Free boundaries, scales and shapes

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Mathematics for the XXI Century

Fundación Ramón Areces

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1. Introduction

A motivation:

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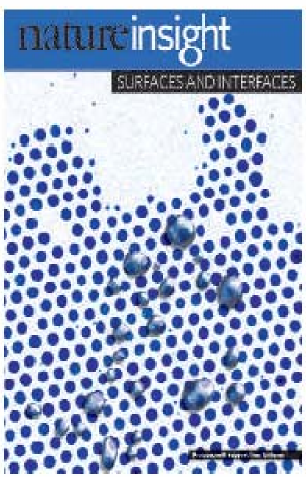
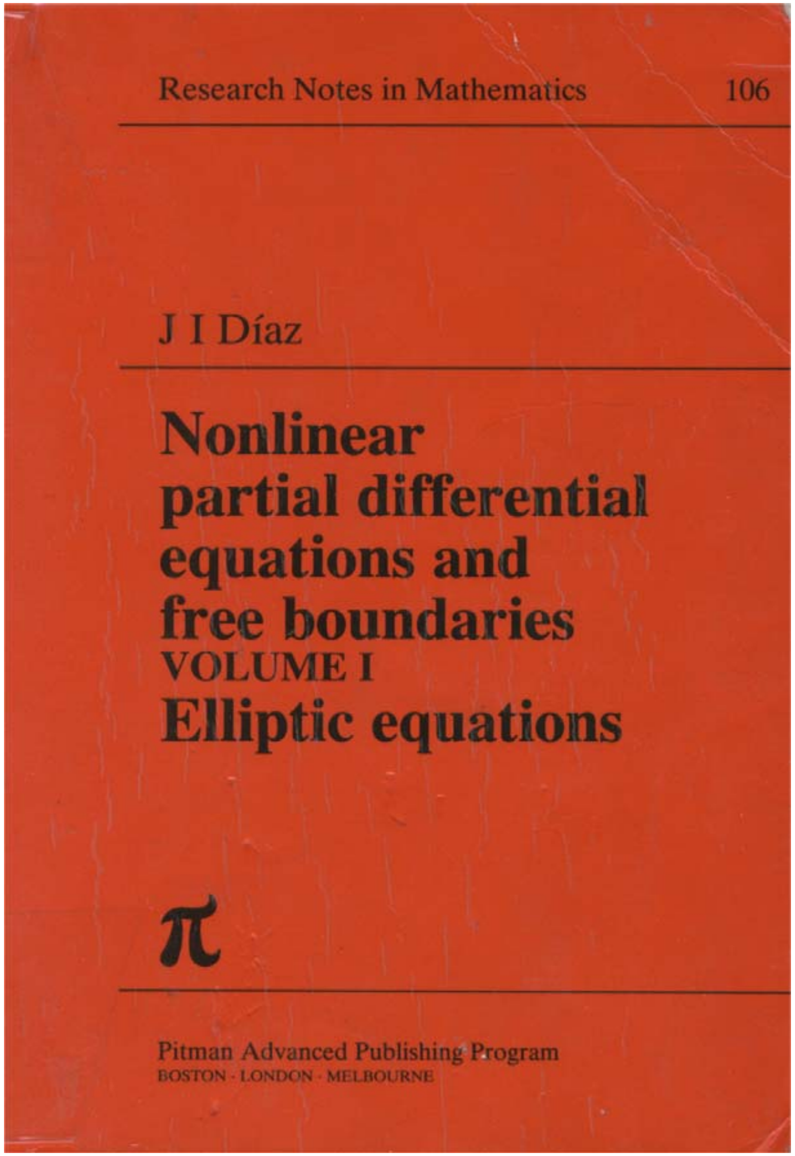


SURFACES AND INTERFACES


Research Notes in Mathematics
106
J I Díaz
Nonlinear partial differential equations and free boundaries
VOLUME I
Elliptic equations
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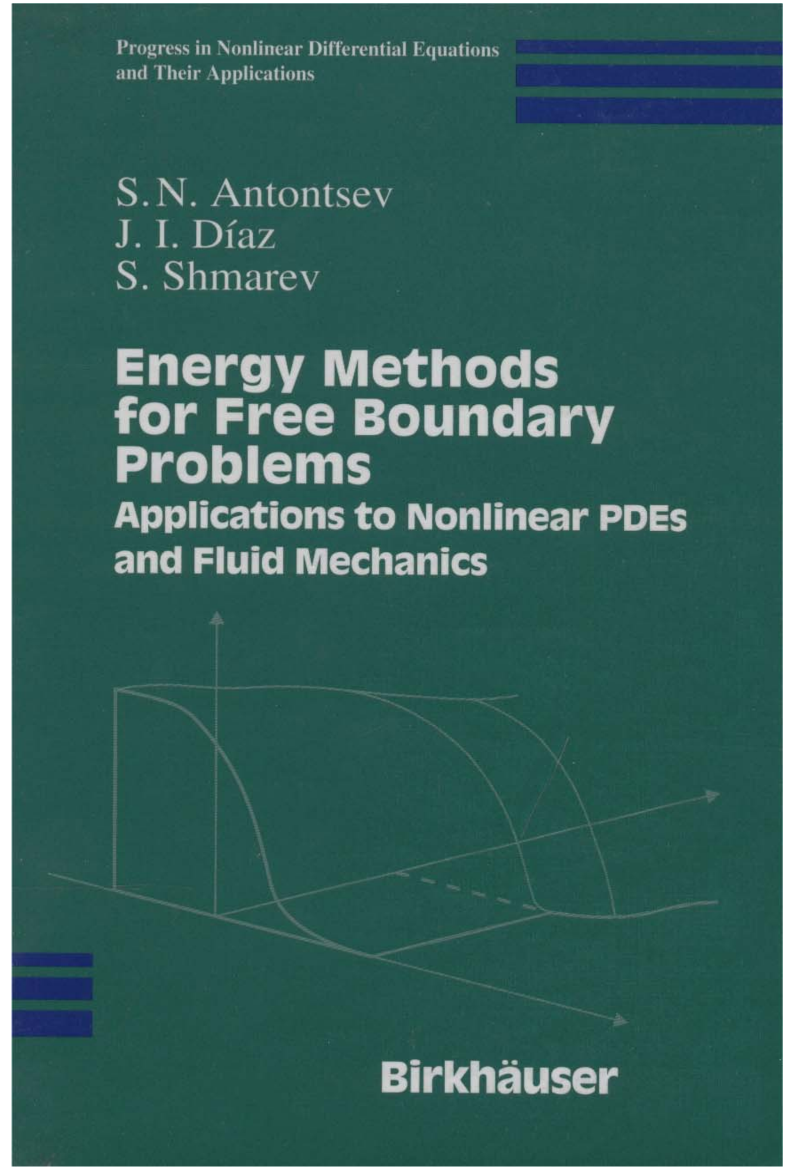
Pitman Advanced Publishing Program
BOSTON · LONDON · MELBOURNE

1985


Progress in Nonlinear Differential Equations and Their Applications
S.N. Antontsev
J. I. Díaz
S. Shmarev
Energy Methods for Free Boundary Problems
Applications to Nonlinear PDEs and Fluid Mechanics
Birkhäuser

2002

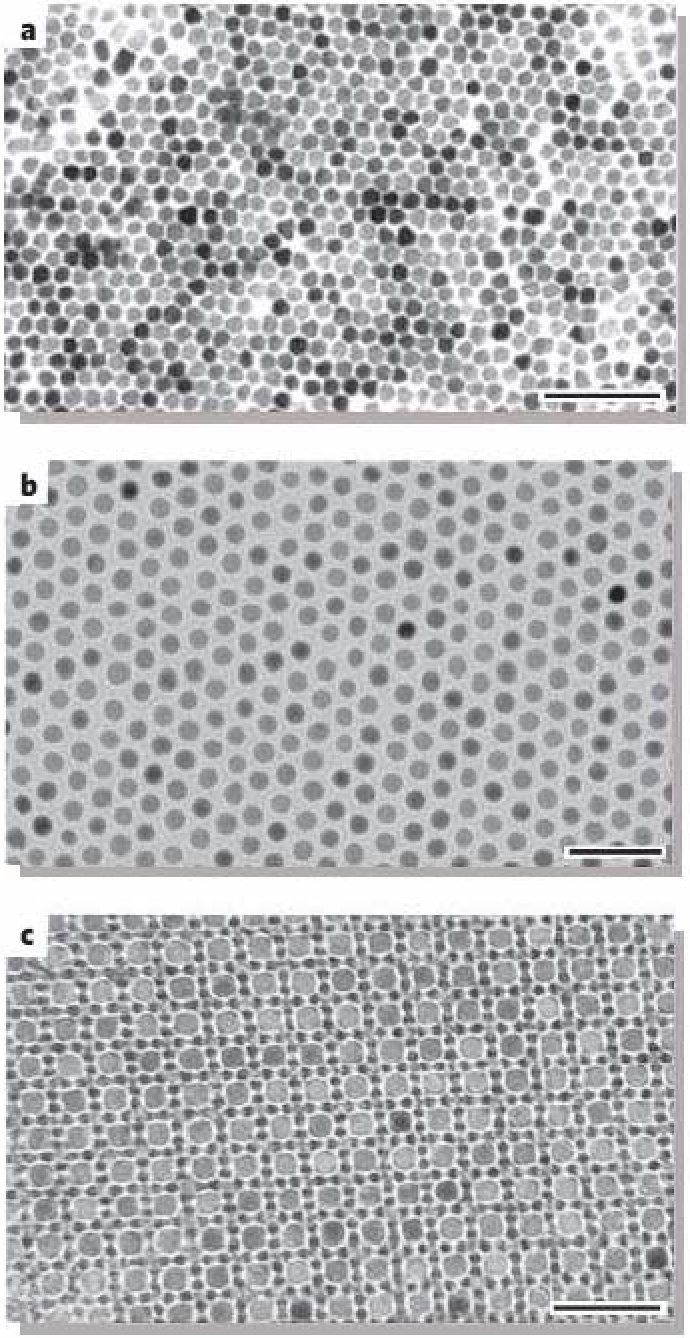


Figure 3 | Monodisperse colloidal nanocrystals synthesized under kinetic size control. **a**, Transmission electron microscopy (TEM) image of CdSe nanocrystals. **b**, TEM image of cobalt nanocrystals. **c**, TEM micrograph of an AB_{13} superlattice of γ-Fe_2O_3 and PbSe nanocrystals. The precise control on the size distributions of both nanocrystals allows their self-assembly into ordered three-dimensional superlattices. Scale bars, 50 nm. Reprinted from ref. 27.

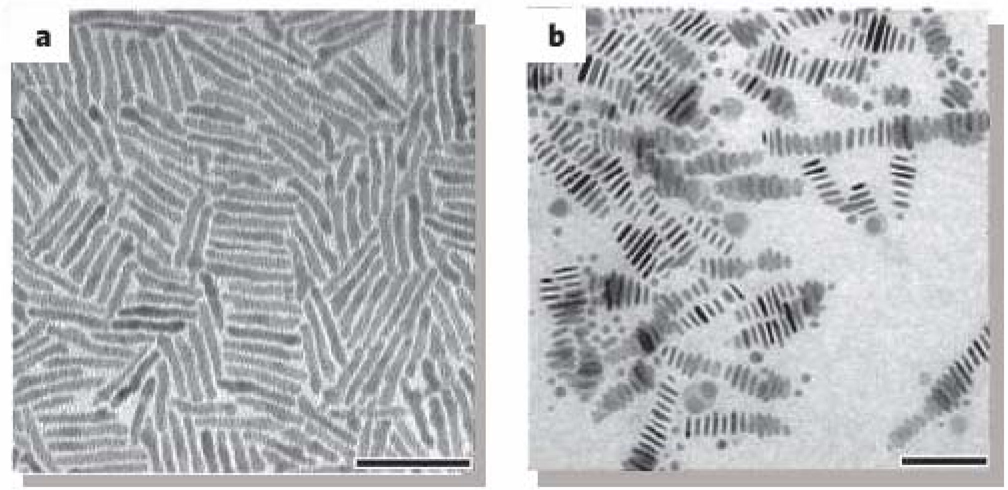


Figure 4 | Anisotropic growth of nanocrystals by kinetic shape control and selective adhesion. **a**, CdSe nanorods (scale bar, 50 nm). Reprinted with permission from ref. 52. **b**, Cobalt nanodisks (scale bar, 100 nm). The organic surfactant molecules selectively adhere to one facet of the nanocrystal, allowing the crystal to grow anisotropically to form a rod or disk.

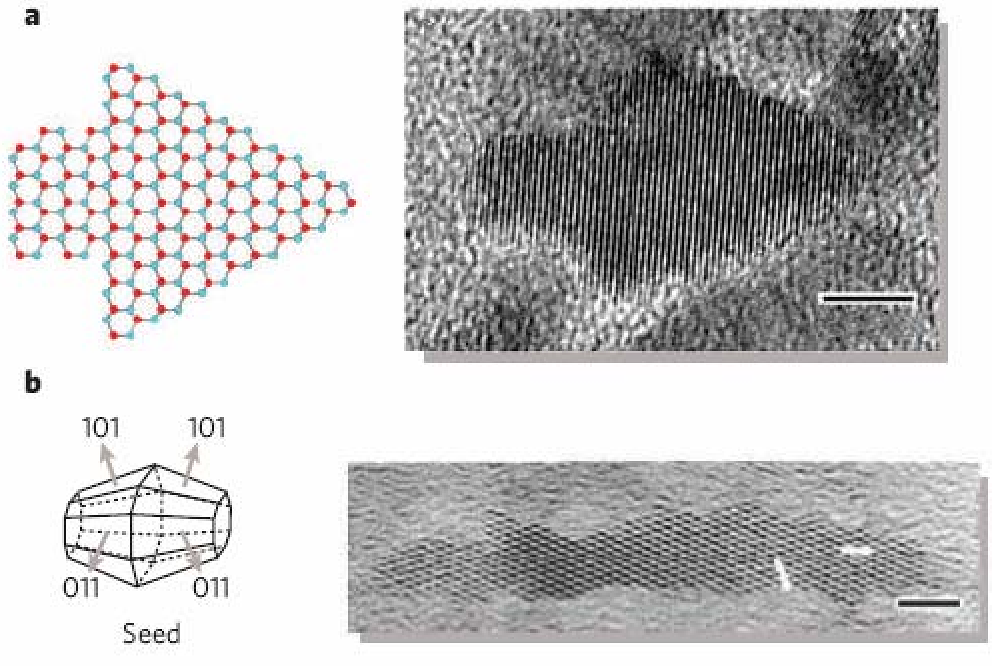


Figure 5 | Nanocrystals with complex shapes prepared by sequential elimination of a high-energy facet. **a**, Two-dimensional representation and a high-resolution TEM image of an arrow-shaped nanocrystal of CdSe. High-resolution TEM characterization shows that each shape of nanocrystal is predominantly wurtzite and that the angled facets of the arrows are the (101) faces. Scale bar, 5 nm. Red and blue dots represent selenium and cadmium atoms, respectively. Reprinted with permission from ref. 22. **b**, Simulated three-dimensional shape and high-resolution TEM analysis of a TiO_2 rod. The long axes of the nanocrystals are parallel to the *c*-axis of the anatase structure, while the nanocrystals are faceted with (101) faces along the short axes. Hexagon shapes (the [010] projection of a truncated octagonal bipyramid) truncated with two (001) and four (101) faces are observed either at the one end or at the centre of the nanocrystals. Scale bar, 3 nm. Reprinted with permission from ref. 35. Copyright (2003) American Chemical Society.

Free boundaries at the micro scale: semiconductors.

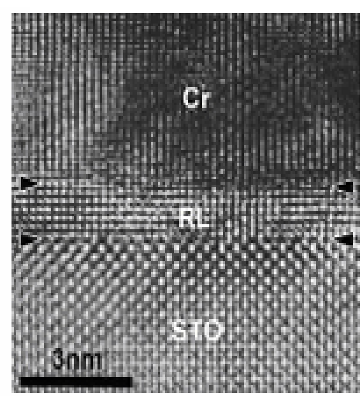
Mock (1972), Markowitch (1986), Friedman (1988), Markowich, Ringofer, Schmeiser (1990), Díaz, Galiano, Jüngel (1999), ...

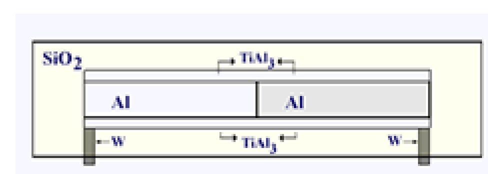
In solid state physics, the drift-diffusion equations are today the most widely used model to describe semiconductor devices. The drift-diffusion models describe the flow of electrons in the conduction band of the semiconductor material and of holes (or defect electrons) in the valence band of the crystal, influenced by the electric field. Mathematically, they form a system of parabolic equations for the electron density n and the hole density p and the Poisson equation for the electric potential V:

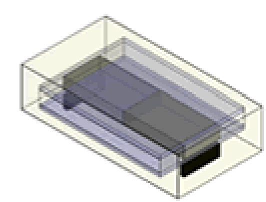
$$\frac{\partial n}{\partial t} - \nabla \cdot (\nabla r(n) - n\nabla V) = -R(n, p),$$

$$\frac{\partial p}{\partial t} - \nabla \cdot (\nabla r(p) + p\nabla V) = -R(n, p),$$

$$\Delta V = n - p - C(x) \quad \text{in } Q_T = \Omega \times (0, T),$$







Different point of view:

Is it possible to show (rigourously) the existence of some free boundary arising at the macroscopic scale but not arising at the microscopic scale?

Some similar philosophy (but not exactly the same) in many other contexts:

Approximate problem (without free boundary) / Límit problem (with a free boundary)

- Obstacle problem (Approximate problem = penalisation)
- Cavitation in hidrodynamic lubrication (Approximate problem = aproximation of the Heaviside function)
- Porous media equation and/or NonNewtonian flows (Approximate problem = uniformly parabolic equations)

Question: the scale as an approximating argument ?

Plan:

2. A model problem arising in Chemical Engineering and a mathematical tool: homogenization.

3. Model problem: effective chemical reactions. Convergence.

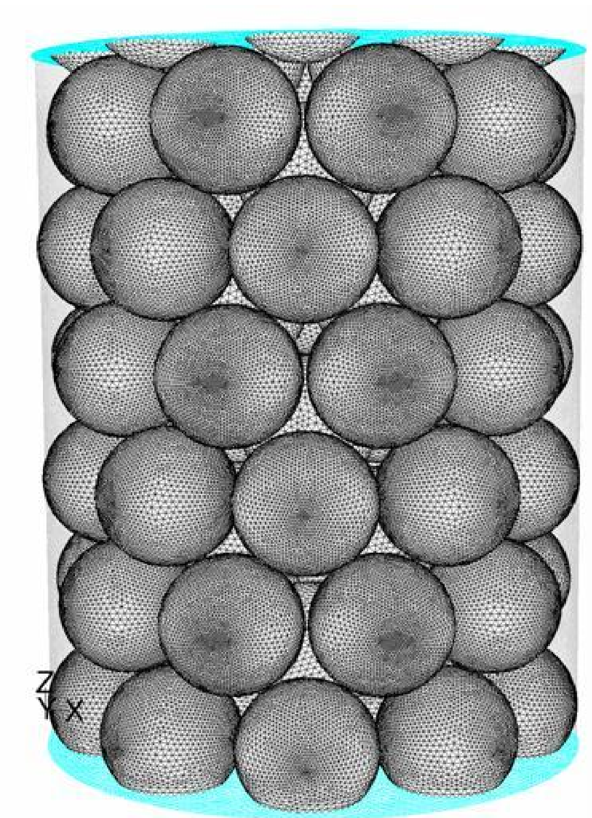
4. Macroscopic “dead core” versus microscopic strictly positive solutions.

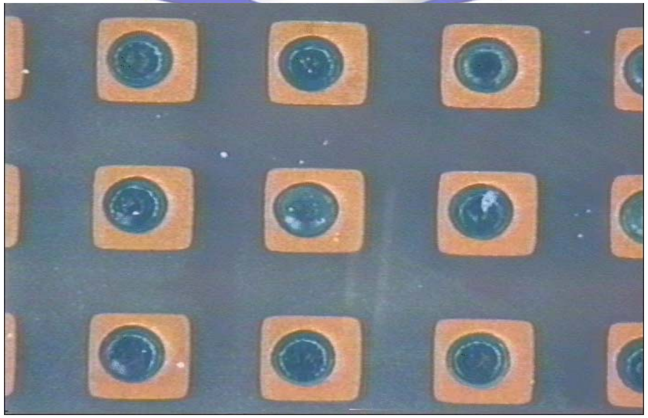
5. An isoperimetric inequality for the (macroscopic) dead core.

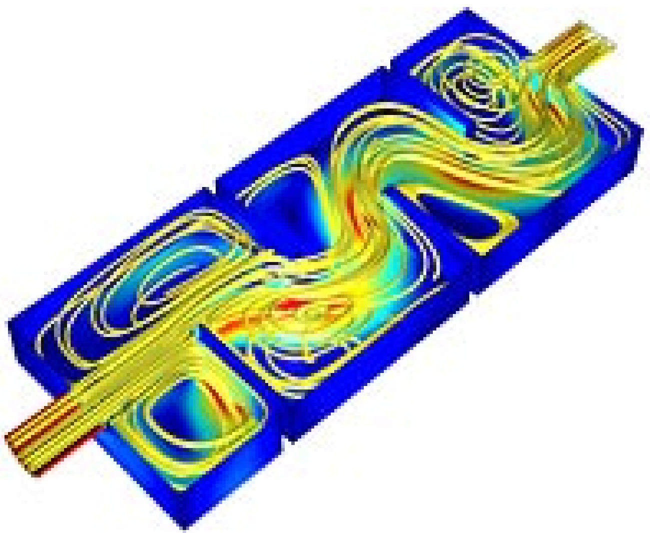
6. Effectiveness as cost fonctional in optimal and controllability shape design problems.

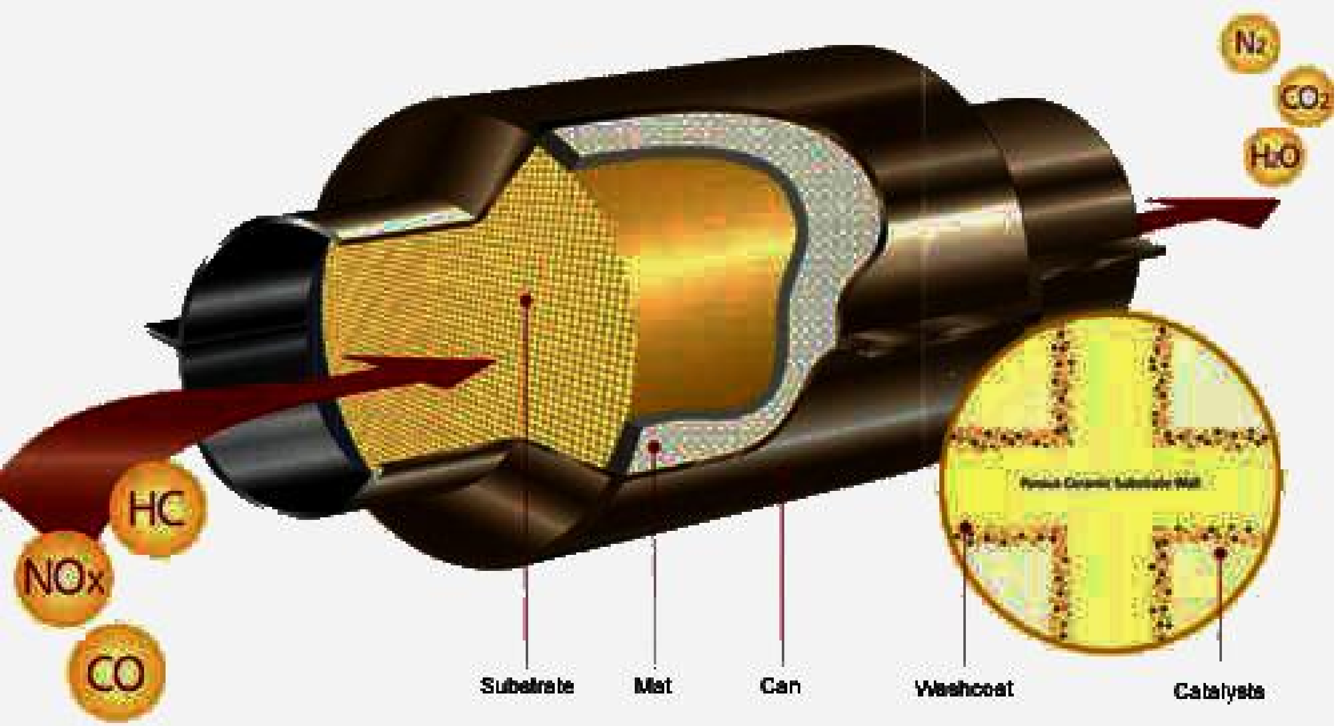
2. A model problem arising in Chemical Engineering and a mathematical tool: homogenization.

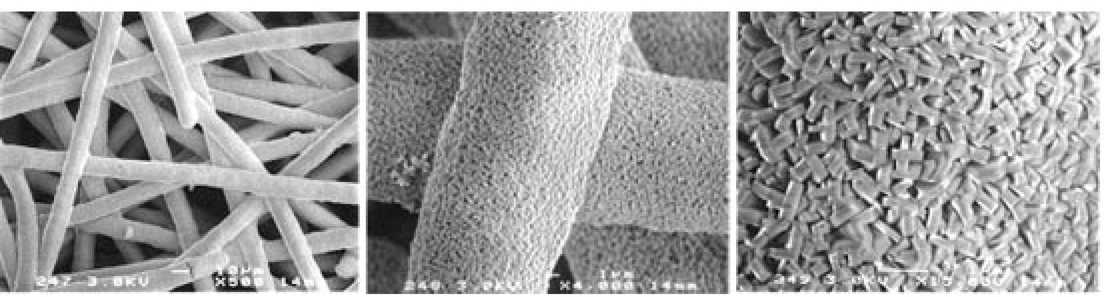
Incompressible flow of a fluid reacting with the exterior of many packed solid particles: Absorption and adsorption phenomena in beds or towers. Of relevance in Chemical Engineering (separation, chemical industry, etc).

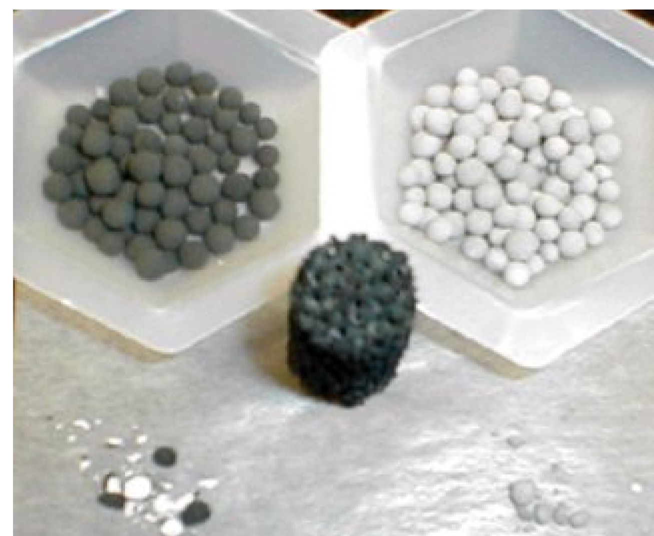


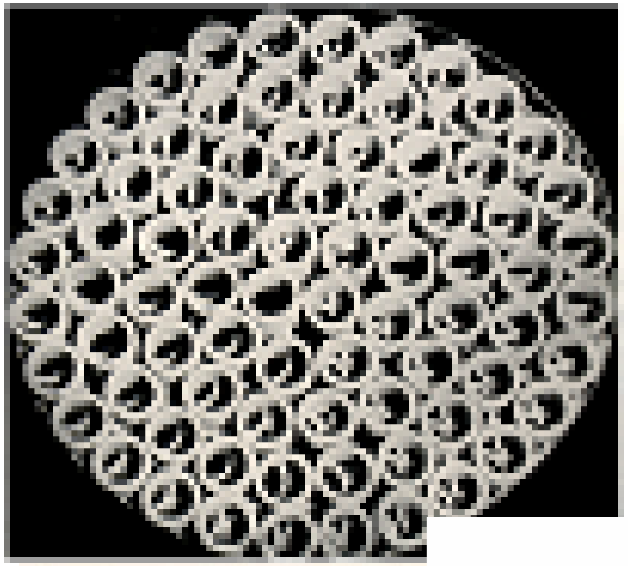


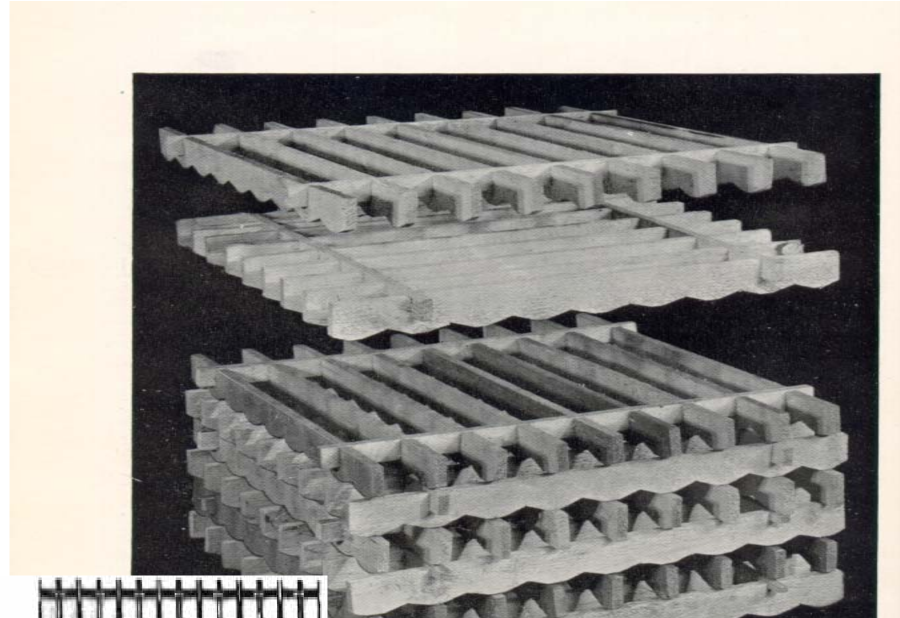


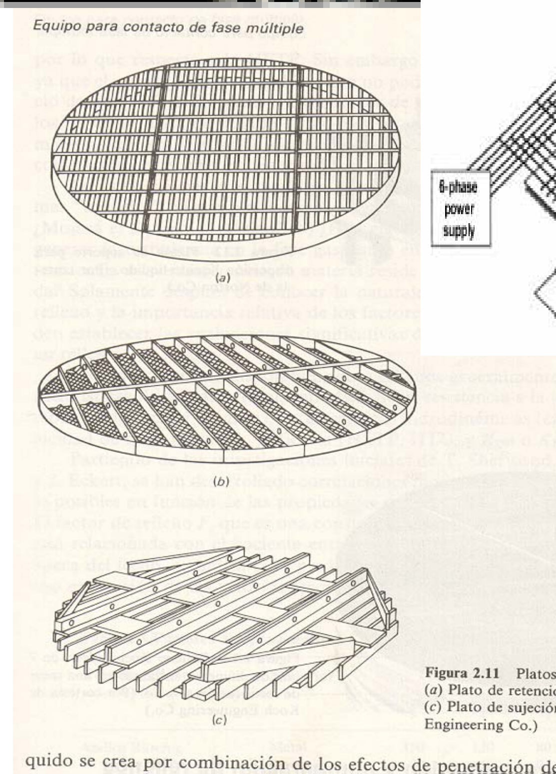

N2
CO2
H2O
HC
NOx
CO
Substrate
Mat
Can
Washcoat
Catalysts



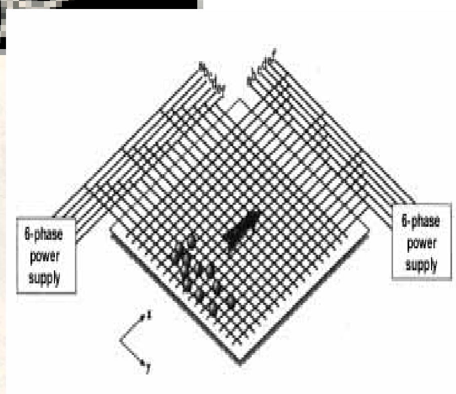




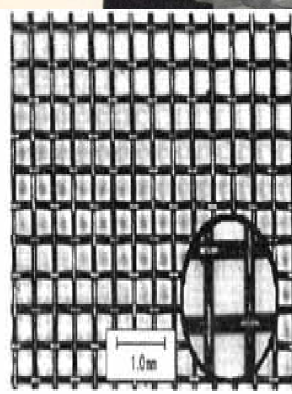




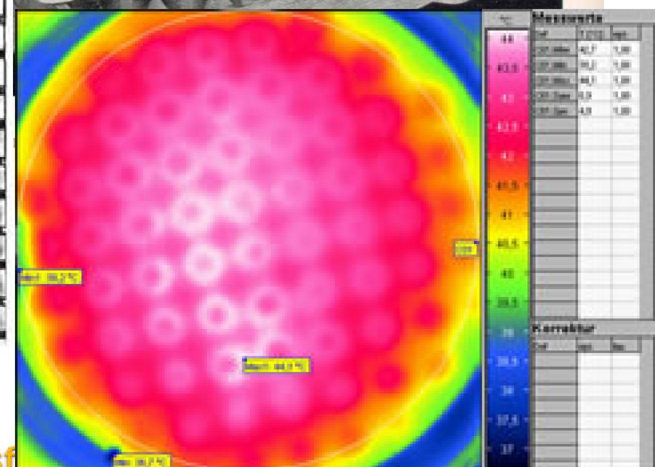
Equipo para contacto de fase múltiple
(a)
(b)
(c)
Figura 2.11 Platos
(a) Plato de retenció
(c) Plato de sujeción
Engineering Co.)
quido se crea por combinación de los efectos de penetración de

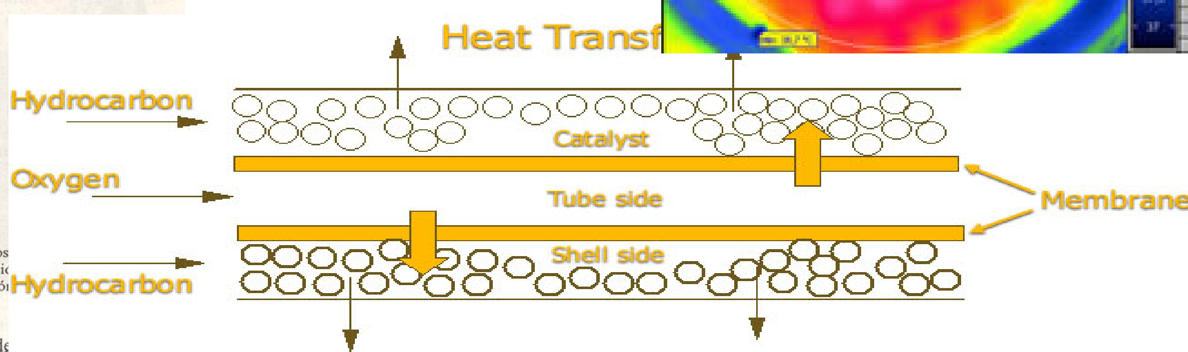


6-phase power supply
6-phase power supply



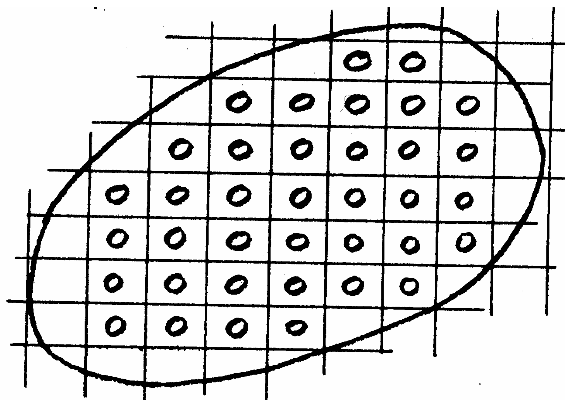
1.0m

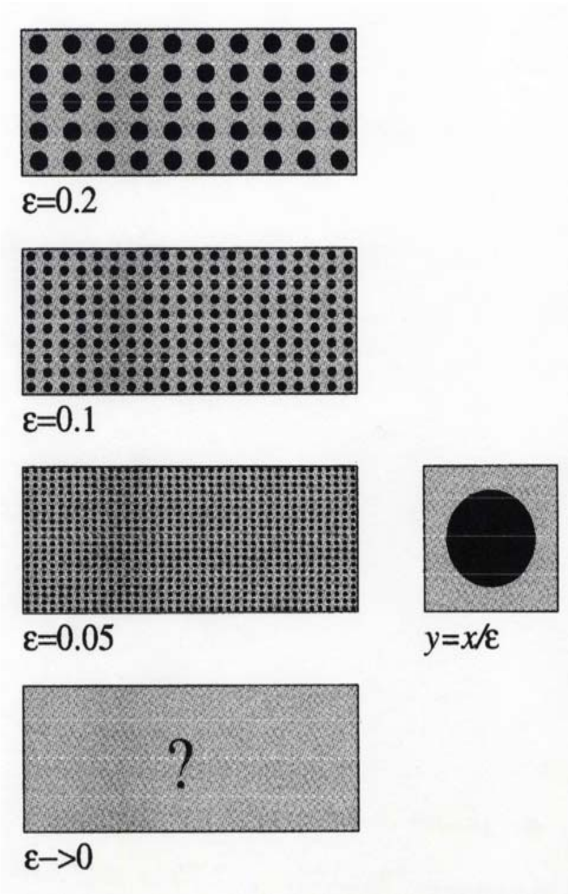




Heat Transf
Hydrocarbon
Catalyst
Oxygen
Tube side
Membrane
Shell side
Hydrocarbon

Homogenization: process related to the overall modelling in presence of a double spatial scale

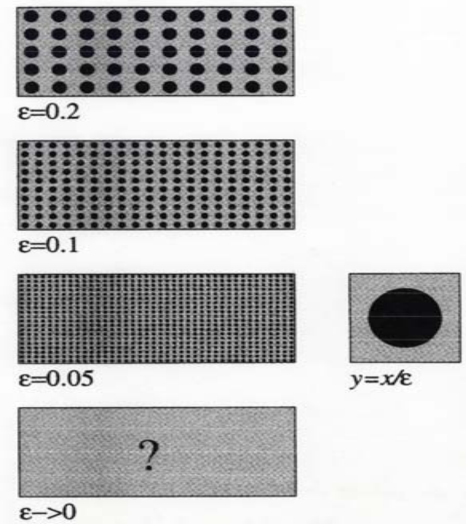




Sánchez-Palencia, Bensoussan-Lions-Papanicolau,...

Many availible convergence methods, variants, applications, …

• **Homogeneization**: Question: $u^\varepsilon \to ?$, as $\varepsilon \to 0$ (homogeneized region Ω)



•Two different steps:

a. *Formal Asymptotic Expansion.* "Ansatz"

$$u^\varepsilon(x) = u^\varepsilon(x,y)\,|_{y=\frac{x}{\varepsilon}} = u_0(x,y) + \varepsilon u_1\,(x,y) + \varepsilon^2 u_2(x,y) +$$

with

$$u_i\,(x,y) \text{ Y-periodic with respect to the } y = \frac{x}{\varepsilon} \text{ variable.}$$

Then $u^\varepsilon \to u_0$, as $\varepsilon \to 0$.

b. *Rigurous proof.* There exists a u_0 such that $u^\varepsilon \to u_0$, in some functional space, as $\varepsilon \to 0$.

3. Model problem: effective chemical reactions. Convergence.

C. Conca, J.I. Díaz, A. Liñan and C. Timofte, Homogenization in Chemical Reactive Flows through Porous Media, *Electr. J. Diff. Eqns.* 2004, 1-22.

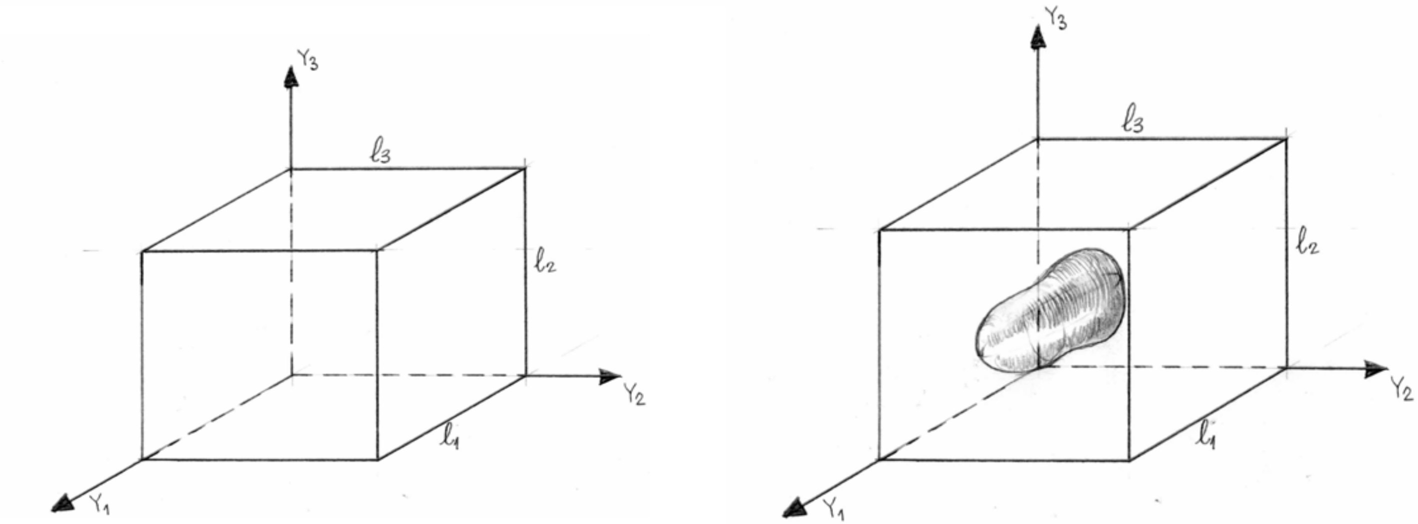
The stationary reactive flow of a fluid confined in Ω^ε, of concentration u^ε, reacting on the boundary of the catalytic particles (obstacles)

$$\begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \dfrac{\partial u^\varepsilon}{\partial \nu} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

$$g(v) = |v|^{p-1} v, \quad 0 < p < 1 \quad \text{(Freundlich kinetics)}$$

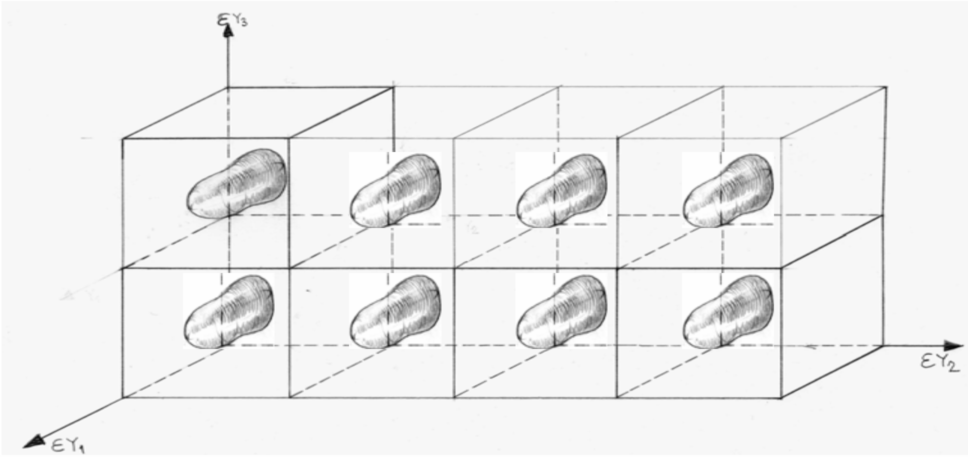
Ω - smooth bounded domain in $\mathbb{R}^n$ $(n \geq 3)$

$Y = [0, l_1[\times ... [0, l_n[$ -the representative cell in $\mathbb{R}^n$



$\overline{T} \subset Y$ - the elementary obstacle $\quad Y^* = Y \setminus \overline{T}, \quad \rho = \dfrac{|Y^*|}{|Y|}.$

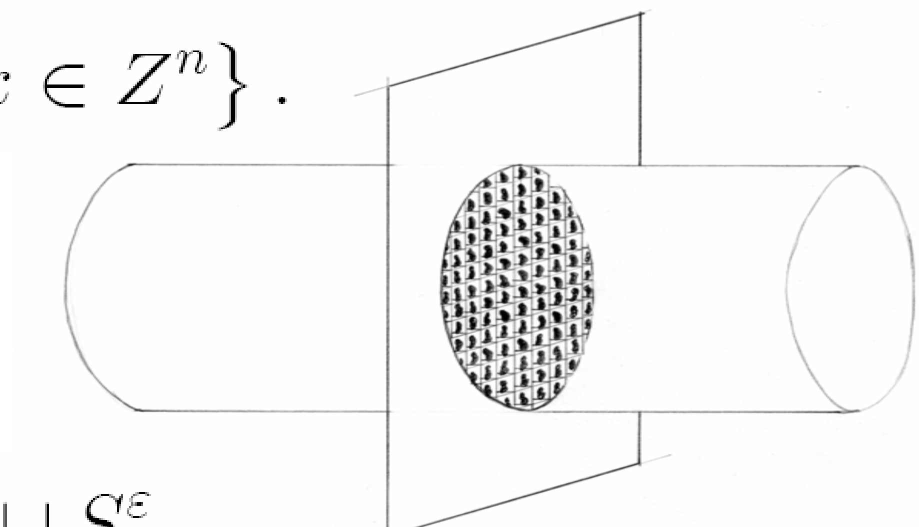
T_k^ε -translated image of εT by the vector $(k_1 l_1, ..., k_n l_n)$



$$T_k^\varepsilon = \varepsilon(kl + T)$$

T^ε the set of all the obstacles contained in Ω, i.e.

$$T^\varepsilon = \bigcup \{T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega, \; k \in Z^n\}.$$



$$\Omega^\varepsilon = \Omega \setminus \overline{T^\varepsilon}, \quad \partial\Omega^\varepsilon = \partial\Omega \cup S^\varepsilon.$$

$$S^\varepsilon = \cup\{\partial T_k^\varepsilon \mid \overline{T_k^\varepsilon} \subset \Omega, \; k \in Z^n\}.$$

Theorem (CDLT 2004).

Assume $f \in L^2(\Omega)$. Then, we can construct an extension $P^\varepsilon u^\varepsilon$ of the solution u^ε such that

$$P^\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in} \quad H^1_0(\Omega)$$

and u is the unique solution of the problem

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{n} q_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + a \frac{|\partial T|}{|Y^*|} g(u) = f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\mathbf{Q} = ((q_{ij}))$ is the "classical homogenized matrix":

$$q_{ij} = D_f \{ \delta_{ij} + \frac{1}{|Y^*|} \int_{Y^*} \frac{\partial \chi_j}{\partial y_i} dy \}$$

with χ_i, $i = 1, ..., n$ solution of the "cell problems"

$$\begin{cases} -\Delta \chi_i = 0 \quad \text{in} \quad Y^*, \\ \dfrac{\partial(\chi_i + y_i)}{\partial \nu} = 0 \quad \text{on} \quad \partial T, \\ \chi_i \quad Y- \quad \text{periodic.} \end{cases}$$

The proof uses an energy method applied to weak solutions (Tartar (1978)), a priori estimates and some properties of maximal monotone graphs (Brezis (1973)).

4. Macroscopic “dead core” versus microscopic strictly positive solutions.

In order to study the formation of the interface given by the boundary of the support (the boundary of the “dead core”) we assume

$$f \in L^{\infty}(\Omega) \text{ and } f(x) \geq 0 \text{ , a.e. } x \in \Omega.$$

$$S(f) := \text{ support of } f \subsetneq \Omega$$

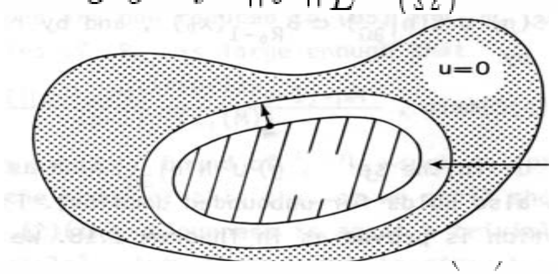
Theorem

i) $\forall\ \varepsilon > 0\ \ u^\varepsilon(x) > 0$ *in* Ω^ε *and so* $S(u^\varepsilon) = \Omega^\varepsilon$.

ii) For fixed Ω *and* $Y = [0, l_1[\times ...[0, l_n[$ *(the representative cell in* R^n*), there exists a continuous function* $\delta = \delta(\|f\|_{L^\infty(\Omega)}, |T|, |\partial T|, \Lambda_0)$*, with* Λ_0 *the larger eigenvalue of the matrix* Q*, i.e. verifying that*

$$\Lambda_0 \left\|\overrightarrow{\xi}\right\|^2 \geq \sum_{i,j=1}^{n} q_{ij}\xi_i\xi_j \ \ \text{for any} \ \ \overrightarrow{\xi} \in \mathbb{R}^n,$$

such that if δ *is small enough then* $S(u) \subsetneq \Omega$ *and so* u *gives rise to the free boundary* $F := \partial S(u)$*. Moreover, function* $\delta(\|f\|_{L^\infty(\Omega)}, |T|, |\partial T|, \Lambda_0)$ *depends increasingly of* $\|f\|_{L^\infty(\Omega)}$ *and decreasingly of* $|T|$, $|\partial T|$ *and* Λ_0.



$S(f) :=$ support of f

Proof. i) By the maximum principle we know that $u^\varepsilon(x) \geq 0$ *in* Ω^ε and that $u^\varepsilon(x) \geq \underline{u}^\varepsilon(x)$ *in* Ω^ε with $\underline{u}^\varepsilon(x)$ solution of the Dirichlet problem

$$\begin{cases} -D_f \Delta \underline{u}^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ \underline{u}^\varepsilon = 0 & \text{on } \partial\Omega \cup S^\varepsilon. \end{cases}$$

Then the conclusion follows by the strong maximum principle for $\underline{u}^\varepsilon$, more precisely

$$\underline{u}^\varepsilon(x) \geq C(\int_\Omega f(y)d(y,\partial\Omega^\varepsilon)dy)d(y,\partial\Omega^\varepsilon),$$

(unpublished result due to J. M. Morel and L. Oswald: see J. I. Díaz, J. M. Morel, L. Oswald, Comm.in Partial Differential Equations,Vol 12,No 12, 1333 1344, 1987).

ii) We know that

$$-Lu+\lambda g(u) = 0 \quad \text{in } \Omega - S(f),$$

with

$$Lu = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(q_{ij}\frac{\partial u}{\partial x_j}) \text{ and } \lambda = a\frac{|\partial T|}{|Y^*|}.$$

Then, by Theorem 1.13 of J. I. Díaz, *Nonlinear Partial Differential Equations and Free Boundaries*, Pitman, London, 1985 (see also J. I. Díaz, J. Hernández, *SIAM J.Math. Anal.* **15**, 670-685, 1984) for $x_0 \in \Omega - S(f)$ we can construct a local barrier function

$$\overline{u}(x : x_0) = C\,\|x - x_0\|^{\frac{2}{1-q}}$$

for some $C = C(n, q, \lambda, \Lambda_0)$ such that

$$-L\overline{u}+\lambda g(\overline{u}) \geq 0 \quad \text{in } B(x_0,\sigma) \cap (\Omega - S(f)) \text{ for any } \sigma > 0.$$

Moreover, since

$$\|u\|_{L^\infty(\Omega)} \leq \left[\frac{1}{\lambda}\|f\|_{L^\infty(\Omega)}\right]^{\frac{1}{q}}$$

we deduce that if $\sigma = \delta$ (as indicated in the statement) then $u(x) \leq \overline{u}(x : x_0)$ on $B(x_0, \delta) \cap (\Omega - S(f))$ and thus $0 \leq u(x_0) = \overline{u}(x : x_0) = 0$ and the conclusion follows by taking $x_0 \in \Omega - S(f)$ such that $d(x_0, S(f)) \geq \delta$.■

Important remark:

If the boundary conditions are not homogeneous on : e.g.

$$P(\varepsilon, 1) \begin{cases} -D_f \Delta u^\varepsilon = f & \text{in } \Omega^\varepsilon, \\ -D_f \frac{\partial u^\varepsilon}{\partial \upsilon} = a\varepsilon g(u^\varepsilon) & \text{on } S^\varepsilon \\ u^\varepsilon = 1 & \text{on } \partial\Omega. \end{cases}$$

then, the same arguments leads to the (weak) convergence of $P^\varepsilon u^\varepsilon$ to u solution of the problem

$$P(1)\begin{cases} -Lu+\lambda g(u) = f & \text{in } \Omega \\ u = 1 & \text{on } \partial\Omega \end{cases}$$

The dead core is now located far from the boundary

Notice that we can reformulate $P(1)$ as a semilinear problem with homogeneous Dirichlet conditions by means of the change $w = 1 - u$ and thus

$$P_0(1)\begin{cases} -Lw+\lambda G(w) = F & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases}$$

$$G(w) = 1 - g(1-w) \qquad F(x) = \lambda - f$$

If $g(u) = |u|^{p-1}\, u$, $0 < p < 1$, now $G'(1) = +\infty$.

Further results: Friedman-Phillips (1984).

5. An isoperimetric inequality for the (macroscopic) dead core.

Besides the usual comparison (or maximum) principle we can use another comparison principle of a different nature: the *symmetrized mass comparison principle*.

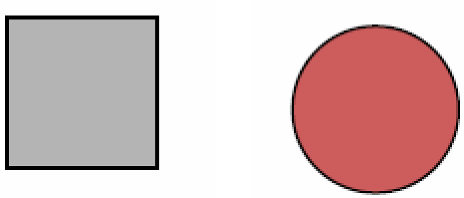
The process of *symmetrization* applied to problem

$$P(0)\begin{cases} -Lu+\lambda g(u)=f & \text{in } \Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

now for g continuous nondecreasing, $g(0)=0$ (so that it can be applied to problem $P_0(1)$).

We start by the *symmetrization of the domain* Ω:

Given Ω, an open bounded set of $\mathbb{R}^N$, the symmetrized version of Ω is the ball centered at the origin having the same measure than Ω. Let us call Ω^* to this ball.



Once we know condition $m(\Omega) = m(\Omega^*)$ we can use the *isoperimetric inequality*

$$L \geq N\omega_N^{\frac{1}{N}} A^{\frac{N-1}{N}}$$

where L is the *length* of $\partial\Omega$ (or $m(\partial\Omega)$), A is the *area* of Ω (or $m(\Omega)$) and

ω_N is the *area* of the unit ball of $\mathbb{R}^N$ (i.e. $\omega_N = m(S^{N-1})$)

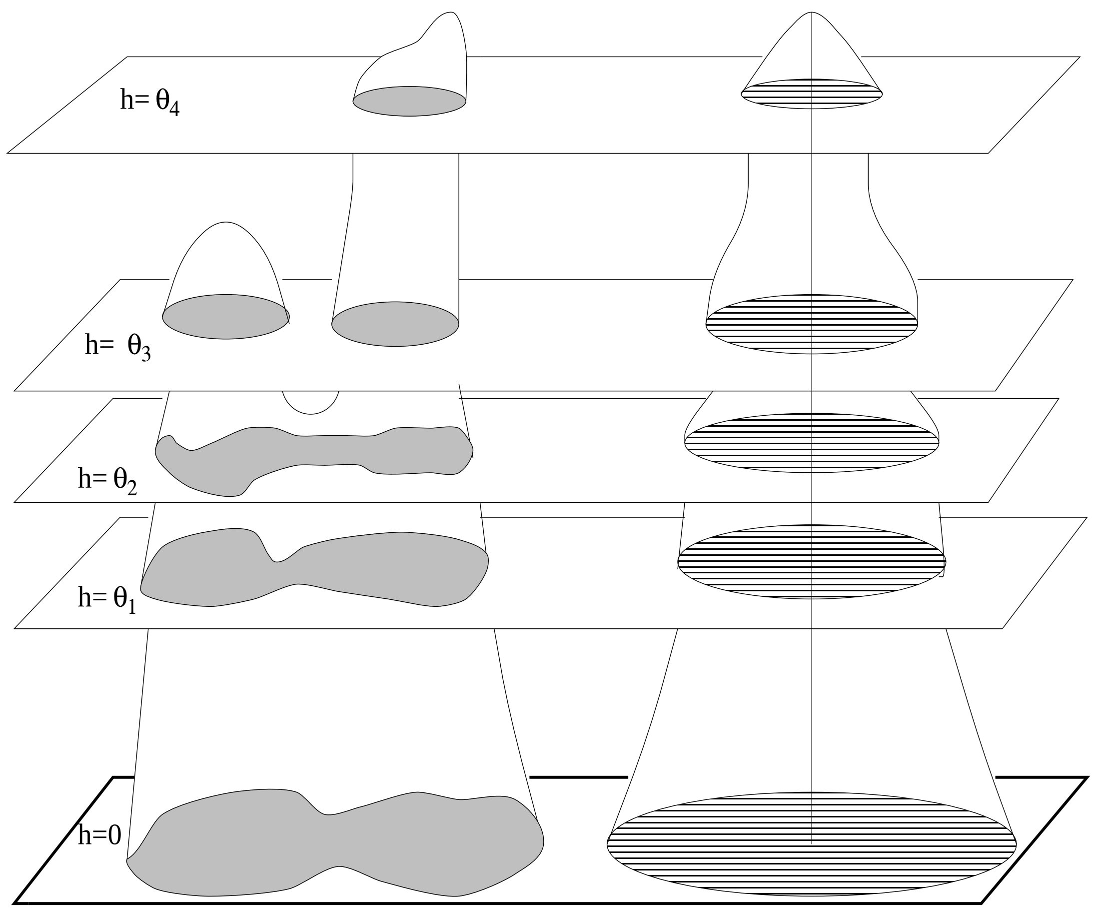
the equality holds if and only if Ω is a ball.

This was a first noted by Dido de Cartago (850 B.C.) (*in* $\mathbb{R}^2$ *the circles are the domains with fixed area having smallest perimeter*). Rigorous proofs of (1) are due to Steiner (1882), Schwarz (1890) and Schmidt (1939).

The second step of the process of symmetrization consists in the *symmetrization of data* f and u_0.

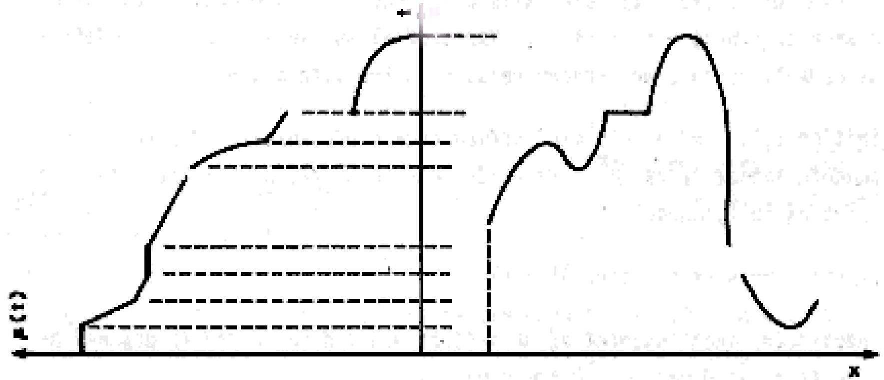
We shall use the notion of the *decreasing symmetric rearrangement* of a function introduced by H.A. Schwarz in 1890:

Given a function $h : \Omega \to I\!R$, $h \in L^1(\Omega)$, we define the *decreasing symmetric rearrangement of h*, h^*, as the (unique) function $h^* : \Omega^* \to I\!R$ such that h^* is symmetric (i.e. $h^*(x) = h^*(\widehat{x})$ if $|x| = |\widehat{x}|$), h^* decreases if $|x|$ decreases and the level sets of h and h^* are equimeasurables (i.e. $m(\{x \in \Omega : h(x) > \theta\}) = m(\{x \in \Omega^* : h^*(x) > \theta\}), \forall \theta \in I\!R$).



A more systematic definition of h^* can be introduced: we first define the *distribution function of* h by

$$\mu : I\!R \to I\!R,\ \mu(\theta) := m\{x \in \Omega : h(x) > \theta\}.$$

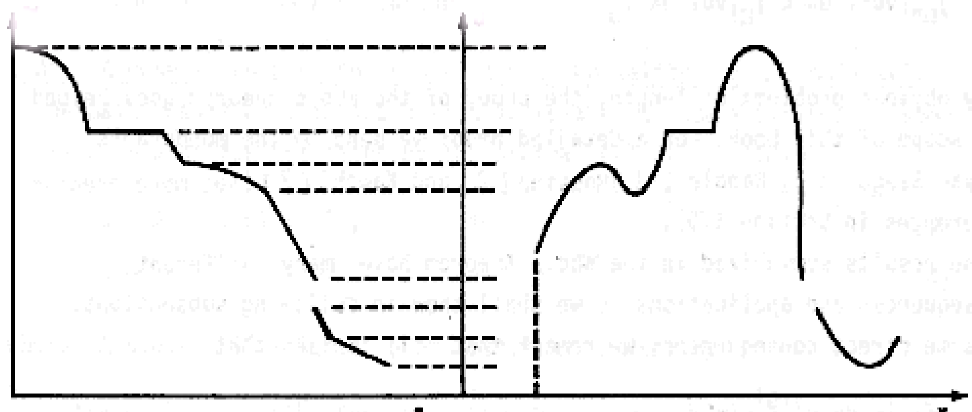


Then we define the *scalar decreasing rearrangement of* h by

$$\widetilde{h} : (0, m(\Omega)] \to I\!R,\ \widetilde{h}(s) := \inf\{\theta \in I\!R : \mu(\theta) \leq s\}$$

(notice that $\widetilde{h}(s) \sim \mu^{-1}(s)$). Finally, we define the *symmetric decreasing rearrangement of* h, by

$$h^* : \Omega^* \to I\!R,\ h^*(x) := \widetilde{h}(\omega_N |x|^N).$$



Notice that, since h^* is symmetric, we can write $h^*(x) = H(|x|)$ with $H : I\!R \to I\!R$. Nevertheless $H \neq \widetilde{h}$ since $H(r) = \widetilde{h}(\omega_N r^N)$. Notice, also, that assumed $h \geq 0$, by construction, we have that

$$h \in L^1(\Omega) \text{ implies that } h^* \in L^1(\Omega^*) \text{ and}$$
$$\int h(x)dx = \int h^*(x)dx \textit{ (the Cavalieri Principle)}$$

and that

$$h \in L^\infty(\Omega) \text{ implies that } h^* \in L^\infty(\Omega^*) \text{ and}$$
$$\operatorname*{esssup}_{x\in\Omega} h(x) = \operatorname*{esssup}_{x\in\Omega^*} h^*(x).$$

The third step of the process is the *symmetrization of the second order operator.*

We must replace the diffusion operator

$$Lu = \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(q_{ij}\frac{\partial u}{\partial x_j})$$

by another *isotropic* diffusion operator, i.e. with the same behavior in any direction x_i.

Once we know that Lu is an elliptic operator

$$\sum_{i,j=1}^{N} q_{ij}\xi_i\xi_j \geq \alpha \left\|\overrightarrow{\xi}\right\|^2 \textit{ for any } \overrightarrow{\xi}\in\mathbb{R}^N, \text{ for some } \alpha > 0.$$

Then we define *as symmetrized operator of* Lu the one given by

$$L^*u = \alpha\Delta u$$

Summarizing, we say that the *symmetrized problem of* (P) is the following one:

Problem $P^*(0)$: Find $U : \Omega^* \to I\!R$ such that

$$P^*(0)\begin{cases} -\alpha\Delta U + \lambda g(U) = f^*(x), & x \in \Omega^*, \\ U = 0, & x \in \partial\Omega^*. \end{cases}$$

Here $f^*(\cdot)$ is the decreasing symmetric rearrangement of $f(\cdot)$.

Some remarks on the statement of the *symmetrized mass comparison principle.*

The first one is that some pioneer authors finding different relations between u and U where Saint-Venant (1856), Poya and Szego (1951) and Weimberger (1962). The inequality

$$u^*(x) \leq U(x) \ , \ x \in \Omega^* \tag{2}$$

was first proved by G. Talenti, in 1976, for the case without absorption term $g \equiv 0$.

Unfortunately, this (pointwise) comparison *fails* to be true in presence of absorption terms ($g \neq 0$).

In those cases we only can compare the *distribution of the mass* of u and U

Theorem (Symmetrized Mass Comparison Principle (SMCP))

$$\int_{B(0,r)} g(u(x))^* dx \leq \int_{B(0,r)} g(U(x))dx, \forall r \in [0, R],$$

assumed that $\Omega^* = B(0, R)$.

Notice that this comparison (..., Vázquez (1982),...) can be, equivalently, expressed in terms of scalar decreasing rearrangement as

$$\int_0^s g(\widetilde{u}(\sigma))d\sigma \leq \int_0^s g(\widetilde{U}(\sigma))d\sigma, \ \forall s \in [0, m(\Omega)].$$

The SMCP has many applications (as we shall see). The main philosophy of the applications is that function U can be easily estimated in many cases and thus, thanks to the SMCP, properties for U can be extended in similar properties for u. Some books dealing with the symmetrization process are the ones by Bandle (1980), Mossino (1984), Kawohl (1985) and Díaz (1985).

Corollary *Among all the domains of given volume the volume of the dead core (for problem $P(1)$) is biggest for the sphere*

Several proofs in the literature: Bandle-Sperb-Stakgold (1984) for u analytic, ..., Díaz (1985) by using a very old result by Hardy, Littlewood and Polya (1929).

The sphere is the worst of the cases.

In fact, in Chemical Engineering the "effectiveness" of the reaction is represented by the positive number

$$\eta := \frac{1}{m(\Omega)} \int_{\Omega} g(u(x))dx$$

u solution of $P(1)$.

As consequence of the *Symmetrized Mass Comparison Principle (SMCP)* we get

Theorem *Let η^* be the effectiveness corresponding to problem $P^*(1)$. Then*

$$\eta^* \leq \eta.$$

Remark. Symmetrization in a partial set of spatial variables (of relevance for transport terms): A. Alvino, G. Trombetti, J. I. Díaz, P. L. Lions. Steiner Symmetrization and Elliptic Equations. *Communications on Pure and Applied Mathematics*, Vol. XLIX, 217-236, 1996.

6. Effectiveness as cost fonctional in optimal and controllability shape design problems.

Given an open set D, with $m(D) > V$, the optimal control problem (with the constraint $m(\Omega) = V$) associated to the cost functional

$$\eta_\Omega = \int_D \frac{g(u_\Omega(x))}{m(\Omega)} dx.$$

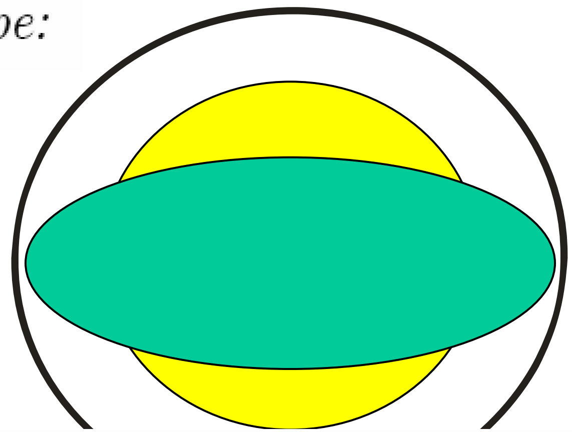
Notice that we extended $u_\Omega(x)$ to D by making $u_\Omega(x) = 0$ on $D - \Omega$. Once we prescribe $V = m(\Omega)$ its is possible to find optimal shapes for the

Theorem *Given an open set D, with $m(D) > V$, there exists a domain Ω (the optimal shape) solution of the optimal control problem*

$$Max_{\substack{\Omega \subset D \\ m(\Omega)=V}} \eta_\Omega.$$

The proof uses that $-\eta_\Omega$ is monotone decreasing with respect to inclusions (use the comparison principle) and $\gamma-$lower semicontinuous (Bucur and Buttazzo: *Variational Methods in Some Shape Optimization Problems*, SNS Pisa, 2002, Chapter 5).

Remark *If, for instance D is a ball of $\mathbb{R}^2$, the optimal shape (biggest effectiveness) is of the following type:*



Let us prove now that, for any fixed $\lambda > 0$, there is always a domain Ω of given volume V such that the dead core $N(u)$ is empty. Moreover, there exists a sequence of domains Ω_k such that

$$m(\Omega_k) = V \text{ and } \lim_{k\to\infty} \eta_{\Omega_k} = 1$$

(*approximate controllability* in the J.-L. Lions sense).

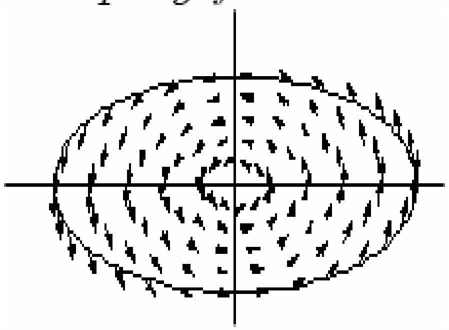
We start by an auxiliary result

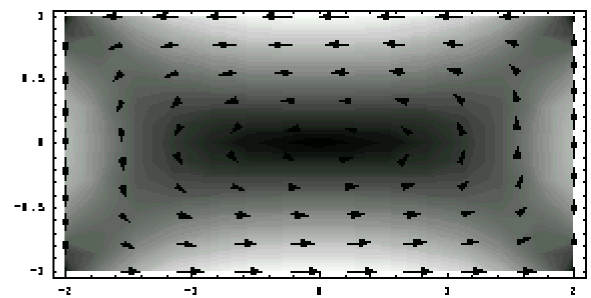
Lemma *Let w be the solution of $P_0(1)$. Then, if $\psi(x)$ is the (unique) solution of the problem*

$$\begin{cases} -L\psi = 1 & \text{in } \Omega \\ \psi = 0 & \text{on } \partial\Omega \end{cases}$$

we have that $w(x) \leq \psi(x)$ a.e. $x \in \Omega$.

Remark *If $N = 2$ and $L = \Delta$, function ψ coincides (up to a constant factor) with the warping function in the torsion of a cylindrical beam.*





By using rearrangement techniques (C. Bandle, 1985) it was proved that, if L is α-elliptic, then $\|\psi\|_{L^\infty(\Omega)}^{1+N/2} \leq \frac{2+N}{2\alpha\omega_N}(2N)^{-N/2}\|\psi\|_{L^1(\Omega)}$ and that if λ_1 is the first eigenvalue

$$\begin{cases} -\Delta\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega \\ \varphi_1 = 0 & \text{on } \partial\Omega \end{cases}$$

we get $\|\psi\|_{L^1(\Omega)} \leq \frac{V}{\alpha\lambda_1}$ *and thus*

$$\|\psi\|_{L^\infty(\Omega)}^{1+N/2} \leq \frac{2+N}{2\alpha^2\omega_N}(2N)^{-N/2}\frac{V}{\lambda_1}$$

Notice that the well-known Rayleigh-Faber-Krahn inequality

$$\lambda_1(\Omega) \geq (\frac{\omega_N}{V})^{2/N} j_{(N-2)/2}$$

(j_k *the first zero of the Bessel function* J_k*) is not of great use for our purpose.*

We must restrict ourselves to "thin" domains and made explicit some lower bounds for λ_1.

We introduce some classes of domains

$\mathcal{P} = \{\Omega \subset \mathbb{R}^N$ lying between two parallel $(N-1)$- dimensional hyperplane at distance 2ρ, that is all domains of breadth $\leq 2\rho\}$,

$\mathcal{C}_N = \{\Omega \subset \mathbb{R}^N$ convex of inradius $\rho\}$

Then, we have (Courant-Hilbert, 1924)

$$\lambda_1(\Omega) \geq (\frac{\pi}{2\rho})^2 \text{ for } \Omega \in \mathcal{P},$$

and (Osserman, 1979)

$$\lambda_1(\Omega) \geq (\frac{1}{2\rho})^2 \text{ for } \Omega \in \mathcal{C}_N.$$

By applying the auxiliary lemma and the above inequalities we arrive to

Theorem *Let Ω belong to either $\mathcal{P}$ or $\mathcal{C}_N$. Then there exists $\rho_0 = \rho_0(\alpha, \lambda, N, V)$ such that meas $N(u) = 0$.*
Moreover, given $\varepsilon > 0$ there exists an open set Ω belonging to either $\mathcal{P}$ or $\mathcal{C}_N$, with $m(\Omega) = V$ and such that

$$\eta_\Omega \geq 1 - \varepsilon.$$

A different study can be made by using the microscale data.

*separation among the holes

*size and shape of the obstacles

Many variants are possible:

Microscopic Signorini boundary conditions, Reactions at the interior of the cell, quasilinear equations, …, systems (fluid transport terms), …

Evolution problem (coupled with an ODE)

Controlling the macroscopic behavior by acting at the microscopic scale

Think **globally** and **act locally**

Thanks for your attention