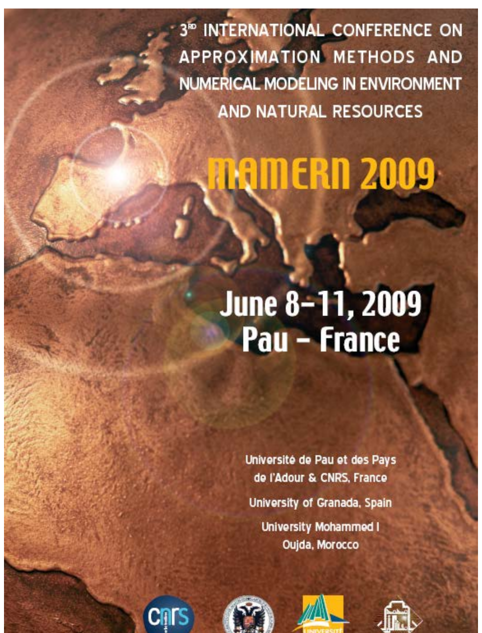
On the mathematical and numerical analysis of a model for the river channel formation

J.I. Díaz

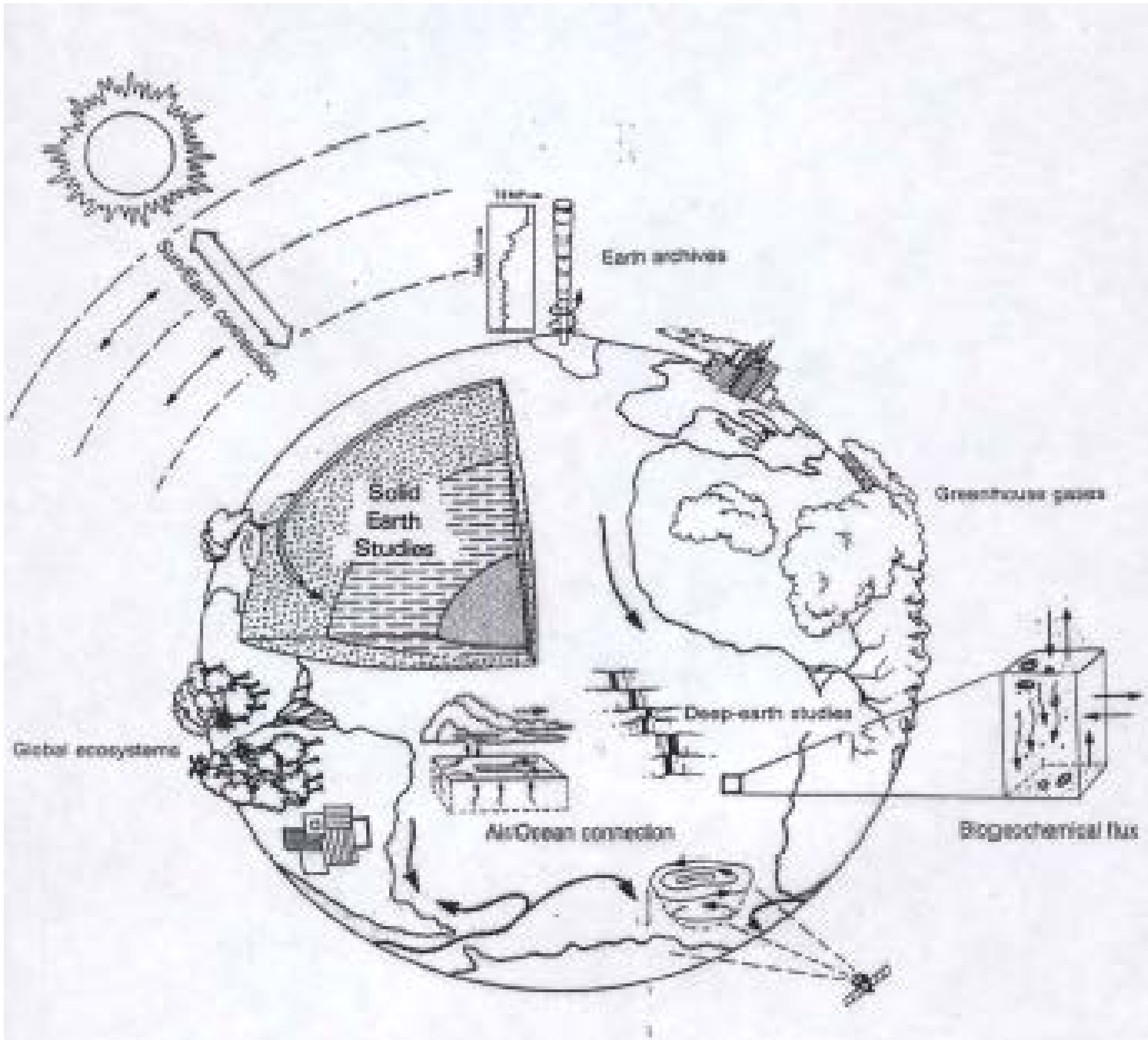
Universidad Complutense de Madrid,

Pau, June, 9, 2009

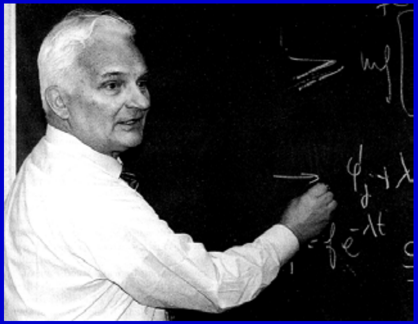


1. Introduction.

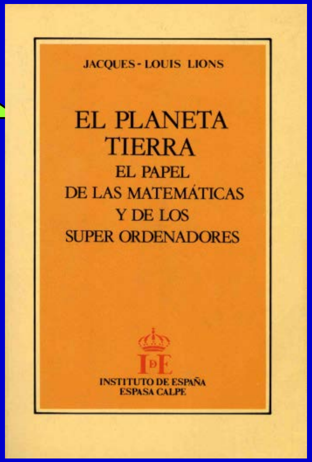
Environment and Natural Resources



“The universal trilogy” in Applied Mathematics



J.L. Lions (1928-2001)



“Real” problem

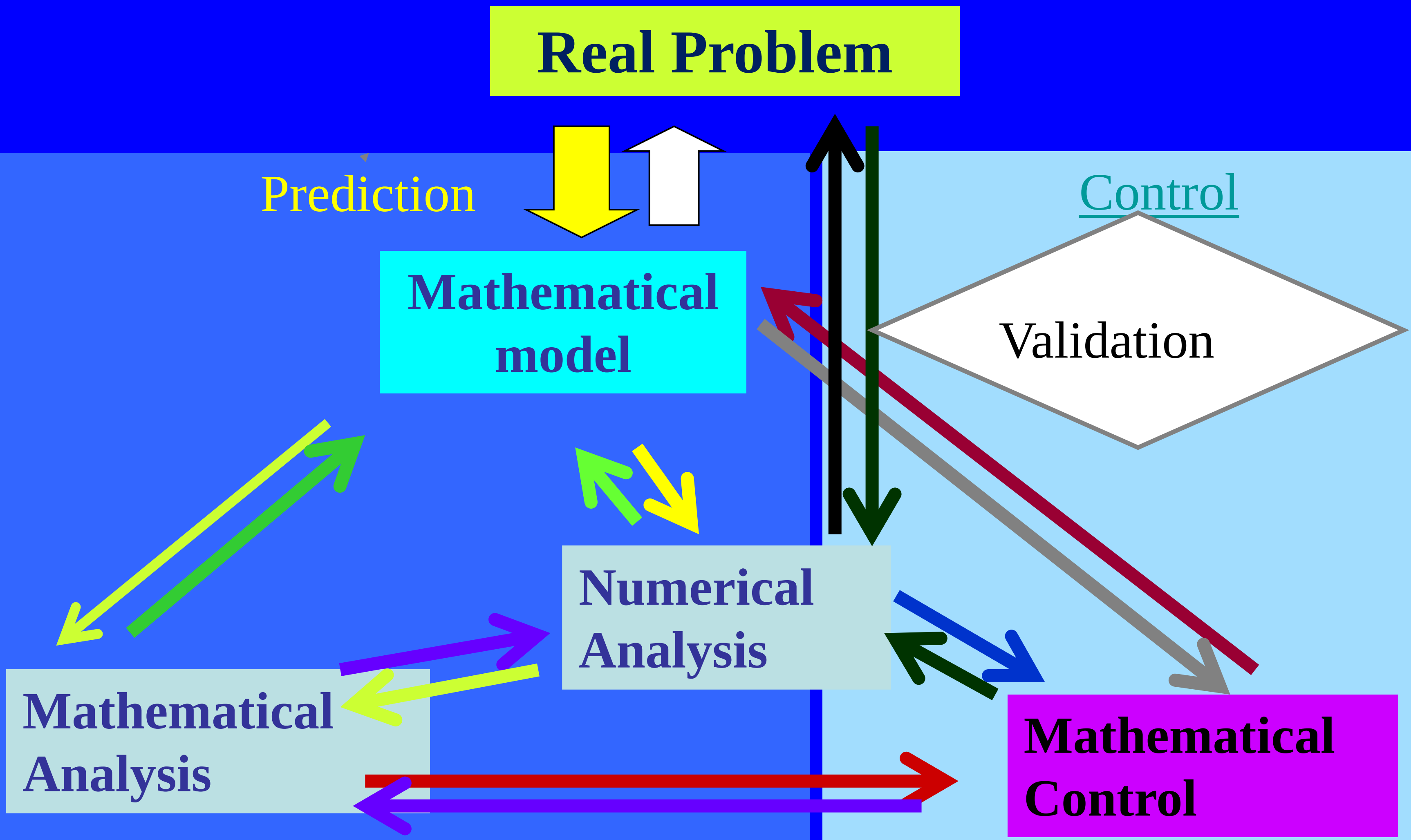
Validation

Mathematical Modelling

Mathematical and Numerical Analysis

Optimization and Control

INTERACTIONS



Main goal:

The mathematical analysis clarifying the modelling of the problem

FOWLER, A. C., KOPTEVA, N., and OAKLEY, C. (2007), *The formation of river channels*, SIAM J. Appl. Math. *67*, 1016–1040.

$$u_t = (u^{3/2})_{\eta\eta} + u^{3/2},$$

$$\int_{-\infty}^{\infty} u^{3/2} d\eta = 1, \quad u \to 0 \quad \text{as} \quad \eta \to \pm\infty, \quad t \to 0$$

- Global formulation?
- Solvability?
- Numerical analysis?

Global formulation and *suitable notion of (weak) solution*

(Hilbert, Paris, 1900)

Joint work with

A. C. FOWLER,[2] A. I. MUÑOZ,[3] and E. SCHIAVI[3]

[2] MACSI, Department of Mathematics and Statistics, University of Limerick, Limerick, Ireland. E-mail: fowler@math.ox.ac.uk.

[3] Departamento de Matemática Aplicada, E.S.C.E.T. Universidad Rey Juan Carlos, E 28933 Móstoles, Madrid, Spain. E-mail: anaisabel.munoz@urjc.es, emanuele.schiavi@urjc.es.

Starting paper: Pure appl. geophys. 165 (2008) 1–20

Pure and Applied Geophysics

EARTH SCIENCES AND MATHEMATICS

Madrid, September, 13, 14 and 15, 2006

2. Modelling
3. On the mathematical analysis of the model: the stationary problem
4. The parabolic problem
5. Numerical results

2. Modelling

River formation: turbulent flow

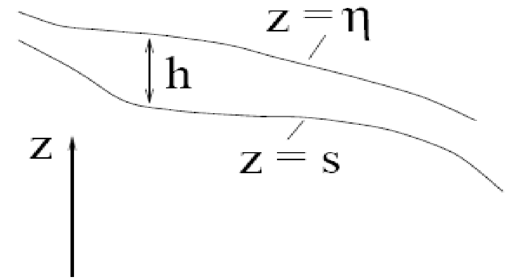


Figure 2: Geometry of overland flow.

$$\eta = s + a + h.$$

s=hill slope elevation
h=water depth
a=bedload layer thickness

Overland flow over an erodible substrate

Non cohesive sediments

Sediment transport

Bedload transport

Model equations

Hydraulic Flow

Conservation of mass and momentum

Sediment transport

Conservation of mobile sediments

Exner equation (land surface evolution)

Constitutive laws

Geometric relation between depths

$$\eta = s + a + h.$$

Dimensional model equations

Non-dimensionalisation

Parameters definition and estimation

Dimensionless equations

Asymptotic limit

Reduced model

Model equations

Conservation of mass and momentum

$$\begin{cases} h_t + \nabla \cdot (h\boldsymbol{u}) &= r \\ \boldsymbol{u}_t + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -g\nabla\eta - \dfrac{f|\boldsymbol{u}|\boldsymbol{u}}{h} \end{cases} \tag{2.1}$$

Conservation of mobile sediments

$$\rho_s(1-\phi)a_t + \nabla \cdot \boldsymbol{q}_b = \rho_s(1-\phi)v_A \tag{2.2}$$

Exner equation

$$\rho_s(1-\phi)s_t = -\rho_s(1-\phi)v_A + \rho_s(1-\phi)U \tag{2.3}$$

Geometric relation between depths

$$\eta = s + a + h \tag{2.4}$$

Turbulent shear stress

$$\boldsymbol{\tau} = f\rho_w|\boldsymbol{u}|\boldsymbol{u}$$

Shields stress

$$\mu = \frac{\tau_e}{\Delta\rho g D_s}$$

Effective stress

$$\boldsymbol{\tau}_e = \boldsymbol{\tau} - \Delta\rho g D_s \nabla s$$

Meyer-Peter and Muller Law (1948)

$$q_b = \left(\frac{\rho_s K}{\rho_w^{1/2}\Delta\rho g}\right)(\tau_e - \tau_c)_+^{3/2}$$

Critical stress

$$\tau_c = \mu_c \Delta\rho g D_s.$$

Bedload velocity

$$v_b = \frac{q_b}{\rho_s(1-\phi)a_0}$$

$$v_b = \left(\frac{K}{\rho_w^{1/2}\Delta\rho g(1-\phi)a_0}\right)(\tau_e - \tau_c)_+^{3/2}$$

Constitutive Laws

$$\boldsymbol{q_b} = \rho_s(1-\phi)a v_b(\tau_e)\boldsymbol{N}, \qquad v_A = k v_b(\tau_e)\left[1 - \frac{a}{a_0}\right]_+ \qquad (2.5)$$

Transport rate

Abrasion rate

Non dimensionalisation

$$r \sim r_D, \quad U \sim U_D, \quad v_b \sim v_D, \quad v_A \sim U_D,$$

$$\eta,\ s \sim d, \quad \mathbf{x} \sim l, \quad t \sim [t] = \frac{d}{U_D}, \quad \tau_e \sim [\tau] = f\rho_w\,[u]^2,$$

$$\mathbf{u} \sim [u] = \left(\frac{g r_D d}{f}\right)^{1/3}, \quad a \sim a_0, \quad h \sim [h] = l\left(\frac{f r_D^2}{g d}\right)^{1/3},$$

$$v_D = \left(\frac{K[\tau]^{3/2}}{\rho_w^{1/2}\Delta\rho\, g(1-\phi)a_0}\right).$$

Parameters

$$F = \frac{[u]}{(g[h])^{1/2}}, \quad \varepsilon = \frac{U_D}{r_D}, \quad \delta = \frac{[h]}{d},$$

$$\nu = \frac{a_0}{[h]}, \quad \alpha = \frac{kl}{a_0} \sim \frac{lU_D}{dr_D}, \quad \beta = \frac{\Delta\rho\, D_s}{\rho_w[h]}.$$

Dimensionless equations

$$
\begin{aligned}
\delta\varepsilon h_t + \boldsymbol{\nabla}.\,(h\mathbf{u}) &= r,\\
\delta F^2[\,\delta\varepsilon\mathbf{u}_t + (\mathbf{u}.\boldsymbol{\nabla})\mathbf{u}] &= -\boldsymbol{\nabla}\eta - \frac{|\mathbf{u}|\mathbf{u}}{h},\\
\eta &= s + \delta h + \delta\nu a,\\
\delta\nu\alpha a_t + \boldsymbol{\nabla}.[aV\mathbf{N}] &= \alpha A,\\
s_t &= -A + U,\\
\boldsymbol{\tau}_e &= |\mathbf{u}|\mathbf{u} - \beta\boldsymbol{\nabla}s,
\end{aligned}
$$

$$V = [\tau_e - \tau_c^*]_+^{3/2}, \quad A = [1-a]_+ V, \qquad \boldsymbol{N} = \boldsymbol{\tau}_e/\tau_e$$

Parameter estimates

$$F^2 \sim 0.1, \quad \varepsilon \sim 10^{-3}, \quad \delta \sim 10^{-5},$$

$$\alpha \sim 0.1, \quad \beta \sim 0.1, \quad \nu \sim 0.05, \quad \tau_c^* \sim 0.5.$$

Reduced model

$$
\begin{aligned}
\boldsymbol{\nabla}.(h\mathbf{u}) &= r, \\
\mathbf{0} &= -\boldsymbol{\nabla}\eta - \frac{|\mathbf{u}|\mathbf{u}}{h}, \\
\eta &= s + \delta h, \\
\boldsymbol{\nabla}.[aV\mathbf{N}] &= \alpha A, \\
s_t &= -A + U, \\
\boldsymbol{\tau}_e &= |\mathbf{u}|\mathbf{u} - \beta\boldsymbol{\nabla}s.
\end{aligned}
$$

$$
\eta = 0 \quad \text{and} \quad \frac{\partial\eta}{\partial n} = 0 \quad \text{on} \quad \partial D.
$$

Downstream flow direction

$$\boldsymbol{n} = -\frac{\nabla \eta}{|\nabla \eta|}$$

Stream slope

$$S = |\nabla \eta|$$

Water flux

$$q = h|\boldsymbol{u}|.$$

Bedload thickness

$$a = 1.$$

$$\begin{aligned} \nabla \cdot [q\boldsymbol{n}] &= r \\ q &= h^{3/2} S^{1/2} \end{aligned}$$

$$\boldsymbol{\tau}_e = -(h+\beta)\nabla \eta + \delta\beta\nabla h.$$

$$s_t = U - \frac{1}{\alpha}\nabla \cdot [V\boldsymbol{N}]$$

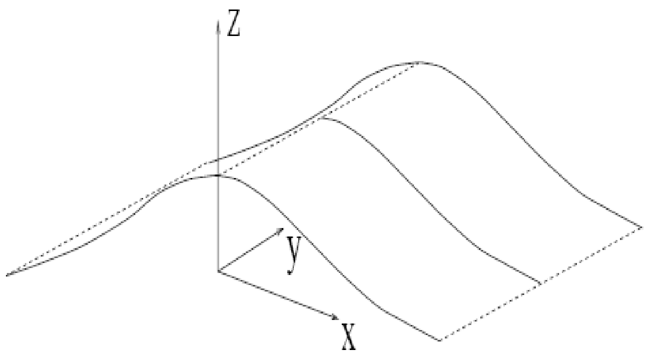


Figure 3: One-dimensional hillslope geometry.

Linearisation of the geometry (only)

Approximate model

$$\frac{\partial q}{\partial x} - \frac{\partial}{\partial y}\left[\frac{q}{S_0}\frac{\partial \tilde{\eta}}{\partial y}\right] = r$$

$$\frac{\partial \tilde{\eta}}{\partial t} - \delta\frac{\partial h}{\partial t} = U - \frac{\partial V}{\partial x} + \frac{\partial}{\partial y}\left[\frac{V}{S_0}\left\{\frac{\partial \tilde{\eta}}{\partial y} - \frac{\beta\delta}{h+\beta}\frac{\partial h}{\partial y}\right\}\right]$$

$$y = \delta^{1/2}Y, \qquad \tilde{\eta} = \delta Z, \qquad t = \delta\tilde{t}$$

$$y = \delta^{1/2} Y, \qquad \tilde{\eta} = \delta Z, \qquad t = \delta \tilde{t}$$

$$\boxed{\begin{aligned} \frac{\partial q}{\partial x} - \frac{\partial}{\partial Y}\left[\frac{q}{S}\frac{\partial Z}{\partial Y}\right] &= r \\ \frac{\partial Z}{\partial \tilde{t}} - \delta\frac{\partial h}{\partial \tilde{t}} &= U - \frac{\partial V}{\partial x} + \frac{\partial}{\partial Y}\left[\frac{V}{S}\left\{\frac{\partial Z}{\partial Y} - \frac{\beta}{h+\beta}\frac{\partial h}{\partial Y}\right\}\right] \end{aligned}}$$

$$\boxed{\tau_e \approx (h+\beta)S, \qquad q = h^{3/2}S^{1/2}}$$

$$\frac{\partial h}{\partial y} = \frac{\partial Z}{\partial y} = 0 \quad \text{on} \quad y = \pm L$$

$$q = V = 0 \quad \text{on} \quad x = 0$$

$$Z = 0 \quad \text{on} \quad x = 1$$

$$h = \frac{H}{\delta^{1/3}}, \quad q = \frac{Q}{\delta^{1/2}}, \quad V = \frac{F}{\delta^{1/2}}, \quad \tau_e = \frac{T_e}{\delta^{1/3}}, \quad \tilde{t} = \delta^{1/6} T$$

$$\boxed{\begin{aligned} \frac{\partial Q}{\partial x} - \frac{\partial}{\partial Y}\left[\frac{Q}{S}\frac{\partial Z}{\partial Y}\right] &= \delta^{1/2} r \\ \delta^{1/2}\frac{\partial Z}{\partial T} - \delta\frac{\partial H}{\partial T} &= \delta^{1/2} U - \frac{\partial F}{\partial x} + \frac{\partial}{\partial Y}\left[\frac{F}{S}\left\{\frac{\partial Z}{\partial Y} - \frac{\beta}{H + \delta^{1/3}\beta}\frac{\partial H}{\partial Y}\right\}\right] \end{aligned}}$$

$$T_e \approx (H + \delta^{1/3}\beta) S, \qquad Q = H^{3/2} S^{1/2}$$

$$F = \left[T_e - \delta^{1/3}\tau_c^*\right]_+^{3/2} = QS + \frac{2}{3}(\delta Q S)^{1/3}(\beta S - \tau_c^*) + \ldots..$$

$$-\delta^{1/2}\frac{\partial Z}{\partial T}+\frac{\partial H}{\partial T}=S'S^{1/2}H^{3/2}+S^{1/2}\frac{\partial}{\partial Y}\left[\beta H^{1/2}\frac{\partial H}{\partial Y}\right]+C\frac{\partial^2 Z}{\partial Y^2}$$

Nonlinear evolution equation

$$\frac{\partial H}{\partial T}=S'S^{1/2}H^{3/2}+S^{1/2}\frac{\partial}{\partial Y}\left[\beta H^{1/2}\frac{\partial H}{\partial Y}\right]$$

$$H\to 0 \qquad \text{as} \quad Y\to\pm\infty$$

$$H\to 0 \qquad \text{as} \quad T\to 0$$

$$\frac{\partial q}{\partial x} - \frac{\partial}{\partial Y}\left[\frac{q}{S}\frac{\partial Z}{\partial Y}\right] = r$$

$$\int_{-L/\delta^{1/2}}^{L/\delta^{1/2}} q dY = 2Lrx/\delta^{1/2}$$

Integral constraint

$$\boxed{\int_{-\infty}^{\infty} H^{3/2} dY = \frac{2Lrx}{S^{1/2}}}$$

In the special case of S'>0 we introduce the following change of variables

$$H = \left(\frac{6}{\beta}\right)^{1/3} (Lrx)^{2/3} u, \quad T = \left(\frac{6}{\beta}\right)^{1/6} \frac{S' S^{1/2}}{(Lrx)^{1/3}} t, \quad Y = \left(\frac{2\beta}{3S'}\right)^{1/2} \eta$$

$$u_t = (u^{3/2})_{\eta\eta} + u^{3/2},$$

$$\int_{-\infty}^{\infty} u^{3/2} d\eta = 1, \quad u \to 0 \quad \text{as} \quad \eta \to \pm\infty, \quad t \to 0$$

FOWLER, A. C., KOPTEVA, N., and OAKLEY, C. (2007), *The formation of river channels*, SIAM J. Appl. Math. *67*, 1016–1040.

-Degenerate nonlinear diffusion equation suggestive of compact support solutions (velocity of the free boundary, Darcy law, ??????)

- Source term suggestive of blow up (??????)

- The flux is discontinuous at the free boundaries (no detail about its role in a global formulation)

Merely, the authors made mention to the book

SAMARSKI, A. A., GALAKTIONOV, V. A., KURDYUMOV, S. P., and MIKHAILOV, A. P., *Blow-up in quasilinear parabolic equations* (Walter de Gruyter, Berlin, 1995).

3. On the mathematical analysis of the model: the stationary problem.

We consider the above problem assuming an initial thickness perturbation $u_0(x)$ satisfying some natural physically based hypothesis.

For the sake of simplicity of the exposition we also assume symmetric the initial data

We replace 3/2 by a general exponent m>1.

We assume $u_0(x)$ be a bounded and non negative function with a compact and connected support $[-\zeta_0, \zeta_0]$ such that $\int_0^{+\infty} u_0^m(x)\, dx = M/2$

Problem: Find a continuous curve $\zeta : [0, +\infty) \to \mathbb{R}^+$ and a function $u : \mathcal{P} \to [0, +\infty)$, $\mathcal{P} = \bigcup_{t>0} \Omega_t$, $\Omega_0 = (0, \zeta_0)$, $\Omega_t = (0, \zeta(t)) \times \{t\}$ such that

$$
(SL)\begin{cases}
u_t = (u^m)_{xx} + u^m, & \text{in } \mathcal{D}'(\mathcal{P}),\\
u(x,0) = u_0(x) & \text{a.e. } x \in \Omega_0,\\
u(x,t) > 0, & \text{a.e. } (x,t) \in \mathcal{P},\\
u(x,t) \equiv 0, & \text{a.e. } (x,t) \notin \mathcal{P},\\
\zeta(0) = \zeta_0 \text{ and } \zeta(t) > 0 & \text{for any } t \geq 0,\\
u(\zeta(t),t) = 0, \quad (u^m)_x(0,t) = 0 & \text{a.e. } t \in (0, +\infty),\\
\displaystyle\int_0^{\zeta(t)} u^m(x,t)\, dx = \frac{M}{2} & \text{a.e. } t \in (0, +\infty).
\end{cases}
$$

Strong local formulation

No classical solution of the nonlinear partial differential equation

Degenerate parabolic equation

Two unknowns: u(x,t) and the *free boundary* $\zeta : [0,+\infty) \rightarrow \mathbb{R}^+$

Global weak formulation $(x,\ t) \in (0,\ +\ \infty) \times (0,\ +\ \infty),$

Usual *weak formulation:* extend u(x,t) by zero outside $\mathcal{P} = \cup_{t>0}\Omega_t$

and use the distributions sense: i.e. $u \in C([0,+\infty) : L^1(0,+\infty))$

and for any test function $\phi \in C^2((0,+\infty)\times(0,+\infty)) \cap C^0((0,+\infty)\times(0,+\infty)),$

$\phi(\cdot,t)$ with compact support in $(0,+\infty)$ and $\phi(\cdot,T) \equiv 0$

$$-\int_0^T\int_0^{+\infty} u\phi_t + \int_0^T\int_0^{+\infty} u^m\phi_{xx} + \int_0^T\int_0^{+\infty} u^m = -\int_0^{+\infty} u(x,0)\phi(x,0).$$

In general, well established theory for weak solutions:

Oleinik, O.A.,Kalashnikov, A.S. and Chzhou Y.-L., The Cauchy problem and boundary problems for equations of the type of non-stationary filtration, *Izv. Akad. Nauk SSR, Ser. Mat.*, **22,** 1958, 667-704 (in russian).

Aronson, D.G., Regularity properties of flows through porous media, *SIAM J. Appl. Math.* **17,** 1969, 461-467.

Bénilan, Ph., *Evolution Equations and Accretive Operators*, Lecture Notes (taken by S. Lenhart), Univ. of Kentucky, 1981.

Gagneux, G. and Madaune-Tort, M., *Analyse mathématique de modèles non linéaires de l'ingenierie petrolière. (Mathematical analysis of nonlinear models of petrol engineering).*, Mathematiques & Applications, **22** , Springer-Verlag, Paris, 1995.

Meirmanov, A.M., Pukhnachov, V.V. and Shmarev, S.I., *Evolution Equations and Lagrangian Coordinates,* Walter de Gruyter, Berlin, 1997.

S.N. Antontsev, J.I. Díaz and S.I. Shmarev, *Energy Methods for Free Boundary Problems: Applications to Nonlinear PDEs and Fluid Mechanics*, Progress in Nonlinear Differential Equations and Their Applications, 48, Birkhäuser, Boston, 2002.

Main difficulty:

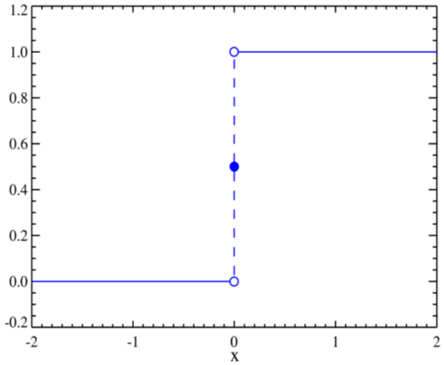
the flux, $u^m(\zeta(t),t)_x$ near the free boundary can not be continuous

SAMARSKI, A. A., GALAKTIONOV, V. A., KURDYUMOV, S. P., and MIKHAILOV, A. P., *Blow-up in quasilinear parabolic equations* (Walter de Gruyter, Berlin, 1995).

If the above global formulation is correct (case of zero flux near the free boundary) then there is blow-up in finite time: contradiction with

$$\int_0^{\zeta(t)} u^m(x,t)\,dx = \frac{M}{2} \qquad \text{a.e. } t \in (0,+\infty).$$

Going forwards to a global formulation:



Oliver Heaviside (1850 –1925)

$$dH(x)\,/\,dx = \delta(x).$$

Paul Adrien Maurice Dirac (1902 – 1984)

$$\delta(x) = \begin{cases} +\infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1.$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-T)\,dt = f(T)$$

The stationary case: $v = u^m$

$$-v_{xx} - v = 0, \quad , \quad v'(0) = 0, \quad \lim_{x\to+\infty} v(x) = 0, \quad \int_0^{+\infty} |v(x)|\, dx = \frac{M}{2}$$

Our first commentary is that the formulation does not correspond to a standar constrained problem of the Calculus of Variation. Indeed, if it let an standard constrained problem the solution would coincide with the solution of the constrained minimization problem:

$$\mathrm{Min}_{v\in X} J(v), \quad \text{such that} \quad v'(0) = 0, \quad \lim_{x\to+\infty} v(x) = 0, \quad \int_0^{+\infty} G(v)dx = \frac{M}{2}$$

$$J(v) = \frac{1}{2}\int_0^{+\infty} |v_x|^2 dx - \frac{1}{2}\int_0^{+\infty} |v|^2 dx, \quad \text{and } G(v) = |v|.$$

$$\mathrm{Min}_{v\in X} J(v) + \lambda \int_0^{+\infty} G'(v)dx, \quad \text{such that} \quad v'(0) = 0, \quad \lim_{x\to+\infty} v(x) = 0,$$

$$-v_{xx} - v = \lambda G'(v), \qquad \boxed{-v_{xx} - v = \lambda}$$

which is not our equation !!!

Discontinuity of the flux near the free boundary:

$$v_{xx} + v = 0 \qquad v(x) = A\cos x + B\sin x$$

$$\{v > 0\} = (0, \zeta_\infty)$$

$$v_x(x) = 0 \text{ if } x \notin \{v > 0\} = (0, \zeta_\infty).$$

$lim_{x \nearrow \zeta_\infty} v_x(x)$ is strictly negative

(since the function is passing from positive values to zero)

v_{xx} is not an integrable function on $(0, +\infty)$

but a measure with a non-zero singular part.

$$\mu = -v_{xx} \in \mathcal{M}(0, +\infty),$$

space of Radon measures   Johann Karl August Radon (1887–1956)

L.C. Evans - R.F. Gariepy, *Measure Theory and Fine Properties of Functions,*. Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

$-v_{xx} = v$ on $\{v > 0\}$ the (signed) Jordan decomposed of μ $\mu = \mu_+ - \mu_-$

Marie Ennemond Camille Jordan (1838 –1922)

$\mu_+ = v$ (with is in $L^1(0,\infty)$) $\mu_- = -c\delta_{\zeta_\infty}$ for some $c > 0$,

alternative notation $\delta_{\zeta_\infty} = \delta_{\partial\{v=0\}}$

$$-v_{xx} - v = c\delta_{\partial\{v=0\}},$$

$$0 = \int_0^{+\infty} d\mu = \int_0^{+\infty} v dx - \int_0^{+\infty} d\mu_- = \frac{M}{2} - c < \delta_{\zeta_\infty}, 1 >= \frac{M}{2} - c,$$

necessarely, $\mu_- = -\frac{M}{2}\delta_{\zeta_\infty}$

$$0 = \int_0^{+\infty} d\mu = -\int_0^{\zeta_\infty} v_{xx} dx + \int_{\zeta_\infty}^{+\infty} d\mu_- = -v_x(\zeta_\infty) - \frac{M}{2}$$

the flux is determined by the integral constraint

$$lim_{x \nearrow \zeta_\infty} v_x(x) = -\frac{M}{2}$$

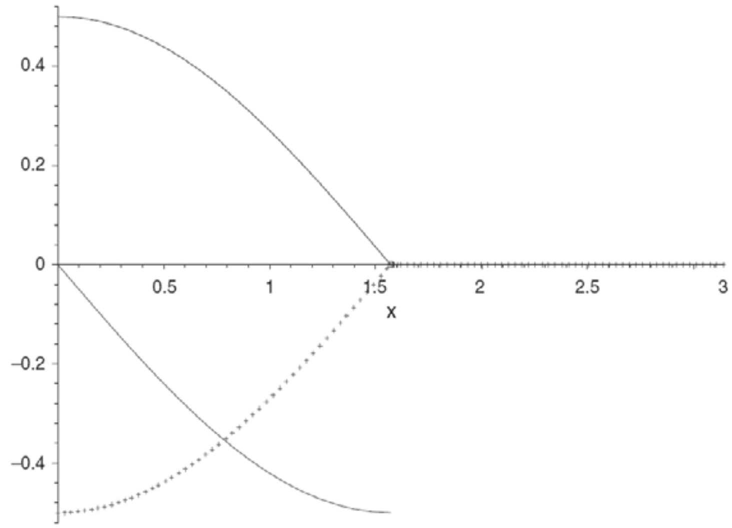
$$(SP) \begin{cases} v_{xx} + v = \frac{M}{2}\delta_{\zeta_\infty}, & \text{a.e. } x \in \mathbb{R}^+, \\ v(x) > 0, & x \in [0, \zeta_\infty), \\ v(x) \equiv 0, & x \geq \zeta_\infty, \\ v_x(0) = 0. \end{cases}$$

Proposition 1. *Given $M > 0$ there exists a unique solution $(v(x), \zeta_\infty)$ of (SP) given by*

$$\zeta_\infty = \frac{\pi}{2} \quad \text{and} \quad v(x) = \frac{M}{2}\cos x\left[1 - H\left(x - \frac{\pi}{2}\right)\right],$$

where $H(x - \pi/2)$ denotes the Heaviside function located at $\pi/2$ i.e.,

$$v(x) = \begin{cases} (M/2)\cos x & \text{if } x \in [0, \pi/2], \\ 0 & \text{if } x \in (\pi/2, +\infty). \end{cases}$$



Remark. Problems of this type arise in fluid mechanics (the Bernoulli problem), combustion and plasma physics.

Díaz, J. I., Padial, J. F., and Rakotoson, J. M. (2007), *On some Bernouilli free boundary type problems for general elliptic operators*, Proc. Roy. Soc. Edimburgh *137A*, 895–911.

4. The parabolic problem.

Next, we shall show that in order to generalize the global formulation, obtained in the stationary case, to the parabolic case, it is not enough to consider the presence of the Dirac delta. In the parabolic case, the Dirac delta does not prevent the blow-up phenomenon.

$$(P_0)\quad \begin{cases} u_t = (u^m)_{xx} + u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0,T)), \\ u(x,0) = u_0(x) & \text{a.e. } x \in (0,\zeta_0) \\ u(x,t) > 0, & \text{a.e. } (x,t) \in \mathcal{P}_T, \\ u(x,t) \equiv 0, & \text{a.e. } (x,t) \notin \mathcal{P}_T, \\ u(\zeta(t),t) = 0, \quad u_x(0,t) = 0 & \text{a.e. } t \in (0,T), \\ \zeta(0) = \zeta_0 \text{ and } \zeta(t) > 0 & \text{for any } t \in [0,T], \end{cases}$$

We shall look for separable solutions of the form

$$u(x,t) = (T_e - t)^{-1/(m-1)}\theta(x).$$

$$\int_0^{+\infty} u^m(x,t)dx = \frac{MT_e^{1/(m-1)}}{2(T_e-t)^{1/(m-1)}} \text{ for } t \in [0, T_e).$$

Theorem. *For any $c > 0$, the problem:*

$$(P_w)\begin{cases} w'' + w - \frac{1}{m-1}w^{1/m} = c\delta_{\zeta_0}, & \mathcal{D}'(0,+\infty), \\ w(x) = 0 & x > \zeta_0, \\ w'(0) = 0, & \end{cases}$$

admits a unique nonnegative solution w such that $\int_0^{+\infty} w(x)dx = c.$

If we take $\theta^m(x) = w(x)$ and $c = \frac{M}{2T_e^{m/(m-1)}}$, then the pair $u_{T_e}(x,t) = (T_e - t)^{-1/(m-1)}\theta(x)$ and $\zeta(t) \equiv \zeta_0$ satisfies (P_0) for $u_0(x) := T_e^{-1/(m-1)} w^{1/m}(x)$.

Díaz, J. I., Fowler, A. C., Muñoz, A. I., and Schiavi, E.,
Monografías de la Real Academia de Ciencias de Zaragoza **31**, 57–66, (2009).

We improved Section 1.1, Chapter IV of

Samarski, A. A., Galaktionov, V. A., Kurdyumov, S. P., and Mikhailov, A. P., *Blow-up in quasilinear parabolic equations* (Walter de Gruyter, Berlin, 1995).

Therefore, we shall need some additional condition in order to give a global formulation to the whole domain $\mathbb{R}^+ \times (0,T)$. Notice that if we define (for a.e. $t \in (0,T)$ fixed) the spatial distribution

$$\mu(t,\cdot) := u_t(t,\cdot) - (u^m)_{xx}(t,\cdot)$$

then we must expect to know that, in fact, such a distribution is a bounded measure $\mathcal{M}(0,+\infty)$ (with compact support) since

$$\mu(t,\cdot) = u^m(t,\cdot) - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}.$$

Now, as in the stationary case we have that the mass constraint

$$\int_0^{+\infty} u^m(x,t)dx = M/2$$

is equivalent to the "zero total measure condition" (ZTMC)

$$\int_0^{+\infty} d\mu(t,\cdot) = 0, \qquad \text{for a.e. } t \in (0,T).$$

$$
(P)\begin{cases}
u_t = (u^m)_{xx} + u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0,T)), \\
u(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty), \\
u_x(0,t) = 0,\ u(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T), \\
\mu(t,\cdot) := u_t(t,\cdot) - (u^m)_{xx}(t,\cdot) \text{ satisfies ZTMC} & \text{a.e. } t \in (0,T).
\end{cases}
$$

Theorem. *Assume that* $u_0(x)$ *satisfies*

$$
u_0(x) > 0 \quad \text{for any } x \in [0, \zeta(0)).
$$

then, there exists a function $C^*(t) > 0$,

$C^* \in L^\infty(0,T)$ *and a function* $u \in C([0,T] : L^1(\mathbb{R}^+))$ *such that*

$$
\begin{cases}
u_t = (u^m)_{xx} + C^*(t)^{m-1}u^m - \frac{M}{2}\delta_{\partial\{u(t,\cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0,T)), \\
u(x,0) = u_0(x) & a.e.\ x \in (0,+\infty), \\
u_x(0,t) = 0,\ u(x,t) \to 0 \text{ as } x \to +\infty & a.e.\ t \in (0,T),
\end{cases}
$$

and

$$
C^*(t)^{m-1}\int_0^{+\infty} u(x,t)^m\,dx = \frac{M}{2}.
$$

Remark. We do not know if $C^*(t) = 1$, but by a rescaling $\widetilde{t} = k(t)$, $y = Y(t,x)$ and $V = V(u, y, \widetilde{t})$ it is possible to reformulate the above equation in the terms

$$V_{\widetilde{t}} = (V^m)_{yy} + V^m - \frac{M}{2}\delta_{\partial\{V(\widetilde{t},\cdot)=0\}}.$$

Idea of the proof.

we use a two-step iterative approximation. The main idea is to construct $\{u_{2n+1}:n = 0, 1, 2\ldots\}$ as solutions of the problem with a semi-implicit linear source term

$$(P_{2n+1}) := \begin{cases} (u_{2n+1})_t = ((u_{2n+1})^m)_{xx} + (u_{2n})^{m-1}(u_{2n+1}) - \frac{M}{2}\delta_{\partial\{(u_{2n+1})(t,\cdot)=0\}}, & \mathcal{D}'(\mathbb{R}^+ \times (0,T)), \\ (u_{2n+1})(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty), \\ (u_{2n+1})_x(0,t) = 0, \ (u_{2n+1})(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T), \end{cases}$$

(where for $n = 0$ we use as u_{2n} the initial condition u_0) and then to construct the sequence $\{u_{2n}:n = 1,2\ldots\}$ by requiring that

$$(P_{2n}) := \begin{cases} u_{2n}(x,t) = C_{2n}(t)u_{2n-1}(x,t) & \text{for a.e. } (x,t) \in \mathbb{R}^+ \times (0,T), \\ \int_0^{+\infty} ((u_{2n}(x,t))^{m-1}(u_{2n-1}(x,t))\, dx = \frac{M}{2} & \text{for a.e. } t \in (0,T), \end{cases}$$

for some $C_{2n}(t) > 0$.

The detailed proof of the convergence of the algorithm is quite technical and will not be presented here.

For instance, many of the a priori estimates on u_{2n+1} are obtained previously for the solutions $u_{2n+1,\varepsilon}$ of the equation obtained by replacing the singular equation of (P_{2n+1}) by the more regular equation

$$(u_{2n+1,\varepsilon})_t = ((u_{2n+1,\varepsilon})^m)_{xx} + (u_{2n})^{m-1}(u_{2n+1,\varepsilon}) - \beta_\varepsilon(u_{2n+1,\varepsilon}),$$

where $\beta_\varepsilon(r)$ is a regular nonnegative and bounded function, approximating $\frac{M}{2}$ times the Dirac delta.

Moreover, it is not difficult to show that

$$u_{2n+1}(x,t) \geq U(x,t) \text{ for any } n \text{ and a.e. } (x,t) \in IR^+ \times (0,T),$$

where $U(x,t)$ is the (bounded) solution of

$$\begin{cases} U_t = (U^m)_{xx} - \frac{M}{2}\delta_{\partial\{U(t,\cdot)=0\}}, & \mathcal{D}'(IR^+ \times (0,T)), \\ U(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty), \\ U_x(0,t) = 0,\ U(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T), \end{cases}$$

obtained by approximating the equation

$$U_{\varepsilon t} = ((U_\varepsilon)^m)_{xx} - \beta_\varepsilon(U_\varepsilon),$$

Notice that we can write that $\{u_{2n+1}\}$ is a set of mild solutions satisfying

$$\begin{cases} (u_{2n+1})_t - ((u_{2n+1})^m)_{xx} = f_{2n+1}(t,x) & \mathcal{D}'(IR^+ \times (0,T)), \\ (u_{2n+1})(x,0) = u_0(x) & \text{a.e. } x \in (0,+\infty), \\ (u_{2n+1})_x(0,t) = 0,\ (u_{2n+1})(x,t) \to 0 \text{ as } x \to +\infty & \text{a.e. } t \in (0,T), \end{cases}$$

with f_{2n+1}uniformely integrable in $\mathcal{M}(0,+\infty)$ in the sense that

$$\|f_{2n+1}\|_{L^1((0,T):\mathcal{M}(0,+\infty))} \leq K \text{ for any } n, \text{ for some } K > 0.$$

Then by the main result of

J. I. Díaz, I. Vrabie,Proprietés de compacité de l'opérateur de Green généralisé pour l'équation des milieux poreux.*Comptes Rendus Acad. Sciences. París*, t.**309**, Série I, (1989) 221-223,

we get that $\{u_{2n+1}\}$ is a relatively compact set of $C([0,T] : L^1(IR^+))$. Then there exist a subsequence strongly convergent $\{u_{2n+1}\} \to u$ in $C([0,T] : L^1(IR^+))$. This, and the special contruction of ´$\{u_{2n}(x,t)\}$ allows to have the a priori estimate

$$0 < C_{2n}(t) \leq \left[\frac{M}{2\int_0^{+\infty}(U(x,t))^m dx}\right]^{1/m} \text{ for a.e. } t \in (0,T).$$

Then $\{C_{2n}(t)\}$ is uniformely bounded in $L^\infty(0,T)$ and so, there exists $C^*(t)$ such that $C_{2n}(.) \rightharpoonup C^*(.)$ weakly-* in $L^\infty(0,T)$. But, as $\{u_{2n+1}\} \to u$ strongly in $C([0,T] : L^1(IR^+))$ we get that

$$\lim u_{2n}(x,t) = \lim \left(C_{2n}(t)u_{2n-1}(x,t)\right) = \lim C_{2n}(t) \lim u_{2n-1}(x,t)) = C^*(t)u(x,t).$$

Moreover, we can read the algorithm as

$$(u_{2n+1})_t = ((u_{2n+1})^m)_{xx} + C_{2n}(t)^{m-1}(u_{2n-1})^{m-1}(u_{2n+1}) - \frac{M}{2}\delta_{\partial\{(u_{2n+1})(t,\cdot)=0\}},$$

and so we get (by the Lebesgue´s dominated convergence theorem) that $(u_{2n-1})^{m-1}(u_{2n+1}) \to u^m$ in $L^1(IR^+ \times (0,T))$.

5. Numerical Results

We first consider a time marching scheme in the coordinate t, of step dt. At each level in the time discretization, we shall employ a semi-implicit scheme in order to deal with the nonlinearities. An iterative (splitting) numerical scheme is implemented in order to impose the mass conservation constraint. In order to discretize with respect to the coordinate x, at each time level l dt, we will employ piecewise linear finite elements

$$L_{l,k} := \{\phi \in C^0([0,+\infty)) : \phi|E \in \mathbf{P}_1,\ \forall\ E \in \mathbf{T}_{l,k}\}$$

in a uniform grid of step k $\mathbf{T}_{l,k}$.

Also, $\mathbf{B}_{l,k}:= \{\phi_i\}$ is a base of finite linear elements in $L_{l,k}$.

Then, the discretized problem is formulated as follows:
Find $(u_{l+1})_k \in L_{l,k}$, $(u_{l+1})_k = \sum_j (u_{l+1})_k^j \phi_j$, *such that*

$$\int_{\mathbf{T}_{l,k}} (u_{l+1})_k \phi_i \, dx = \int_{\mathbf{T}_{l,k}} (u_l)_k \phi_i \, dx - \frac{3dt}{2} \int_{\mathbf{T}_{l,k}} (u_l)_k^{\frac{1}{2}} ((u_{l+1})_k)_x \phi_{ix} \, dx$$
$$+ dt \int_{\mathbf{T}_{l,k}} (u_{l+1})_k^{\frac{3}{2}} \phi_i \, dx - dt \int_{\mathbf{T}_{l,k}} \frac{M}{2} \delta(u_l) \phi_i \, dx, \forall \phi_i \in \mathbf{B_{l,k}}.$$

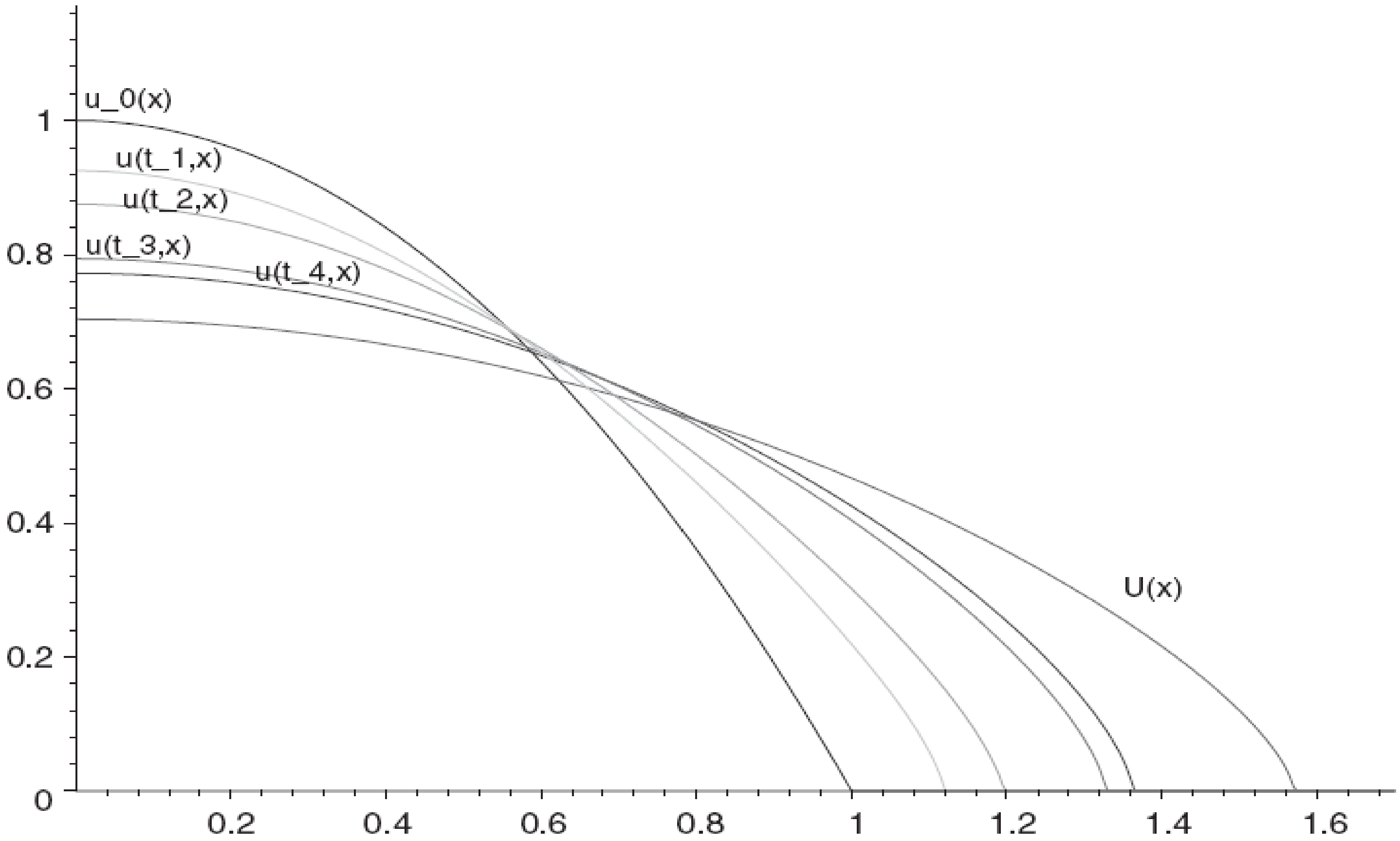
for p = 2n + 1 from 1 to N, $n = 0, 1, 2\ldots$, and N an odd number to be fixed, we

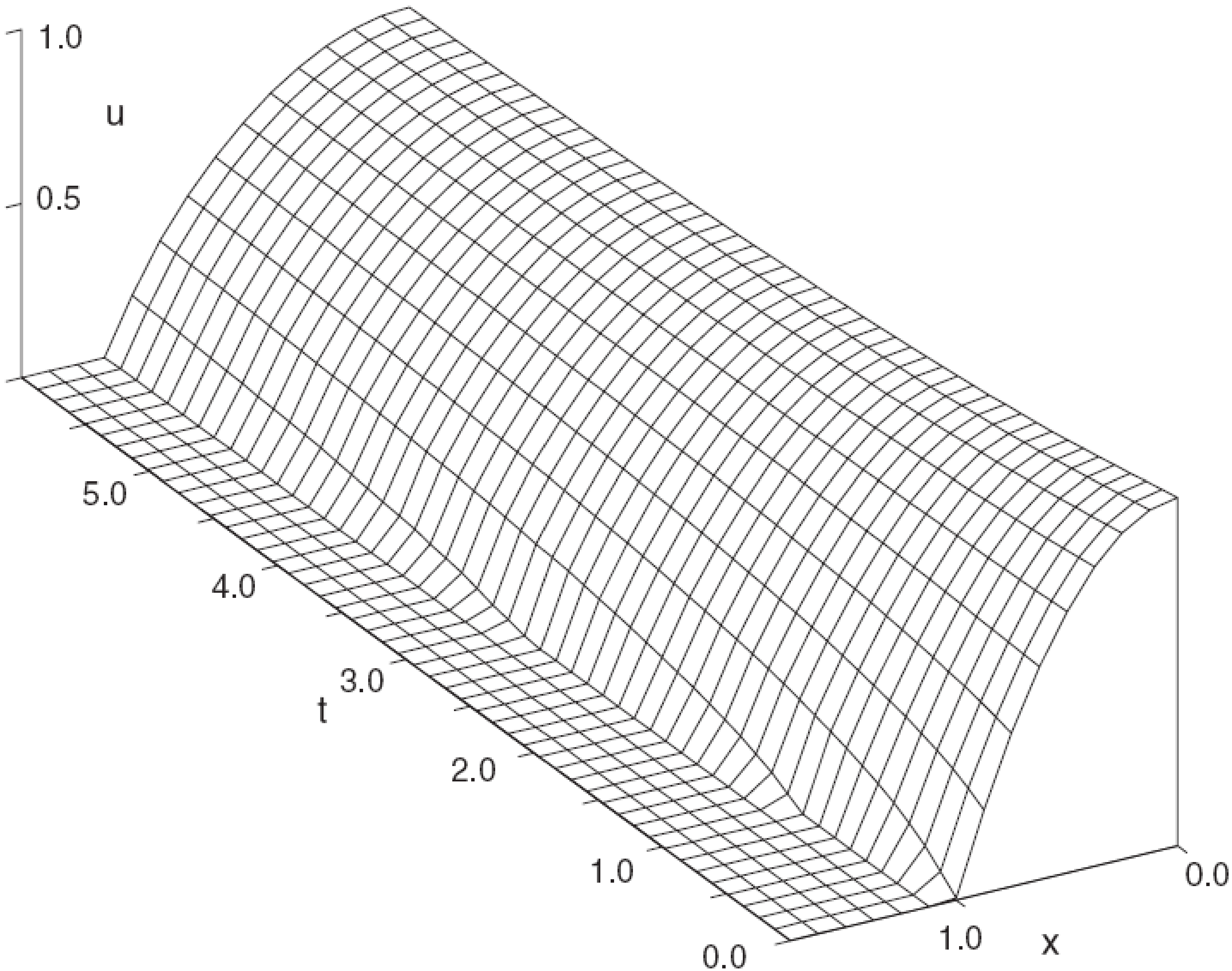
consider the problem,

$$\int_{\mathbf{T}_{l,k}} (u_{l+1,2n+1})_k \phi_i \, dx = \int_{\mathbf{T}_{l,k}} (u_l)_k \phi_i \, dx - \frac{3dt}{2} \int_{\mathbf{T}_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}} ((u_{l+1,2n+1})_k)_x \phi_{ix} \, dx$$
$$+ dt \int_{\mathbf{T}_{l,k}} (u_{l+1,2n})_k^{\frac{1}{2}} (u_{l+1,2n+1})_k \phi_i \, dx - dt \int_{\mathbf{T}_{l,k}} \frac{M}{2} \delta(u_l) \phi_i \, dx, \forall \phi_i \in \mathbf{B_{l,k}},$$

$\int(u_{l+1,2n})_k^3/^3 = M/2$, according to ($P_{2n}$), i.e., $(u_{l+1,2n})_k = C_{l+1,2n}(u_{l+1,2n\text{-}1})_k$. The resulting system of equations for the nodal values at the $(2n+1)$th-step is solved with the Gauss Seidel method. In order to initiate the iterative scheme, one can take as $(u_{l+1,p=1})_k$ the values obtained in the previous time step, that is to say, $(u_{l+1,p=1})_k = u_l$. The scheme finishes assuming the values for the $(l+1)$-time level given by $u_{l+1} = (u_{l+1,p=N})_k$.

$$u_0(x) = 1 - x^2,$$





1.0
u
0.5
5.0
4.0
3.0
t
2.0
1.0
0.0
1.0
x
0.0

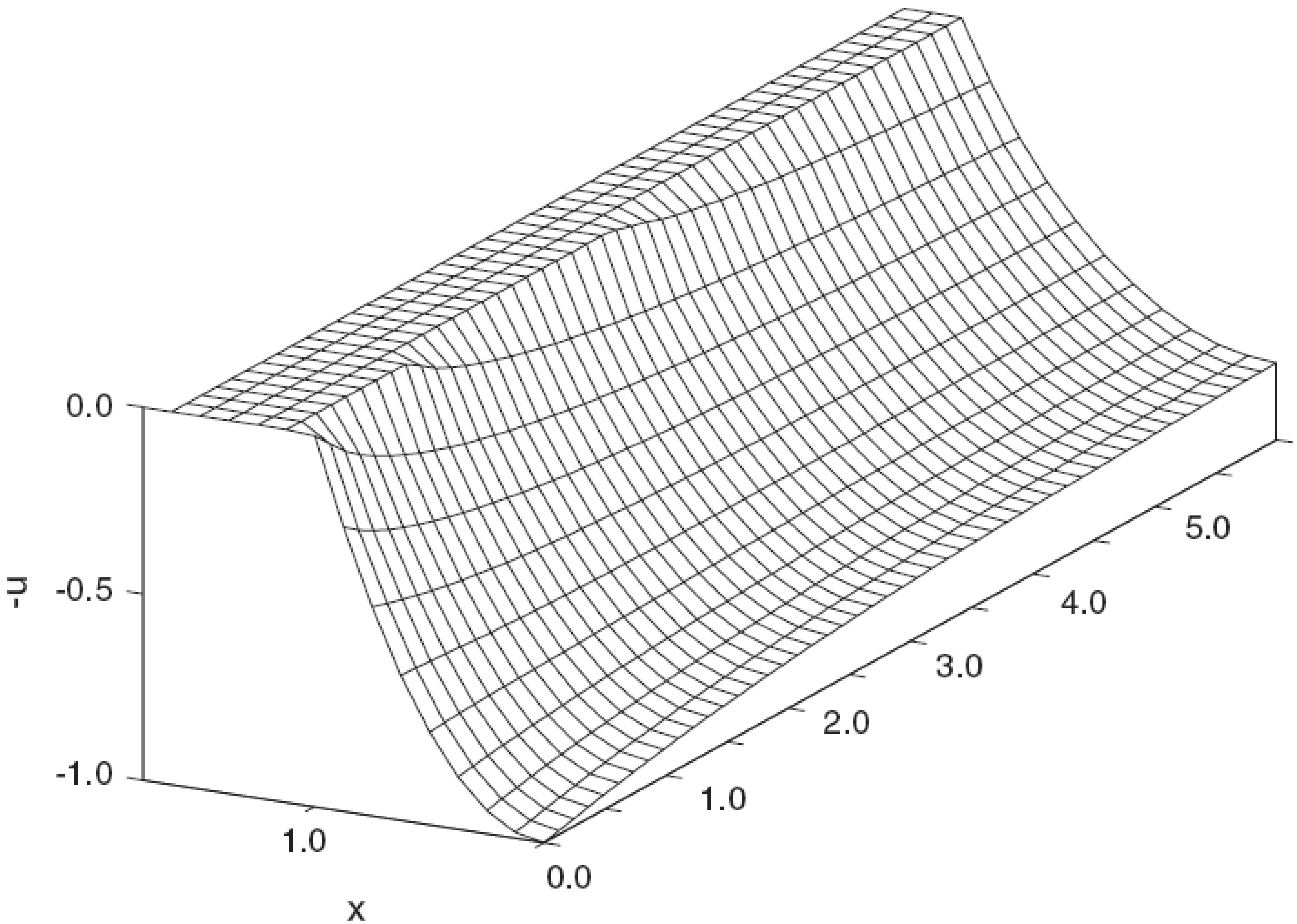

0.0
-0.5
-1.0
-u
1.0
0.0
x
1.0
2.0
3.0
4.0
5.0

Table 1

Time (t)	Results for C_N
1.0	1.00000071770833
2.0	1.00000061614692
3.0	1.00000056130953
4.0	1.00000051022114
5.0	1.00000050094979
6.0	1.00000046809999

Time (t)	Location of the boundary (x)
1.0	1.31999997049570
2.0	1.36999996937811
3.0	1.37499996926036
4.0	1.38849999690422
5.0	1.38999996893108
6.0	1.39499996881932

6. Summary and conclusions.

A coupled model describing the evolution of the topographic elevation and the depth of the overland water film is studied here when considering the overland flow of water over an erodible sediment. The instability of the spatially uniform solution corresponds to the formation of rills, which in reality then grow and coalesce to form large-scale river channels.

We started by considering the deduction and mathematical analysis of a deterministic model describing river channel formation and the evolution of its depth. We complete the previous modeling of the problems by SMITH and BRETHERTON (1972) and FOWLER *et al.* (2007), obtaining a model which involves a degenerate nonlinear parabolic equation (satisfied on the interior of the support of the solution) with a superlinear source term and a prescribed constant mass. We propose here a global formulation of the problem (formulated in the whole space, beyond the support of the solution) which allows us to show the existence of a solution and leads to a suitable numerical scheme for its approximation. As we show, the solution does not blow up despite of the presence of the superlinear forcing term at the equation thanks to the mass constraint.

A particular feature of the model for channel evolution which we have studied is that the degeneracy of the equation causes the channel width to be self-selecting. This is of some interest in the geomorphological literature, since the issue of channel width determination is one that has caused some difficulties (e.g., PARKER, 1978).

PARKER, G. (1978), *Self-formed straight rivers with equilibrium banks and mobile bed, Part 1. The sand-silt river*, J. Fluid Mech. *89*, 109–125.

Some open problems:

- Uniqueness and continuous dependence in the parabolic case
- Dynamics of the free boundary

The mathematical analysis clarifying the modelling of the problem

Think **globally** and **act locally**

Thanks for your attention