Higher order parabolic potential formulation of stationary shells with some rigid constraints.

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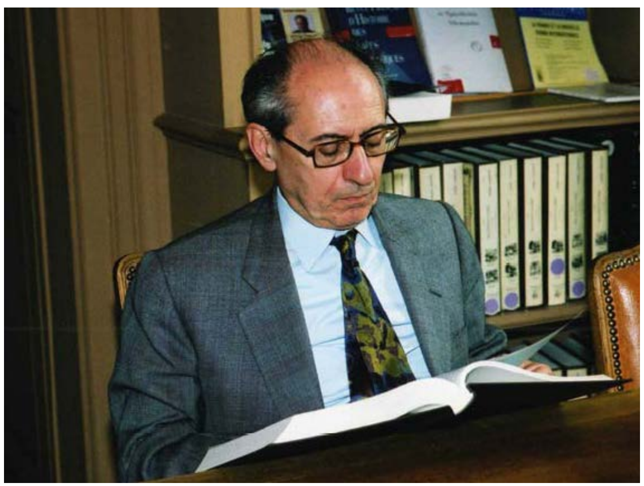

Colloque en l'honneur du 65ème anniversaire d'E. Sanchez-Palencia

Université Pierre et Marie Curie – Paris 6

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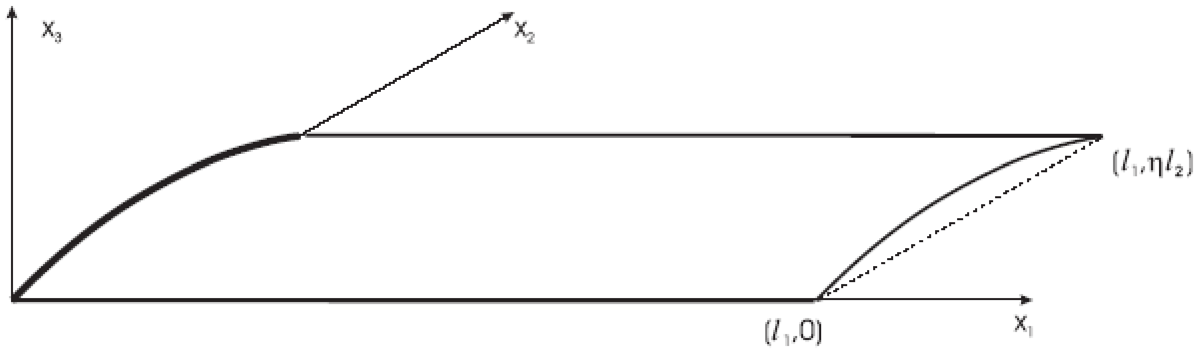
1. Introduction

The present talk, based on a joint paper (to appear in *Asymptotic Analysis*, 2007) with **E. Sanchez-Palencia,**

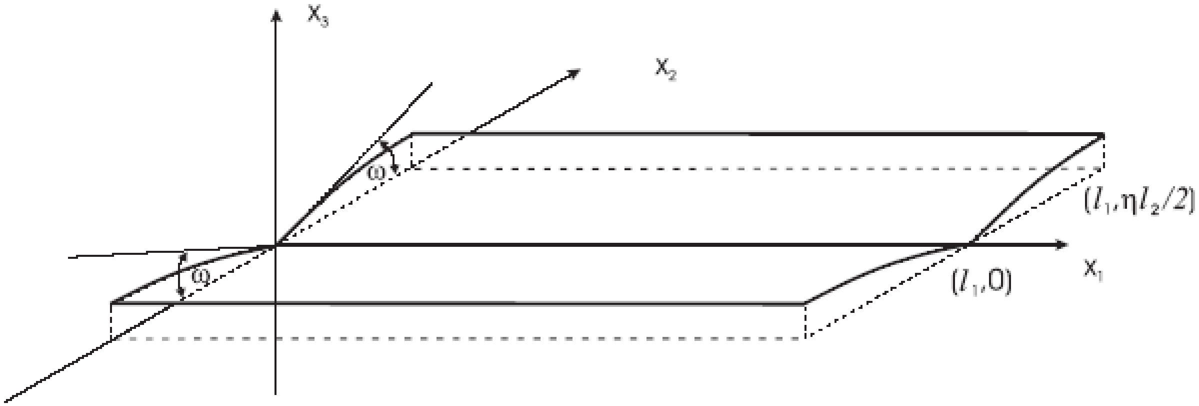
The experience shows that when considering a slender or shell a small curvature in the transversal direction to the main length supply an extra rigidification with respect to the planar case:

* flexible steel retractable meter tape measure, …,

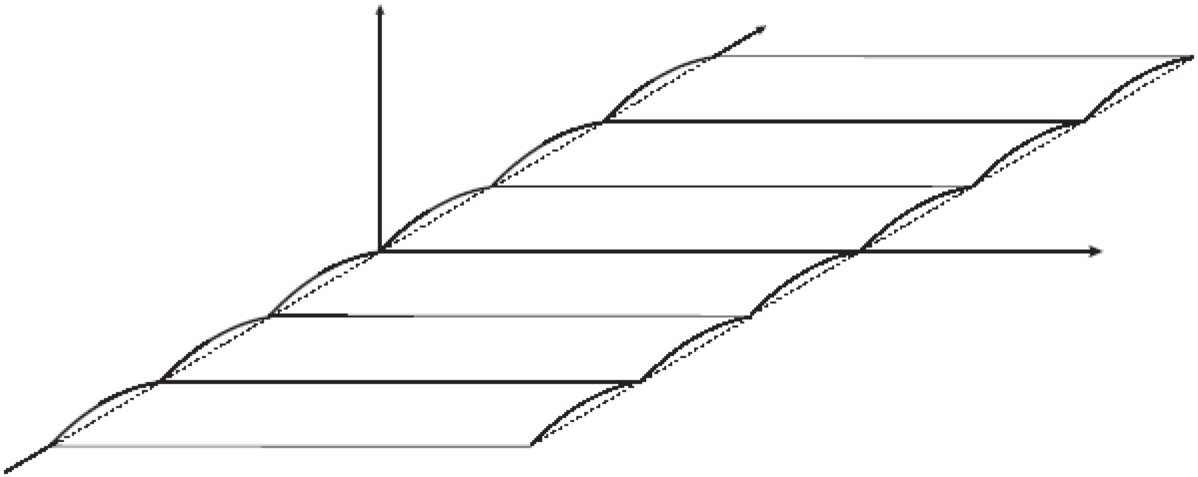
We shall carry out the study of the asymptotic modelling of such kind of shell structures



We also will consider more sophisticated structures formed by coupling two of such basic shells by means of an edge with slight folding



as well as the case of an infinity set of shells obtained by the periodic repetition of the basic structure



The consideration of this type of periodic structures is motivated by some of the structures designed by the outstanding engineer Eduardo Torroja (Madrid, 1899-1961).

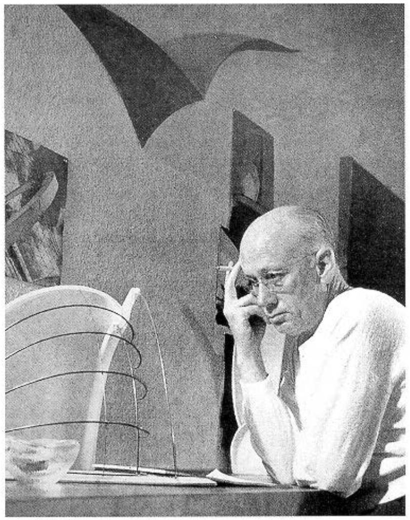
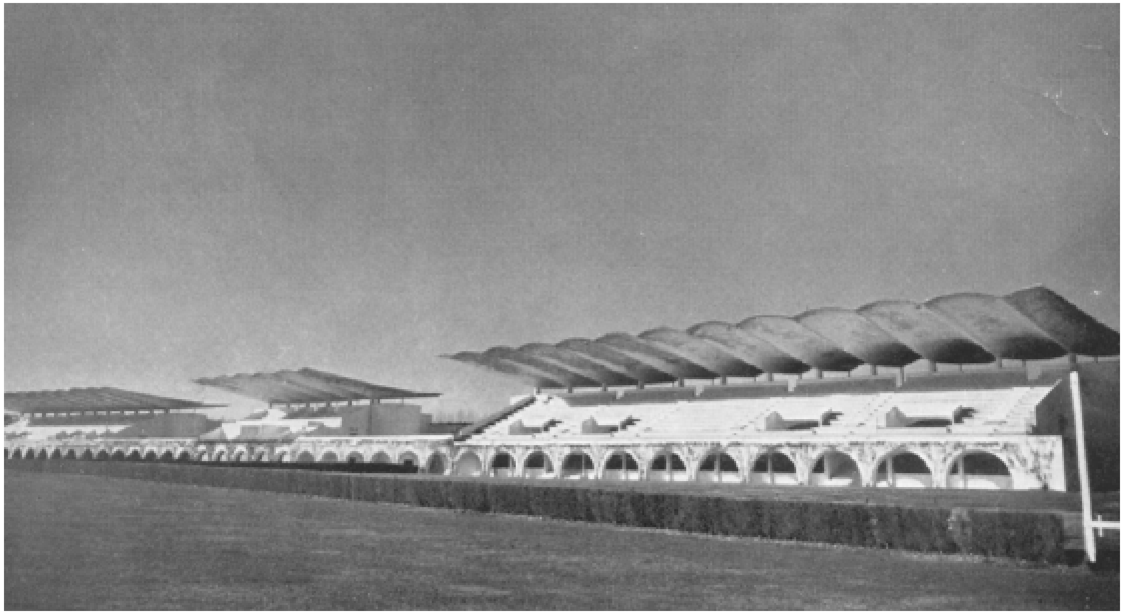


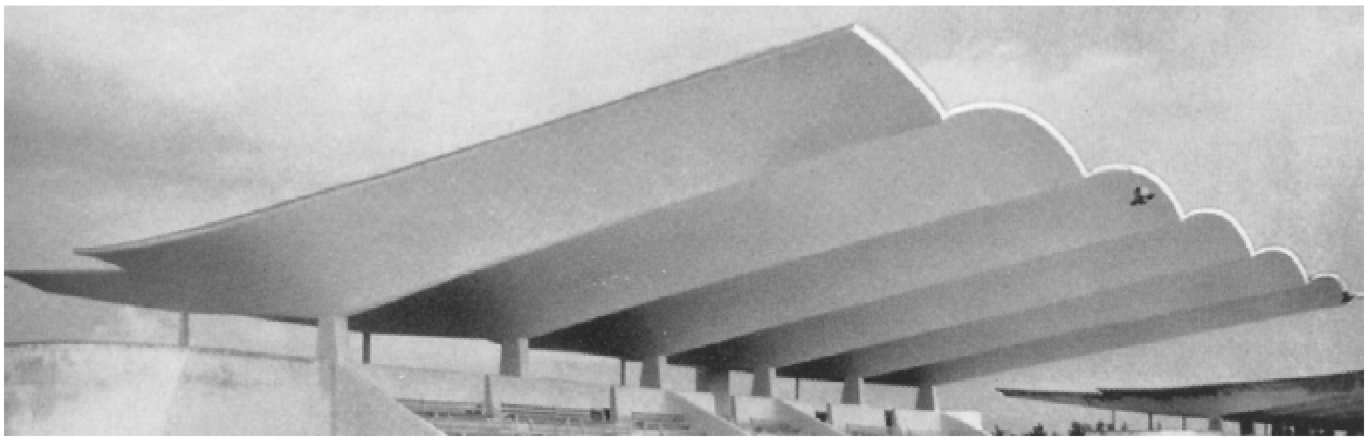
Foto: M. Garcia Moya.

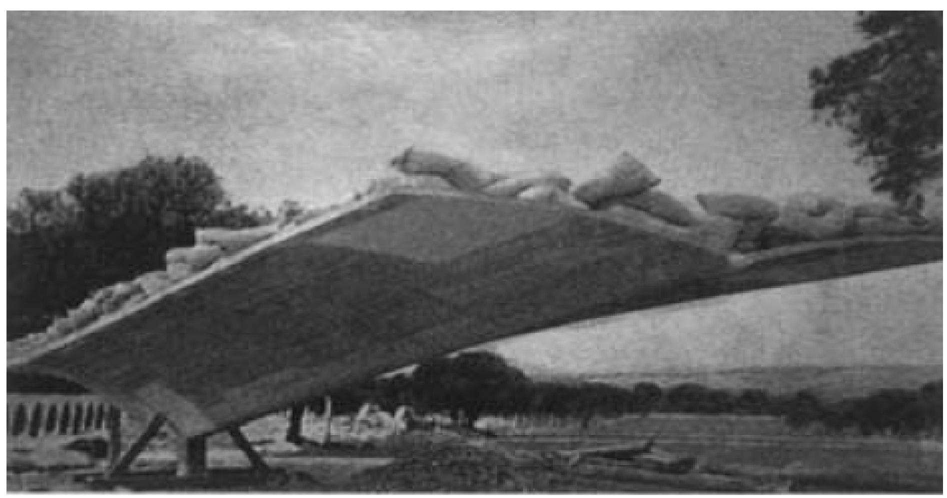


HIPÓDROMO DE LA ZARZUELA. MADRID, 1935.

C. Arniches, L. Domínguez y Eduardo Torroja. Con la empresa constructora Agroman E.C.

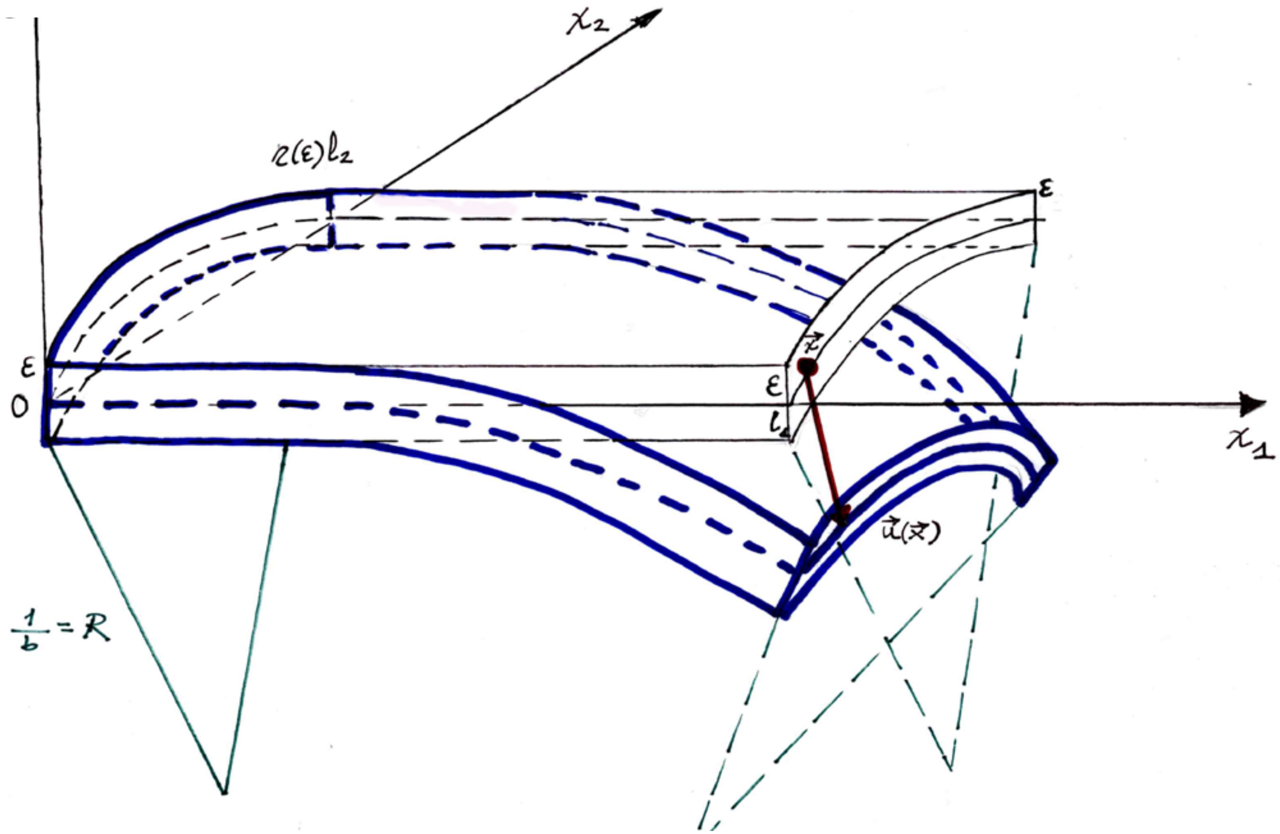
The shell roofs of the Madrid Racecourse (1935) are a brilliant result of the forms of the reinforced concrete consisting of a system of portal frames, spread at 5 m intervals and connected longitudinally by small reinforced concrete double curvature vaults. The cantilever roof, with a minimum thickness of 5 cm, overhangs to a distance of 12,8 m.



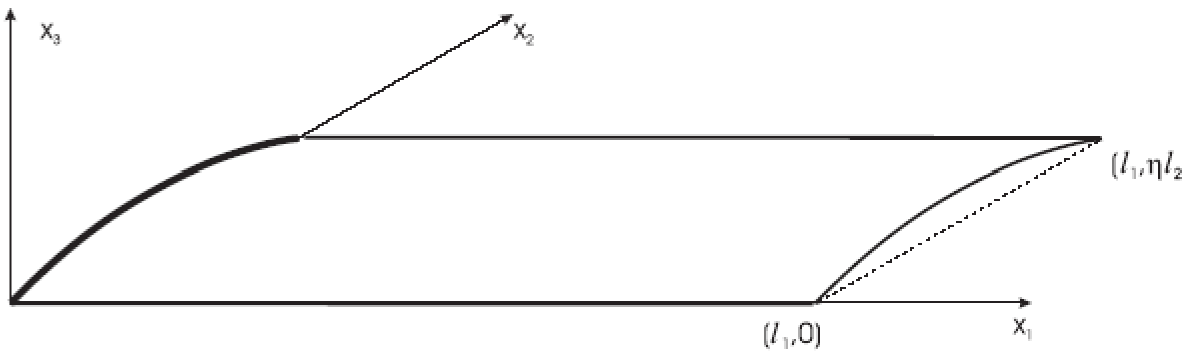


2. The basic problem.

We consider a slender cylindrical shell



According to standard notations in cylindrical shell theory the "plane of parameters x_1, x_2" is merely the middle surface (cylinder) of the shell developed into a plane. We chose x_1 in the direction of the generators and x_2 normal to them, so that the principal curvatures are zero in the direction x_1 and $b = 1/R$ in the direction x_2, where R denotes the radius of the cross section

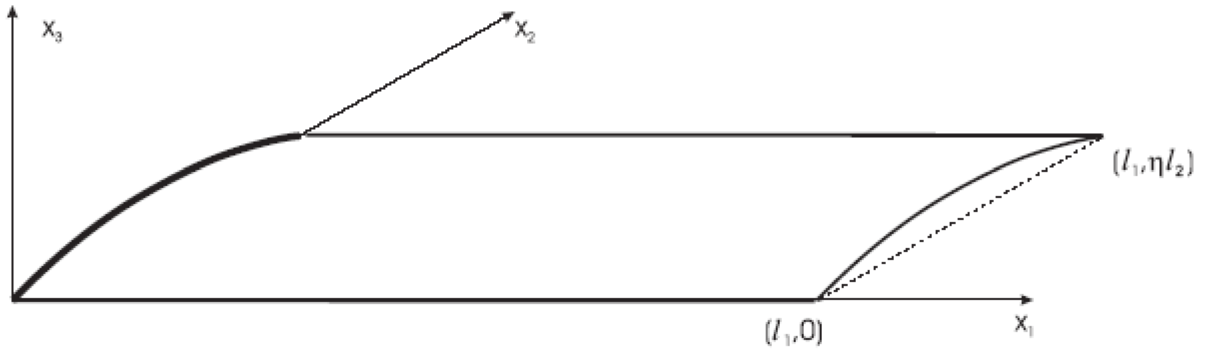


Accordingly, the second fundamental form of the surface has components $b_{11} = b_{12} = 0$ and $b_{22} = b$, Moreover, the Christoffel symbols of the surface vanish identically, so that covariant and classical differentiation coincide. Since $b_{12}^2 - b_{11}b_{22} = 0$ the surface is parabolic, i.e. the directions of the principal curvatures coincide

Let ε be a small parameter, the relative thickness of the plate. Let $\eta = \eta(\varepsilon)$ be a new small parameter satisfying

$$\varepsilon^{1/3} \leq \eta \leq 1.$$

the typical example will be $\eta = \varepsilon^{1/4}$, Let us denote the shell domain by $\Omega_\varepsilon = (0, l_1) \times (0, \eta l_2)$, with $\eta l_2 \leq 2R$.



The corresponding tangential displacements are $\tilde{u}_1, \tilde{u}_2$, whereas $\tilde{u}_3$ is the displacement normal to the shell. Some times we shall use the notation $\tilde{\mathbf{u}} = \tilde{\mathbf{u}}^\varepsilon$ to indicate explicitly the ε-dependence. We shall admit, in this section, that the shell is clamped by the "small curved boundary" $(\{0\} \times [0, \eta l_2]$ and free by the rest This implies the kinematic boundary conditions:

$$0 = \tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = \tilde{\partial}_1 \tilde{u}_3 \quad \text{on } \{0\} \times [0, \eta l_2],$$

where

$$\tilde{\partial}_\alpha = \frac{\partial}{\partial x_\alpha}.$$

The space of configuration will be denoted by V_ε. It is the subspace of $H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$ formed by the functions satisfying the kinematic boundary conditions

Although it is possible to write the complete system of equations modeling the above elastic problem (the "strong formulation": see. e.g.

F. Niordson, *Shell theory*, North Holland, Amsterdam, 1985

here we shall follow a "variational or weak formulation"

$$\varepsilon a(\mathbf{u}^{\varepsilon}, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^{\varepsilon}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$$

where the coefficients ε and ε^3 account for the fact that the membrane and flection rigidities are proportional to the thickness of the plate and to its cube, respectively. Moreover, the two bilinear forms $a(\mathbf{u}^{\varepsilon}, \mathbf{v})$ and $b(\mathbf{u}^{\varepsilon}, \mathbf{v})$ on the space $\mathbf{V}$ are defined thought the previous expressions (membrane strains in shell theory):

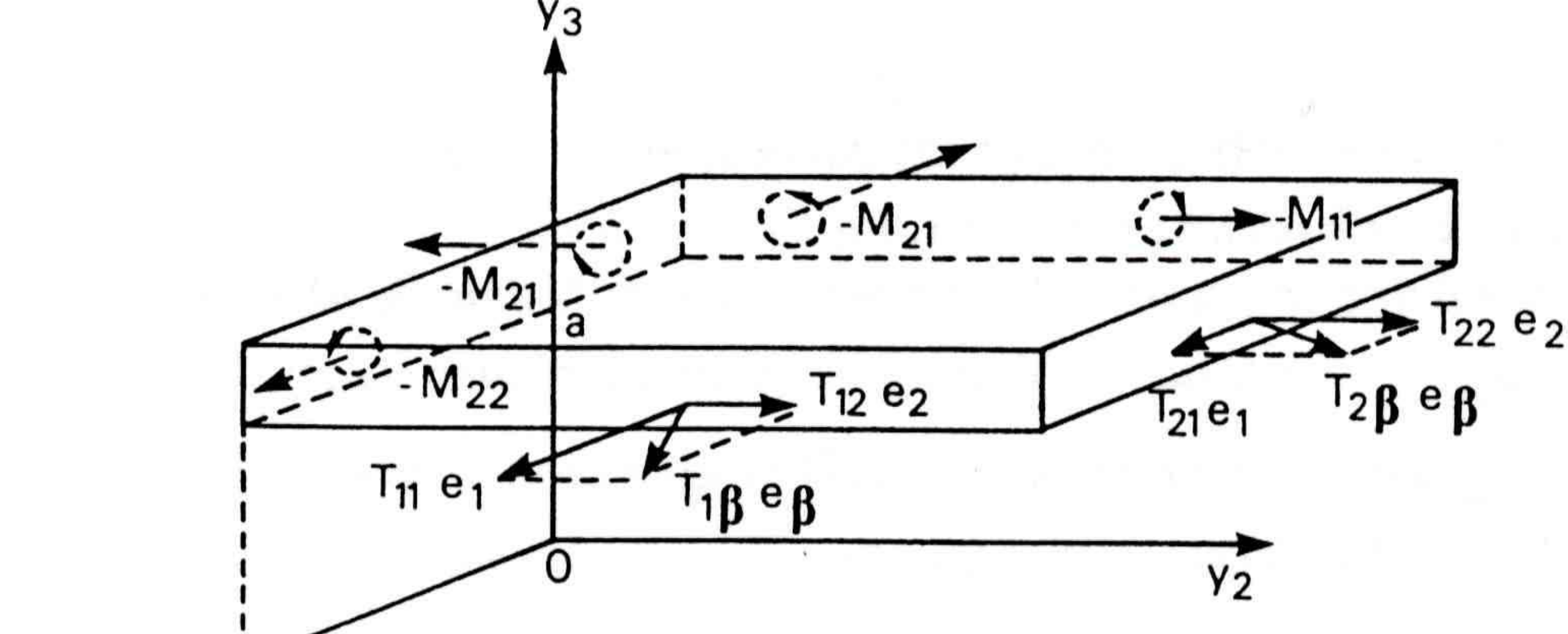
$$\tilde{T}^{\alpha\beta}(\tilde{\mathbf{u}}) = \tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}}) = A^{\alpha\beta\lambda\mu}\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}})$$

$$\begin{cases} \tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \tilde{\partial}_1 \tilde{v}_1 \\ \tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \tilde{\partial}_2 \tilde{v}_2 + b_3 \tilde{v}_3 \\ \tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \frac{1}{2}(\tilde{\partial}_2 \tilde{v}_1 + \tilde{\partial}_1 \tilde{v}_2) \end{cases}$$

and

$$\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}) = \tilde{\partial}_{\alpha\beta} \tilde{v}_3 \qquad \text{curvature variation tensor}$$

for the triplets $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)$.



Uncoupling phenomenon (due to the symmetry with respect to $x_3 = 0$):
the total energy is the addition of membrain enery and flection energy

The two bilinear forms on $\mathbf{V}$ are then defined by:

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}) d\mathbf{x}$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}) d\mathbf{x},$$

where the coefficients $A^{\alpha\beta\lambda\mu}$ and $B^{\alpha\beta\lambda\mu}$ satisfy the symmetry and positivity conditions

$$A^{\alpha\beta\lambda\mu} = A^{\beta\alpha\lambda\mu} = A^{\lambda\mu\alpha\beta}$$

$$A^{\alpha\beta\lambda\mu} \theta_{\alpha\beta} \theta_{\lambda\mu} \geq c \theta_{\alpha\beta} \theta_{\alpha\beta} \quad \text{for} \quad \theta_{\alpha\beta} = \theta_{\beta\alpha} \qquad \text{with some } c > 0.$$

As applied forces, we shall give a normal loading depending on ε by the factor ε^3

$$\langle \mathbf{f}, \mathbf{v} \rangle = \varepsilon^3 \int_{\Omega_\varepsilon} F_3(x_1, x_2/\eta) \tilde{v}_3(x_1, x_2) d\mathbf{x},$$

We note that the shape of the profile of the applied loading in x_2 is independent of ε

but applied to the points x_2/η,

We shall admit in the sequel that

$$F_3 \in L^2(\Omega)$$

where

$$\Omega = (0, l_1) \times (0, l_2).$$

Problem P_ε. *Find* $\tilde{\mathbf{u}}^\varepsilon \in \mathbf{V}_\varepsilon$ *satisfying*

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \qquad \forall \tilde{\mathbf{v}} \in \mathbf{V}_\epsilon.$$

Remark

Since the bilinear forms $a(\mathbf{u}, \mathbf{v})$ *and* $b(\mathbf{u}, \mathbf{v})$ *are symmetric, from well known results we deduce that, in fact,* $\tilde{\mathbf{u}}^\varepsilon$ *is the unique solution of the minimization problem*

$$Min_{\mathbf{V}} \widetilde{J}_\varepsilon(\mathbf{v})$$

where

$$\widetilde{J}_\varepsilon(\mathbf{v}) = \frac{\varepsilon}{2} a(\mathbf{v}, \mathbf{v}) + \frac{\varepsilon^3}{2} b(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v} \rangle.$$

The objective of the rest of the section is to study its asymptotic behavior as $\varepsilon \downarrow 0$.

3. Scaling and a priori estimates in the basic problem.

Let us perform the change of variables :

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2), \\ y_1 = x_1, \qquad y_2 = \eta^{-1} x_2 \end{cases}$$

so, the domain Ω_ε is transformed into Ω and

$$\partial_1 = \tilde{\partial}_1, \qquad \partial_2 = \eta \tilde{\partial}_2; \qquad \partial_\alpha = \frac{\partial}{\partial y_\alpha}.$$

Moreover, we shall perform the change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^\theta u_1(\mathbf{y}), \\ \tilde{u}_2(\mathbf{x}) = \eta^{\theta-1} u_2(\mathbf{y}), \\ \tilde{u}_3(\mathbf{x}) = \eta^{\theta-2} b^{-1} u_3(\mathbf{y}), \end{cases}$$

As θ is not defined, the total level of the scaling is not specified, only the mutual ratios of dilatation of the three components are fixed. They are chosen in analogy with layers in parabolic shells. Specifically, the ratio between the components 1 and 2 is fixed in order that the new form of the shear membrane strain $\tilde{e}_{12}$ be formed by two terms of the same order (which, on the other hand, are asymptotically large, forming a constraint for the limit problem). The ratio between the components 2 and 3 is also fixed in such a way that the new form of the membrane strain $\tilde{e}_{22}$ be formed by two terms of the same order.

We then perform the previous change for $\tilde{\mathbf{u}}^\varepsilon$ as well as for $\tilde{\mathbf{v}}$ in P_ε and we have

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^\theta \partial_1 v_1$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2),$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-2}(\partial_2 v_2 + v_3),$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-2} b^{-1}\partial_1^2 v_3,$$

$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-3} b^{-1}\partial_1\partial_2 v_3, \qquad \tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\partial_2^2 v_3.$$

It will prove useful to define

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \partial_1 v_1$$

$$\gamma_{12}^\varepsilon(\mathbf{v}) = \gamma_{21}^\varepsilon(\mathbf{v}) = \eta^{-1}\frac{1}{2}(\partial_2 v_1 + \partial_1 v_2),$$

$$\gamma_{22}^\varepsilon(\mathbf{v}) = \eta^{-2}(\partial_2 v_2 + v_3);$$

$$\rho_{11}^\varepsilon(\mathbf{v}) = \eta^2\partial_1^2 v_3,$$

$$\rho_{12}^\varepsilon(\mathbf{v}) = \rho_{21}(\mathbf{v}) = \eta\partial_1\partial_2 v_3,$$

$$\rho_{22}^\varepsilon(\tilde{\mathbf{v}}) = \partial_2^2 v_3.$$

so that:

$$\tilde{\gamma}_{11}(\tilde{\mathbf{v}}) = \eta^\theta \gamma_{11}^\varepsilon(v)$$

$$\tilde{\gamma}_{12}(\tilde{\mathbf{v}}) = \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \eta^\theta \gamma_{12}^\varepsilon(\mathbf{v})$$

$$\tilde{\gamma}_{22}(\tilde{\mathbf{v}}) = \eta^\theta \gamma_{22}^\varepsilon(\mathbf{v})$$

$$\tilde{\rho}_{11}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_{11}^\varepsilon(\mathbf{v})$$

$$\tilde{\rho}_{12}(\tilde{\mathbf{v}}) = \tilde{\rho}_{21}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_{12}^\varepsilon(\mathbf{v}) \qquad \tilde{\rho}_{22}(\tilde{\mathbf{v}}) = \eta^{\theta-4} b^{-1}\rho_2^\varepsilon(\mathbf{v}).$$

We recall that the spatial domain is now $\Omega = (0, l_1) \times (0, l_2)$. The space of configuration, after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega) \times H^1(\Omega) \times H^2(\Omega)$$

formed by the functions satisfying the kinematic boundary conditions

$$0 = u_1 = u_2 = u_3 \text{ on } \{0\} \times [0, l_2].$$

The expression $\quad \varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle \quad$ then becomes:

$$P \int_\Omega A^{\alpha\beta\lambda\mu} \gamma^\varepsilon_{\alpha\beta}(\mathbf{u}_\varepsilon) \gamma^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} + Q \int_\Omega B^{\alpha\beta\lambda\mu} \rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon) \rho^\varepsilon_{\lambda\mu}(\mathbf{v}) d\mathbf{y} = R \int_\Omega F_3(y_1, y_2) v_3(y_1, y_2) d\mathbf{y},$$

with

$$P = \varepsilon \eta^{2\theta+1}$$

$$Q = \varepsilon^3 \eta^{2\theta-7} b^{-2}$$

we shall determine the $b(\varepsilon)$ and θ as functions of ε and the function $\eta(\varepsilon)$ using the two equations

$$P = Q = R.$$

This gives $\qquad b = \varepsilon/\eta^4 \qquad$ and $\qquad \eta^{\theta-2} = \varepsilon.$

b is always small with respect to η^{-1} , and equal to 1 (or rather 0(1)) in the "typical example" $\eta = \varepsilon^{1/4}$ we have $\theta = 6$.

Once θ is determined, the scaling $\tilde{u}_3(\mathbf{x}) = \eta^{\theta-2}b^{-1}u_3(\mathbf{y})$, is perfectly defined. We then observe that the factor $\eta^{\theta-2}b^{-1}$ takes the form: η^4 which is always small. It means that the scaling of the component u_3^ε is such that, after scaling, it is asymptotically large with respect to the case before scaling. As we shall prove in the sequel, the scaled unknown u_3^ε has a non zero limit; it follows that the initial unknown $\tilde{u}_3^\varepsilon$ tends to 0 at the ratio η^4. We shall come again on this property, which amounts to the rigidification of the plate with respect to the plane case.

Summing up, the problem P_ε becomes after scaling:

Problem Π_ε. *Find* $\mathbf{u}^\varepsilon \in \mathbf{V}$ *satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \int_\Omega F_3(y_1, y_2)v_3(y_1, y_2)d\mathbf{y}.$$

$\forall v \in \mathbf{V}$, *where*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) \stackrel{def}{=} \int_\Omega A^{\alpha\beta\lambda\mu}\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y}.$$

It should be emphasized that, by virtue of the definitions the coefficients involve various powers of η, running from -4 to $+4$. The terms in η^{-4} to η^{-1} are "penalty terms", whereas those in η^1 to η^4 are "singular perturbation terms". Only the terms of order 1 will remain in the limit expression.

Remark $\mathbf{u}^\varepsilon$ *is the unique solution of the minimization problem*

$$Min_{\mathbf{V}} J_\varepsilon(\mathbf{v})$$

where

$$J_\varepsilon(\mathbf{v}) = \frac{1}{2}a^\varepsilon(\mathbf{v}, \mathbf{v}) - \langle \mathbf{f}, \mathbf{v}\rangle.$$

Let us proceed to the a priori estimates. From the expression of $a^\varepsilon(\mathbf{v},\mathbf{v})$ with $\mathbf{u}^\varepsilon=\mathbf{v}$,

Lemma *The estimates:*

$$\|\partial_1 v_1\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 v_1+\partial_1 v_2)\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^{-2}(\partial_2 v_2+v_3)\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\partial_2^2 v_3\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta\partial_1\partial_2 v_3\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\eta^2\partial_1^2 v_3\|^2_{L^2(\Omega)} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

hold true for a certain $c>0$ independent of ε and $\mathbf{v}\in\mathbf{V}$.

Now, in order to prove that the functional in the right hand side is bounded independently of ε, we need an estimate on u_3 itself.

Lemma *The estimate:*

$$\|v_3\|^2_{L^2((0,l_1);H^2(0,l_2))} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

holds true for a certain $c>0$ independent of ε and $\mathbf{v}\in\mathbf{V}$.

Proof. Discarding the factors in η and differentiating we have:

$$\|\partial_2^2 v_1+\partial_2\partial_1 v_2)\|^2_{L^2((0,l_1);H^{-1}(0,l_2))} \le ca^\varepsilon(\mathbf{v},\mathbf{v})$$

$$\|\partial_1\partial_2 v_2+\partial_1 v_3)\|^2_{H^{-1}((0,l_1);L^2(0,l_2))} \le ca^\varepsilon(\mathbf{v},\mathbf{v}).$$

using the fact that v_1 vanishes on $\{0\} \times [0, l_2]$, by using the Poincaré's inequality we obtain:

$$\|v_1\|^2_{H^1((0,l_1);L^2(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

and differentiating,

$$\|\partial_2^2 v_1\|^2_{H^1((0,l_1);H^{-2}(0,l_2)))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}).$$

taking the weaker norm, it follows that

$$\|\partial_2\partial_1 v_2\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

$$\|\partial_1 v_3\|^2_{H^{-1}((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})$$

or even (integrating with respect to y_1 on account of the vanishing of the trace on $\{0\} \times [0, l_2]$):

$$\|v_3\|^2_{L^2((0,l_1);H^{-2}(0,l_2))} \leq ca^\varepsilon(\mathbf{v}, \mathbf{v}).$$

The conclusion follows.

Lemma *The estimate*

$$|\int_\Omega F_3 v_3 d\mathbf{y}| \leq ca^\varepsilon(\mathbf{v}, \mathbf{v})^{1/2}$$

holds true for a certain $c > 0$ independent of ε and $\mathbf{v} \in \mathbf{V}$.

Now, taking $\mathbf{v} = \mathbf{u}^\varepsilon$ we get the energy estimate:

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The estimates*

$$\|\gamma^\varepsilon_{\alpha\beta}(u^\varepsilon)\| \leq C \qquad \alpha, \beta = 1, 2 \qquad \|\partial_1 u_1^\varepsilon\|^2_{L^2(\Omega)} \leq C$$

$$\|\eta^{-1}\frac{1}{2}(\partial_2 u_1^\varepsilon + \partial_1 u_2^\varepsilon)\|^2_{L^2(\Omega)} \leq C \qquad \|\eta^{-2}(\partial_2 u_2^\varepsilon + u_3^\varepsilon)\|^2_{L^2(\Omega)} \leq C$$

$$\|\partial_2^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \le C \qquad \|\eta\partial_1\partial_2 u_3^\varepsilon\|^2_{L^2(\Omega)} \le C \qquad \|\eta^2\partial_1^2 u_3^\varepsilon\|^2_{L^2(\Omega)} \le C$$

hold true for a certain $C > 0$ independent of ε.

We shall need an estimate on u_2^ε itself. We shall obtain it by differentiating with respect to y_2 and integrating in y_1.

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The estimates*

$$\|u_1^\varepsilon\|_{H^1((0,\mathrm{l}_1);L^2(0,\mathrm{l}_2))} \le C \qquad \|u_2^\varepsilon\|_{\widetilde{H}_0^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2))} \le C \qquad \|u_3\|^2_{L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2))} \le C,$$

holds true for a certain $C > 0$ independent of ε, where

$$\widetilde{H}_0^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) = \{w \in H^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) \text{ such that } w(0,\cdot) = 0\}.$$

A first result of convergence is

Lemma *Let $\mathbf{u}^\varepsilon$ be the solution of Π_ε. The following convergences (as $\varepsilon \to 0$) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \quad \text{weakly in } \widetilde{H}_0^1((0,\mathrm{l}_1);L^2(0,\mathrm{l}_2)) \qquad u_2^\varepsilon \to u_2^* \quad \text{weakly in } \widetilde{H}_0^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2))$$

$$u_3^\varepsilon \to u_3^* \quad \text{weakly in } L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2))$$

where $\mathbf{u}^* = (u_1^*, u_2^*, u_3^*)$ are distributions on Ω, belonging to the spaces specified

$$\partial_2 u_1^* + \partial_1 u_2^* = 0$$

$$\partial_2 u_2^* + u_3^* = 0.$$

Finally,

$$\gamma^{\varepsilon}_{\alpha\beta}(\mathbf{u}^{\varepsilon}) \to \gamma^{*}_{\alpha\beta} \qquad \textit{weakly in } L^2(\Omega), \quad \alpha,\beta = 1,2,$$

for some $\gamma^{*}_{\alpha\beta} \in L^2(\Omega)$.

4. Limit problem and convergence in the basic problem.

Let us define the space $\mathbf{G}$ for the definition of the limit problem:

$$\mathbf{G} = \{\mathbf{v} = (v_1, v_2, v_3) \in \widetilde{H}^1_0((0,l_1); L^2(0,l_2)) \times \widetilde{H}^1_0((0,l_1); H^{-1}(0,l_2)) \times L^2((0,l_1); H^2(0,l_2)),$$

$$\partial_2 v_1 + \partial_1 v_2 = 0, \quad \partial_2 v_2 + b v_3 = 0\},$$

where we observe that v_1 defines completely v_2 and then v_3.

Clearly, $\mathbf{G}$ is a Hilbert space with the norm

$$\begin{cases} \|\mathbf{v}\|^2_{\mathbf{G}} = \|v_1\|^2_{\widetilde{H}^1_0((0,l_1);L^2(0,l_2))} + \|\partial_2^2 v_3\|^2_{L^2(\Omega)} \\ \simeq \|\partial_1 v_1\|^2_{L^2(\Omega)} + \|\partial_2^3 v_2\|^2_{L^2(\Omega)} \end{cases}$$

Remark *A straightforward comparison with the space* $\mathbf{V}$ *shows that the space* $\mathbf{G}$ *for the limit problem incorporates the two constraints corresponding to the "penalty terms" in* Π_ε *whereas the boundary conditions for* u_3*, which are concerned with the "singular perturbation terms" in* Π_ε *are lost.*

It is worthwhile to state an equivalent definition of the space $\mathbf{G}$ where the functions are defined in terms of a scalar "potential ψ":

Lemma *The space* $\mathbf{G}$ *may equivalently be defined as the space of the triplets* $\mathbf{v} = (v_1, v_2, v_3)$ *such that:*

$$v_1 = \partial_1 \psi, \qquad v_2 = -\partial_2 \psi, \qquad v_3 = -\partial_2^2 \psi.$$

where ψ *is an element of*

$$\widetilde{G} = \tilde{H}_0^2((0, l_1); L^2(0, l_2)) \cap L^2((0, l_1); H^4(0, l_2))$$

where

$$\tilde{H}_0^2((0, l_1); L^2(0, l_2)) = \{\psi \in H^2((0, l_1); L^2(0, l_2)); \psi(0, y_2) = \partial_1 \psi(0, y_2) = 0\}.$$

Remark The introduction of the scalar potential φ seems to be new in the shell literature.

D. Caillerie, A. Raoult and E. Sanchez Palencia, On internal and boundary layers with unbounded energy in thin shell theory. Parabolic characteristic and non-characteristic cases, *Asymptotic Analysis* **46** (2006), 221-249.

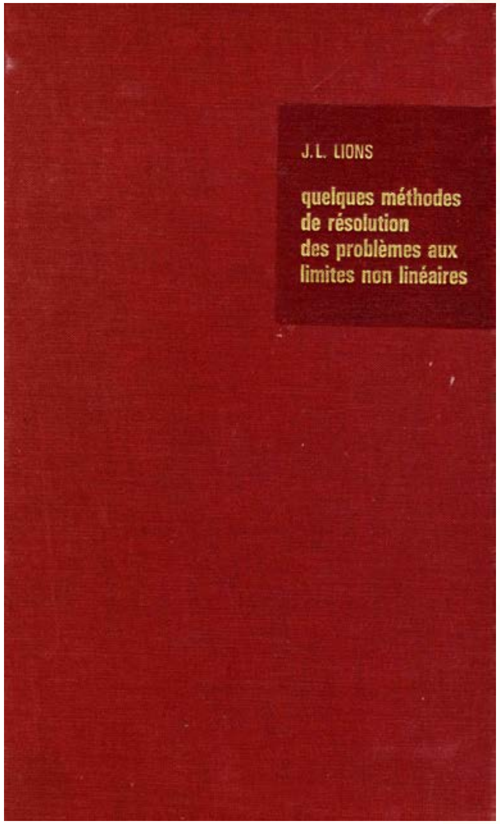
Some closed, but different, ideas can be associated with the *stress function* introduced by G.B. Airy (1801-1892)

It should prove useful to prove a lemma on density in $\mathbf{G}$.

Lemma *The subspace of* $\mathbf{G}$ *formed by the elements* $\mathbf{v} = (v_1, v_2, v_3)$ *which are smooth, vanish in a neighborhood of* $\{0\} \times [0, l_2]$ *and derive from a "potential"* ψ *is dense in* $\mathbf{G}$.

We are now defining the limit problem. It involves the numerical coefficient $1/C_{1111}$, and B^{2222} where $C_{\alpha\beta\lambda\mu}$ is the matrix inverse of $A^{\alpha\beta\lambda\mu}$, i. e. the matrix of membrane compliances, and $\mathbf{B}$ is the matrix of flection rigidities. They are both strictly positive.

$$\tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{u}}) = C_{\lambda\mu\alpha\beta} \tilde{T}^{\beta\alpha}(\tilde{\mathbf{u}})$$



312 MÉTHODES DE RÉGULARISATION ET DE PÉNALISATION [CHAP. 3]

L'idée générale de la régularisation elliptique est la suivante : soit à résoudre la classique équation de la chaleur :

$$\frac{\partial u}{\partial t} - \Delta u = f \text{ dans } Q\,, \tag{1.1}$$

avec $u(x, 0) = 0$, $u = 0$ sur Σ.

On « *approche* » l'équation (1.1) par l'équation *elliptique*

$$-\,\varepsilon \frac{\partial^2 u_\varepsilon}{\partial t^2} + \frac{\partial u_\varepsilon}{\partial t} - \Delta u_\varepsilon = f \text{ dans } Q,\ \varepsilon > 0\,, \tag{1.2}$$

les conditions aux limites existantes étant maintenues et en *ajoutant* une condition aux limites pour $t = T$.

On résout (1.2) (par des « méthodes elliptiques ») puis l'on passe à la limite ($\varepsilon \to 0$).

J.-L. Lions (CIME 1963),.., O. A. Oleinik (1966), C. Bardos- H. Brezis (1968),…

Problem Π_0. *Find* $\mathbf{u} \in \mathbf{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y} = \int_\Omega F_3 v_3 d\mathbf{y}.$$

$\forall \mathbf{v} \in \mathbf{G}$, *or equivalently, in terms of the potential, find* $\varphi \in \widetilde{G}$ *such that*

$$\int_\Omega \frac{1}{C_{1111}} \partial_1^2 \varphi \partial_1^2 \psi d\mathbf{y} + \int_\Omega B^{2222} \partial_2^4 \varphi \partial_2^4 \psi d\mathbf{y} = -\int_\Omega F_3 \partial_2^2 \psi d\mathbf{y},$$

$\forall \psi \in \widetilde{G}$.

Obviously, this problem is in the Lax - Milgram framework, as the right hand side is a continuous functional on $\mathbf{G}$. We then have

Theorem *Under the assumption* $F_3 \in L^2(\Omega)$, *Problem* Π_0 *has a unique solution.*

Our main convergence result is:

Theorem *Let* $\mathbf{u}_\varepsilon$ *and* $\mathbf{u}$ *be the solutions of* Π_η *and* Π_0 *respectively. Then, for* $\varepsilon \downarrow 0$, *we have:*

$$\mathbf{u}^\varepsilon \to \mathbf{u}$$

In other words, the limit $\mathbf{u}^*$ *is the solution of the limit problem*

The corresponding higher order partial differential equation for φ is obviously

$$(\frac{1}{C_{1111}}\partial_1^4 + B^{2222}\partial_2^8)\varphi = -\partial_2^2 F_3.$$

parabolic according the theory of linear partial differential equations

Remark *if we define the bilinear form*

$$a^0(\mathbf{u},\mathbf{v}) \stackrel{def}{=} \int_\Omega \frac{1}{C_{1111}}\partial_1 u_1 \partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3 \partial_2^2 v_3 d\mathbf{y},$$

then the symmetry of $a^0(\mathbf{u},\mathbf{v})$ shows that the (unique) solution $\mathbf{u}$ of problem Π_0 can be characterized as the unique element of $\mathbf{G}$ *solving the minimization problem*

$$Min_{\mathbf{G}} J_0(\mathbf{v})$$

where

$$J_0(\mathbf{v}) = \frac{1}{2}a^0(\mathbf{v},\mathbf{v}) - \int_\Omega F_3 v_3 d\mathbf{y}.$$

We can formulate, equivalently, this property in terms of the potential φ
So, the (unique) solution $\varphi \in \widetilde{G}$ of problem Π_0 can be characterized as the unique element of $\widetilde{G}$ solving the minimization problem

$$Min_{\widetilde{G}} \widetilde{J}_0(\psi)$$

$$\widetilde{J}_0(\psi) = \frac{1}{2C_{1111}}\int_\Omega \left|\partial_1^2\psi\right|^2 d\mathbf{y} + \frac{B^{2222}}{2}\int_\Omega \left|\partial_2^4\psi\right|^2 d\mathbf{y} + \int_\Omega F_3 \partial_2^2 \psi d\mathbf{y}$$

Remark *It seems important to point out that the a priori estimates does not allow to conclude the identification* $\gamma^*_{12} = 0$ *and* $\gamma^*_{22} = 0$ *in spite to know that* $\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)$ *weakly converge to* $\gamma^*_{\alpha\beta}$ *and that necessarily* $\partial_2 u^*_1 + \partial_1 u^*_2 = 0$ *and* $\partial_2 u^*_2 + u^*_3 = 0$. *The reason is due to the presence of the terms* η^{-1} *(respectively* η^{-2}*) in the definition of* γ^ε_{12} *(respectively* γ^ε_{22}*). Notice that, in fact, in most of the cases we must have that* $\gamma^*_{12} \neq 0$ *or* $\gamma^*_{22} \neq 0$, *since otherwise we could get that* $T^{11\varepsilon}(\mathbf{u}^\varepsilon) = A^{1111}\gamma^\varepsilon_{11}(\mathbf{u}^\varepsilon) + 2A^{1112}\gamma^\varepsilon_{12}(\mathbf{u}^\varepsilon) + A^{1122}\gamma^\varepsilon_{22}(\mathbf{u}^\varepsilon)$ *converges (weakly in* $L^2(\Omega)$*) to* $T^{11*}(\mathbf{u}^*) = A^{1111}\gamma_{11}(\mathbf{u}^*) = A^{1111}\partial_1 u^*_1$ *and this would imply (thanks to Theorem 2.2) that necessarily*

$$A^{1111} = \frac{1}{C_{1111}},$$

which is not necessarily true since it depends of the constitutive assumptions made on the elastic medium.

Our next result improves the convergence under some additional condition on the coefficients.

Theorem *Assume that*

$$A^{11\lambda\mu} = 0 \text{ if } \lambda > 1 \text{ or } \mu > 1.$$

Let $\mathbf{u}^\varepsilon$ *and* $\mathbf{u}$ *be the solutions of* Π_ε *and* Π_0 *respectively. Then*

$$u^\varepsilon_1 \to u^*_1 \qquad \text{strongly in } \widetilde{H}^1_0((0,l_1);L^2(0,l_2))$$

$$u^\varepsilon_2 \to u^*_2 \qquad \text{strongly in } \widetilde{H}^1_0((0,l_1);H^{-1}(0,l_2))$$

$$u^\varepsilon_3 \to u^*_3 \qquad \text{strongly in } L^2((0,l_1);H^2(0,l_2))$$

for $\varepsilon \downarrow 0$.

Idea of the proof of the identification theorem

From the definition of $\mathbf{G}$ we see that the only non vanishing $\gamma^\varepsilon_{\alpha\beta}$ is $\gamma^\varepsilon_{11}(\mathbf{v}) = \partial_1 v_1$

and we have

$$\int_\Omega T^{11\varepsilon}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{11}(\mathbf{v})d\mathrm{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathrm{y} = \int_\Omega F_3 v_3 d\mathrm{y}.$$

As a result, the limit is

$$\int_\Omega T^{11*}\partial_1 u_1 \partial_1 v_1 d\mathrm{y} + \int_\Omega B^{2222}\partial_2^2 u_3^* \partial_2^2 v_3 d\mathrm{y} = \int_\Omega F_3 v_3 d\mathrm{y}.$$

We are now transforming the term in T^{11*} in the previous equation. To this end, let us take

$$w_1 = w_2 = 0, \qquad w_3 \in C_0^\infty(\Omega)$$

and let us take in (2.44) the test function

$$\mathbf{v} = \eta^2 \mathbf{w}$$

$$\int_\Omega T^{22*}(-w_3)d\mathrm{y} = 0$$

so that

$$T^{22*} = 0.$$

Let us now take

$$w_1 \in C_0^\infty((0,l_1); C^\infty(0,l_2)), \qquad w_2 = w_3 = 0,$$ the test function $$\mathbf{v} = \eta\mathbf{w}$$

$$\int_\Omega T^{12*}(\frac{1}{2}\partial_2 w_1)d\mathrm{y} = 0$$

so that, as $\partial_2 w_1$ is "arbitrary",

$$T^{12*} = 0.$$

As the only non zero $T^{\alpha\beta*}$ is T^{11*}, the $\gamma^*_{\alpha\beta}$ are given by the expressions

$$\gamma^*_{\alpha\beta} = C_{\alpha\beta 11}T^{11*}$$

and in particular

$$\gamma^*_{11} = C_{1111}T^{11*}.$$

Idea of the proof of the strong convergence

$$\begin{aligned} a^\varepsilon(\mathbf{u}^\varepsilon,\mathbf{v}) &= \int_\Omega A^{\alpha\beta\lambda\mu}\gamma^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\gamma^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} + \int_\Omega B^{\alpha\beta\lambda\mu}\rho^\varepsilon_{\alpha\beta}(\mathbf{u}^\varepsilon)\rho^\varepsilon_{\lambda\mu}(\mathbf{v})d\mathbf{y} \\ &= \mathfrak{a}_0(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon,\mathbf{v}) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon,\mathbf{v}), \end{aligned}$$

$$\mathfrak{a}_0(\mathbf{u},\mathbf{v}) = \int_\Omega A^{1111}\partial_1 u_1\partial_1 v_1 d\mathbf{y} + \int_\Omega B^{2222}\partial_2^2 u_3\partial_2^2 v_3 d\mathbf{y}.$$

$$\begin{aligned} &\mathfrak{a}_0(\mathbf{u}^\varepsilon-\mathbf{u}^*,\mathbf{u}^\varepsilon-\mathbf{u}^*) + \varepsilon^{1/2}\mathfrak{a}_{1/2}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon\mathfrak{a}_1(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon^{-1/4}\mathfrak{a}_{-1/4}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) + \varepsilon^{-1/2}\mathfrak{a}_{-1/2}(\mathbf{u}^\varepsilon,\mathbf{u}^\varepsilon) \\ = &\int_\Omega F_3(y_1,y_2)u_3^\varepsilon(y_1,y_2)d\mathbf{y} - 2\mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^\varepsilon) + \mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^*) \to \\ \to &\int_\Omega F_3(y_1,y_2)u_3^*(y_1,y_2)d\mathbf{y} - \mathfrak{a}_0(\mathbf{u}^*,\mathbf{u}^*) = 0. \end{aligned}$$

Remark the solution) vanishes when F_3 is affine with respect to y_2

$$\left(\frac{1}{C_{1111}}\partial_1^4 + B^{2222}\partial_2^8\right)\varphi = -\partial_2^2 F_3.$$

$$\begin{cases} B^{2222}\partial_2^7\varphi = -\partial_2 F_3, \quad B^{2222}\partial_2^6\varphi = -F_3 & \text{on } \Gamma_l, \\ \partial_2^5\varphi = \partial_2^4\varphi = 0 & \text{on } \Gamma_l, \end{cases}$$

occupying a region which, for simplicity, we shall assume of the form

$$\mathcal{S}_\varepsilon := (\underline{s}_1, \overline{s}_1) \times (\eta \underline{s}_2, \eta \overline{s}_2) \subset \Omega_\varepsilon,$$

i.e. with

$$\begin{cases} 0 < \underline{s}_1, \overline{s}_1 \leq l_1, \\ 0 \leq \underline{s}_2, \overline{s}_2 \leq l_2. \end{cases}$$

We assume that the upper surface of the obstacle has the same cylindrical geometry than the shell in such a way that the possibility of contact between the shell and the obstacle is merely formulated in terms of

$$\tilde{u}_3(x_1, x_2) \geq 0 \text{ a.e. } (x_1, x_2) \in \mathcal{S}_\varepsilon.$$

By introducing the change of variables (2.21), the above constrain can be stated as

$$u_3(y_1, y_2) \geq 0 \text{ a.e. } (y_1, y_2) \in \mathcal{S},$$

where

$$\mathcal{S} := (\underline{s}_1, \overline{s}_1) \times (\underline{s}_2, \overline{s}_2) \subset \Omega.$$

Problem OP_ε. *Find* $\mathbf{u}^\varepsilon \in \mathbf{K}$ *satisfying*

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v} - \mathbf{u}^\varepsilon) \geq \int_\Omega F_3(y_1, y_2)(v_3(y_1, y_2) - u_3^\varepsilon(y_1, y_2))d\mathbf{y} \textit{ for any } v \in \mathbf{K},$$

where F_3, a^ε *and* $\mathbf{V}$ *were given in Subsection 2.2, and*

$$\mathbf{K} = \{\mathbf{v} \in \mathbf{V} \textit{ such that } v_3(y_1, y_2) \geq 0 \textit{ a.e. } (y_1, y_2) \in \mathcal{S}\}.$$

Lemma 6.1 *Let* $\mathbf{u}^\varepsilon$ *be the solution of* OP_ε. *The following convergences (as* $\varepsilon \to 0$*) hold true (in the sense of subsequences, the limits being not necessarily unique):*

$$u_1^\varepsilon \to u_1^* \qquad \textit{weakly in } \widetilde{H}_0^1((0,\mathrm{l}_1);L^2(0,\mathrm{l}_2)) \tag{6.4}$$

$$u_2^\varepsilon \to u_2^* \qquad \textit{weakly in } \widetilde{H}_0^1((0,\mathrm{l}_1);H^{-1}(0,\mathrm{l}_2)) \tag{6.5}$$

$$u_3^\varepsilon \to u_3^* \qquad \textit{weakly in } L^2((0,\mathrm{l}_1);H^2(0,\mathrm{l}_2)) \tag{6.6}$$

We introduce the subspace $\mathbf{G}$, the potential φ and the subspace $\widetilde{G}$ as in Subsection 2.3. We define the convex and closed subset of $\mathbf{G}$ and $\widetilde{G}$ by

$$\mathbf{K_G} = \{\mathbf{v} \in \mathbf{G} \text{ such that } v_3(y_1,y_2) \geq 0 \text{ a.e. } (y_1,y_2) \in \mathcal{S}\},$$

$$K_{\widetilde{\mathbf{G}}} = \{\ \psi \in \widetilde{G} \ \text{ such that } \partial_2^2\psi(y_1,y_2) \leq 0 \text{ a.e. } (y_1,y_2) \in \mathcal{S}\}.$$

)

Notice that, obviously $\mathbf{u}^* \in \mathbf{K_G}$. We define (which can be understood as the limit problem) as follows
Problem OP_0. *Find* $\mathbf{u} \in K_{\mathbf{G}}$ *such that*

$$a^0(\mathbf{u},\mathbf{v}-\mathbf{u}) \geq \int_\Omega F_3(v_3 - u_3)d\mathbf{y} \text{ for any } \mathbf{v} \in \mathbf{K_G}, \tag{6.8}$$

or equivalently, in terms of the potential, find $\varphi \in K_{\widetilde{\mathbf{G}}}$ *such that*

$$\widetilde{a}^0(\varphi,\psi-\varphi) \geq -\int_\Omega F_3(\partial_2^2\psi - \partial_2^2\varphi)d\mathbf{y},\ \text{for any } \psi \in K_{\widetilde{\mathbf{G}}}, \tag{6.9}$$

with the bilinear forms a^0 and $\widetilde{a}^0$ given by (2.103) and (2.109).

This problem is in the Lions-Stampacchia framework (see, for instance, [18]), as the right hand side of (2.79) is a continuous functional on $\mathbf{G}$. We then have

Theorem 6.1 *Under the assumption $F_3 \in L^2(\Omega)$, Problem OP_0 has a unique solution.*

The proof of the identification of OP_0 as the limit problem of OP_ε is now more delicate than in Section 2. As in Theorem 2.3, we can obtain a positive result at least for a not too restrictive class of coefficients:

Theorem 6.2 *Assume (2.93). Let $\mathbf{u}^\varepsilon$ be the solutions of OP_ε and let $\mathbf{u}^*$ be the weak limit given in Lemma 6.1. Then $\mathbf{u}^*$ verifies problem OP_0. Moreover,*

$$u_1^\varepsilon \to u_1^* \qquad \textit{strongly in } \widetilde{H}_0^1((0,l_1);L^2(0,l_2)) \tag{6.10}$$

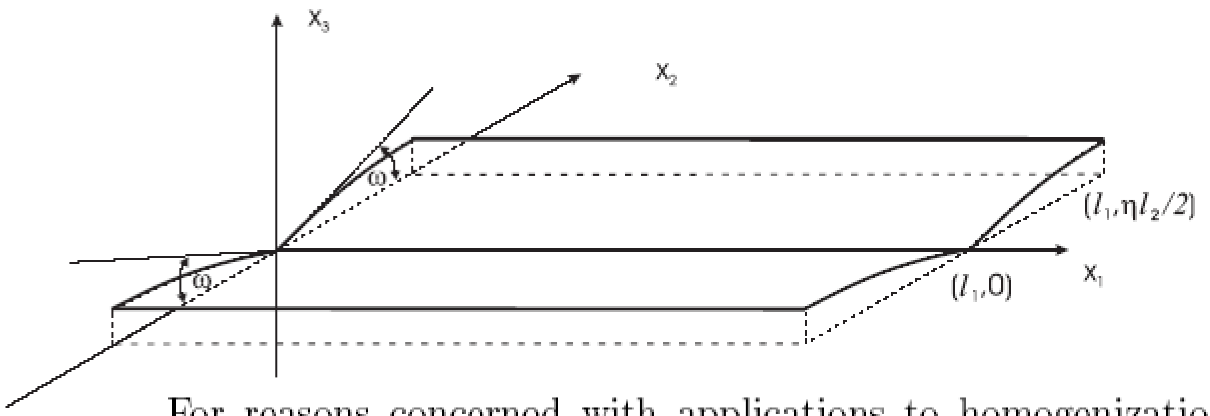
$$u_2^\varepsilon \to u_2^* \qquad \textit{strongly in } \widetilde{H}_0^1((0,l_1);H^{-1}(0,l_2)) \tag{6.11}$$

$$u_3^\varepsilon \to u_3^* \qquad \textit{strongly in } L^2((0,l_1);H^2(0,l_2)) \tag{6.12}$$

for $\varepsilon \downarrow 0$. So, in particular, OP_0 can be identified as the limit problem of OP_ε for $\varepsilon \downarrow 0$.

5. The shell has an edge with slight folding …

In this section we consider a case slightly more complicated than the basic problem, when the section by $y_1 = const.$ is as sketched in Fig 2.



For reasons concerned with applications to homogenization problems

tangent plane on $y_2 = -l_2/2$ and $y_2 = l_2/2$ is horizontal. This amounts to saying that the angle of the folding is 2ω, with $\omega = b\eta l_2/2$ (see Fig 2) where b always denote the (constant) curvature. Denoting by $\tilde{u}_i^-$ and $\tilde{u}_i^+$ the traces on $x_2 = 0$, the continuity of the displacement $\bar{\mathbf{u}}$ at the folding gives in the projections along x_1, its normal in the "base plane" and the axis Z (see Fig. 2) respectively:

In order to avoid irrelevant and cumbersome expressions, as ω is small, we shall take $cos\omega = 1$, $sin\omega = \omega$. Moreover, we shall see in the sequel that the components u_3 are asymptotically larger than u_2, and we shall neglect $\omega^2 u_2$ with respect to u_3. Then we shall consider

$$\left\{ \begin{array}{l} \tilde{u}_1^+ = \tilde{u}_1^- \\ -\omega\tilde{u}_3^+ + \tilde{u}_2^+ = \omega\tilde{u}_3^- + \tilde{u}_2^- \\ \tilde{u}_3^+ = \tilde{u}_3^- \end{array} \right.$$

so that we merely may keep in mind that $\tilde{u}_1$ and $\tilde{u}_3$ are continuous across $x_2 = 0$ and

$$\tilde{u}_2^+ - \tilde{u}_2^- = 2\omega\tilde{u}_3.$$

Let us denote

$$\Omega_\varepsilon^+ = (0, l_1) \times (0, \eta l_2/2) \text{ and } \Omega_\varepsilon^- = (0, l_1) \times (-\eta l_2/2, 0)$$

and we shall also denote by Ω_ε the union of Ω_ε^+ and Ω_ε^-. The space of configuration will be denoted by $\mathbf{V}_\varepsilon$. It is the subspace of

$$H^1(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^+) \times H^2(\Omega_\varepsilon^+) \times H^1(\Omega_\varepsilon^-) \times H^1(\Omega_\varepsilon^-) \times H^2(\Omega_\varepsilon^-)$$

formed by the functions satisfying the kinematic boundary conditions

$$\begin{aligned} 0 &= \tilde{u}_1^+ = \tilde{u}_2^+ = \tilde{u}_3^+ \text{ on } \{0\} \times [0, \eta l_2/2], \\ 0 &= \tilde{u}_1^- = \tilde{u}_2^- = \tilde{u}_3^- \text{ on } \{0\} \times [-\eta l_2/2, 0], \end{aligned}$$

and the transmission conditions

The "variational or weak formulation" of the elasticity problem for this structure takes again the form

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$$

with

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon^+} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}^-) d\mathbf{x}$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon^+} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^+) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^+) d\mathbf{x} + \int_{\Omega_\varepsilon^-} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}^-) \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}^-) d\mathbf{x},$$

where we are using the obvious decomposition $\tilde{\mathbf{v}} = (\tilde{\mathbf{v}}^+, \tilde{\mathbf{v}}^-)$ for any element of the energy space $\mathbf{V}_\varepsilon$.

The scaling and other developments are then analogous to those of the "basic problem". The space of configuration after scaling will be denoted by $\mathbf{V}$. It is the subspace of

$$H^1(\Omega^+) \times H^1(\Omega^+) \times H^2(\Omega^+) \times H^1(\Omega^-) \times H^1(\Omega^-) \times H^2(\Omega^-)$$

formed by the functions satisfying the transmission and kinematic boundary conditions

$$u_2^+ - u_2^- = l_2 u_3$$

and

$$0 = u_1 = u_2 = u_3 \ \text{ on } \ \{0\} \times [-l_2/2, l_2/2],$$

Theorem *Let $\mathbf{u}_\varepsilon$ and $\mathbf{u}$ be the solutions of the above coupled problems respectively. Then, for $\varepsilon \downarrow 0$, we have:*

$$\mathbf{u}^\varepsilon \rightarrow \mathbf{u}$$

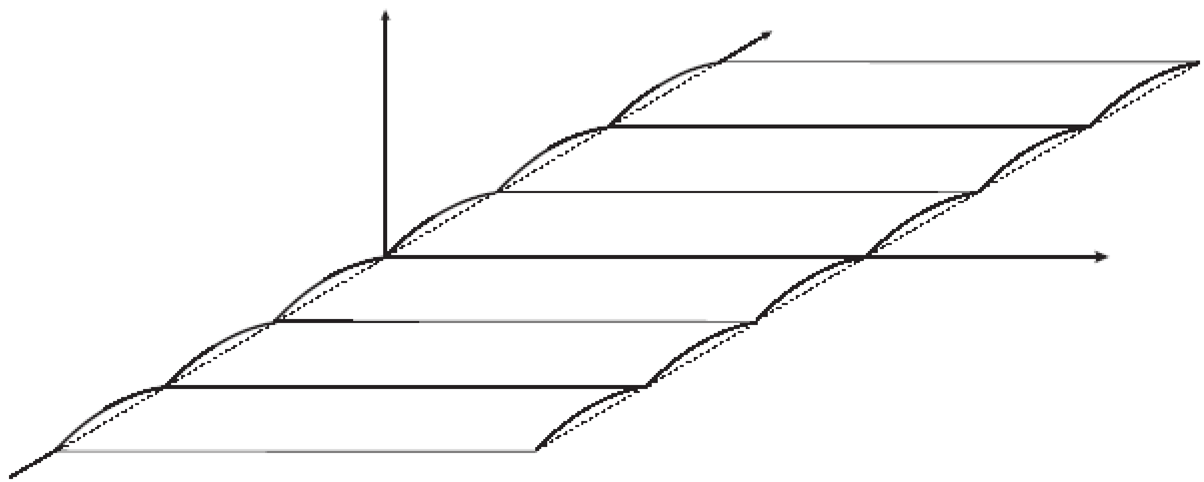
with convergence of each component $\mathbf{u}^\varepsilon = (\mathbf{u}^{\varepsilon+}, \mathbf{u}^{\varepsilon-})$

Equivalently, the limit $\mathbf{u}$ can be obtained through its potential $\varphi = (\varphi^+, \varphi^-) \in \widetilde{G}$, solution of

$$\int_{\Omega+} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ d\mathbf{y} + \int_{\Omega-} \frac{1}{C_{1111}} \partial_1^2 \varphi^+ \partial_1^2 \psi^+ d\mathbf{y}$$
$$+ \int_{\Omega+} B^{2222} \partial_2^4 \varphi^+ \partial_2^4 \psi^+ d\mathbf{y} + \int_{\Omega-} B^{2222} \partial_2^4 \varphi^- \partial_2^4 \psi^- d\mathbf{y}$$
$$= - \int_{\Omega+} F_3 \partial_2^2 \psi^+ d\mathbf{y} - \int_{\Omega-} F_3 \partial_2^2 \psi^- d\mathbf{y},$$

$\forall \psi = (\psi^+, \psi^-) \in \widetilde{G}$.

We consider now the case in which the shell is $2\eta l_2$−periodic with respect the section by $x_1 = const.$projected on the band $(0, l_1) \times (-\infty, +\infty)$ and having a slight folding at any section of the form $(0, l_1) \times \{k\eta l_2\}$ with $k \in \mathbb{Z}$, as sketched in Fig 3.

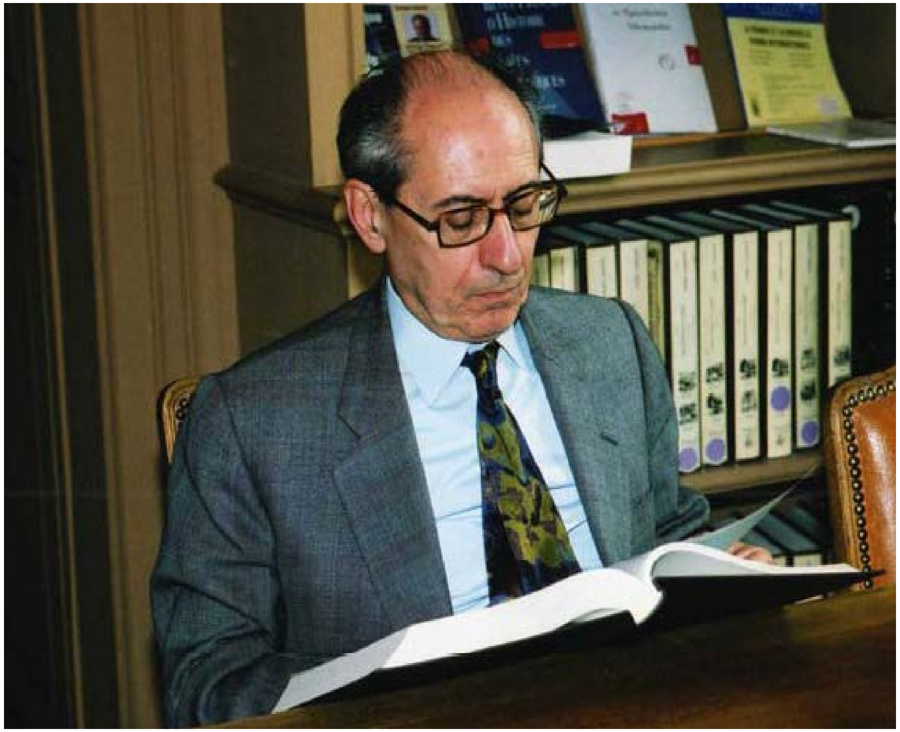


We can consider as "unit shell" the shell defied trough the rectangle $(0, l_1) \times (-\eta l_2, +\eta l_2)$, clamped along the "small sides" at $\{0\} \times [-\eta l_2, \eta l_2]$, which implies again kinematic boundary conditions similar to those indicated

We then consider periodic loadings and search for periodic solutions.

The convergence arguments follows as in previous Subsections with easy modifications.

Remark. Many other results (global properties, the obstacle problem, ...), work in progress (homogeneization, hyperbolic surfaces, ...). Many open problems.



Francisco de **Goya** y Lucientes

1746 Fuendetodos (Zaragoza)

1828 Bordeaux (France)

Thanks