# On the approximate controllability of Stackelberg-Nash strategies for some environmental problems

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**Mathématique et environnement**
Paris 23 et 24 mai 2002

# 1. Introduction

$$\frac{\partial \mathbf{y}}{\partial t} + A(\mathbf{y}) = \text{sources}+\text{sinks}+\text{actions}$$
$$\mathbf{y}(.,0) = \mathbf{y}_0(.),$$

("actions" = active controls=unknowns)

Suppose we are not satisfied with the initial state and that it would be "better" to be near (ideal target state) $\mathbf{y}^T$ : Given T>0,
can we "drive the system" (by choosing the actions) in such a manner that **y**(.,T) let as close as possible to $\mathbf{y}^T$ ?

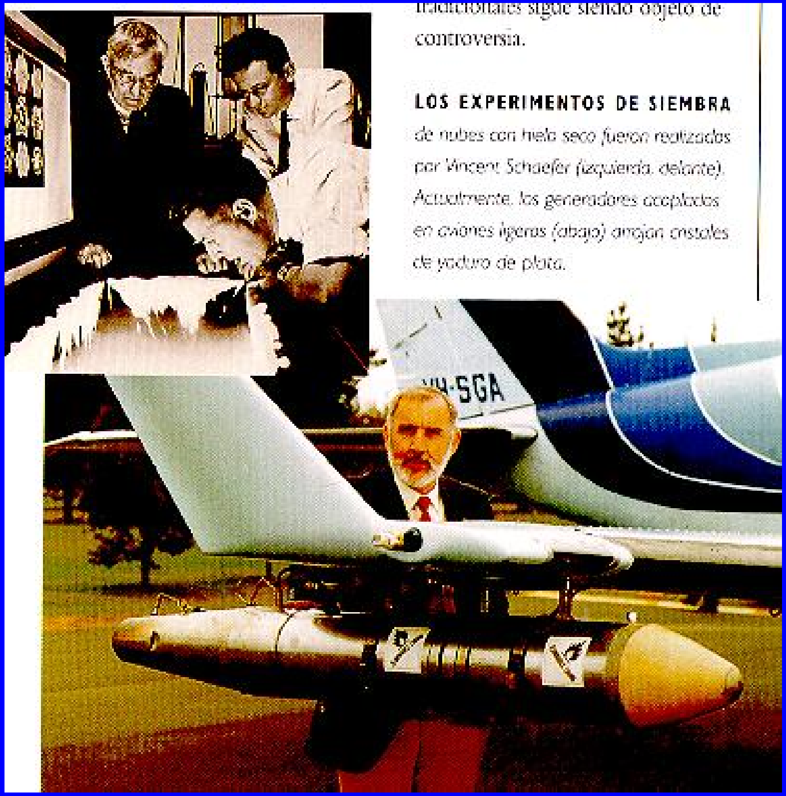
* a concrete $\mathbf{y}^T$: **inverse problem**

* $\mathbf{y}^T$ arbitrary in a functional space X: **approximate controllability** in X

91B76: Environmental economics

# Some examples of antropogenerated actions

## Local actions: cloud seeding



tradicionales sigue siendo objeto de controversia.

**LOS EXPERIMENTOS DE SIEMBRA** de nubes con hielo seco fueron realizados por Vincent Schaefer (izquierda, delante). Actualmente, los generadores acoplados en aviones ligeros (abajo) arrojan cristales de yoduro de plata.

EL PAÍS, jueves 13 de mayo de 1999 — DEPORTES

# El Parma se corona sin rival

El **Marsella** se rinde al primer contratiempo y acaba goleado en la final de la **Copa de la UEFA**

| | |
|---|---|
| MARSELLA | 0 |
| PARMA | 3 |

**Olímpico de Marsella:** Porato; Blondeau, Issa, Blanc, Domoraud, Da Silva (Camara, m. 46); Brando, Bravo, Gourvennec, Pires; y Maurice.

**Parma:** Buffon; Thuram, Sensini, Cannavaro; Fuser, Dino Baggio, Boghossian, Vanoli; Verón (Fiore, m. 76); Chiesa (Balbo, m. 72) y Crespo.

**Goles:** 0-1. M. 26. Blanc cabecea pifiado hacia su portero, Hernán Crespo adivina la cesión, se anticipa y bate a Porato por arriba.

0-2. M. 36. Vanoli ajusta un cabezazo al palo izquierdo tras un centro preciso de Fuser.

0-3. M. 55. Verón centra desde la derecha, Crespo deja pasar el balón y Chiesa fusila a la escuadra.

**Árbitro:** Dallas (Escocia). Mostró tarjeta amarilla a Blondeau.

65.000 espectadores en el estadio Luzhniki de Moscú. Final de la Copa de la UEFA. Campeón, el Parma.

JOSÉ MIGUÉLEZ

No es el Parma un equipo voraz, de esos que siempre quieren más y más. Por eso la final de Moscú concluyó en 3-0. Simplemente en 3-0. La superioridad italiana fue mucho más grande que el resultado. Pero se sintió tan seguro, tan dueño de

Los jugadores del Parma celebran el triunfo con la Copa de la UEFA. / REUTERS

## El alcalde de Moscú ordenó quitar las nubes

# 1972: Cambridge, EE.UU.

10:37 LST–16,100'

11:12 LST–14,250'
26 min. después de la inseminación

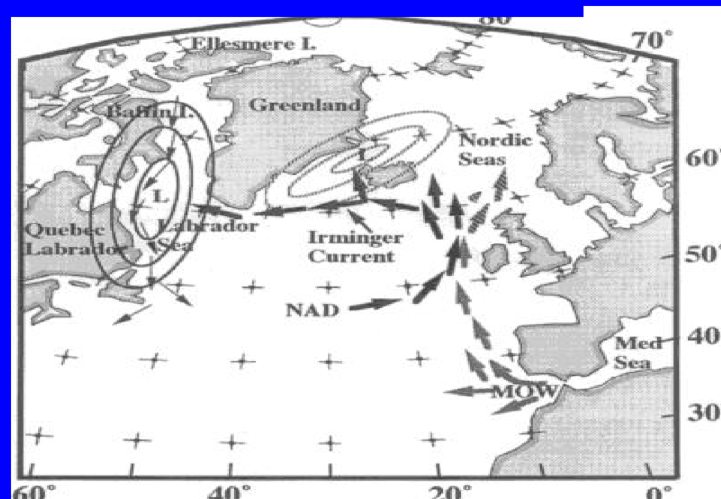
11:20 LST–16,100'
34 min. después de la inseminación

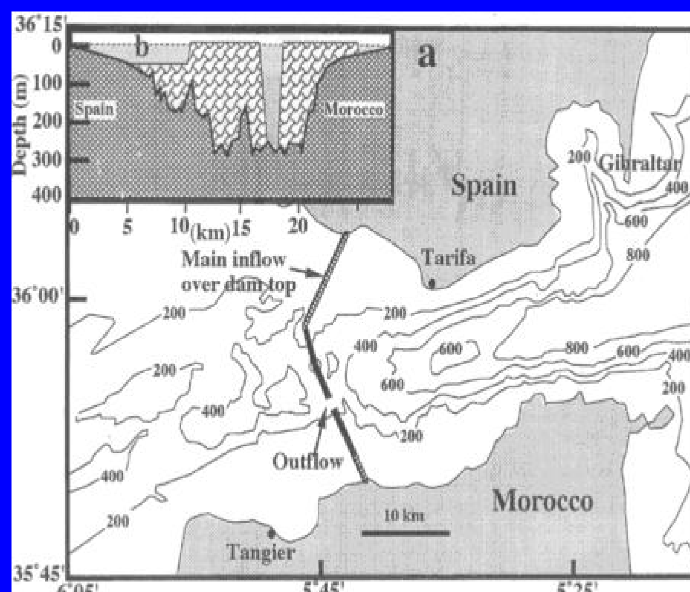
11:31 LST–16,200'
45 min. después de la inseminación

# Global actions: a dam near Gibraltar

**R.G. Thomson**

**American Geophysical Union, 1997**




- Milankovitch insolation at 25°N decreases
- ↓ African monsoons weaken, Nile discharge diminishes
- ↓ Mediterranean hydrologic deficit increases
- ↓ Mediterranean salinity rises, Outflow at Gibraltar increases
- ↓ Outflow water upwelling off Scotland increases
- ↓ Warm surface water diverted to Labrador Sea, cold upwelling water enters Nordic Seas
- ↓ Warmer Labrador Sea, greater advection of moisture into Canada
- ↓ Candian ice-sheet growth begins as Nordic Seas and Europe cool

# The von Neumann problem


John von Neumann (1955 ): *Probably intervention in atmospheric and climate matters will come in a few decades, and will unfold on a scale difficult to imagine at present*

To act on the atmospheric climate by acting on the albedo : **the von Neumann problem**

A single “player” and a single control v (terminology of Control Theory and Games Theory).

In a first part of the lecture, I will report some recent results (Díaz, 2001) on a mathematical formulation (raised to me by J.L. Lions: fax of 15 may 1999).

Energy Balance Models, M.I. Budyko and W. D. Sellers, (1969):

$$y_t - \Delta y + By + C = QS(x)(\beta(y) + v(x)H(y)) \text{ in } (\mathcal{M} - \mathcal{I}) \times (0,$$

$$y = u_i \qquad \Gamma \times (0,T)$$

$$y(0,\cdot) = y_0(\cdot) \qquad \text{on } (\mathcal{M} - \mathcal{I}),$$

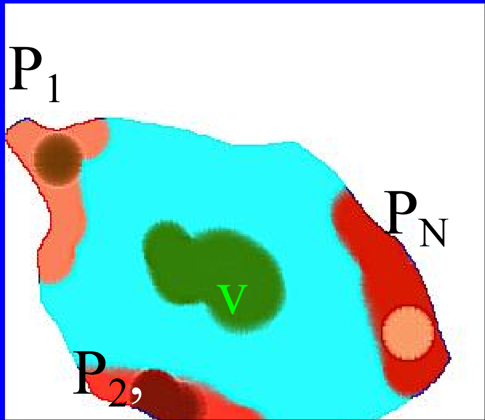
find v(x) such that $y(T:v)=y^T$.

**The second part of the lecture:** **a different problem**, many players (nonnecessarely cooperating among them).

**An example** (an academic scenario) : to mantain "clean " a resort lake), $\Omega$. Now, the state of the system is $\mathbf{y} = (y_1,\ldots,y_n),\ y_i(x,t)$ concentrations of various chemical products in the lake or of living organisms (solutions of a set of diffusion-convection equations).

Several local agents or local plants $P_1, P_2, \ldots, P_N$.

Each plant decide its policy $w_i$. But, there is a different action v made by a representative autority or general manager (**leader** ) in contrast to the rest of the players **followers**).



Formulation introduced in Economy by H von Stackelberg (1934)

The general goal of the manager (control v): to "drive the system" at time T, **y**(T:v), as close as possible to an ideal state $\mathbf{y}^T$. Each plant has (essentially) the same goal but it will be particularly careful to the state **y** near its location.

Let $\rho_i$ be a smooth function given in $\Omega$.

$$\rho_i(x) \geq 0, \rho_i = 1 \text{ near } P_i.$$

Then $P_i$ try to choose such that **y**(T:v), will be "close" to $\rho_i \mathbf{y}^T$ and to achieve this at minimum cost.

$$J_i(v; w_1, .., w_N) = \frac{1}{2} |||w_i|||^2 + \frac{\alpha_i}{2} \left\| \rho_i(\mathbf{y}(.,T) - \mathbf{y}^T) \right\|^2,$$

The "local" controls $w_1, w_2, \ldots, w_N$ assume that the leader has made a choice $v$ and try to find an equilibrium (J.F.Nash,1951) of their cost $J_i$, i.e., one looks for $w_1, w_2, \ldots, w_N$ (as functions of $v$) such that

$$\left.\begin{array}{l} J_i(v; w_1, .., w_{i-1}, w_i, w_{i+1}, .., w_N) \leq J_i(v; w_1, .., w_{i-1}, \widehat{w_i}, w_{i+1}, .., w_N) \\ , \forall \widehat{w_i}, \text{ for } i = 1, .., N. \end{array}\right\}$$

The leader $v$ wants now that the global state (i.e. the state $\mathbf{y}(.,T)$ in the whole $\Omega$ ) let as close as possible from $\mathbf{y}^T$

But,..., not always there exists a Nash equilibrium. We shall prove later some sufficient conditions for the existence and uniqueness of it.

We shall show is that if there is existence and uniqueness of a Nash equilibrium for the followers, then the leader can control the system (in the sense of approximate controllability).

**Remark.**

Stackelberg`s type strategy is not the only one possible! One could also replace the Nash equilibrium by a Pareto equilibrium for the followers (see, for instance, J.L. Lions (1986). Here all the controls $w_i$ agree to work in a strategy where $v$ is the leader, and they agree to work in the context of a Nash equilibrium. Their personal (selfish) interest are expressed in the cost functions $J_i$ as we shall see later.

**Remark.** Rio (1992), Agenda 21, International Scientific and Technological Community, IPGC,...

**Remark.** A general reference:

J.I. Díaz y J.L. Lions, *Matemáticas, superordenadores y control para el planeta Tierra*, Editorial Complutense (Oxford Univ. Press), Madrid, 2002

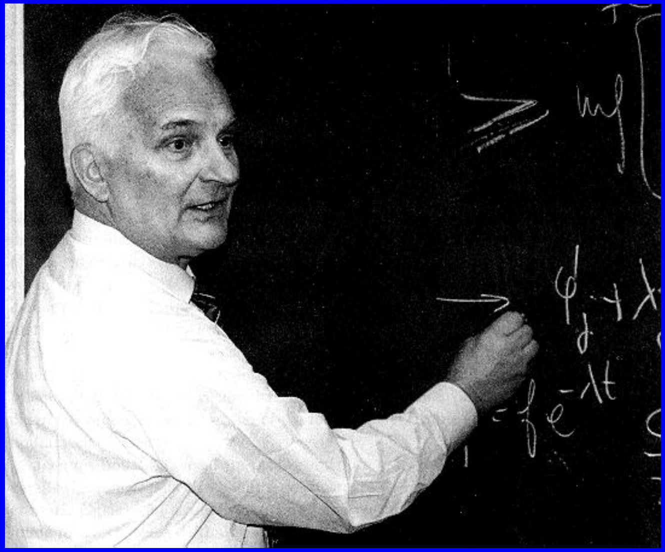
The plan of the rest of this lecture is the following:

2. The von Neumann problem

J.I. Díaz: Modelos matemáticos en Climatología; la conjetura de von Neumann, In *Les Matematiques y les seus aplicacions*, Editorial de la UPV, Valencia, 2001, pp. 67-98

3. The Stackelberg-Nash strategies: the optimal leader action.

J. I. Díaz and J.L. Lions, On the Approximate Controllability of Stackelberg-Nash Strategies. In Proceedings of the Video-conference Amsterdam-Madrid-Venice (December, 1999) on Environment and Mathematics, EMS, Springer Verlag, 2002 (J.I. Díaz, ed.).



# 2. The von Neumann problem

**Energy Balance Models** have a diagnostic character and intended to understand the evolution of the global climate on along time scale.
Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters

Used in the study of the Milankovitch theory of the ice-ages (R.North, 1984)
The distribution of temperature y(x,t) is expressed pointwise after a standard average process, where the spatial variable x is in the Earth's surface

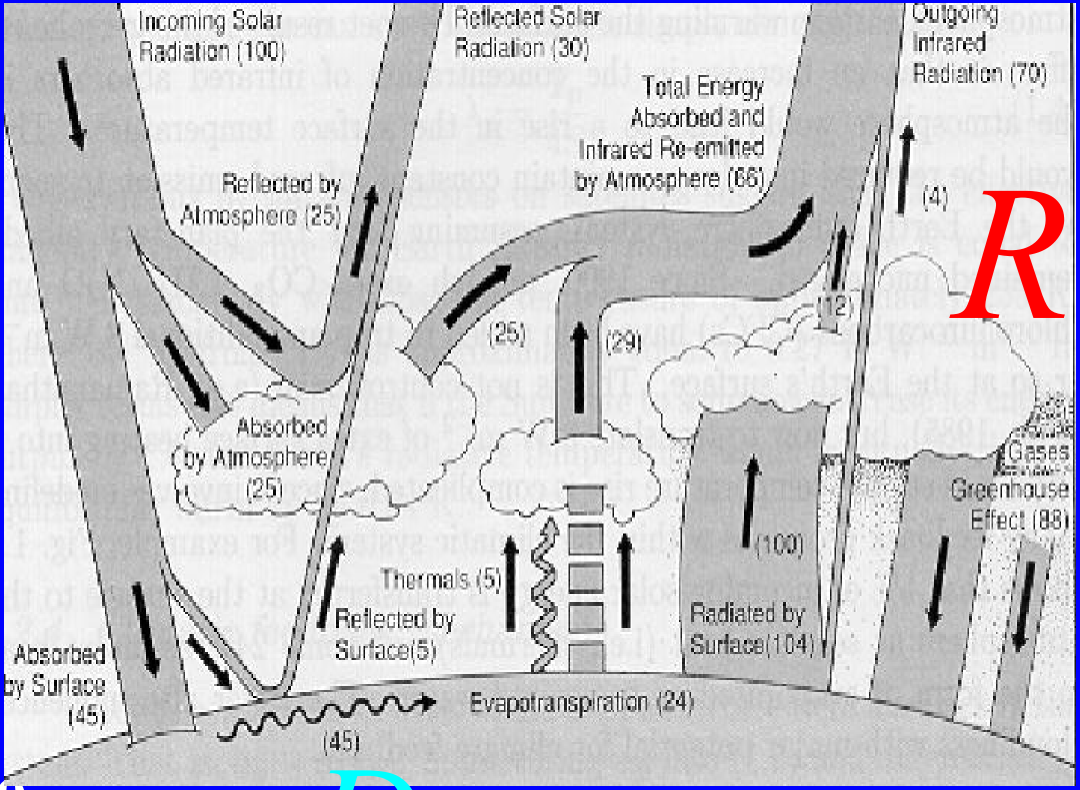
$$y(x,t) = \frac{1}{2\tau|B(x)|}\int_{t-\tau}^{t+\tau}\int_{B(x)} T(y,s)\,dy\,ds$$

**S. Arrhenius (1896), ....**

**W.D. Sellers(1969), M.I. Budyko (1969),....**

The Energy Balance on the Earth's surface:

$$c\frac{\partial y}{\partial t} = R_a - R_e + D$$




$R_e$

**Greenhouse effect**

$R_a$

**Albedo**

$D$

# Constitutive laws

$$R_a = QS(x)\beta(u)$$

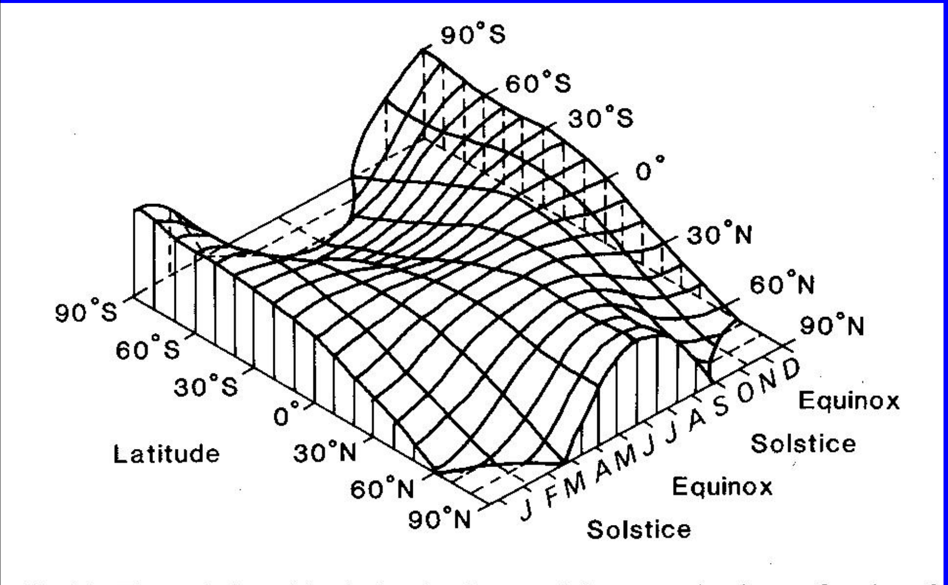
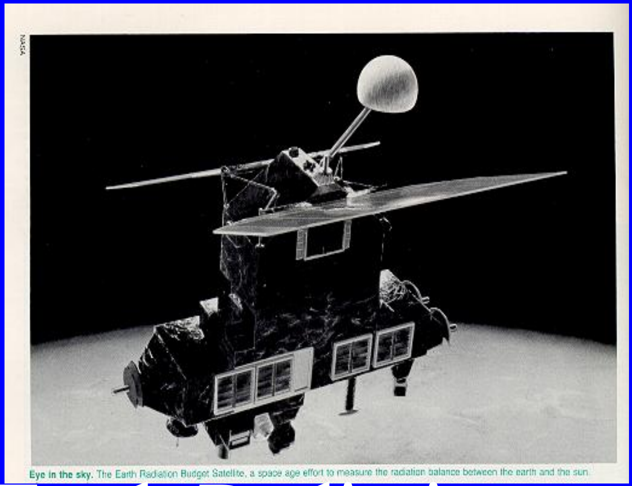
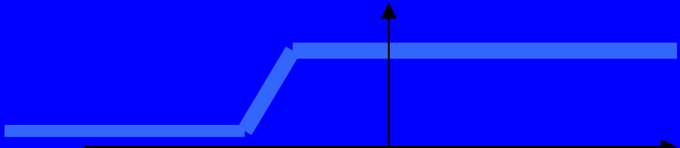



Fig. 2.8. The variation of insolation (at the top of the atmosphere) as a function of



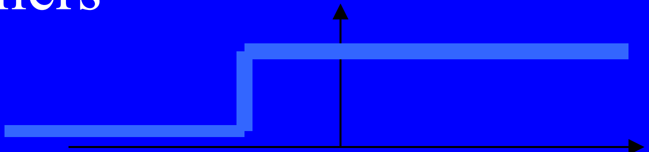

Earth Radiation Budget Satellite

$$\beta(u) = (1 - a(u)) \text{ coalbedo}$$



Sellers

$$\beta(u) = \begin{cases} 0.38 \text{ if } u << -10 \\ 0.71 \text{ if } u >> -10 \end{cases}$$



Budyko **u= -10º**

On $R_e$ :

**Sellers** $R_e = \sigma u^4$ Stefan-Boltzman

**Budyko** $R_e = A + Bu$ Newton

**Empirical relation, Depends of the greenhaus gases, anthropogénicos changes,... (internal variables)**

# On the diffusion operator, *D*: a jerarchy.

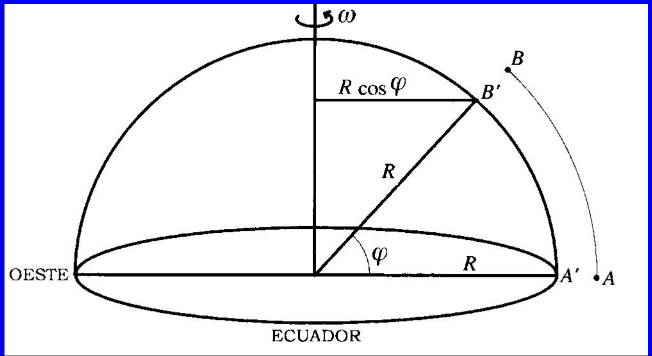
**0-dimensional model *D*=0**

$$c\frac{du}{dt} = Q\beta(u) - R_e(u)$$

**1-dimensional model**

$$D = \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left(k\cos\varphi\frac{\partial u}{\partial\varphi}\right) = \frac{\partial}{\partial x}\left(k(1-x^2)\frac{\partial u}{\partial x}\right)$$

$$x = \cos\varphi$$




In planetary scales O($10^4$ Km) the velocity is eliminated by using the eddy diffusive approximation

# Bidimensional model

$$D = div(k(x)\nabla u)$$

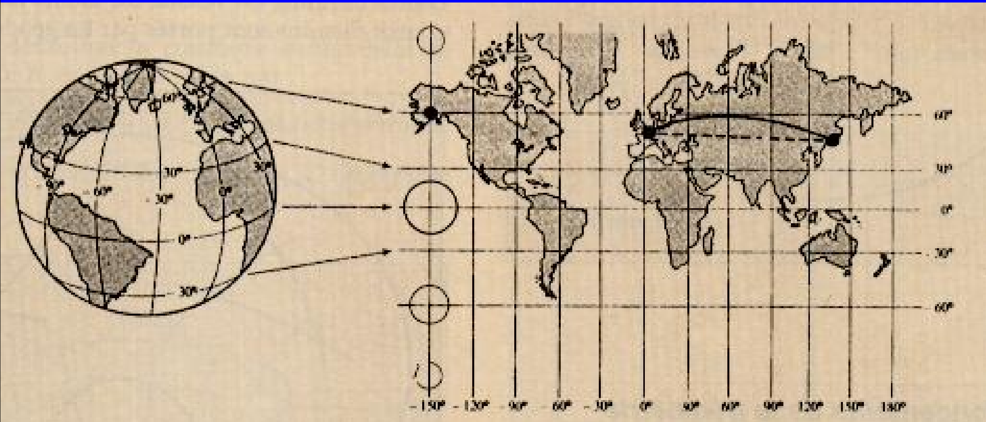
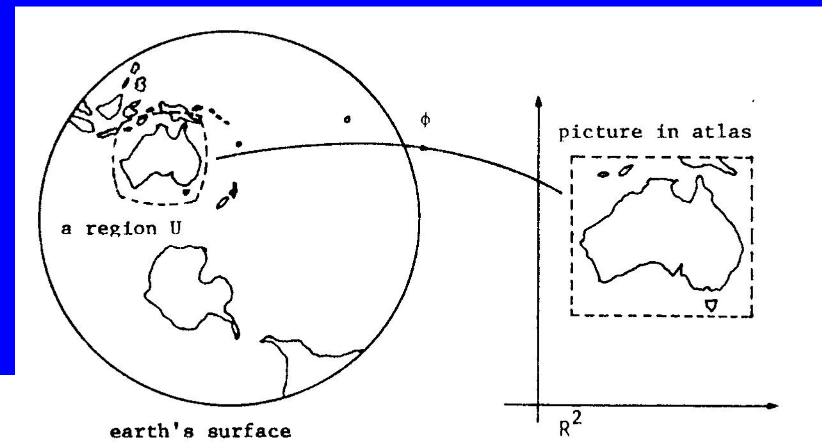
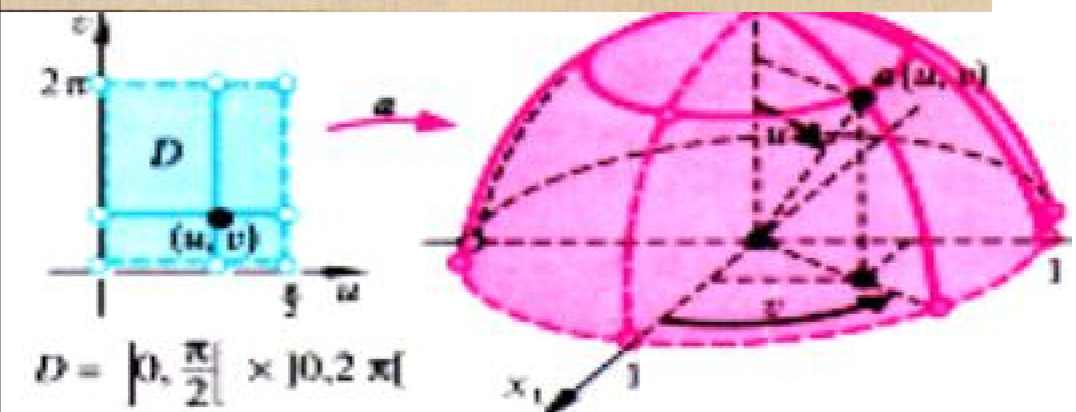



*Figure 10* Projection de *Mercator* : $(\varphi,\theta) \longrightarrow (u = \varphi, v = log(tg(\frac{\theta}{2} + \frac{\pi}{4})))$

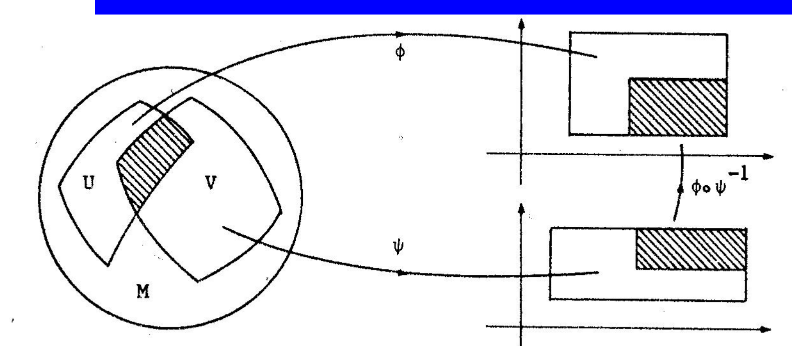



$D = ]0, \frac{\pi}{2}[ \times ]0, 2\pi[$

$a : D \to \mathbb{R}^3$ déf. par $(u, v) \mapsto a(u, v)$

$$a(u, v) = \begin{pmatrix} \sin u \cos v \\ \sin u \sin v \end{pmatrix}$$



$$c\frac{\partial y}{\partial t} = Q\beta(y) + R_e(x,y) + div(k(x)\nabla y)$$

$$y(x,t_0) = y_0(x)$$

Mathematical treatment (North, Hetzer, Drazin,D, Tello,Badii, Arcoya, Hrnandez,...

J.I. Díaz and J. L. Lions (eds.), Mathematics, Climate and Environement Masson, Paris, 1993.

J.I. Díaz and J. L. Lions (eds.), Environement, Economics and Their Mathematical Models}, Masson, Paris, 1994.

J.I. Díaz (ed), The Mathematics of Models for Climatology and Environement, NATO ASI Series, Springer Verlag, 1997.

# The von Neumann conjecture

Artificial modification of the coalbedo near the ice cups.
To simplify the exposition

$$R_e(x,r)=Br+C, \ B>0.$$

x-depending Sellers type coalbedo fuction (Henderson-Sellers).

$$\beta(x,u)=\begin{cases}\beta_i(x) & u<u_i,\\ \beta_i(x)+\left(\frac{u-u_i}{u_w-u_i}\right)(\beta_w(x)-\beta_i(x)) & u_i\le u\le u_w,\\ \beta_w(x) & u>u_w,\end{cases}$$

The modification proposed by von Neumann corresponds to

$$\beta(x, u: v) = \beta(u) + v(x)\chi_\omega(x)H(u)$$

$$H(u) = \begin{cases} 0 & u < u_i, \\ (\frac{u - u_i}{u_w - u_i}) & u_i \leq u \leq u_w, \\ 1 & u > u_w. \end{cases}$$

We assume that the modification is out side the ice cups

$$\omega = \mathcal{M} - \mathcal{I}.$$

$$\begin{aligned} y_t - \Delta y + By + C = QS(x)(\beta(y) + v(x)H(y)) \quad & \text{in } (\mathcal{M} - \mathcal{I}) \times (0, T), \\ y = u_i \quad & \Gamma \times (0, T) \\ y(0, \cdot) = y_0(\cdot) \quad & \text{on } (\mathcal{M} - \mathcal{I}), \end{aligned}$$

**Inverse Problem**: find v(x) such that $y(T:v)=y^T$

**Theorem**

$$Let\ y_0 \in C^{2,\alpha}(\overline{\mathcal{M}-\mathcal{I}}),\ y_0 = u_i\ on\ \Gamma\ and$$

$$\Delta y_0 - By_0 - C + QS(x)\beta(y_0) \leq 0 \quad \mathcal{M}-\mathcal{I}.$$

$$Let\ y^T \in C^{2,\alpha}(\overline{\mathcal{M}-\mathcal{I}}),\ y^T = u_i\ on\ \Gamma\ and$$

$$\Delta y(T:0) \leq \Delta y^T \quad \mathcal{M}-\mathcal{I}.$$

$$\Delta y^T \leq \Delta y(T:0) + QS(x)(\beta_w - \beta_i)(H(y(T:0)) + \delta) \quad \mathcal{M}-\mathcal{I}.$$

Then, there exists

$$v \in C^{0,\alpha}(\overline{\mathcal{M}-\mathcal{I}})\ with\ v(x) \in [-\mu, 0],\ \forall x \in (\mathcal{M}-\mathcal{I})$$

such that $y(T:v)=y^T$ on M-I

**Remark.** To decrease the temperature

$$v(x) \in [-\mu, 0],\ x \in \mathcal{M} - \mathcal{I} \qquad \mu = \beta_w - \beta_i.$$

and so, $y(T : -\mu) \leq y^T \leq y(T : 0)$ on $(\mathcal{M} - \mathcal{I})$.

**Idea of the proof.**

The existence of v(x) is reduced to find a fixed point for

$$(\mathcal{T}v)(x) = \frac{y_t(T : v) + By(T : v) + C - \beta(y(T : v))) - \Delta y_d + QS(x)(v(\gamma y(T : v) + \delta)}{QS(x)(H(y(T : v)) + \gamma y(T : v) + \delta)}$$

$$\gamma > \frac{\beta_w - \beta_i}{u_w - u_i}. \qquad \delta > \gamma(-u_i)$$

*T is a decreasing operator* on a suitable "interval": Amann fixed point theorem on Hölder spaces.

# 3. The Stackelber-Nash strategies for the approximate controllability

Let A be a second order elliptic operator

$$A\varphi = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial \varphi}{\partial x_j}\right) + \sum_{i=1}^{N} a_i(x)\frac{\partial \varphi}{\partial x_i} + a_0(x)\varphi$$

with smooth coefficients and

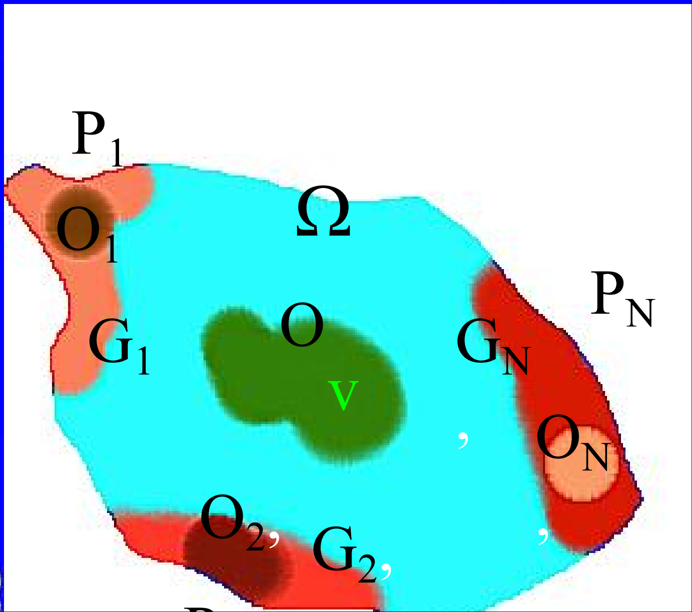
$$\sum_{i,j=1}^{N} a_{i,j}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{N} \xi_i^2, \quad \alpha > 0, \quad x \in \overline{\Omega}.$$

We assume the state equation

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^{N} w_i \chi_i$$

$\chi$ is the characteristic function of $\mathcal{O} \subset \Omega$

$\chi_i$ is the characteristic function of $\mathcal{O}_i \subset \Omega$



**Remark**. The control of the leader is distributed in O and the control of the follower i is distributed in $O_i$.

**Remark**. Since the system is linear there is no restriction in assuming the initial state, sources and sinks to be zero.

We assume that the boundary conditions $y = 0 \text{ on } \partial\Omega \times (0, T)$.

We introduce

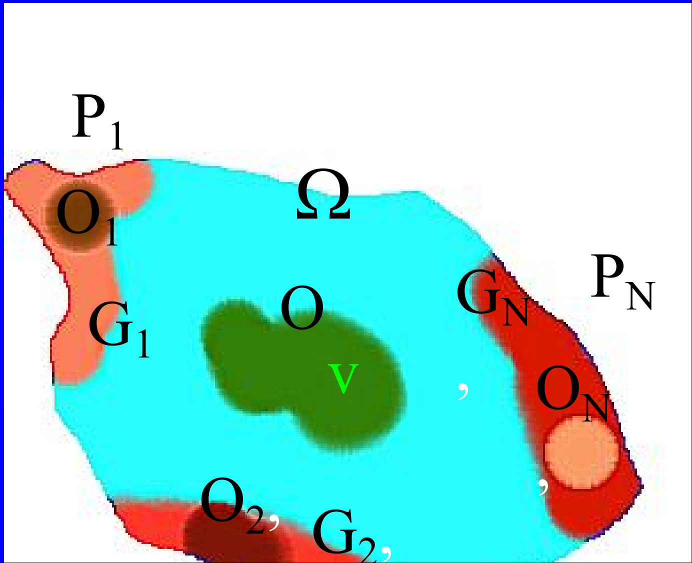
$$\left.\begin{array}{c} \rho_i \in L^\infty(\Omega),\ \rho_i \geq 0, \\ \rho_i = 1 \text{ in a domain } G_i \subset \Omega. \end{array}\right\}$$

$G_i$ represents the region of the lake the plant $P_i$ is particularly interested (the place near $P_i$ for instance!). If $P_i$ is selfish, then $\rho_i = 0$ outside.

We define the cost function $J_i$

$$J_i(v; w_1, .., w_N) = \frac{1}{2} \int_0^T \int_{O_i} w_i^2 dx dt + \frac{\alpha_i}{2} \left\| \rho_i y(T; v, \mathbf{w}) - \rho_i y^{\mathbf{T}} \right\|^2,$$

$||\cdot||$ is the norm in $L^2(\Omega)$



We assume

$$v \in L^2(\mathcal{O}\times(0,T)), w_i \in L^2(\mathcal{O}_i \times (0,T))$$

Given $v \in L^2(\mathcal{O} \times (0,T))$, we now define

$$\left.\begin{array}{c}\mathbf{w} = \{w_1, .., w_N\},\ \text{a } \textit{Nash equilibrium} \text{ for the cost} \\ \text{functions } J_1, .., J_N\ .\end{array}\right\}$$

This Nash equilibrium can be defined as a function of v:

$$\mathbf{w} = \mathbf{w}(v) \text{ or } w_i = w_i(v),\ i = 1, .., N.$$

$$\frac{\partial y}{\partial t} + Ay = v\chi + \sum_{i=1}^{N} w_i(v)\chi_i$$

The initial problem for this system admits a unique solution y(x,t;v,w(v))

**Theorem**. Assume that there exists a unique Nash equilibrium. Then, when v spands $L^2(O)\times(0,T))$, the functions y(x,t;v,w(v)) describe a dense subset of $L^2(\Omega)$. In other words, there is approximate controllability when a Stackelberg-Nash strategy is followed.

**A non constructive proof of the theorem.**

The Nash equilibrium is characterized by

$$\int_0^T \int_{O_i} w_i \widehat{w_i} \, dx dt + \alpha_i \int_\Omega \rho_i^2 (y(T; v, \mathbf{w}) - y^T) \widehat{y_i}(T) dx = 0 \qquad \forall \widehat{w_i},$$

where

$$\left.\begin{array}{l} \dfrac{\partial \widehat{y_i}}{\partial t} + A\widehat{y_i} = \widehat{w_i}\chi_i \ , \\ \widehat{y_i}(0) = 0, \text{ in } \Omega, \ \widehat{y_i} = 0 \text{ in } \partial\Omega \times (0,T). \end{array}\right\}$$

We introduce the adjoint state $p_i$

$$\left.\begin{array}{l} -\dfrac{\partial p_i}{\partial t} + A^* p_i = 0 \text{ in } \Omega \times (0,T), \\ p_i(x,T) = \rho_i^2(x)(y(x,T;v,\mathbf{w}) - y^T(x)) \text{ in } \Omega, \\ p_i = 0 \text{ in } \partial\Omega \times (0,T), \end{array}\right\}$$

$A^*$ adjoint of A.

Multiplying and integrating by parts

$$\int_\Omega \rho_i^2(y(T;v,\mathbf{w}) - y^T)\widehat{y_i}(T)dx = \int_0^T \int_\Omega p_i \widehat{w_i} \chi_i dx\,dt,$$

so that

$$\int_0^T \int_{O_i} (w_i + \alpha_i p_i)\widehat{w_i} dx dt = 0, \ \forall \widehat{w_i},$$

i.e., $w_i + \alpha_i p_i \chi_i = 0.$

We want to show that the set described by y(x,t;v,w(v)) is dense.

By the Hanh-Banach theorem, it suffices to show that if f is given in $L^2(\Omega)$, and we assume that

$$(y(.,T;v), f) = 0, \ \forall v \in L^2(\Omega)$$ then, f=0.

Let us define the auxiliary adjoint problem

$$
\left.\begin{array}{c}
-\dfrac{\partial \varphi}{\partial t} + A^* \varphi = 0, \\
\dfrac{\partial \psi_i}{\partial t} + A \psi_i = -\alpha_i \varphi \chi_i, \\
\varphi(T) = f + \sum_i \psi_i(T) \rho_i^2, \\
\psi_i(0) = 0, \\
\varphi = 0,\ \psi_i = 0 \text{ in } \partial\Omega \times (0,T).
\end{array}\right\}
$$

Multiplying the first (resp. the second) equation by y (resp. $p_i$) we get

$$
\begin{aligned}
&-(f + \sum_i \psi_i(T)\rho_i^2, y(T)) + \\
&\int_0^T \int_\Omega \varphi\left(\tfrac{\partial y}{\partial t} + Ay\right) dx\,dt + \\
&\sum_i (\psi_i(T), p_i(T)) + \\
&+ \sum_i \int_0^T \int_\Omega \psi_i\left(-\frac{\partial p_i}{\partial t} + A^* p_i\right) dx\,dt \\
&= -\sum_i \alpha_i \int_0^T \int_\Omega \varphi p_i \chi_i \,dx\,dt.
\end{aligned}
$$

Then

$$-(f, y(T)) + \int_0^T \int_\Omega \varphi v \chi \, dx dt = 0.$$

Therefore $\varphi = 0$ on $\mathcal{O} \times (0, T)$.

Using Mizohata's Unique Continuation theorem

$$\varphi = 0 \text{ on } \Omega \times (0, T).$$

and $\psi_i = 0$ in $\partial\Omega \times (0, T)$ imply that

$$\psi_i = 0 \text{ in } \Omega \times (0, T), \ \forall \ i = 1, .., N,$$

so that f= 0 and the proof ends.

**A criterion for the existence and uniqueness of Nash equilibria:**

Let

$$\left.\begin{array}{c} \mathcal{H}_i = L^2(\mathcal{O}_i \times (0,T)), \\ \mathcal{H} = \Pi_{i=1}^N \mathcal{H}_i, \\ L_i \widehat{w_i} = \widehat{y_i}(T) \\ \text{which defines } L_i \in L(\mathcal{H}_i; L^2(\Omega)). \end{array}\right\}$$

Since v is fixed, one can write

$$y(T; v, \mathbf{w}) = \sum_{i=1}^{N} L_i w_i + z^T, \ \ z^T \text{ fixed.}$$

With these notations we can rewrite

$$J_i(v; \mathbf{w}) = \frac{1}{2} \|w_i\|^2_{\mathcal{H}_i} \, dx dt + \frac{\alpha_i}{2} \left\| \rho_i (\sum_j L_j w_j - \eta^T) \right\|^2$$

$\eta^T = y^T - z^T$. Then $\mathbf{w} \in \mathcal{H}$ is a *Nash equilibrium* iff

$$(w_i, \widehat{w_i})_{\mathcal{H}_i} + \alpha_i (\rho_i (\textstyle\sum_j L_j w_j - \eta^T), \rho_i L_i \widehat{w_i}) = 0$$
$$\forall\, i = 1, .., N, \ \forall \widehat{w_i}$$

Or, equivalently

$$w_i + \alpha_i L_i^*(\rho_i^2 \sum_{j=1}^{N} L_j w_j) = \alpha_i L_i^*(\rho_i^2 \eta^T), \forall\, i = 1,..,N,$$

$L_i^* \in \mathcal{L}(L^2(\Omega); \mathcal{H}_i)$ is the adjoint of $L_i$

Or, vectorially

$$\left.\begin{array}{c} \mathbf{Lw} = \text{given in } \mathcal{H} \\ \mathbf{L} \in \mathcal{L}(\mathcal{H};\mathcal{H}) \\ (\mathbf{Lw})_i = w_i + \alpha_i L_i^*(\rho_i^2 \sum_{j=1}^{N} L_j w_j). \end{array}\right\}$$

**Proposition**. Assume that $\alpha = \alpha_i \quad \forall i$ and

$\alpha \left\|\rho_i - \rho_j\right\|^2_{L^\infty(\Omega)}$ *is small enough, for any* $i, j = 1,..,N.$

Then **L** is invertible. In particular there is a unique Nash equilibrium

**Proof.** We can write

$$(\mathbf{Lw},\mathbf{w})=\sum_{i=1}^{N}\|w_i\|_{\mathcal{H}_i}^2+\alpha\left\|\sum_{i=1}^{N}\rho_i L_i w_i\right\|^2 \;.$$
$$+\alpha\sum_{i,j=1}^{N}(\rho_i-\rho_j)^2(L_j w_j, L_i w_i)$$

Then, by applying Young's inequality

$$(\mathbf{Lw},\mathbf{w})\geq\gamma\,\|\mathbf{w}\|_{\mathcal{H}}^2,\ \text{for some } \gamma>0.$$

The conclusion is now consequence of the Lax-Milgram theorem.

**Remark.** It is possible to show (see Díaz-J.L. Lions (1999)) that the assumptions are optimal in some suitable sense.

**Remark:** There are infinite controls **v** leading to the approximate controllability.

**The optimal leader action: a contructive proof .** Given $\delta > 0$

we want to find the best leader control **v** in the sense that

$$\inf_{v\in L^2(\omega\times(0,T))} \left\{ \frac{1}{2}\int_{O\times(0,T)} |v|^2\, dx dt, \quad y(T,v) \in y^T + \delta B_{L^2(\Omega)} \right\},$$

B unit ball.

**Theorem.** i) The minmum **v** is given by $v = \varphi\chi$

from the unique solution $\{y, p_1, \varphi, \psi_1\}$ of the Optimality System

$$\frac{\partial y}{\partial t} + Ay + \sum_{i=1}^{N} \alpha_i p_i \chi_i = \varphi\chi,$$

$$-\frac{\partial p_i}{\partial t} + A^* p_i = 0$$

$$-\frac{\partial \varphi}{\partial t} + A^* \varphi = 0,$$

$$\frac{\partial \psi_i}{\partial t} + A\psi_i = -\alpha_i \varphi \chi_i$$

$$\varphi(T) = f + \sum \psi_i(T)\rho_i^2, \ \ p_i(T) = \rho_i^2(y(T) - y^T),$$

$$y(0) = 0, \psi_i(0) = 0$$

$$y = p_i = \varphi = \psi_i = 0 \text{ on } \Sigma$$

with f given by the minimization problem

$$\inf\{I(f) : f \in L^2(\Omega)\},$$

$$I(f) = \frac{1}{2}\int_{O\times(0,T)} |\varphi|^2 \, dxdt + \delta \left\| \widehat{f} \right\|_{L^2(\Omega)} - \int_\Omega f y^T dx$$

ii) The minimization dual problem has a unique solution

**Idea of the Proof.** i) Let

$$F(v) = \frac{1}{2}\int_{O\times(0,T)} |v|^2 \, dxdt$$

$$G(f) = \left\{ \begin{array}{cc} 0 & \text{if } f \ \in y^T + \delta B_{L^2(\Omega)} \\ +\infty & \text{otherwise on } L^2(\Omega) \end{array} \right\}$$

Then, an equivalent formulation is

$$\inf_{v\in L^2(O\times(0,T))} (F(v) + G(Lv)),$$ where Lv=y(T:v).

By Fenchel and Rockafellar's (1967) duality

$$\inf_{v\in L^2(O\times(0,T))}(F(v)+G(Lv)) = -\inf_{f\in L^2(\Omega)}(F^*(L^*f)+G^*(-f))$$

where $L^*$ is the adjoint operator and $F^*$ the conjugate function

$$F^*(\varphi) = \sup_{\widehat{\varphi}}((\varphi,\widehat{\varphi}) - F(\widehat{\varphi})).$$

But, $F^*(\varphi) = \varphi$, $G^*(f) = \delta \left\|\widehat{f}\right\|_{L^2(\Omega)} + \int_\Omega f y^T dx$, and $L^*f = \varphi\chi$

which gives conclusion i). The proof of ii) comes from the fact that I(f) is is strictly convex, continuous and coercive (by the Unique Continuation theorem).

**Remark.** f is characterized as the unique solution of the Variational Inequality

$$(y(T;f) - y^T, \widehat{f} - f) + \delta\left\|\widehat{f}\right\| - \delta\|f\| \geq 0,$$
$$\forall \widehat{f} \in L^2(\Omega).$$

# Remark: Non linear state equations 1. Sublinear case

A) Controllability via lienarisation and fixed point arguments for states equations with A given by .

$$Ay = -\Delta y + f(y), \text{ or } Ay = -\Delta y + div\,\mathbf{f}(y)$$

$$f, \mathbf{f} \text{ sublinear at the infinity,}$$

$$|f(s)| \leq C_1 + C_2|s| \quad \forall s \in \mathbb{R},\ |s| > \overline{M}$$

Henry (1976), Fabre, Puel, Zuazua (1992),...

B) Controllability via a penalized optimal control problem.

$$J_k(v) = \frac{1}{2}\|v\|^2_{L^2(O\times(0,T))} + \frac{k}{2}\|y(T;v) - y_T\|^2_{L^2(\Omega)}$$

passing to the limit, as k increases to infinity.

(Idea of J.L. Lions, Málaga, 1991)

# **Non linear state equations 2. Superlinear case.**

## **Obstruction**

Semilinear equation: Henry (Bamberger),1976, energy method

Universal super and subsolutions over the exterior to the control subdomains

Burger equation

J.I.Díaz, *Comptes Rendus de l'Academie des Sciences de Paris*,t. 1 312,Série I, 519-522, 1991, and J.I.Díaz and A.M.Ramos, *Revista de la Real Academia de Ciencias Exactas, Físicas y Nat. de Madrid,* Tomo LXXXIX,11-30,1995

H. Brezis and E.H. Lieb, Long Range Atomic Potentials in Thomas-Fermi Theory, Commun. Math. Phys,. 65,(1979), 234-246.

Remark. Some related numerical experiences: **control for EBM**

J.I. Díaz and A. M. Ramos, In CD-Rom Proceedings European Congress Computational Methods in Applied Sciences and Engineering (ECCOMAS 2000).

$$y_t - y_{xx} + y^3 = arctgy + u(t)\delta_{1/2} \text{ in } (0,1)\times(0,T),$$

$$y(0,t) = y(1,t) = 0 \qquad t \in (0,T),$$

$$y(x,0) = y^0(x) \qquad x \in (0,1),$$

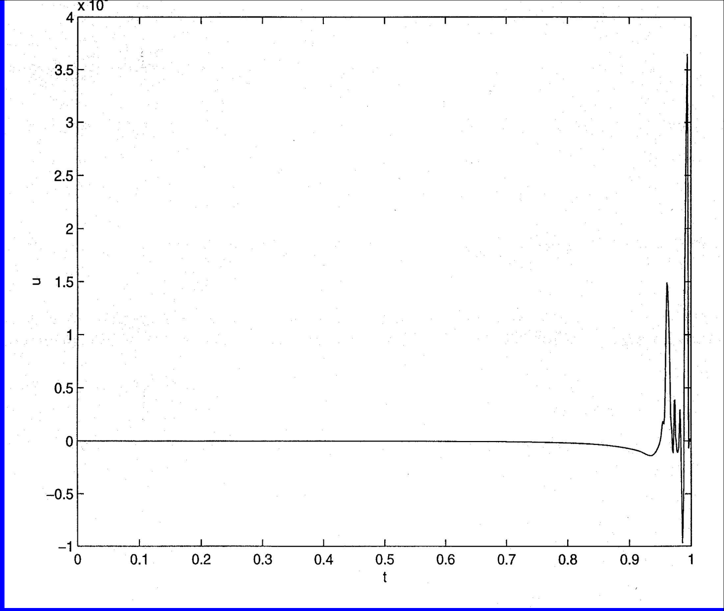
$$J_k(u) = \frac{1}{2}\|u\|_{L^2(0,T)} + \frac{k}{2}\|y(T,.:u) - y_d\|_{L^2(0,1)}$$

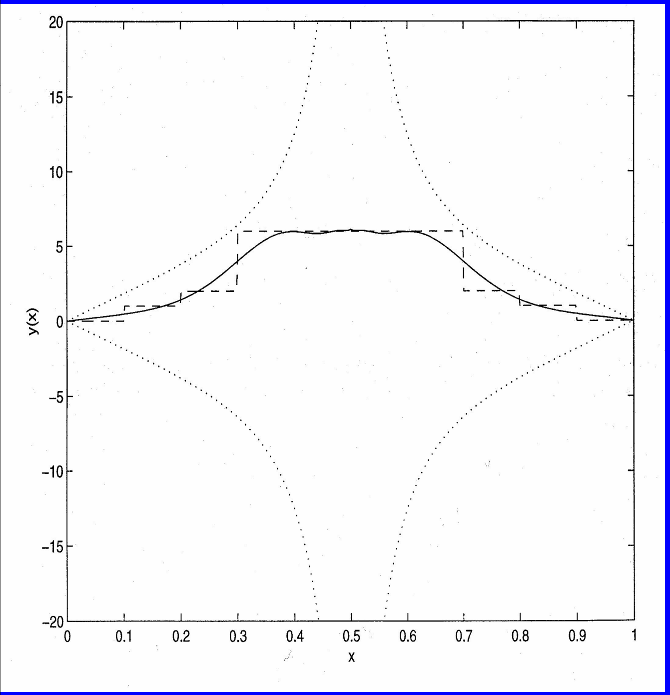
$$k = 10^{12}$$

*The cost of control decreases with complexity:*

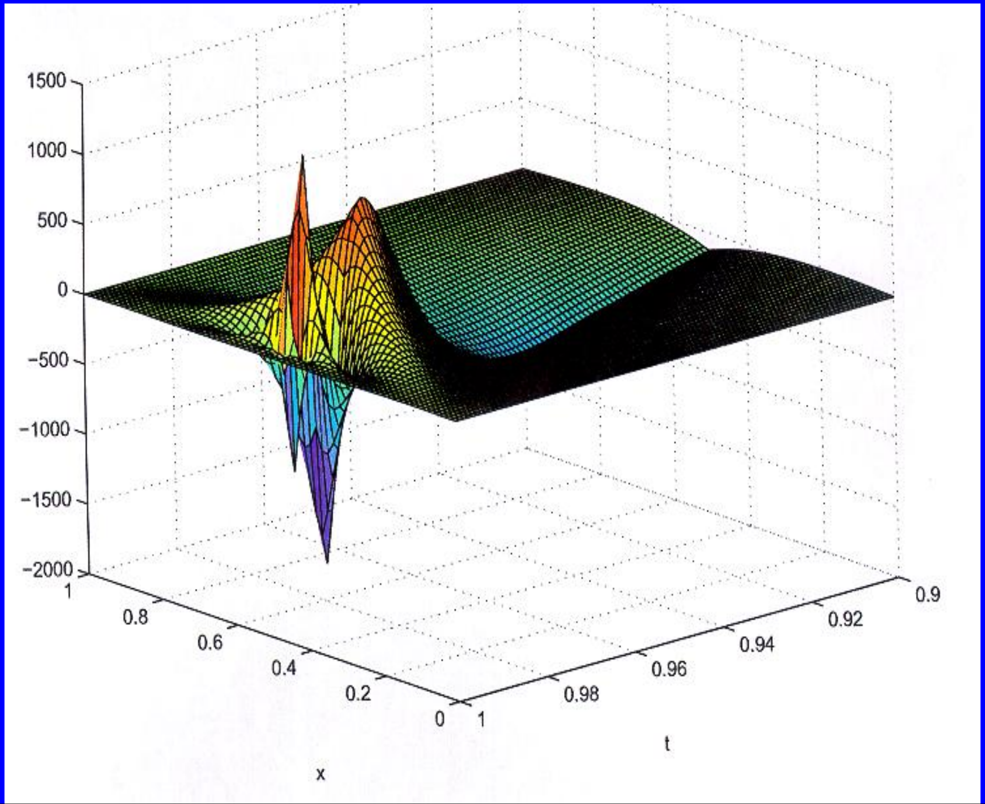
J.I. Díaz and J.L. Lions (Semilinear heat equation with blow-up)

CRAS, 2000, IFIP (Chemonix, 2000: Birkhauser) control




State




Target state