On the confinement of a viscous fluid by means of a feedback external field

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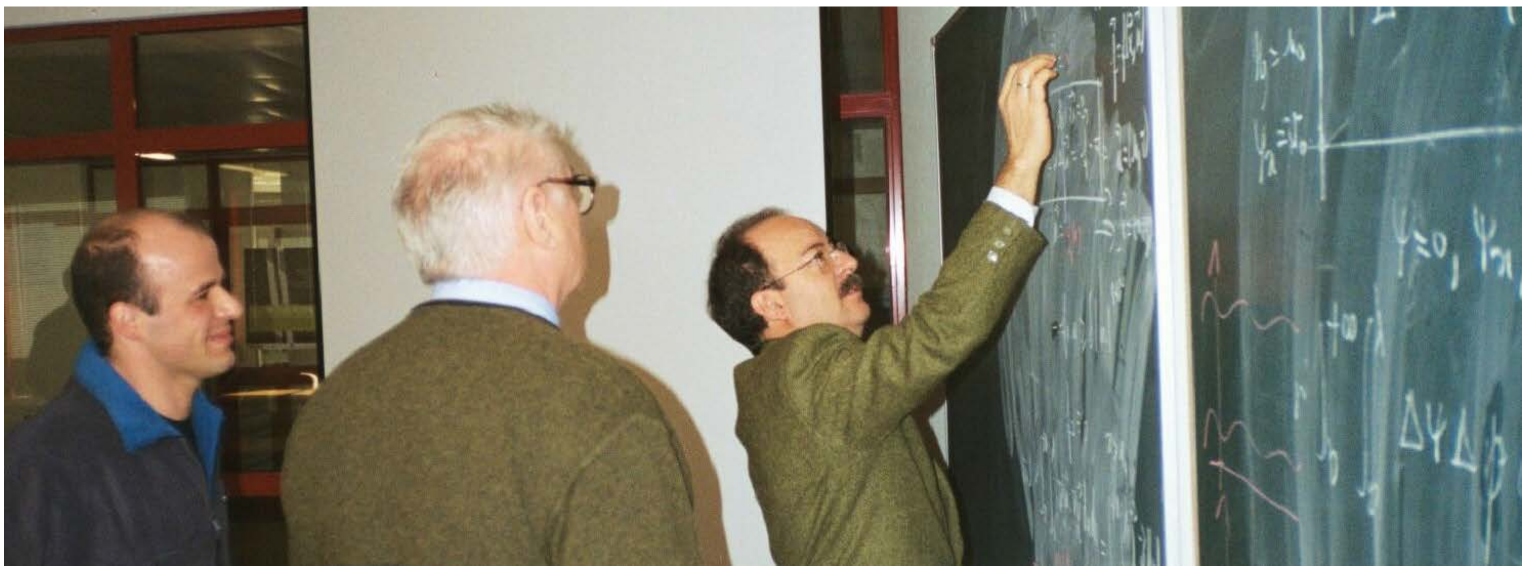

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planar stationary flow of an incompressible viscous fluid

$$\Omega = (0,\infty)\times(0,L), \qquad \mathbf{u}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x})) \quad \mathbf{x} = (x,y)\in\Omega,$$

$\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$

$$-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \mathbf{f} - \nabla p \quad \text{in } \Omega,$$

$$\mathrm{div}\mathbf{u} = 0 \quad \text{in } \Omega,$$

$$\mathbf{u}(0,y) = \mathbf{u}_*(y),\; y\in(0,L)$$

$$\mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\; x\in(0,\infty).$$

$$|\mathbf{u}(x,y)| \to 0,\; \text{as } x\to\infty \text{ and } \; y\in(0,L).$$

Problem: *can we find an external localized forces field* $\mathbf{f}$ *stopping the fluid at a finite distance, i.e., such that*

$$\mathbf{u}(x,y) = \mathbf{0} \;\text{ for } x > x_{\mathbf{u}} \text{ and } y\in(0,L), \; \textbf{?}$$

$$\mathbf{f}(\mathbf{x},\mathbf{u}) = \mathbf{0} \;\text{ for } x \geq x_{\mathbf{f}} \;\text{ and } y\in(0,L),$$

compatibility conditions

$$\mathbf{u}_*(0) = \mathbf{u}_*(L) = \mathbf{0}.$$

$$\int_0^L u_*(s)ds = 0. \qquad \mathbf{u}_*(y) = (u_*(y), v_*(y))$$

Remarks:

1. Resemblance with the question of the (magnetic) confinement of a plasma (ideal gas), typical of the magnetohydrodynamics (MHD).

2. For the classical Navier-Stokes problem ($\mathbf{f} \equiv \mathbf{f}(\mathbf{x})$) the decay is exponential and so the *localization effect* fails (Horgan and Wheeler, Amick, Ladyzhenskaya and Solonnikov, Ames and Payne, Iosif'yan: see book by Galdi (1994)). More recently: *Unique Continuation results*: L. Brandolese (CRAS, 2001), C. Fabre and G. Lebau (Comm. in PDE, 2002), and Y. Meyer (book in preparation).

Main result: Positive answer for $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{x}, \mathbf{u}(\mathbf{x}))$ (external body forces given in a feedback dissipative form)

$$\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2, \ \mathbf{f}(\mathbf{x}, \mathbf{u}) = (f_1(\mathbf{x}, \mathbf{u}), f_2(\mathbf{x}, \mathbf{u})),$$

and such that,

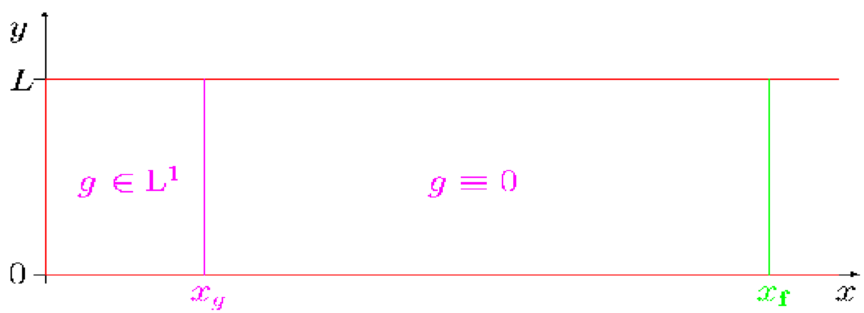
$$-\mathbf{f}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \geq \delta \, \chi_{\mathbf{f}}(\mathbf{x}) \, |u|^{1+\sigma} - g(\mathbf{x}) \tag{1}$$

$$\chi_{\mathbf{f}}(\mathbf{x}) := \chi_{(0,x_{\mathbf{f}})}(\mathbf{x}) = \begin{cases} 1 & \text{if } x \in (0, x_{\mathbf{f}}), \\ 0 & \text{if } x \notin (0, x_{\mathbf{f}}). \end{cases}$$

for some $0 < \sigma < 1$ and $\delta > 0$, $\forall \, \mathbf{u} \in \mathbb{R}^2$, $\mathbf{u} = (u, v)$, and a.e. $\mathbf{x} \in \Omega$, with

$$g \in L^1\left(\Omega^{x_g}\right), \ g \geq 0, \ g(\mathbf{x}) = 0 \ \text{ a.e. in } \ \Omega_{x_g} \tag{2}$$

for some x_g, with $0 \leq x_g \leq x_{\mathbf{f}} \leq \infty$ and $x_{\mathbf{f}}$ large enough, where $\Omega^{x_g} = (0, x_g) \times (0, L)$ and $\Omega_{x_g} = (x_g, \infty) \times (0, L)$.

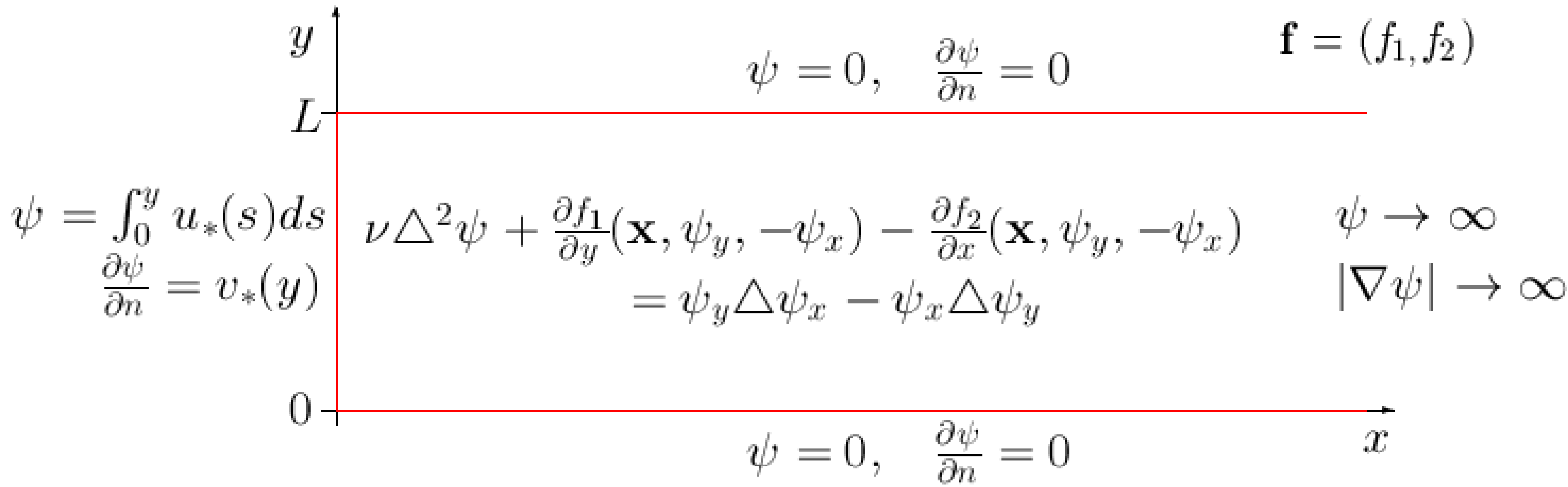


Example: $\mathbf{f}(\mathbf{x}, \mathbf{u}) = -\delta \chi_{\mathbf{f}}(\mathbf{x})(|u|^{\sigma-1}u, 0)$

Main idea to prove the localization property: introducing the current function ψ,

$$u = \psi_y \quad \text{and} \quad v = -\psi_x \text{ in } \Omega, \tag{3}$$

we reduce $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ to the scalar quasilinear fourth order problem $(\mathcal{P}_\psi)$



• Our starting point was a previous unidirectional result for second order anisotropic equations proved in S. N. Antontsev, J. I. Díaz and S. I. Shmarev, Birkhäuser, 2002, (Section 1.4.2).

• For the adaptation to four order equations we apply the technique introduced by F. Bernis (1984,...).

We assume

$$\mathbf{u}_* \in \mathbf{H}^{\frac{1}{2}}(0, L).$$

We introduce the functional spaces

$$\widetilde{\mathbf{H}}(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathrm{div}\mathbf{u} = 0,\ \mathbf{u}(0,.) = \mathbf{u}_*(.),\ \mathbf{u}(x, L_1(x)) = \mathbf{u}(x, L_2(x)) = \mathbf{0},\ x \in (0,\infty),\ \lim_{x\to+\infty} |\mathbf{u}| = 0\},$$

$$\widetilde{\mathbf{H}}_0(\Omega) = \{\mathbf{u} \in \mathbf{H}^1(\Omega) : \mathrm{div}\mathbf{u} = 0,\ \mathbf{u}(0,.) = \mathbf{0},\ \mathbf{u}(x, L_1(x)) = \mathbf{u}(x, L_2(x)) = \mathbf{0},\ x \in (0,\infty),\ \lim_{x\to\infty} |\mathbf{u}| = 0\}.$$

Assumptions for the existence of solutions:

$$\boxed{\mathbf{f}(\mathbf{x},\mathbf{u}) = -\delta\chi_{\mathbf{f}}(\mathbf{x})(|u|^{\sigma-1}u(\mathbf{x}), 0) - \mathbf{h}(\mathbf{x},\mathbf{u})}$$

$$\mathbf{h}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \geq -g(\mathbf{x}) \text{ for every } \mathbf{u} \in \mathbb{R}^2 \text{ and a.e. } \mathbf{x} \in \Omega,$$

$$H_M(\mathbf{x}) = \sup_{|\mathbf{u}|\leq M} |\mathbf{h}(\mathbf{x},\mathbf{u})| \qquad H_M \in L^1(\Omega^{xf}) \cap L^2(\Omega^{xf}) \text{ for all } M > 0,$$

no upper restriction on the growth of $|\mathbf{f}(\mathbf{x},\mathbf{u})|$ with respect to $\mathbf{u}$ is imposed this type of non-linear terms are called *strongly non linear*.

Definition. A function $\mathbf{u}$ is a weak solution of problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ if:
(i) $\mathbf{u} \in \widetilde{\mathbf{H}}(\Omega)$, $\mathbf{f}(\mathbf{x}, \mathbf{u}) \in \mathbf{L}^1_{loc}(\Omega)$ and (ii) for every $\boldsymbol{\varphi} \in \widetilde{\mathbf{H}}_0(\Omega) \cap \mathbf{L}^\infty(\Omega)$ with compact support

$$\nu \int_\Omega \boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\varphi \, d\mathbf{x} + \int_\Omega (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \cdot \varphi \, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \varphi \, d\mathbf{x}.$$

Theorem 1. There exists, at least, one weak solution of problem $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$.

Moreover, $\mathbf{h}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u} \in L^1(\Omega)$, $\mathbf{h}(\mathbf{x}, \mathbf{u}) \in \mathbf{L}^1(\Omega)$ and $\mathbf{u}$ satisfies to the energy estimate

$$\int_\Omega \left(|\boldsymbol{\nabla}\mathbf{u}|^2 + \chi_{\mathbf{f}}|u|^{1+\sigma} + |\mathbf{h}(\mathbf{x}, \mathbf{u}) \cdot \mathbf{u}|\right) d\mathbf{x} \leq C\left(\|\mathbf{u}_*\|^4_{\mathbf{H}^{\frac{1}{2}}(L_1(0), L_2(0))} + \|g\|_{L^1(\Omega^{x_g})} + 1\right) \tag{4}$$

where $C = C(L, \delta, \nu, \sigma)$ is a positive constant. If, in addition, we assume

$$(\mathbf{f}(\mathbf{x}, \mathbf{u}_1) - \mathbf{f}(\mathbf{x}, \mathbf{u}_2)) \cdot (\mathbf{u}_1 - \mathbf{u}_2) \leq 0,$$

and ν large enough then $\mathcal{P}(\Omega, \mathbf{u}_*, \mathbf{f})$ has only one solution.