On time periodic free boundaries in distributed systems: the last paper by Maurizio Badii

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ILL-POSED PROBLEMS
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PEOPLE

Dedicated to the memory of

Maurizio Badii (1944-2009)

1. Introduction

When Maurizio Badii passed away (November 2, 2009) it was only some months after we got some joint results dealing with the study of the formation and qualitative properties of time periodic free boundaries given rise by the solutions of nonlinear parabolic equations.

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Research Article

On the Time Periodic Free Boundary Associated to Some Nonlinear Parabolic Equations

M. Badii[1] and J. I. Díaz[2]

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Universidad Complutense Madrid

Badii, Maurizio

MR Author ID: **29090**
Earliest Indexed Publication: 1978
Total Publications: **37**
Total Citations: **7**

⊞ Also published as: Badii, M....

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Co-authors (by number of collaborations)

Croce, Antonio Díaz Díaz, Jesús Ildefonso Schiaffino, Andrea Tesei, Alberto
Webb, Glenn F.

- It is well known that many natural and social phenomena are modeled in terms of non-linear parabolic partial differential equations with time periodic data (source forcing terms and/or boundary conditions).

- For instance, in the framework of natural and biological phenomena, the periodicity property is related to daily or seasonal fluctuations. Some more complicated periodicity cycles arises, for instance, in Celestial Mechanics (as we shall indicate later) and have been used as one of the theoretical justifications of the past Glaciations.

- In the technology framework, the periodicity plays also a fundamental role in the design of motors, electrical and magnetic devices and plants.

• On 1991, it was by attending an international meeting at Pont à Mousson (European Conference on Elliptic and Parabolic Problems. June 14-21, 1991) how we started our joint collaboration in the study of periodic solutions of degenerate quasilinear parabolic equations, as a fruitful general formulation for so many different and relevant applications.

• Curiously enough, when we meet personally by first time, on June 1991, we already have published a joint paper in collaboration with Alberto Tesei:

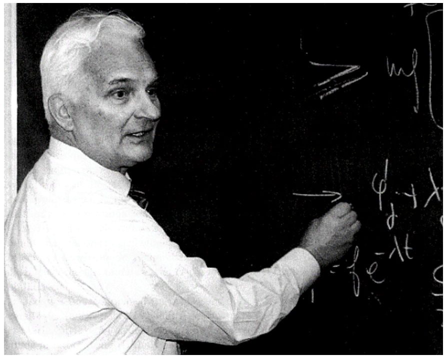
M. Badii, J. I. Díaz, A. Tesei, Existence and attractivity results for a class of degenerate functional parabolic problems. *Rend. Sem. Mat. Univ. Padova*, Vol **78** (1987) 109-124.

• Our first joint paper on periodic solutions was related to the question of surface runoff and seasonal infiltration of rain in porous soils:

M. Badii, J. I. Díaz, Periodic solutions of a quasilinear parabolic boundary value problem arising in unsaturated flow through a porous medium. *Applicable Analysis*, **56**, 1994, 279-301.

• It received some (laudatory) comments and citations by several authors: O.A. Oleinik and I. Stakgold, among them.

• Those were years in which my relationship with Jacques-Louis Lions became closer and we started a fruitful collaboration. He began to be interested in Environment and Climatology (which, with no doubt, it was the reason why I did so also).



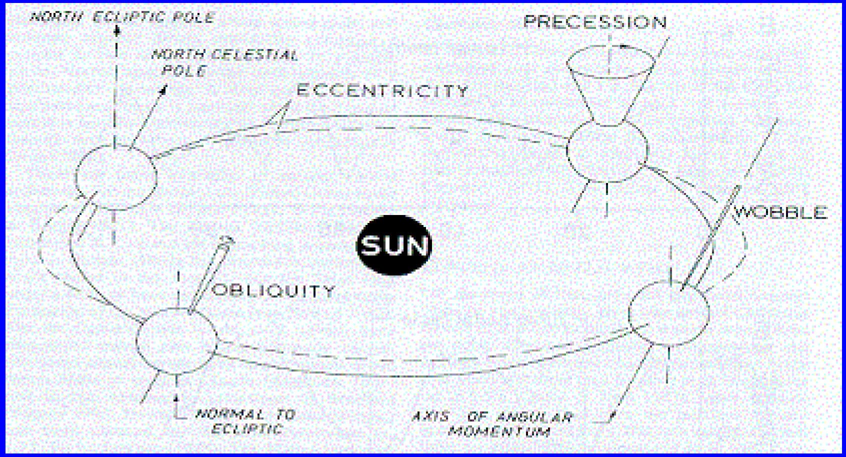
J.L. Lions (1928-2001)

• In contrast with the special interest on the primitive equations of the atmosphere and Oceans by J.L. Lions (with R. Temam and S. Wang), I devoted many forces to the study of some simpler parabolic models: the so called ***Energy Balance Models*** proposed (independently) by M.I. Budyko and W. D. Sellers, in1969, dealing with the global superficial temperature of the Earth.

• ***Energy Balance Models*** have a diagnostic character and intended to understand the evolution of the global climate on long time scales. Their main characteristic is the high sensitivity to the variation of solar and terrestrial parameters.

$$(P)\begin{cases} c(x)u_t - div(k(x)\,|\nabla u|^{p-2}\nabla u) + \mathcal{G}(x,u) \in QS(t,x)\beta(x,u) + f(t,x) & \text{on } (0,T)\times\mathcal{M} \\ u(0,x)=u_0(x) & \text{on } \mathcal{M}, \end{cases}$$

Used in the study of the Milankovitch theory of the ice-ages (R.North, 1984)



Milutin Milankovitch (1879-1958)

This is why we studied (rigurously) the periodic solutions of EBMs:

M. Badii, J. I. Díaz, Time Periodic Solutions for a Diffusive Energy Balance Model in Climatology *J. Mathematical Analysis and Applications,* **233** (1999) 713-729.

• But, from the begining of our collaboration, on 1991, we were specially interested in the study of possible time periodic free boundaries raised by the solutions.

• At this time, several deep studies on some related problems were available in the literature:

Kruzhkov (1970), Ishi (1980), Meirmanov-Kaliev (1983), DiBenedetto-Friedman, (1986), Cannon-Yin (1990),…

• But this kind of “strong formulations” start by assuming the existence of the free boundary (their main concern was to study the regularity of the solution).

• The literature concerning the existece and uniqueness of periodic solutions to nonlinear PDEs (even under some abstract formulations) was very large:

Kranosel’ski, Browder, Brezis, Benilan, Seidman, Amann, Hess, Kenmochi, Kawhol, Rüll, Mawhin, …

• No free boundary spatial estimates (location, measure of the support of the solution) except in some special cases:

P. Hess, M.A. Pozio and A. Tesei, Time periodic solutions for a class of degenerate parabolic problems, *Houston J. Math*. **21** (1995) 367-394.

$\partial_t u = \Delta u^m + a(t,x)u$ in $Q_T = \mathbf{R} \times \Omega$, $\partial u^m / \partial n = 0$ on $\mathbf{R} \times \partial\Omega$, where $m > 1$ and $\Omega \subset \mathbf{R}^N$

$\Omega^+ = \{x \in \Omega: \int_0^T a(t,x)dt > 0\}$. Ω^+ is nonempty and $\iint_{Q_T} a(t,x)dtdx < 0$,

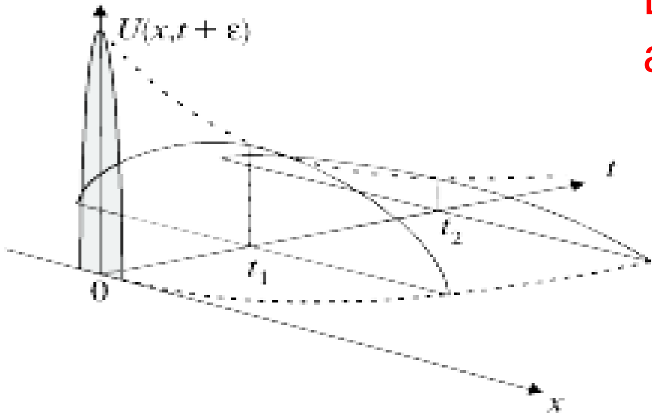
Then, support u(t,.) does not depend on t.

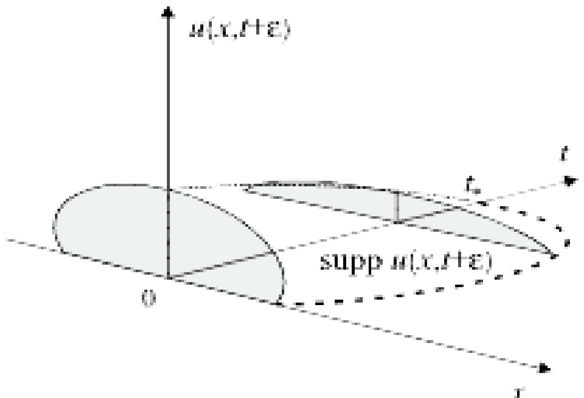
• Some important mathematical dificulties: the solutions of “most” of the initial value problems satisfy:

Positivity property: if $u(t_0, x_0) > 0$ then $u(t, x_0) > 0$ for all $t > t_0$

For every x_0 there exists $T = T(x_0)$ such that $u(t, x_0) > 0$ for every $t \geq T(x_0)$

Except in the case of the presence of some absorption terms at the equation !!!!!!!





Formulation in Badii-Díaz (2010):

$$(P)\begin{cases} u_t - \triangle_p u + \lambda f(u) = g & \text{in } Q := \Omega \times \mathbb{R}, \\ u(x,t) = h(x,t) & \text{on } \Sigma := \partial\Omega \times \mathbb{R}, \\ u(x,t+T) = u(x,t) & \text{in } Q. \end{cases}$$

(H_f): $f \in C(\mathbb{R})$, $f(0) = 0$ and there exist two nondecreasing continuous functions f_1, f_2 such that $f_2(0) = f_1(0) = 0$ and

$$f_2(s) \leqslant f(s) \leqslant f_1(s), \forall s \in \mathbb{R},$$

(H_g): $g \in C(\mathbb{R}; L^\infty(\Omega))$ and g is T-periodic,
(H_h): $h \in C(\Sigma)$ and h is T-periodic.

Analogously, we can consider the doubly nonlinear diffusion-absorption equation

$$(P_b)\begin{cases} b(u)_t - \Delta_p u + \lambda f(u) = g & \text{in } Q := \Omega \times \mathbb{R}, \\ u(x,t) = h(x,t) & \text{on } \Sigma := \partial\Omega \times \mathbb{R}, \\ u(x,t+T) = u(x,t) & \text{in } Q, \end{cases}$$

$b \in C(\mathbb{R})$ is a nondecreasing function such that $b(0) = 0$

2. Sufficient conditions for the existence of a periodic free boundary

Together with problem (P) we consider the following stationary problems

$$(\underline{SP})\begin{cases} -\Delta_p v + \lambda f_1(v) = g_1 & \text{in } \Omega, \\ v = h_1 & \text{on } \partial\Omega, \end{cases}$$

$$(\overline{SP})\begin{cases} -\Delta_p w + \lambda f_2(w) = g_2 & \text{in } \Omega, \\ w = h_2 & \text{on } \partial\Omega, \end{cases}$$

$$g_1(x) \leqslant g(x,t) \leqslant g_2(x), \ \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega,$$

$$h_1(x) \leqslant h(x,t) \leqslant h_2(x), \ \forall (x,t) \in \Sigma.$$

Existence and uniqueness of weak solutions of (P)

$$W_{T-per} := \{u - h \in L^p(0,T;W_0^{1,p}(\Omega)),\ u_t - h_t \in L^q(0,T;W^{-1,p'}(\Omega)) \text{ and } u(.,t+T) = u(.,t)\ \forall t \in \mathbb{R}\ \}$$

Suitable modifications for (P_b): Badii-Díaz (2010).

Some previous general notations:

Given $\varphi : Q \to \mathbb{R}$, $\varphi \in C\left([0,T] : L^1_{loc}(\Omega)\right)$,
$S(\varphi(.,t)) =$ support $\varphi(.,t)$,
$N(\varphi(.,t)) = \overline{\Omega} - S(\varphi(.,t))$ null set $\varphi(.,t)$

Theorem 1. *Assume* (H_f), (H_g), (H_h) *and let* $g_1, h_1 \geq 0$. *Let* $F_i(s) = \int_0^s f_i(s)ds$ *and assume that*

$$\int_{0+} \frac{ds}{F_i(s)^{1/p}} < +\infty,\ i = 1, 2.$$

Then, if $u(x,t)$ *denotes the unique periodic solution of problem* (P) *we have that* $N(u_1) \supset N(u(.,t)) \supset N(u_2)$ $\forall t \in \mathbb{R}$. *In particular,* $N(u(.,t))$ *contains, at least, the set of* $x \in N(h_2) \cup N(g_2)$ *such that*

$$d(x, \partial(N(h_2) \cup N(g_2)) > \Psi_{2,N}(\|u_2\|_{L^\infty(\Omega)})$$

where

$$\Psi_{2,N}(\tau) = \left(\frac{N(p-1)}{p}\right)^{1/p} \int_0^\tau \frac{ds}{F_2(s)^{1/p}}.$$

Nevertheless, if $\min_{\partial\Omega} h_1 \geq k > 0$ *and if*

$$R < \Psi_{1,1}(k),$$

then $N(u(.,t))$ *is empty since we have* $0 < u_1(x) \leqslant u(x,t)$ $\forall t \in \mathbb{R}$ *and a.e.* $x \in \Omega$. *Here* R *is the radius of the smaller ball containing* Ω *and*

$$\Psi_{1,1}(\tau) = \left(\frac{(p-1)}{p}\right)^{1/p} \int_0^\tau \frac{ds}{F_1(s)^{1/p}}.$$

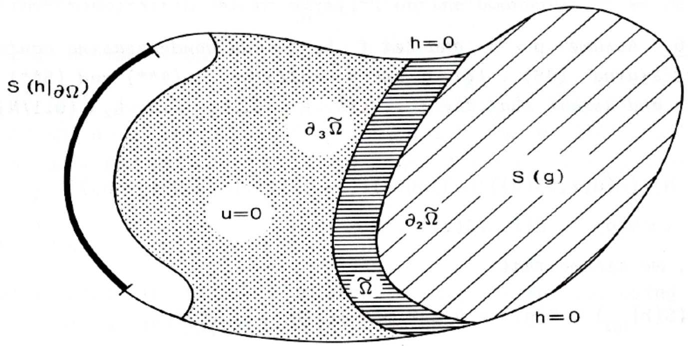
- Balance between the size of the domain and the « size » of the data.

- A different negative criterion (Vázquez 1986):

Theorem 2. *Under assumptions* (H_f), (H_g) *and* (H_h), *if* $g_1, h_1 \geq 0$ *and*

$$\int_{0^+} \frac{ds}{F_1(s)^{1/p}} = +\infty,$$

then $N(u(.,t))$ *is empty.*



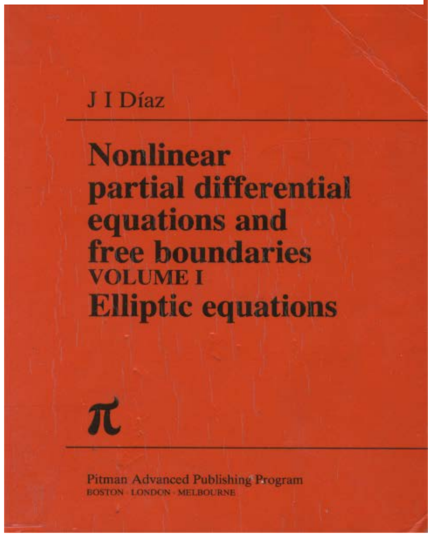
Lemma 1. *Assume* (H_f), (H_g) *and* (H_h). *Let* $u(x,t)$ *be the unique periodic solution of problem* (P). *Then*

$$u_1(x) \leqslant u(x,t) \leqslant u_2(x),\ \forall t \in \mathbb{R} \text{ and a.e. } x \in \Omega.$$

Remarks

1. The above results apply to some ill-posed problems since they are established under a great generality which does not require a complete information on the equation neither on the type of boundary condition (for instance, it is enough to know some L^∞ estimates outside the support of the data). Neumann problem,…

2. Application under non-uniqueness of solutions (D-Hernández (1999), (2009),…



Some different techniques: Energy Methods

$$(P)\begin{cases} \frac{\partial b(u)}{\partial t} - \operatorname{div} \mathbf{A}(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) = g & \text{in } B_\rho \times \mathbb{R}, \\ b(u(x,t+T)) = b(u(x,t)) & \text{in } B_\rho \times \mathbb{R}, \end{cases}$$

$$|\mathbf{A}(x,t,r,\mathbf{q})| \le C_1 |\mathbf{q}|^{p-1}, C_2\, |\mathbf{q}|^p \le \mathbf{A}(x,t,r,\mathbf{q}) \cdot \mathbf{q},$$

$$|B(x,t,r,\mathbf{q})| \le C_3\, |r|^\alpha\, |\mathbf{q}|^\beta,\ \ C_0 |r|^{q+1} \le C(x,t,r)\, r,$$

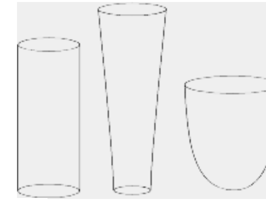
$$C_6\, |r|^{\gamma+1} \le G(r) \le C_5\, |r|^{\gamma+1},\ \text{ where}$$

$$G(r) = b(r)\, r - \int_0^r b(\tau)\, d\tau,$$

Definition. A function $u(x,t)$, with $b(u) \in C\left([0,T] : L^1_{loc}(B_\rho)\right)$, is called a local weak solution of the above problem if $b(u(x,t+T)) = b(u(x,t))$ in $B_\rho \times \mathbb{R}$, for any domain $\Omega' \subset \mathbb{R}^N$ with $\overline{\Omega'} \subset B_\rho$ we have $u \in L^\infty(0,T;\, L^{\gamma+1}(\Omega')) \cap L^p(0,T;\, W^{1,p}(\Omega'))$, $\mathbf{A}(\cdot,\cdot,u,Du)$, $B(\cdot,\cdot,u,Du)$, $C(\cdot,\cdot,u) \in L^1(B_\rho \times \mathbb{R})$ and for every test function $\varphi \in L^\infty\left(0,T; W^{1,p}(B_\rho)\right) \cap W^{1,2}(0,T; L^\infty(B_\rho))$ with $\varphi(x,t+T) = \varphi(x,t)$ in $B_\rho \times \mathbb{R}$ and for any $t \in [0,T]$ we have

$$\int_0^t \int_{B_\rho} \{b(u)\varphi_t - \mathbf{A}\cdot D\varphi - B\,\varphi - C\,\varphi\}\, dx dt\, - \int_\Omega b(u)\varphi\, dx\Big|_0^t = - \int_0^t \int_{B_\rho} g\,\varphi\, dx\, dt.$$

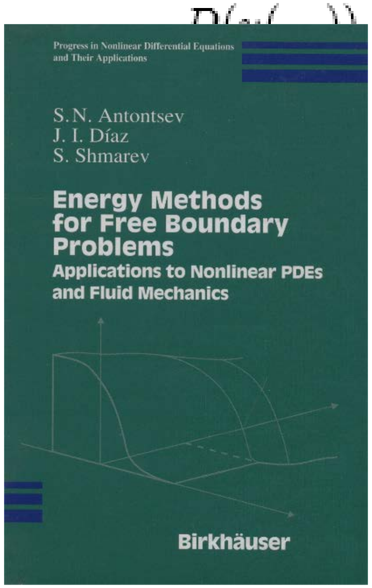
$$P(t) \equiv P(t;\vartheta,\upsilon) = \{(x,s) \in : |x-x_0| < \rho(s) \equiv \vartheta(s-t)^{\upsilon}, s \in (t,T]\}.$$



$$E(P) := \int_{P(t)} |Du(x,\tau)|^p \, dx d\tau, \quad C(P) := \int_{P(t)} |u(x,\tau)|^{q+1} dx d\tau$$

$$\Lambda(T) := ess \sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^{\upsilon}} |u(x,s)|^{\gamma+1} \, dx.$$

$$D(u(\cdot,\cdot)) := ess \sup_{s\in(0,T)} \int_{\Omega} |u(x,s)|^{\gamma+1} \, dx + \int_{Q} \left(|Du|^p + |u|^{q+1}\right) dx dt.$$



We assume the following conditions

$$q < \gamma, \, 1 + q < \frac{\gamma p}{p-1}, \qquad g(x,t) \equiv 0 \text{ on } B_{\rho}(x_0), \text{ a.e. } t \in \mathbb{R},$$

$$\begin{cases} \alpha = \gamma - (1+\gamma)\beta/p, \\ C_3 < \left(C_0 \frac{p}{p-1}\right)^{(p-\beta)/p} \left(C_2 \frac{p}{\beta}\right)^{\beta/p} \text{ if } 0 < \beta < p, \\ C_3 < C_0 \text{ if } \beta = 0 \text{ (respectively } C_0 < C_2 \text{ if } \beta = p). \end{cases}$$

Theorem 3 *Any periodic weak solution satisfies that* $u(x,t) \equiv 0$ *on* $B_{\rho_*} \times \mathbb{R}$, *for some suitable* $\rho_* \in (0,\rho_0)$, *assumed that the global energy* $D(u)$ *is small enough.*

$$Y^{\varepsilon} \leq c \frac{\partial Y}{\partial \rho},$$

Remark 3.

The above energy method applies to suitable higher order equations of Cahn-Hilliard type (see details for the associated initial BVP in Antontsev-Diaz-Shmarev (2002)).

3. Periodical time connection between stationary episodes and on disconnected free boundaries

It is possible to connect (in a finite time) two different solutions of the stationary problem

$$(SP)\begin{cases} -\Delta_p v + \lambda f(v) = g^* & \text{in } \Omega, \\ u = h^* & \text{on } \partial\Omega, \end{cases}$$

by means of a transient periodic solution of the associated parabolic problem (P).

As example we have:

Theorem 4. *Let* $n = 1$, $\Omega = (-L, L)$ *and* $f(s) = |s|^{q-1} s$ *with* $q < p - 1$. *Then the function*

$$u(x,t) = C\left[|x| - \tau(t)\right]_+^{p/(p-1-q)}$$

with

$$\tau(t) = \begin{cases} l_0 + \frac{(l_1 - l_0)t}{t_1} & \text{if } 0 \leqslant t \leqslant t_1, \\ l_1 & \text{if } t_1 \leqslant t \leqslant t_2, \\ l_1 + \dfrac{(l_0 - l_1)}{T - t_2}(t - t_2) & \text{if } t_2 \leqslant t \leqslant T, \end{cases} \tag{1}$$

for some l_0, l_1 *nonnegative given constants*, $0 \leq t_1 \leqslant t_2 \leq T$, *such that*

$$\max\left\{\frac{(l_1 - l_0)}{t_1}, \frac{(l_0 - l_1)}{T - t_2}\right\} \leq \frac{C^{q-1}(\lambda - C^{(p-1-q)})(p - 1 - q)}{p},$$

is a T*-periodic solution of problem* (P) *with* $h(\pm L, t) = C(L - \tau(t))^{p/p-1-q} > 0$ *and*

$$g(x,t) = \left(\lambda C^q - \frac{p}{(p-1-q)} C\tau'(t) - C^{p-1}\right)\left[|x| - \tau(t)\right]_+^{pq/(p-1-q)}.$$

In particular, it connects the stationary solutions corresponding to the data

$$\begin{cases} g^*(x) = \left(\lambda C^q - \frac{p}{(p-1-q)} C^{\frac{(l_1-l_0)}{t_1}} - C^{p-1}\right) [|x| - l_0]_+^{pq/(p-1-q)}, \\ h^*(\pm L) = C(L-l_0)^{p/p-1-q}, \end{cases}$$

and

$$\begin{cases} g^*(x) = \left(\lambda C^q - C^{p-1}\right) [|x| - l_1]_+^{pq/(p-1-q)}, \\ h^*(\pm L) = C(L-l_1)^{p/p-1-q}. \end{cases}$$

Exponent of the non-diffusion of the support, waiting time

Álvarez-Díaz (1993, 2003).

Remark. This behavior (heteroclinic connection in a finite time) is very exceptional: for instance it cannot hold in the case of linear parabolic problems. In particular, this solution can be used for different purposes in the study of controllability problems (see, e.g., Coron 2002).

We shall end this section by showing that it is possible to construct nonnegative periodic solutions of (P_b) giving rise to *disconnected free boundaries*, i.e. with free boundaries given by closed hypersurfaces of the space $\mathbb{R}^{n+1}$.

For instance, we have

Theorem 5. *Assume $\Omega = (-L, L)$, $f(s) = |s|^{q-1} s$ with $q < \min(1, p-1)$. Let $u(x,t)$ be the unique T-periodic solution of problem (P) corresponding to data $h(\pm L, t)$ and $g(x,t)$*

$$0 \leq U(t) \leq h(\pm L, t) \leq C\,(L - \tau(t))^{p/(p-1-q)} + U(t)$$

and

$$G(t) \leq g(x,t) \leq \left(\frac{\lambda}{2} C^q - \frac{p}{(p-1-q)} C\tau'(t) - C^{p-1}\right) [|x| - \tau(t)]_+^{pq/(p-1-q)} + G(t),$$

with $\tau(t)$ given as before, with $0 \leq l_0 < l_1 = L$, $0 < t_1 < t_2 < T$, and $C > 0$ such that

$$\frac{(l_1 - l_0)}{t_1} \leq C^q \left(\frac{\lambda}{2} - C^{(p-1-q)}\right).$$

where $U(t)$ is given by

$$U(t) = \begin{cases} \underline{w}(t) & \text{if } t \in [0, t^*], \\ w(t : \varsigma, t^*) & \text{if } t \in [t^*, T], \end{cases} \tag{2}$$

with $\underline{w} \in C([0,t^*])$ *such that* $\underline{w}' \in L^1(0,t^*)$, $\underline{w} \geq 0$, $\underline{w}'(t) + \frac{\lambda}{2} f(\underline{w}) \geq 0$ *on* $(0,t^*)$, $\underline{w}(0) = 0$, $\underline{w}(t^*) = \varsigma$, *(for some* $\varsigma > 0$*) and* $w(t:\varsigma,t^*)$ *the unique solution of the Cauchy problem*

$$\begin{cases} w'(t) + \frac{\lambda}{2} f(w) = 0 & t > t^*, \\ w(t^*) = \varsigma, \end{cases}$$

for some $t^* \in (t_1, t_2)$*. Here*

$$G(t) = \begin{cases} 0 & \textit{if } t \in [0,t_1], \\ \underline{w}'(t) + \frac{\lambda}{2} f(\underline{w}) & \textit{if } t \in [t_1,t^*], \\ 0 & \textit{if } t \in [t^*,t_2], \\ 0 & \textit{if } t \in [t_2,T]. \end{cases}$$

Finally take $t_2 = t^* + \frac{2\Psi(\varsigma)}{\lambda}$*, with*

$$\Psi(\tau) := \int_0^\tau \frac{ds}{f(s)} \textit{ for any } \tau > 0.$$

Then $U(t) \leq u(x,t) \leq C\left[|x| - \tau(t)\right]_+^{p/(p-1-q)} + U(t)$ *on* $\Omega \times \mathbb{R}$*. In particular the null set* $\cup_{t\in[0,T]} N(u(.,t))$ *has at least two connected components since it contains the set*

$$\{(x,t) \in (-L,L)\times[0,t_1] : |x| \leq l_0 - \frac{l_0 t}{t_1}\} \cup \{(x,t) \in (-L,L)\times[t_2,T] : |x| \leq \frac{l_1}{T-t_2}(t-t_2)\},$$

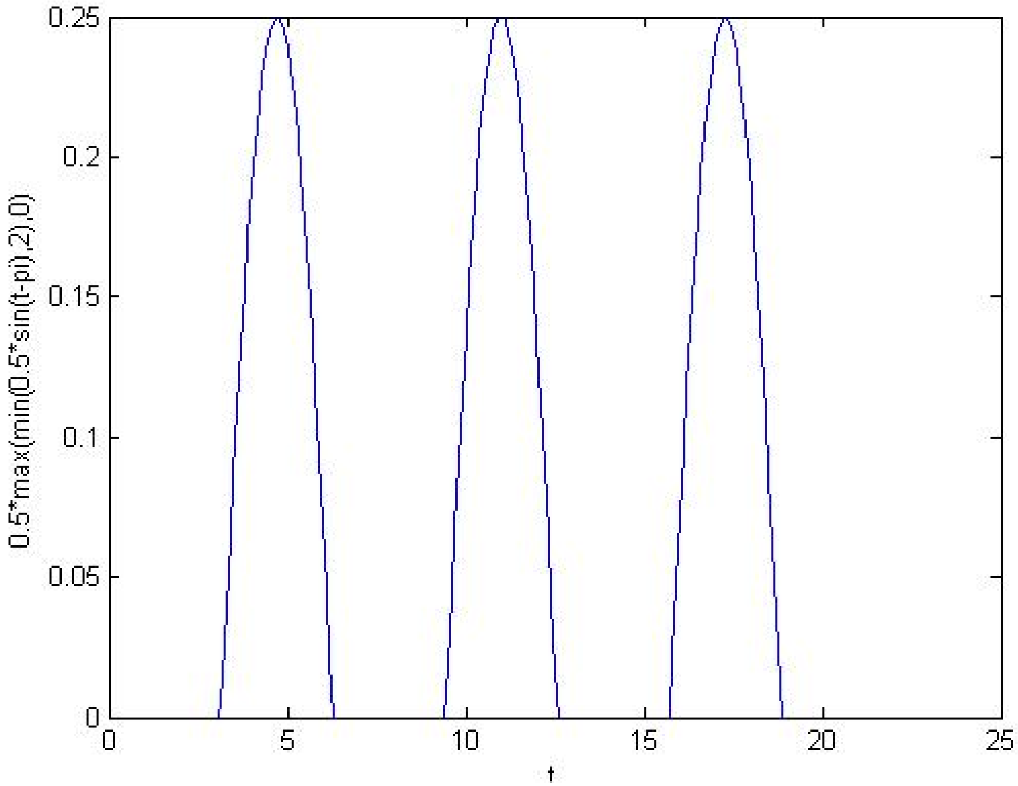
and $u(x,t) > 0$ *on the set* $(-L,L) \times (t_1,t_2)$*.*

3. Some numerical experiences

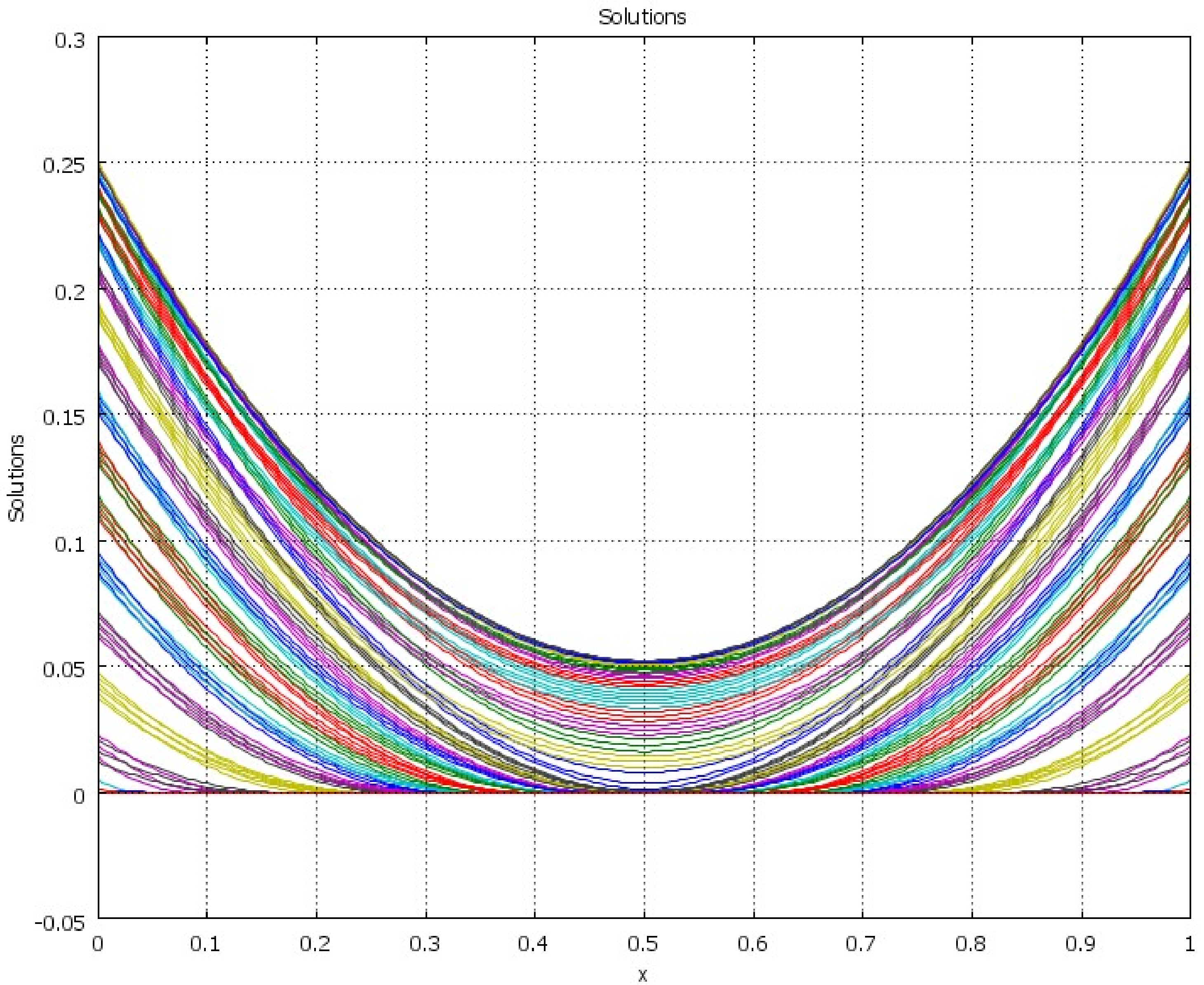
Disconnected support and reproductive periodic phenomena:

D-Ramos (2010).

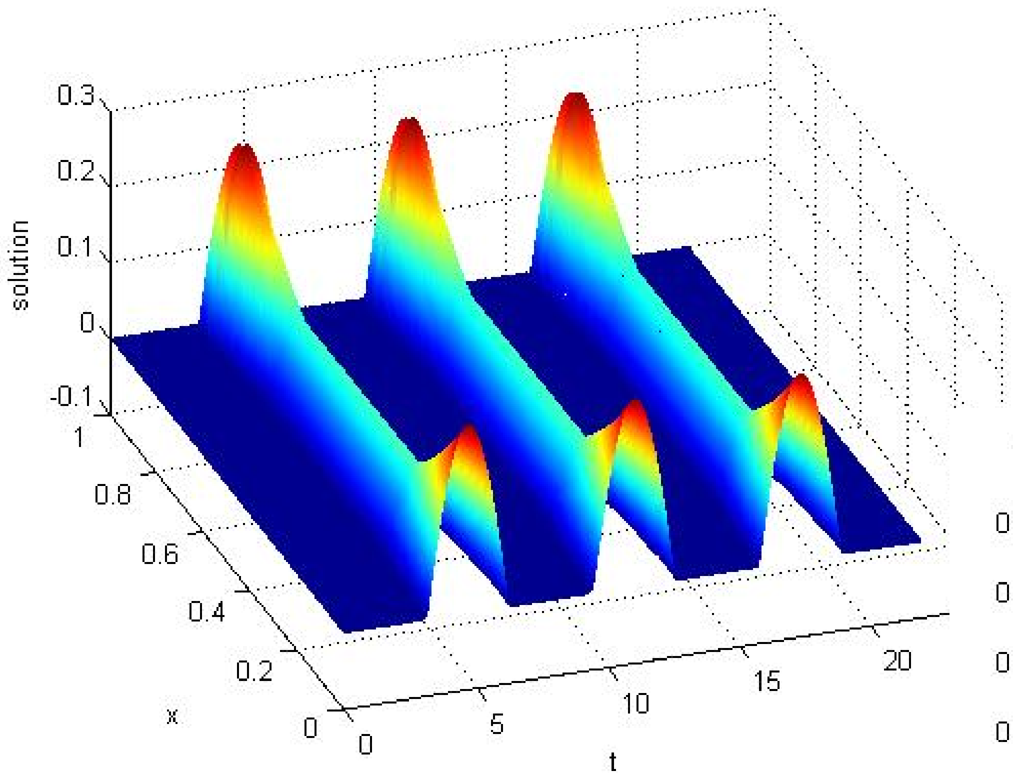
Motivation: ***Food Engineering*** (intermittent heat treatment to reduce the period of maturation of beef: Valin-Koop (1978),…, Grajales-Ruiz (2004))



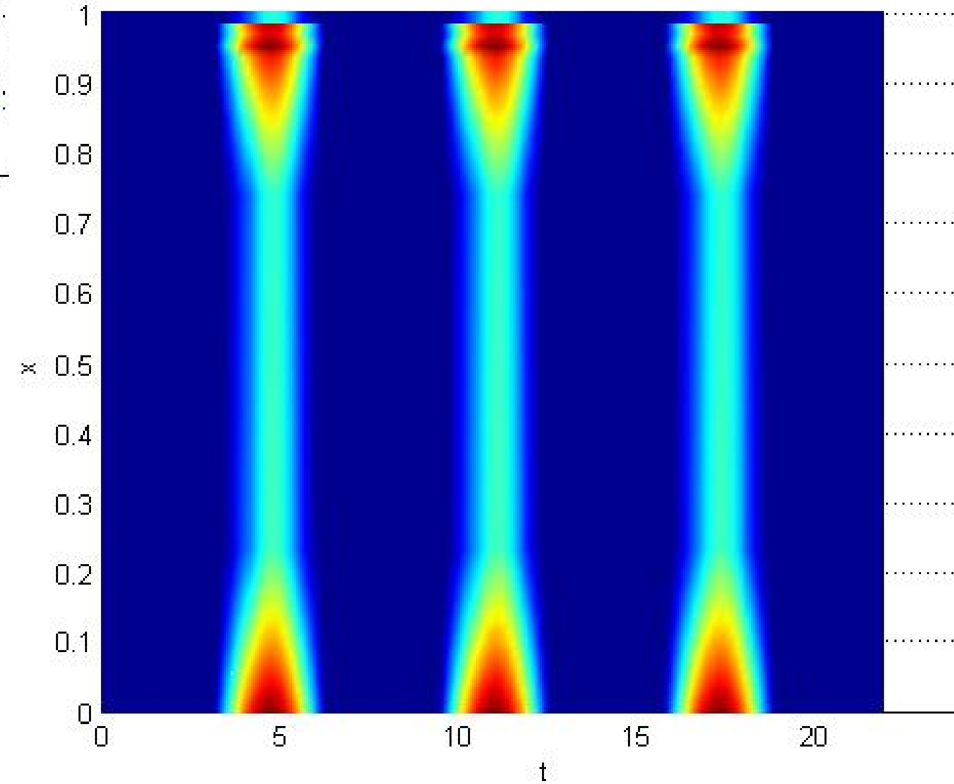
g(t,x)=0 and f'(0)$\uparrow\infty$


Solutions
0.3
0.25
0.2
0.15
0.1
0.05
0
-0.05
Solutions
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
x

Periodic solution: **reproductive phenomenon**



Disconnected support



Animation: periodic solution.

Thank you very much

He will be always alive in my memory