k(f)]$ and $u = D[f]$). Then $u_k^N \to u$, in $L^1(\Omega,\delta)$, as k and $N \to +\infty$.

Proof. We have

$$\left\|D^N[T_k(f)] - D[f]\right\|_{L^1(\Omega,\delta)} \leq \left\|D^N[T_k(f)] - D[T_k(f)]\right\|_{L^1(\Omega,\delta)} + \|D[T_k(f)] - D[f]\|_{L^1(\Omega,\delta)}.$$

Then, the first term goes to zero (and even when the norm is replaced by the L^∞ norm) as $N \to +\infty$ thanks to the application of the result by Ciarlet and Varga (1970) and the continuity of the function ϕ^{-1}. Moreover, the second term goes to zero as $k \to +\infty$ thanks to the estimate (2.3) obtained in Theorem 2.1 (because $T_k(f) \to f$ in $L^1(\Omega,\delta)$ as $k \to +\infty$).

Some additional results on the discrete maximum principle for the discrete fourth order problem are presently in progress.

Remark 4.2. Another remark on the numerical approach of problem (B_{SS}) has a different nature.

One of the main points in the approximation of the solution u of a boundary value problem by a finite differences algorithm giving u_h is the study of the convergence $u_h \to u$ when the step h of the discretization goes to zero.

It is well know (see, for instance, P.G. Ciarlet, *Introduction à l'analyse numerique matricielle et l'optimisation*, Masson, Paris, 1982, page 41) that, at least, in the case of linear problems as for instance

$$\begin{cases} m''(x) = f(x) & \text{in } \Omega = (0,1), \\ m(0) = m(1) = 0, \end{cases}$$

that if $\boldsymbol{\varphi}_h = (\varphi_i)_{i=0}^{i=N+1}$, $h = \frac{1}{N+1}$ is the solution of the approximate problem

$$\frac{\varphi_{i+1} - 2\varphi_i + \varphi_{i-1}}{h^2} = b_{h,i}$$

with $b_{h,i} = f(x_i)$, $x_i = ih$, then in fact we have

$$m(x_i) = \varphi_i$$

if we assume that $f(x)$ is a polynomial function and $m(x)$ is a polynomial function of degree 3. Indeed, in that case $m'''(x) = 0$ and then it is enough to apply Taylor formula.

A curious fact, which seems to be very few analyzed in the literature, is that we can produce other finite differences algorithms $\{u_h\}$ for which $u_h(x_i) \equiv u(x_i)$ for any i (and any h small enough) even when f is merely integrable (and in fact for very weak solutions of the problem). The prize we must pay to that is to replace the discrete values $b_{h,i} = f(x_i)$ of the data $f(x)$ on the points x_i by the values of the nonlocal operator $D(f)$ (given in Theorem 3.1) in these points $D(f)(x_i)$.

Corollary 4.2 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a continuous strictly increasing function such that $\phi(0) = 0$ and let*

$$f \in L^1(\Omega, \delta), \text{ i.e. } \int_0^1 |f(x)|\, x(1-x)dx < +\infty. \tag{4.41}$$

Let $h \in (0,1)$. Consider the finite difference algorithm associated to problem (B_{SS}) :

$$\Delta_h^* \phi(\Delta_h^* u) = H_h,$$

where Δ_h^* denotes the progressive difference operator ($\Delta_h^* u(x) = u(x+h) - u(x)$) and

$$H_h(x) = \Delta_h^* \phi(\Delta_h^* D[f])(x)$$

where $D : L^1(\Omega, \delta) \to L^1(\Omega)$ *is the nonlocal operator given in Theorem 3.1. Then*

$$u_h(x_i) \equiv u(x_i) \text{ for any } i \text{ (and any } h \in (0,1)).$$

In the linear case we get the "fourth order" functional equation

$$\begin{aligned} u(x+4h) = \quad & 4u(x+h) - 6u(x+2h) + 4u(x+3h) - u(x) + \\ & -4D(x+h) + 6D(x+2h) - 4D(x+3h) + D(x+4h), \end{aligned} \tag{4.42}$$

where

$$D(y) = D[f])(y).$$

In fact, if we denote by $u(x), w(x), m(x), q(x)$ and $f(x)$ to the deflection, the rotation of the beam, the curvature, the shear and the load density at point x, respectively, then we can prove similar results for the associate finite differences schemes of "third", "second" and "first" order for $w(x)$, $m(x)$ and $q(x)$, respectively. For instance, in the linear case we get

$$\begin{aligned} w(x+3h) = \quad & w(x) - 3w(x+h) + 3w(x+2h) \\ & +3C(x,h) - 3C(x,2h) + C(x,3h), \end{aligned}$$

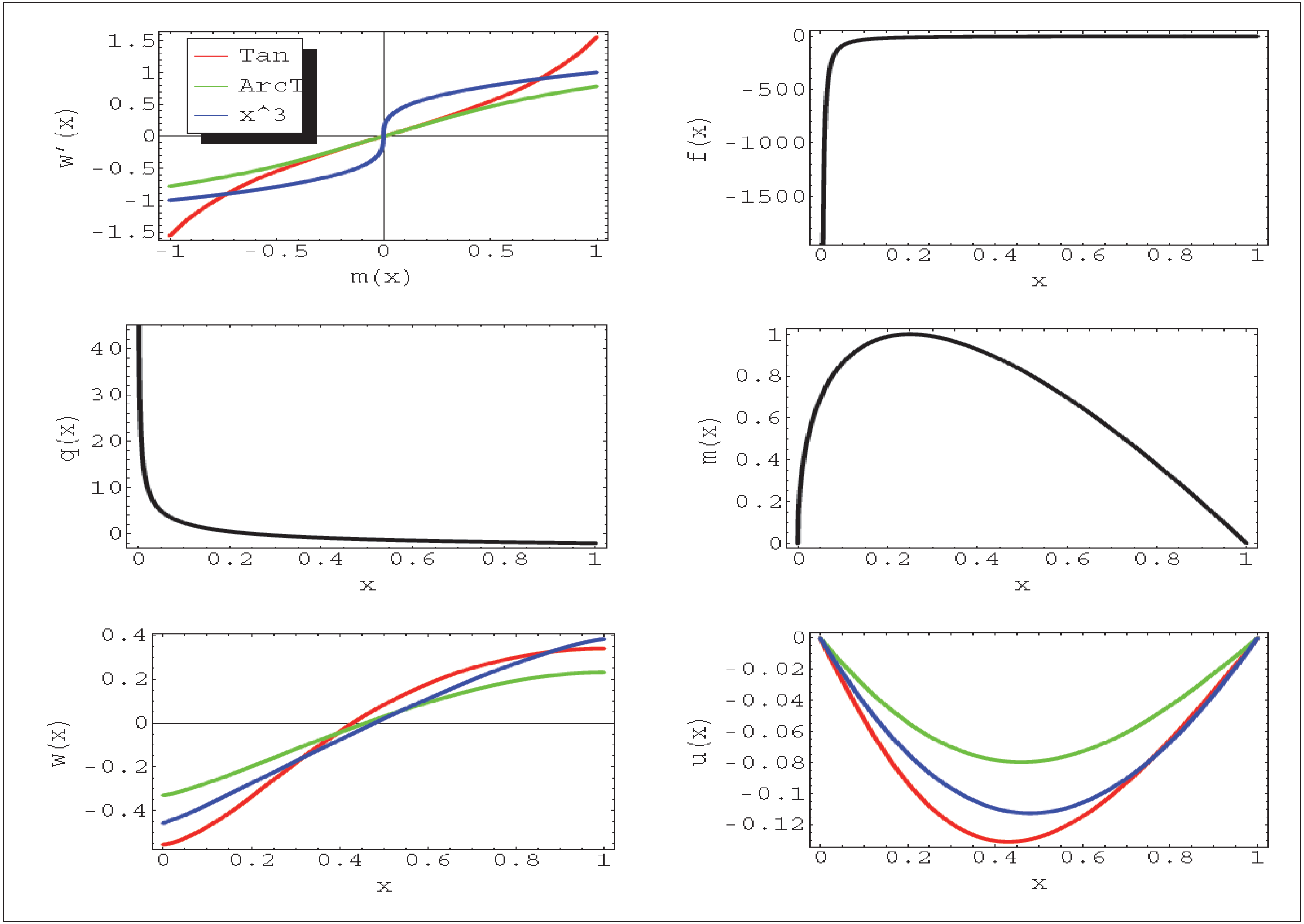
$$m(x+2h) = 2m(x+h) - m(x) - 2B(x,h) + B(x,2h), \tag{4.43}$$

$$q(x+h) = q(x) + A(x,h), \tag{4.44}$$

for the $L^1(\Omega)$ functions $A(x,h)$, $B(x,h)$ and $C(x,h)$ given in the proof of Theorem 3.1.Such functional equations give the exact solution at the interpolating points for any $h = \Delta x$.

Remark 4.3. *Some numerical experiences on very weak solution for the problem* (B_{SS}) *corresponding to different constitutive laws functions* ϕ

{Simply supported beam, load = -x^(-3/2)}



5.1 Some other boundary conditions

In the cantilever case we have:

Theorem 5.1. i) *Assume (3.10). Then the unique very weak solution u of (B_{Cant}) is given by the representation formula*

$$u(x) = \int_0^x \left[\int_0^\sigma \phi^{-1} \left((1-s) \int_0^1 f(t)dt - \int_0^1 (1-t)f(t)dt + \int_0^s (s-r)f(r)dr \right) ds \right) d\sigma, \tag{5.45}$$

ii) *In the more general case of*

$$\phi^{-1}[\int_x^1 \int_s^1 f(\sigma)d\sigma ds] \in L^1(0,1), \tag{5.46}$$

the unique very weak solution u of (B_{Cant}) is given by the representation formula

$$u(x) = \int_0^x \left(\int_0^\sigma \left\{ \phi^{-1}[\int_s^1 \int_t^1 f(r) dr dt] \right\} ds \right) d\sigma. \qquad (5.47)$$

Proof of Theorem 5.1. Part i). We argue as in the proof of part i) of Theorem 4.1 but now for the cantilever beam. We have $u(0) = w(0) = 0$ and $m(1) = q(1) = 0$. In this case the system (3.26)-(3.29) becomes

$$0 = q(0) + \int_0^1 f(t) dt, \qquad (5.48)$$

$$0 = m(0) + 1q(0) + \int_0^1 (1-t) f(t) dt, \qquad (5.49)$$

$$w(1) = \int_0^1 \phi^{-1} \left(m(0) + tq(0) + \int_0^t (t-r) f(r) dr \right) dt, \qquad (5.50)$$

$$u(1) = \int_0^1 \int_0^s \phi^{-1} \left(m(0) + tq(0) + \int_0^t (t-r) f(r) dr \right) dt ds, \qquad (5.51)$$

To obtain $m(0)$, $q(0)$, $u(1)$ and $w(1)$ from (5.48)-(5.51) we proceed as follows: we obtain $q(0)$ from (5.48)

$$q(0) = -\int_0^1 f(t)dt.$$

Then we obtain $m(0)$ from (5.49)

$$m(0) = 1\int_0^1 f(t)dt - \int_0^1 (1-t)f(t)dt.$$

After that we get $w(1)$ from (5.50)

$$w(1) = \int_0^1 \phi^{-1}\left((1-s)\int_0^1 f(t)dt - \int_0^1 (1-t)f(t)dt + \int_0^s (s-r)f(r)dr\right) ds.$$

We obtain $u(1)$ from (5.51)

$$u(1) = \int_0^1 \left[\int_0^\sigma \phi^{-1}\left((1-s)\int_0^1 f(t)dt - \int_0^1 (1-t)f(t)dt + \int_0^s (s-r)f(r)dr\right) ds\right) d\sigma$$

Finally, we use the closed expressions (3.30)-(3.33) for the shear, moment, rotation or vertical displacement of the beam. In particular we get the representation formula (5.45).

Part ii. As in part ii) of Theorem 4.1, it is clear that the hypothesis (3.9) does not allow to justify the above process (for instance, $q(0) = -\int_0^1 f(t)dt$ and so the boundary value of the shear $q(0) = \phi(u''(0))'$ could be infinite). It seems possible to adapt the arguments of the proof of part i) to the case of the general hypothesis (3.11), nevertheless, as in the proof of part ii) of Theorem 4.1 we can get a shorter representation formula (the one given by (5.47)) by a more direct method. Indeed, since

$$\begin{cases} m''(x) = f(x) & \text{in } \Omega = (0,1), \\ m(1) = m'(1) = 0, \end{cases}$$

and f satisfies (5.46) (and in particular f is integrable near $x = 1$) we know, by simply integration, that

$$m(x) = \int_x^1 \left(\int_t^1 f(r)dr \right) dt \text{ for any } x \in [0,1].$$

Thus

$$\begin{cases} u''(x) = \phi^{-1}[\int\limits_x^1 \left(\int\limits_t^1 f(r)dr \right) dt] & \text{in } \Omega = (0,1), \\ u(0) = u'(0) = 0, \end{cases}$$

and so, since $\phi^{-1}(\int\limits_x^1 \left(\int\limits_t^1 f(r)dr \right) dt)$ is integrable near $x = 0$ (thanks (5.47)), formula (5.47) is well justified (notice that the function

$$x \to \int\limits_0^x \left\{ \phi^{-1}[\int\limits_s^1 \int\limits_t^1 f(r)dr dt] \right\} ds$$

is automatically integrable near $x = 0$ once we impose that (5.47) holds).

5.2 Concentrated charges: measures as right hand side

The existence result holds also in the more general class of Radon measures $f \in M(0,1:\delta)$: something very useful to justify the engineers study in with the weight on the beam is concentrated in isolated points. Notice that although the usual Radon measure space (without weight) $M(0,1)$ is a subset of the dual space $H^{-2}(0,1)$ it is not always true that the duality $\langle f, \zeta \rangle_{H^{-2}(0,1), H_0^2(0,1)}$ coincides with the $\langle f, \zeta \rangle_{M(0,1), C^0([0,1])} = \int_0^1 \zeta(x) df$ duality.

5.3 The associated parabolic problems

The above results lead to interesting results on the existence and and asymptotic behaviour of solutions of parabolic problems of the type

$$(HP) \begin{cases} \frac{\partial u}{\partial t} + \frac{\partial^2}{\partial x^2}\phi(\frac{\partial^2}{\partial x^2}u) = f(t,x) & t \in (0,T), x \in (0,L), \\ + \text{ boundary conditions}, & t \in (0,T), \\ u(0,x) = u_0(x) & x \in (0,L). \end{cases}$$

5.4 General higher (2mth-order) equations

The results remains valid for 2mth-order equations with similar nonlinearities (for instance on the derivatives of the mth-order).

5.5 Perturbed nonlinear problems

Many applications can be obtained to nonlinear perturbed problems of the type

$$(NLSP) \begin{cases} \phi(u''(x))'' + \beta(u) = f(x) & x \in \Omega = (0,L), \\ + \text{ boundary conditions } (BC), \end{cases}$$

even when the nonlinear term $\beta(u)$ becomes singular at $u = 0$

6. On the N-dimensional formulation and its numerical approximation.

For the N-dimensional problem on a bounded open set Ω of $\mathbb{R}^N$

$$(P_{Nd})\begin{cases} -\Delta\phi(-\Delta u(x)) = f(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

the notion of very weak solution can be stated in the following terms:

Definition 6.1. Given $f \in L^1(\Omega : \delta)$, with $\delta = dist\,(x, \partial\Omega)$, a function $u \in W^{2,1}_{loc}(\Omega)$ is a "very weak solution" of (P_{Nd}) if $u \in W^{2,1}(\Omega) \cap W^{1,1}_0(\Omega)$, $\phi(-\Delta u)) \in L^1(\Omega)$ and for any $\zeta \in W^{2,\infty}(\Omega) \cap W^{1,\infty}_0(\Omega)$ we have

$$\int_\Omega \phi(-\Delta u(x))(-\Delta\zeta(x))dx = \int_\Omega f(x)\zeta(x)dx.$$

The main result of this section is the following:

Theorem 6.1. *a)* ***Sufficiency.*** *Assume (2.2) as well as*

$$|r| \leq C_1 |\phi(r)| + C_2 \text{ for any } r \in \mathbb{R}. \tag{6.52}$$

Then, for any $f \in L^1(\Omega : \delta)$ there exists a unique very weak solution of (P_{Nd}). Moreover, the (nonlocal) operator $D : L^1(\Omega, \delta) \to L^1(\Omega)$ defined by $D(f) = u$ satisfies that if $D(g) = v$ then the weak maximum principle holds:

$$
\begin{aligned}
f(x) &\leq g(x) \text{ a.e. on } \Omega \textit{ implies} \\
-\Delta u(x) &\leq -\Delta v(x) \textit{ and so } u(x) \leq v(x) \quad \textit{a.e. } x \in \Omega.
\end{aligned}
$$

Moreover, we have the estimate

$$
\begin{aligned}
&\int_\Omega [u(x - v(x)]_+ \, dx &(6.53)\\
\leq &\int \left[\int_\Omega \phi^{-1}\left(\int_\Omega [f(\sigma) - g(\sigma)]_+ G_\Omega(s,\sigma)d\sigma\right) G_\Omega(x,s)ds\right] dx &(6.54)
\end{aligned}
$$

where, in general, $h_+ = \max(0, h)$, *and* $G_\Omega(x, \xi)$ *is the Green function associated to the operator* $-\Delta$ *with homogeneous boundary conditions on* $\partial\Omega$. *Moreover* u *is smoother than said at Definition 2.2 since, at least,* $u \in W_0^{1,s}(\Omega)$ *for any* $1 \leq s < (N-1)$ *and that if* $f \in L^1(\Omega, \delta^\alpha)$ *for some* $0 \leqslant \alpha < 1$ *then* $|\nabla\phi(-\Delta u(x))|$ *belongs to the Lorentz space* $L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$.

b) ***Strong maximum principle.*** *Let* $f \in L^1(\Omega : \delta)$ *with* $f \geq 0$ *a.e.* $x \in \Omega$, $f \neq 0$. *Then the very weak solution satisfies that*

$$\phi(-\Delta u)(x) \geq C\left(\int_\Omega \left[\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(x) > 0, \qquad (6.55)$$

for a.e. $x \in \Omega$, *and*

$$u(x) \geq C(\int_\Omega \phi^{-1}\left\{C\left(\int_\Omega \left[\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right]\delta(s)ds\right)\delta(y)dy\right\}\delta(x) > 0, \qquad (6.56)$$

for a.e. $x \in \Omega$, *and for some positive constant* C *independent of* f.

c) **Necessity.** *Assume that* $f \in L^1_{Loc}(\Omega)$, *such that* $f \geq 0$ *a.e.* $x \in \Omega$. *Then if* $\int_\Omega f(x)\delta(x)dx = +\infty$ *it can not exists any very weak solution of* (P_{Nd}).

The proof follows the same type of arguments than the proof of Theorem 2.1. Indeed, the representation formula now becomes

$$u(x) = \int_\Omega \phi^{-1}\left(\int_\Omega f(\sigma)G_\Omega(s,\sigma)d\sigma\right)G_\Omega(x,s)ds \text{ for a.e. } x \in \Omega, \qquad (6.57)$$

once we know the existence (and positivity) of the Green function $G_\Omega(x,\xi)$ (see, e. g. the books by Stakgold (1998) and Friedman (1964)). Which is now much harder than in the one-dimensional case is the question of the regularity of the very weak solution. Nevertheless, since the function $m(x) := \phi(-\Delta u(x))$ is a very weak solution of the second order problem

$$(P_2)\begin{cases} -\Delta m = f(x) & \text{in } \Omega \subset \mathbb{R}^N, \\ m = 0, & \text{on } \partial\Omega, \end{cases}$$

we can apply the results by J.I. D., J.M. Rakotoson (2009), (2010), which justify the regularity stated $\phi(-\Delta u(x))$. Thanks to the assumption (6.52) we know that $-\Delta u(x) = F$ with $F \in L^1(\Omega)$ given by $F := \phi^{-1}(m)$ and (in fact it is enough to know that $F \in L^1(\Omega, \delta)$) that so we can apply well known results in the literature to end the proof of the regularity stated in Theorem 6.1. The rest of the arguments (the strong maximum and its consequences) remain valid in the N-dimensional case.

Remark 6.1. We can avoid the additional growth condition (6.52), and so getting the existence for any ϕ satisfying (2.2) if we know $m \in L^\infty(\Omega)$ since then $F := \phi^{-1}(m) \in L^\infty(\Omega)$. Obviously this requires some additional information on the datum $f(x)$. For instance, it is enough to know that $f \in L^p(\Omega, \delta)$ for $p > (N-1)$ (see Souplet 2008) or, for instance that

$$0 \leq f(x) \leq \delta(x)^{-\beta} \text{ for some } \beta < 2, \text{ a.e. } x \in \Omega,$$

because then $0 \leq m(x) \leq \delta(x)^\theta$ for some $\theta > 0$ (Diaz-Hernandez-Rakotoso 2012). Of course that, in that case u becomes much more regular than stated in Theorem 6.1. We also point that some additional regularity can be obtained by applying some results due to J.M. Rakotoson, A few natural extension of the regularity of a very weak solution, *Differential and Integral Equations*, **24** 11-12, (2011), 1125-1140.

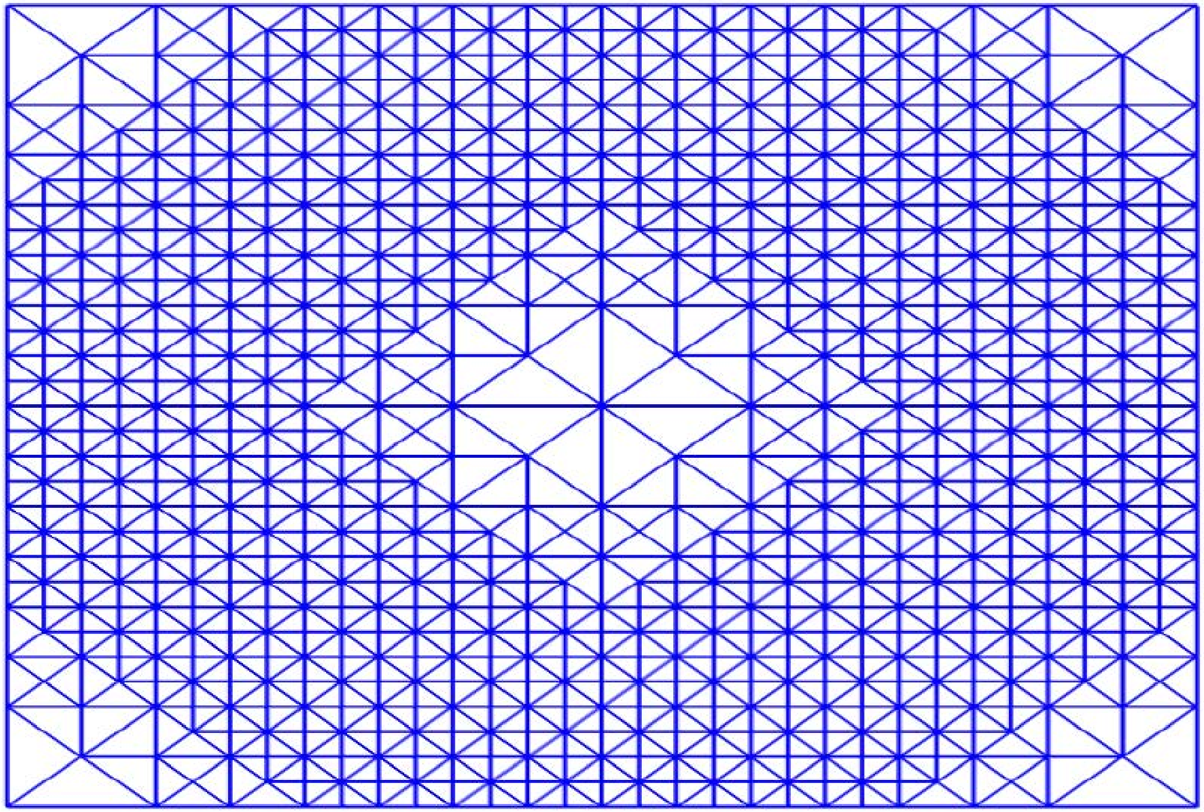
Some numerical experiences

$$\begin{cases} \Delta^2 u = f, & \text{en } \Omega = (0,1)\times(0,1) \\ u = \Delta u = 0, & \text{en } \partial\Omega \end{cases}$$

$$f(x,y) = \frac{1}{|x+\varepsilon|^k}\,\frac{1}{|1+\varepsilon-x|^k}\,\frac{1}{|y+\varepsilon|^k}\,\frac{1}{|1+\varepsilon-y|^k}$$

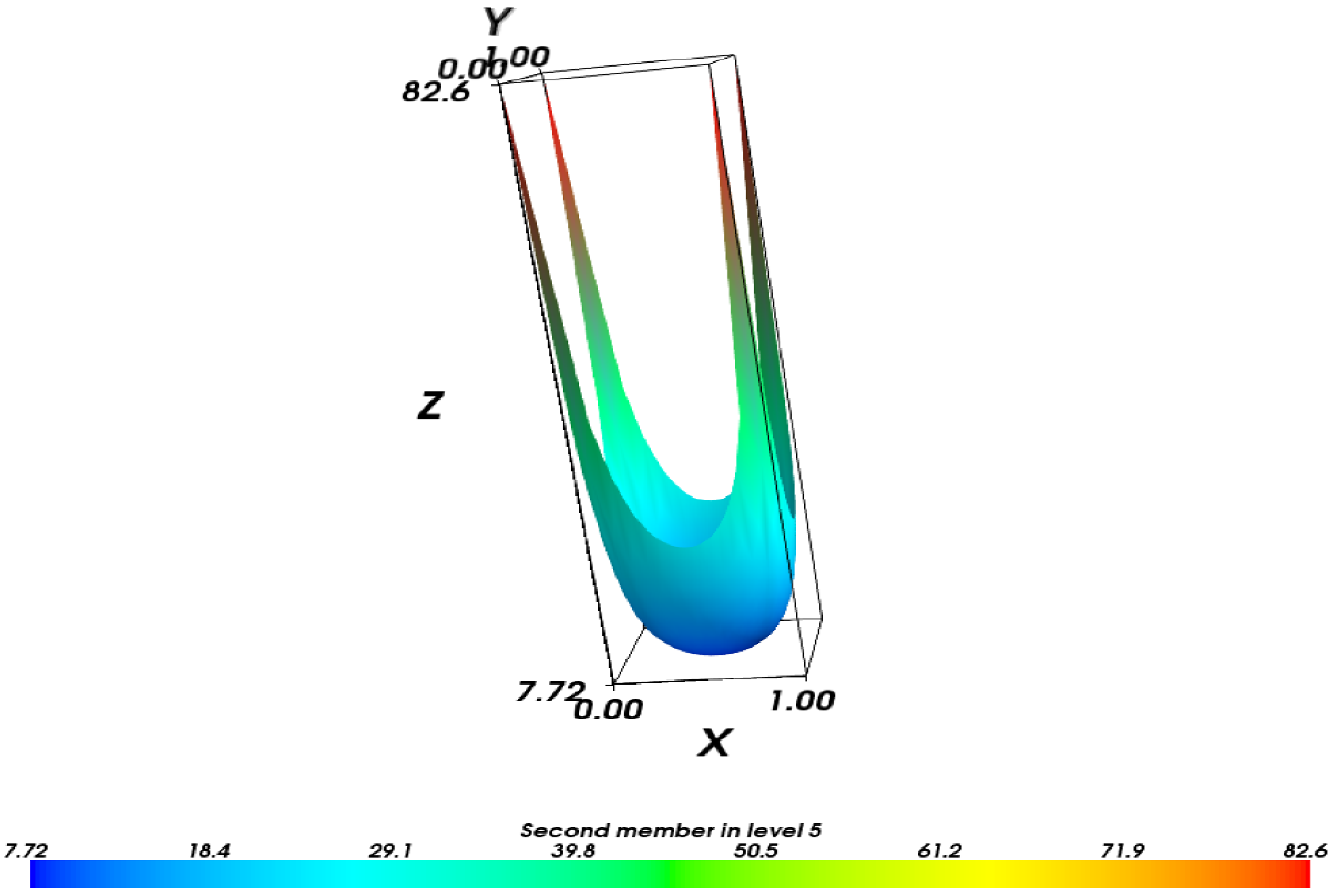
$$k = 1$$

$$\varepsilon = 10^{-1}$$

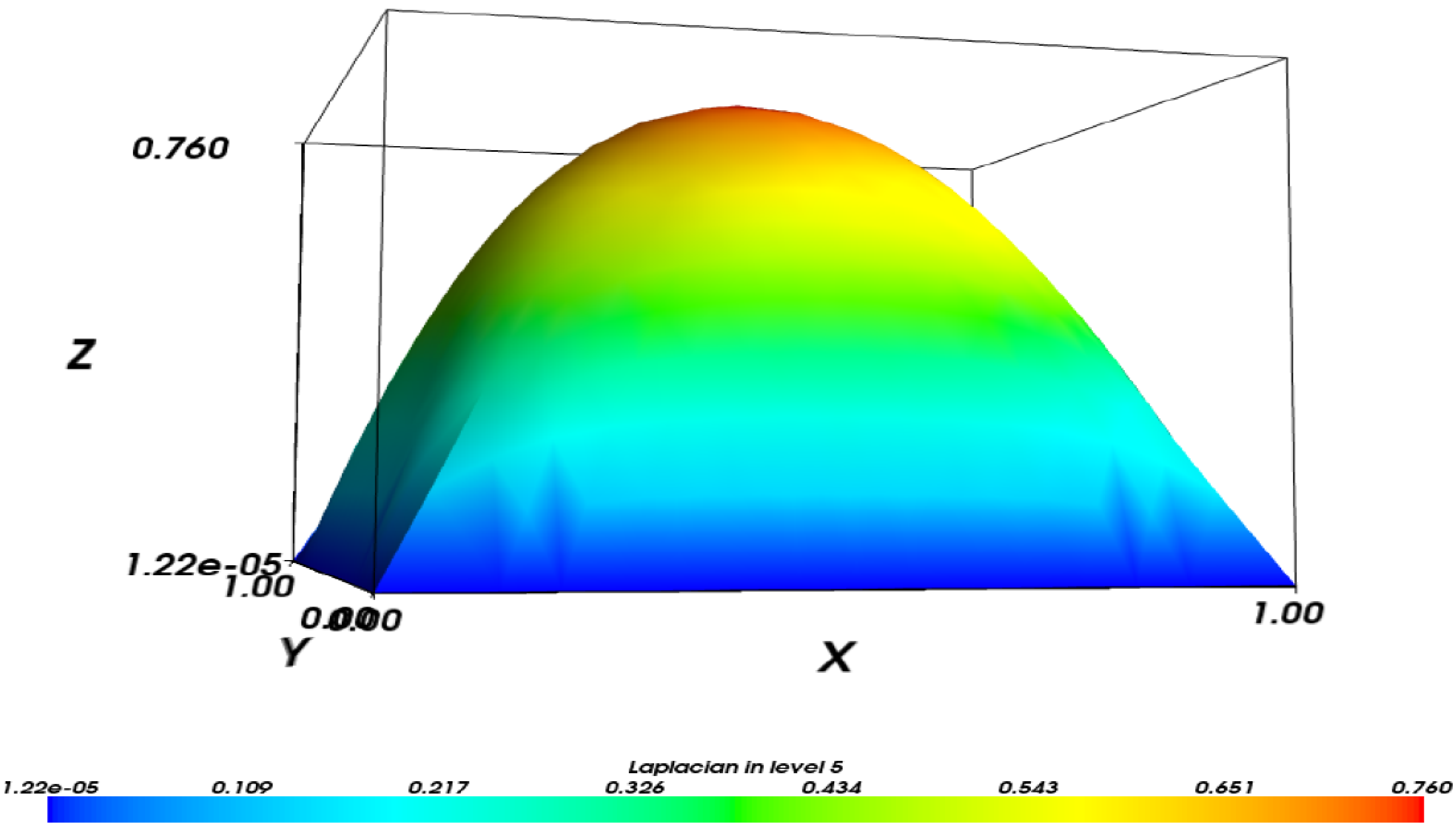


Mesh 5: 901 nodes, 1712 elements

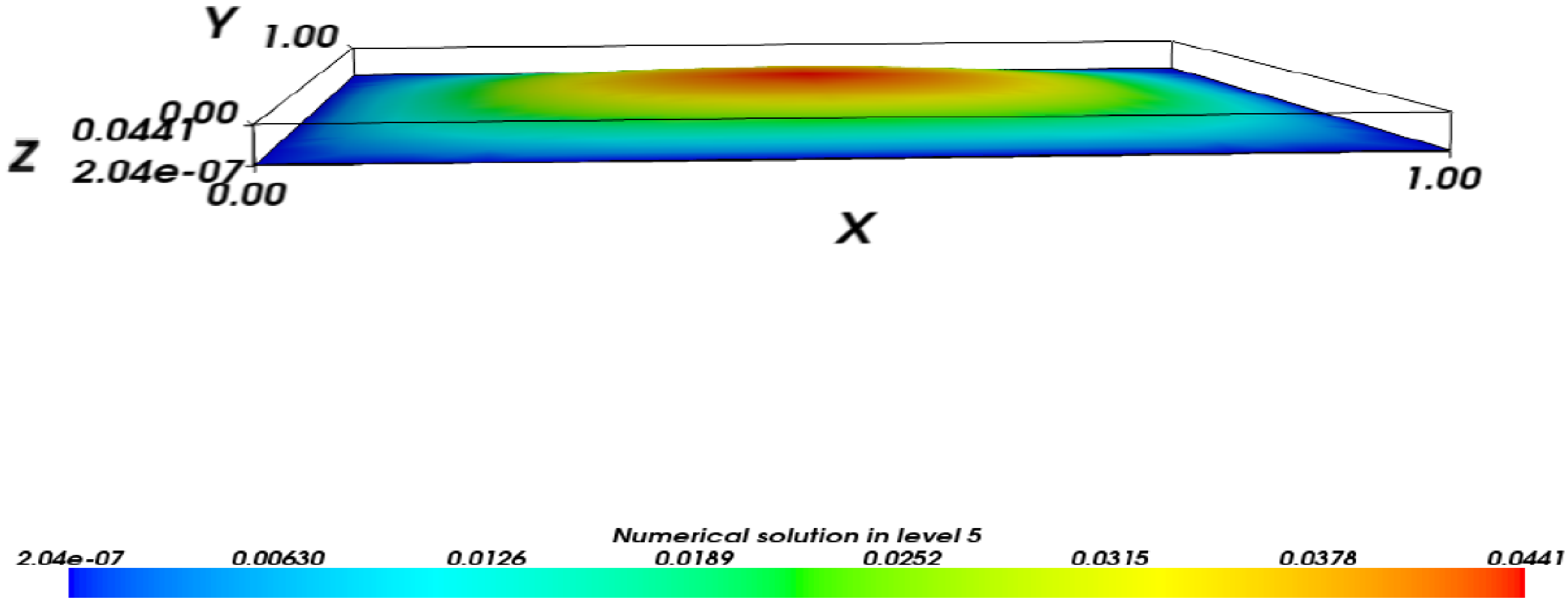
Segundo miembro, $\varepsilon = 10^{-1}$



Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-1}$

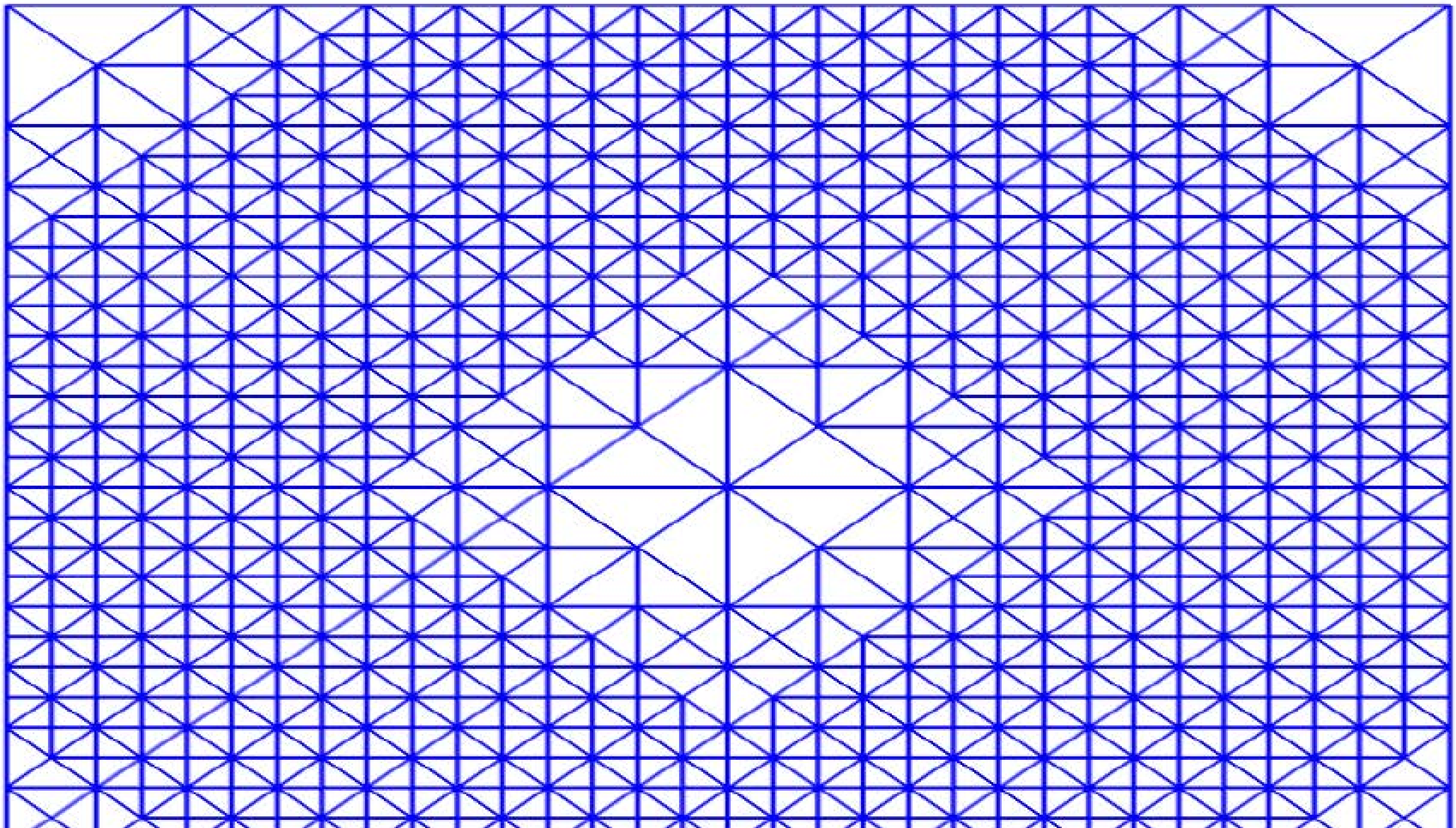


Aproximación numérica de la solución, $\varepsilon = 10^{-1}$

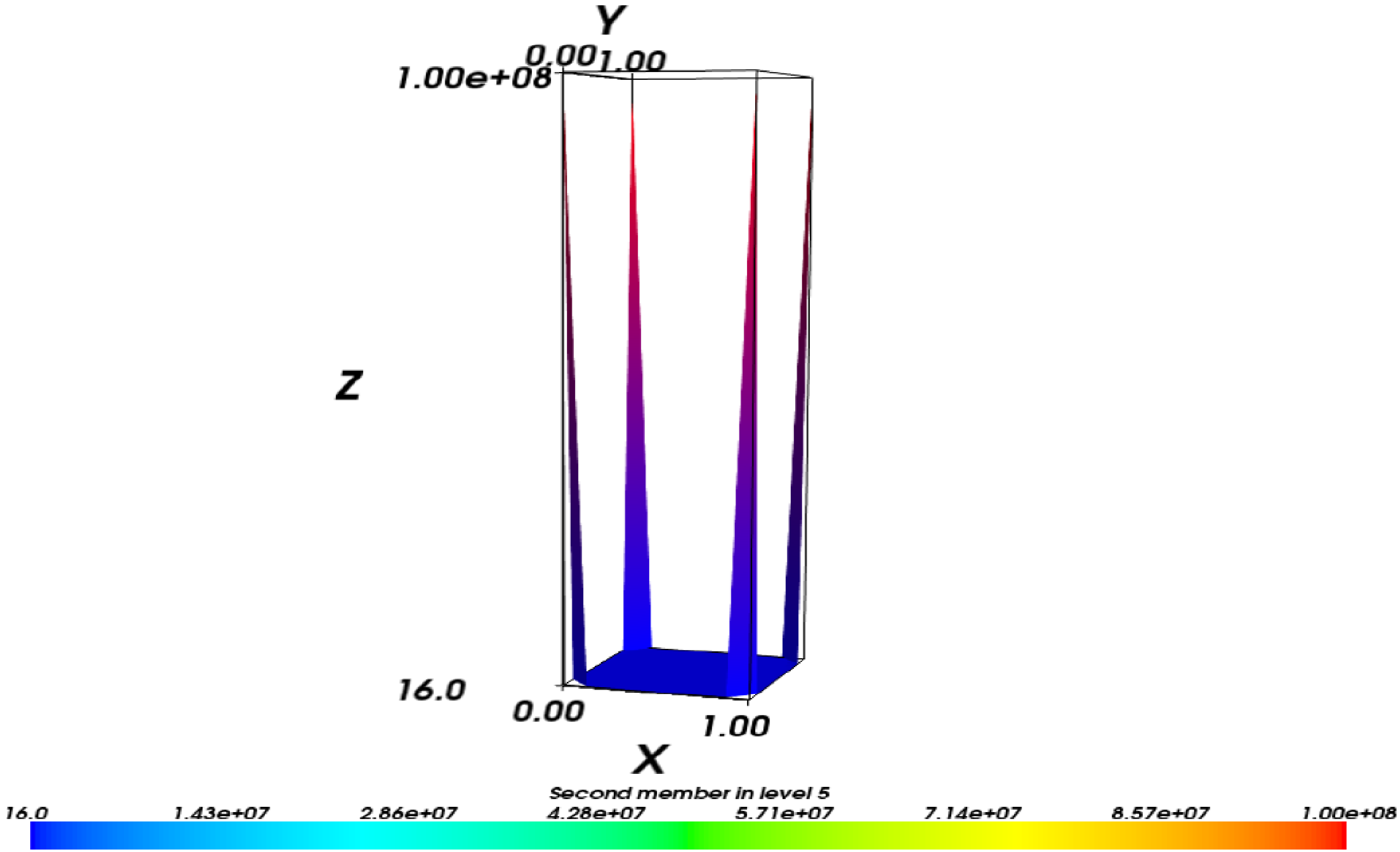


$$\varepsilon = 10^{-4}$$

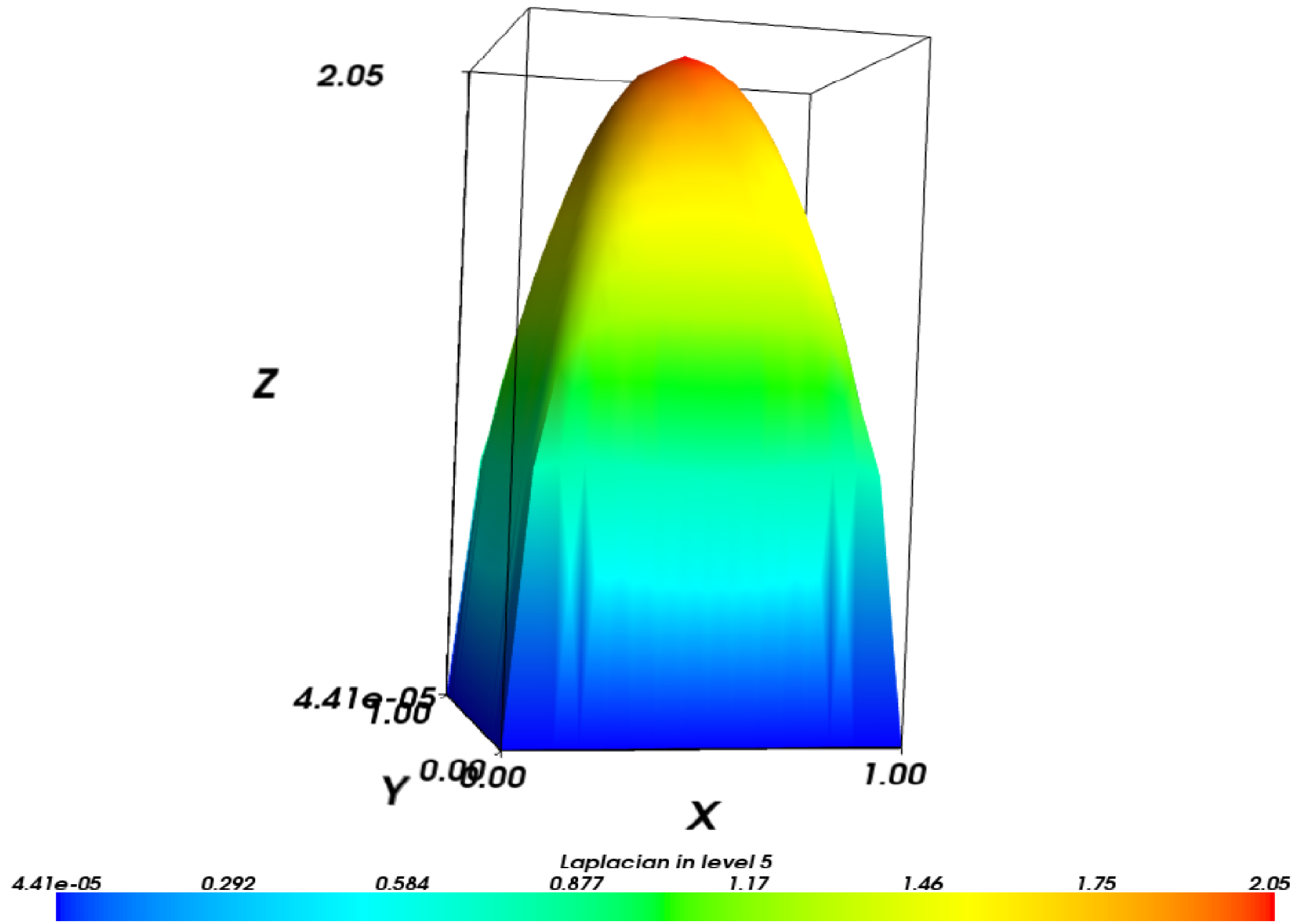
Mesh 5: 901 nodes, 1712 elements



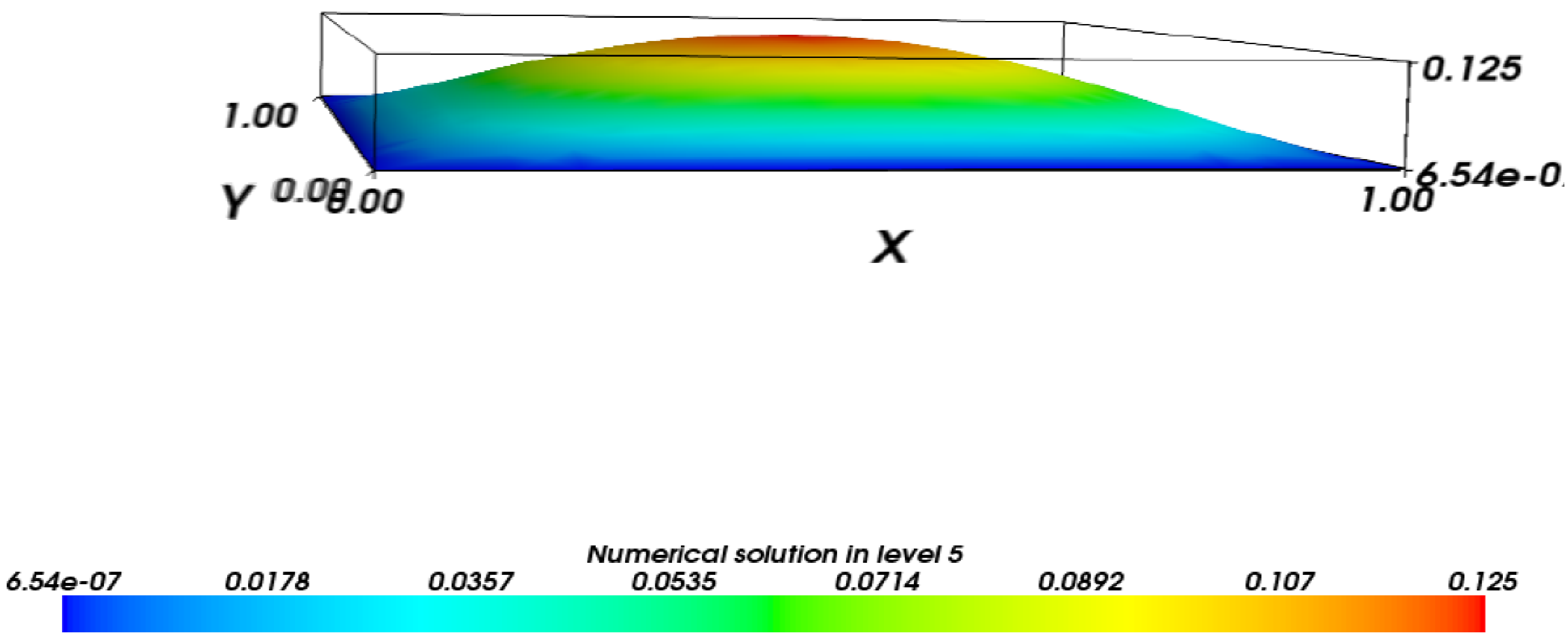
Segundo miembro, $\varepsilon = 10^{-4}$



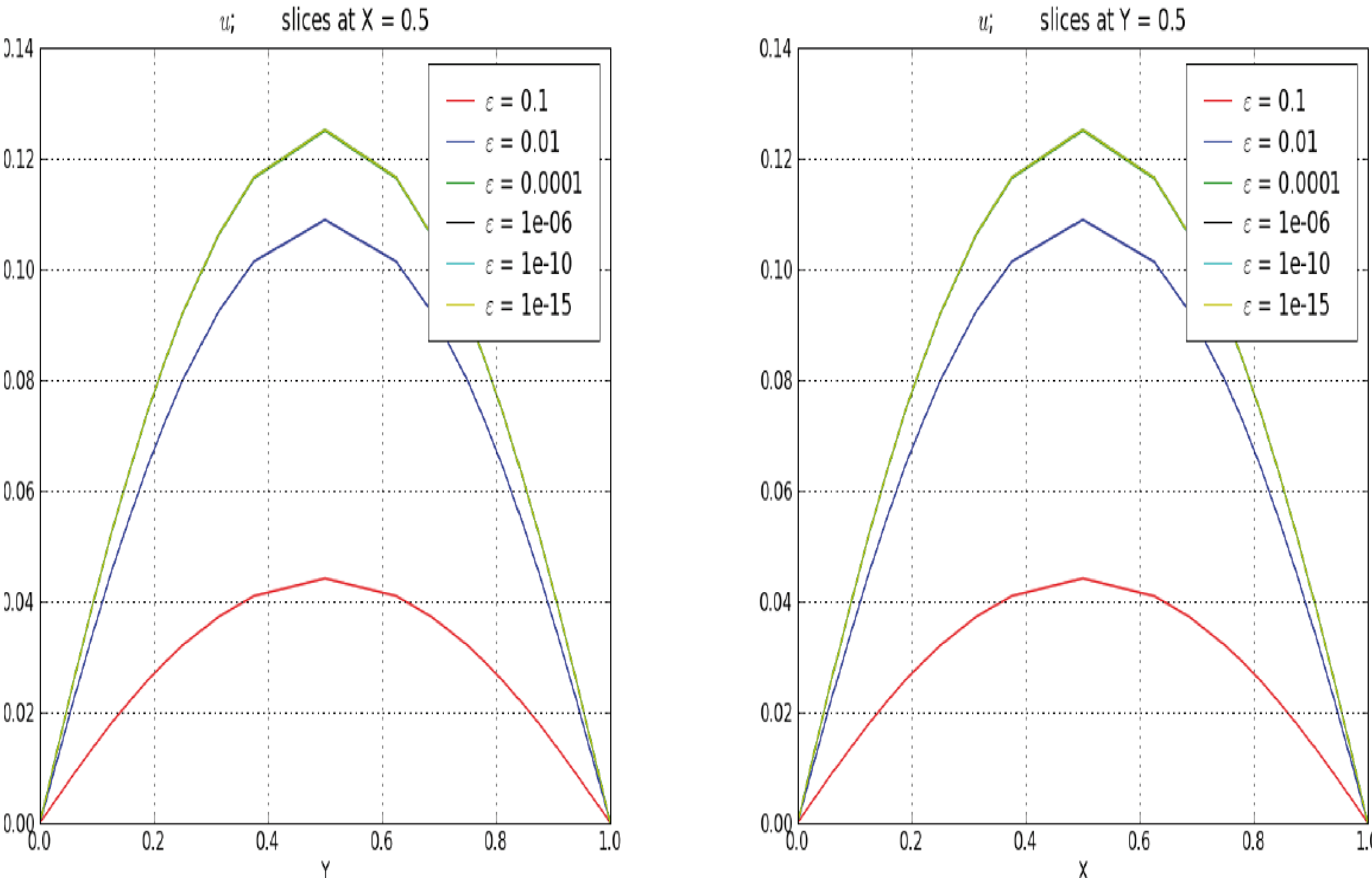
Aproximación numérica del laplaciano de la solución, $\varepsilon = 10^{-4}$



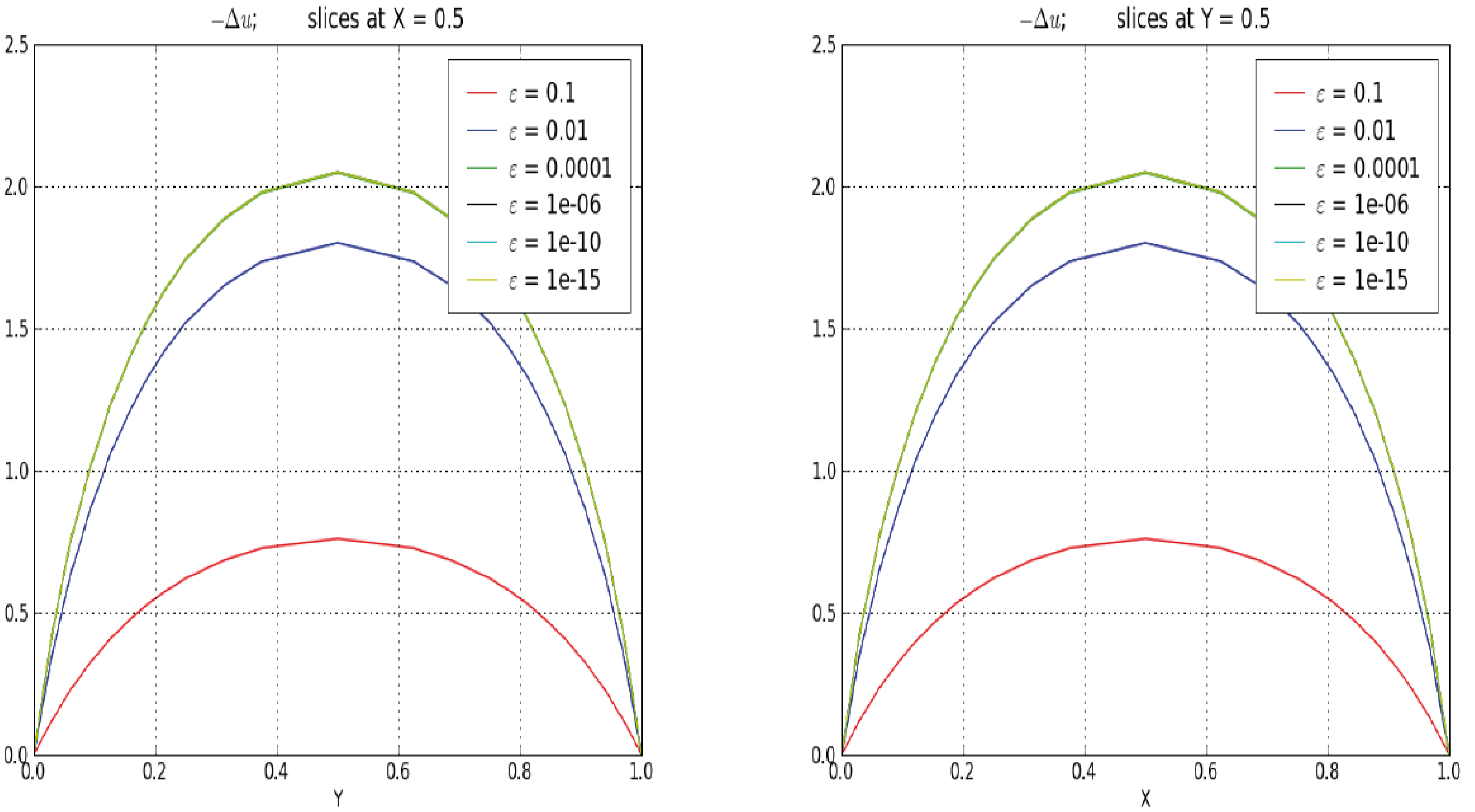
Aproximación numérica de la solución, $\varepsilon = 10^{-4}$



Tests with rhs exploiting on the whole boundary



Tests with rhs exploiting on the whole boundary



Thanks for your attention