1 On the Bingham stationary model

We shall study some qualitative properties on the spatial structure of solutions of problem

$$BS\begin{cases} -\Delta u - g\,\mathrm{div}\left(\frac{Du}{|Du|}\right) = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

Given $f \in L^2(\Omega)$, the existence and uniqueness of a solution $u \in H_0^1(\Omega)$ was shown by Duvaut- Lions (1969). The regularity $H^2(\Omega)$ was obtained later by Brezis (1971). Let us define the *plastic region* by

$$\Omega_0 = \{x \in \Omega : |Du| = 0\}.$$

Theorem 4 *Assume $f \equiv c$, a positive constant. Let* $.\omega_N = |B(0,1)|$, *i) if* $|\Omega| \leq \omega_N(\frac{Ng}{c})^N$ *then* $u(x) = 0$, *a.e.* $x \in \Omega$,

ii) if $|\Omega| > \omega_N(\frac{Ng}{c})^N$ *then* $|\Omega_0| \geq \omega_N(\frac{Ng}{c})^N$

The main ingredients of the proof are the consideration of the special case $\Omega = B(0, R)$ and a comparison in terms of the *decreasing symmetric rearrangement*

Proposition 2. *Let* $\Omega = B(0, R)$,

i) if $R \leq \frac{Ng}{c}$ *then* $u(x) = 0$, *a.e.* $x \in \Omega$,

ii) if $R > \frac{Ng}{c}$ *then* $\Omega_0 = B(0, \frac{Ng}{c})$.

Idea of the proof of Proposition 2. By the equivalent formulation in terms of a *Lagrange multiplier*, there exists $\mathbf{p} \in \Lambda := \{\mathbf{q} \in L^\infty(\Omega)^N : \|\mathbf{q}\|_\infty \leq 1\}$ such that

$$\begin{cases} -\Delta u - g\,\mathrm{div}\,\mathbf{p} = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \mathbf{p}\cdot Du = |Du| & a.e. \text{ in } \Omega. \end{cases}$$

Then, by approaching the solutions (when $p \searrow 1$) by the solutions of

$$BS_p \begin{cases} -\Delta u - g\Delta_p u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

we prove that if $R \leq \dfrac{Ng}{c}$ then $\|\mathbf{p}\|_\infty < 1$, and so $u(x) = 0$, *a.e.* $x \in \Omega$. If $R > \dfrac{Ng}{c}$ it is possible to construct (explicitly) the solution. So, for instance, for $N = 2$,

$$u(r) = \begin{cases} (R-r)(\frac{c}{4}(R+r) - g) & \text{if } \frac{2g}{c} \leq r \leq R, \\ \frac{c}{4}(R - \frac{2g}{c})^2 & \text{if } 0 \leq r \leq \frac{2g}{c}, \end{cases}$$

(see also Glowinski, R. Lions, J.L. and Tremolières, R., *Numerical Analysis of Variational Inequalities,* North-Holland, Amsterdam, 1981).

Proposition 3. *Let* $f \in L^2(\Omega)$, $f \geq 0$. *Let* $f^* \in L^2(\Omega^*)$ *its decreasing symmetric rearrangement. Let* U *be the solution of* BS *associated to* Ω^* *and* f^*. *Then*

$$u^*(x) \le U(x), \text{ a.e. } x \in \Omega^*$$

and

$$|Du^*(x)| \le |DU(x)|\,, \text{ a.e. } x \in \Omega^*.$$

(The proof is an easy variation of J.I.D: Desigualdades de tipo isoperimétrico para problemas de Plateau y capilaridad, *Revista de la Academia Canaria de Ciencias*, Vol. III, No.1, 127-166, 1991: see also Abourjaily and Benilan, Symmetrization of quasilinear parabolic problems, *Revista Union Matemática Argentina,* **41,** 1, 1999).

Remarks.

8. The proof of Theorem 4 is now immediate from Propositions 3 and 4.

9. The radial solutions can be used as super and subsolutions in order to get pointwise estimates on the location of the plastic region.

10. In the radial case we conclude that $|\Omega_0| = \omega_N (\frac{Ng}{c})^N$ independently of R (once that $R > \frac{Ng}{c}$). This is entirely different to the case of the free boundary for

$$\begin{cases} -\Delta_p u + u = 1 & \text{in } \Omega = B(0, R), \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

assumed $p > 2$ (in that case the "solid region" is $\Omega_1 = \{x \in \Omega : u = 1\}$ and $|\Omega_1| \nearrow +\infty$ if $R \nearrow +\infty$).