Remarks on the Monge–Ampère equation: some free boundary problems in Geometry

Gregorio Díaz and Jesús Ildefonso Díaz

Departamento de Matemática Aplicada. Universidad Complutense de Madrid
Parque de las Ciencias 3, 28040–Madrid. Spain
gdiaz@mat.ucm.es & ildefonso.diaz@mat.ucm.es

... of a beautiful friendship
Rick Blaine (Casablanca)

ABSTRACT

This paper deals with several qualitative properties of solutions of some stationary and parabolic equations associated to the Monge–Ampère operator. Mainly, we focus our attention in the occurrence of a free boundary (separating the region where the solution u is locally a hyperplane, and so were the Hessian D^2u is vanishing, from the rest of the domain). Among other things, we take advantage of these proceedings to give a detailed version of some results already announced long time ago when dealing with other fully nonlinear equations (see the 1979 paper by the authors on other parabolic equations [23], Remark 2.25 of the 1985 monograph by the second author [26] and the 1985 paper by the first author [20]). In particular, our results apply to suitable formulations of the Gauss curvature flow and of the worn stones problems intensively studied in the literature.

Key words: Elliptic and parabolic Monge–Ampère equation, Gauss curvatures with flat regions, free boundary problem

2010 Mathematics Subject Classification: 35J96, 35K96, 35R35, 53C45

1. Introduction

Since the pioneering works by Gaspard Monge, Comte de Peluse, (1746-1818) and André–Marie Ampère (1775- 1836), among others, it is well known that Geometry has been an extremely rich source of interesting problems in partial differential equations (see, e.g. [37] and [5]). Many properties of the most varied geometric structures are studied trough suitable differential equations as it has been explained in many classical books as, for instance, the famous Goursat treatises [31] dating from the last years of the XIX century.

The research of the authors was partially supported the Research Group MOMAT (Ref. 910480) supported by UCM and by the project MTM2011-26119, (DGISPI, Spain). The research of J.I.D. has received funding from the ITN FIRST of the Seventh Framework Programme of the European Community. (Grant Agreement number 238702).

Here we shall concentrate our attention in several second order partial differential equations involving the Hessian determinant (what we could call as the Monge-Ampère operator) of the scalar unknown function u. Several concrete problems can be mentioned as source of our motivations. For instance, we can mention the series of works by L. Nirenberg and coauthors starting around the middle of the past century (see *e.g.* Nirenberg [38]) on some geometric problems, as isometric embedding whose most familiar one is the classical Minkowski problem, in which the Monge–Ampère equation arises in presence of a nonlinear perturbation term on the own unknown u. Nevertheless, today it is well-known that the Monge–Ampère operator has many applications, not only in Geometry, but also in applied areas: optimal transportation, optimal design of antenna arrays, vision, statistical mechanics, front formation in meteorology, financial mathematics (see *e.g.* the references [4, 10, 29, 44], mainly for optimal transportation). But, in fact, we shall formulate the parabolic and elliptic problems of this paper in connection to a different problem which attracted the attention of many authors since 1974: the shape of worn stones.

Such as it was shown by Fiery ([28]), in 1974, the idealized wearing process for a convex stone, isotropic with respect to wear, can be described by

$$\frac{\partial \mathbf{P}}{\partial t} = \mathrm{K}^{\mathrm{p}}\mathbf{n}$$

where the points $\mathbf{P}$ of the N-dimensional convex hyper-surface $\Sigma^{\mathrm{N}}(t)$ embedded in $\mathbb{R}^{\mathrm{N}+1}$ (in the physical case, $\mathrm{N} = 3$) under Gauss curvature flow K with exponent $\mathrm{p} > 0$ moves in the inward direction $\mathbf{n}$ to the surface with velocity equal to the p–power of its Gaussian curvature (see also the important paper [34]). In the special case in which we express locally the surface $\Sigma^{\mathrm{N}}(t)$ as a graph $x_{\mathrm{N}+1} = u(x,t)$, with $x \in \Omega$, a convex open set of $\mathbb{R}^{\mathrm{N}}$, then the function u satisfies the parabolic Monge–Ampère equation

$$u_t = \frac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{\big(1 + |\mathrm{D}u|^2\big)^{\frac{(\mathrm{N}+2)\mathrm{p}-1}{2}}}.$$

As a matter of fact, to simplify the exposition, in this paper we shall assume always that this expression is not only local but global. Moreover the exact form of the above denominator will not be relevant (once we assume some suitable conditions). Then, our global formulation will be the following: given a bounded open set Ω of $\mathbb{R}^{\mathrm{N}}$, a continuous function φ on $\partial\Omega$, a locally convex function u_0 on Ω, $\mathrm{p} > 0$, and a continuous function $g \in \mathcal{C}([0, +\infty))$ such that

$$g(s) \geq 1 \text{ for any } s \geq 0, \tag{1}$$

find a convex function satisfying, in some sense to be defined, the problem

$$\begin{cases} u_t = \dfrac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{g(|\mathrm{D}u|)} & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = \varphi(x), & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x). & x \in \Omega. \end{cases} \tag{2}$$

In what follows the power like nonlinearity $f(t) = t^{\mathrm{p}}$ and its inverse function $f^{-1}(t) = t^{\frac{1}{\mathrm{p}}}$ must be understood as the restriction to $\overline{\mathbb{R}}_+$ of the odd functions $f(t) = |t|^{\mathrm{p}-1}t$ and

$f^{-1}(t) = |t|^{\frac{1-\mathrm{p}}{\mathrm{p}}} t$, respectively. Also, in order to simplify the exposition we assumed that the boundary datum $\varphi(x)$ is time–independent. The details in which $\varphi(x) = \varphi(x,t)$ and the case in which the above representation of the hypersurface $\Sigma^{\mathrm{N}}(t)$ is merely local will be presented elsewhere.

One of our goals is to prove that the above problem can be solved thanks to the semigroup theory for accretive operators $\mathcal{A}$ by applying the Crandall–Liggett generation theorem (see *e.g.* [15]) over the Banach space $\mathbb{X} = \mathcal{C}(\overline{\Omega})$ equipped with the supremum norm. Indeed, we shall show (see Section 4) that problem (2) can be regarded as the Cauchy problem

$$\begin{cases} u_t + \mathcal{A}u = 0 & t > 0, \\ u(0) = u_0, \end{cases}$$

for a suitable definition of the operator $\mathcal{A}$ which, at least formally, is given by

$$\mathcal{A}u = -\frac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{g(|\mathrm{D}u|)},$$

where $u \in \mathcal{C}^2$ is a locally convex function on $\overline{\Omega}$ and $u = \varphi$ on the boundary $\partial\Omega$. We shall give a precise definition of the operator $\mathcal{A}$ and show that it is accretive and satisfies $R(\mathrm{I} + \varepsilon\mathcal{A}) \supset \overline{\mathrm{D}(\mathcal{A})}$ for any $\varepsilon > 0$. The so called *mild solution* u of the above Cauchy problem is found by solving the implicit Euler scheme

$$\frac{u_n - u_{n-1}}{\varepsilon} + \mathcal{A}u_n = 0, \quad \text{for } n \in \mathbb{N},$$

or

$$\det \mathrm{D}^2 u_n = \left(g\big(|\mathrm{D}u_n|\big) \frac{u_n - u_{n-1}}{\varepsilon} \right)^{\frac{1}{\mathrm{p}}} \quad \text{in } \Omega. \tag{3}$$

This is why among the many different formulations of elliptic problems to which we can apply our techniques we pay an special attention to the following stationary problem: with the above assumption on Ω, φ, p and g, find a convex function u satisfying, in some sense to be defined, the problem

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) \Big[\big(u - h\big)^{\frac{1}{\mathrm{p}}} \Big]_+ & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

where h is a given continuous function on $\overline{\Omega}$. Certainly if we want to return to (3) we must replace $g\big(|\mathrm{D}u|\big)$ by $\big(g\big(|\mathrm{D}u|\big)\big)^{\frac{1}{\mathrm{p}}}$. Since the Monge–Ampère operator is only elliptic on the set of symmetric definite positive matrices, a compatibility is required on the structure of the equation. In fact, the operator is degenerate elliptic on the symmetric definite nonnegative matrices (see the comments at the end of this Introduction). As it will be proved in Theorem 2.8 (see also Remark 2.9), the compatibility is based on

$$h \text{ is locally convex on } \overline{\Omega} \text{ and } h \le \varphi \text{ on } \partial\Omega. \tag{4}$$

We also emphasize that if $\mathrm{Np} \le 1$ and $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ then the problem (5) is elliptic non degenerate in path-connected open sets Ω, as it is deduced from our Corollary 3.11.

The paper is organized as follows. In Section 2 some weak maximum principles are obtained for the boundary value problem (5). The main consequence of the Weak Maximum Principle is the comparison result for which one deduces $h \leq u$ on $\overline{\Omega}$, provided (4), thus, h behaves as a kind of lower "obstacle" for the solution u (see Remark 2.9 below). Therefore, under (4) the problem becomes

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big)\big(u-h\big)^{\frac{1}{\mathrm{p}}} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where the usual restriction on the non negativity of the right hand side is here supplied by (4). In particular, the inequalities

$$u_0 \leq \ldots \leq u_{n-1} \leq u_n \leq \ldots \leq u \quad \text{on } \overline{\Omega} \tag{6}$$

hold for the scheme (3). We emphasize that since the right hand side of the equation needs not strictly positive in some region of Ω, the ellipticity of the Monge–Ampère operator and the regularity $\mathcal{C}^2$ of solutions cannot be "a priori" guaranteed. The so called "viscosity solutions" or the "generalized solutions" are adequate notions in order to remove the non-degeneracy hypothesis on the operator. In fact, it is shown in [33] for convex domains Ω that both notions coincide. By using the Weak Maximum Principle and well known methods we prove, in Theorem 2.8, the existence of a unique generalized solution of (5). By a simple reasoning we obtain estimates on the gradient $\mathrm{D}u$. Bounds for the second derivatives $\mathrm{D}^2 u$ can be deduced from (22) as we shall prove in [24] (see Remark 2.9).

Since $h \leq u$ holds on $\overline{\Omega}$, the junction $\mathcal{F}$ between the regions where $[u = h]$ and $[h < u]$ is a free boundary (it is not known a priori). This free boundary can be defined also as the boundary of the set of points $x \in \Omega$ for which $\det \mathrm{D}^2 u(x) > 0$. Obviously, since the interior of the regions $[u = h]$ and $[\det \mathrm{D}^2 u = 0]$ coincide, if $h \in \mathcal{C}^2$ we must have that $\mathrm{D}^2 h = 0$. Motivated by the applications, as well as by the structure of the equation, the occurrence and localization of a the free boundary is studied in Section 3 whenever $h(x)$ has flat regions

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \{x \in \Omega : h(x) = \langle \mathbf{p}_\alpha, x \rangle + a_\alpha,\ \mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}},\ a_\alpha \in \mathbb{R}\} \neq \emptyset,$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product in $\mathbb{R}^{\mathrm{N}}$. As it will be proved, the free boundary $\mathcal{F}$ does exist under two different kind of conditions on the data: a suitable behavior of zeroth order term ($\mathrm{Np} > 1$) and a suitable balance between the "size" of the regions of Ω where $h(x)$ is flat and the "size" of the data φ and h. For this last reason, we rewrite the equation making rise a positive parameter λ,

$$\det \mathrm{D}^2 u = \lambda g\big(|\mathrm{D}u|\big)\big(u-h\big)^{\frac{1}{\mathrm{p}}} \quad \text{in } \Omega. \tag{7}$$

We shall show here how the theory on free boundaries (essentially the boundary of the support of the solution u), developed for a class of quasilinear operators in divergence form, can be extended to the case of the solution of (7) inside of flat regions of h, where $u_h = u - h$ solves

$$\det \mathrm{D}^2 u_h = \lambda g\big(|\mathrm{D}u|\big) u_h^{\frac{1}{\mathrm{p}}}.$$

We send the reader to the exposition made in the monograph [26] for details and examples (among many other references on this topic in the literature we mention here the more recent monograph [39] and the paper [20] for the case of other fully nonlinear operators).

As it was suggested in [26] for the Monge–Ampère operator, the appearance of the free boundary is strongly based on the condition

$$\mathrm{Np} > 1. \tag{8}$$

In a more detailed version of this paper [24] we shall prove that (8) corresponds to the power like choice of the more general condition

$$\int_{0^+} \big(\mathrm{F}(t)\big)^{-\frac{1}{\mathrm{N}+1}} dt < \infty, \tag{9}$$

where $\mathrm{F}(t) = \displaystyle\int_0^t f(s)ds$, relative to when we replace the nonlinear expression $\big(u-h\big)^{\frac{1}{\mathrm{p}}}$ by $f(u-h)$ for a continuous increasing functions f satisfying $f(0) = 0$ (see [24]). Because the strict convexity must be removed, a critical size of the data is required, the parameter λ governs these kind of magnitude (see (33) below). For instance, it is satisfied if λ is large enough. In Theorem 3.3 below we prove the occurrence of the free boundary $\mathcal{F}$ and give some estimates on its localization. We also prove that if $h(x)$ growths moderately (in a suitable way) near the region where it ceases to be flat then the free boundary region associated to the flattens of u (*i.e.* the region where $u_h = u - h$ vanishes) may coincide with the own boundary of the set where h is flat (see Theorem 3.6). The estimates on the localization of the free boundary are optimal, in the class of nonlinearities $f(s)$ satisfying (9), as it will be proved in [24].

By means of a Strong Maximum Principle for u_h we prove that the condition $\mathrm{Np} > 1$ is a necessary condition for the existence of such free boundary (see Theorem 3.10, Corollary 3.11 and Remark 3.12 below). More precisely, we shall prove that under the condition $\mathrm{Np} \leq 1$, (or more general, if

$$\int_{0^+} \frac{dt}{\mathrm{F}(t)^{\frac{1}{\mathrm{N}+1}}} = \infty$$

as we prove in [24]) the solution cannot have any flat region. This can be regarded as an extension of [45] to the non divergence case (see also [20], [26] and [39]). As it was pointed out, the condition $\mathrm{Np} \leq 1$ implies the ellipticity non degenerate of the problem (5) under very simple assumptions, as $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ for path-connected open set Ω (see Corollary 3.11).

Section 4 is devoted to the study of the parabolic problem (2). We prove the existence and uniqueness of solution by means of the semigroup theory already mentioned. Among other things we also show that the existence conditions imply that $u_t \geq 0$ in some weak sense, as it follows from (6). As in the stationary case, we study the free boundary involved to the degeneracy of the equation once we assume that the initial datum u_0 have some flat regions. More precisely, we focus the attention on the evolution of the flat region of the solution. We prove that if $\mathrm{Np} > 1$ then the initial flat region persists at least for small times. Moreover, we study some conditions on u_0

which guarantee that, in fact, the flat region becomes static during a small time. It is the so called "finite waiting time" effect which was already considered in [14] under a different formulation. We also study the associated self–similar solution in order to show that any flat region must disappear after a time large enough. Concerning the asymptotic behavior for large t, if $\mathrm{Np} \geq 1$, it is proved that if a flat "obstacle" does not coincide with u_0 in a set with positive measure the same occurs in any $t > 0$. Thus the solution never is flat in these region (see (69)). In fact, we deduce this "flattened retention" from an asymptotic behavior (see Theorem 4.14). By the contrary, when

$$\mathrm{Np} < 1 \tag{10}$$

the solution becomes globally flat at a finite time by conciding with upper flat "obstacles" h (see Theorem 4.15 below). In [25] we extend this last result to general case in which we replace the nonlinear power function $f(t) = t^{\mathrm{p}}$ by a general continuous increasing function f satisfying $f(0) = 0$ and

$$\int_{0+} \frac{ds}{f^{-1}\big(s^{\mathrm{N}}\big)} < \infty. \tag{11}$$

We end this introduction by pointing out that our methods can be applied to the borderline cases for (9) and (11). This will be made in the future papers [24, 25] in which the Monge–Ampère operator is replaced by other nonlinear operators of the Hessian of the unknown such as the k^{th} elementary symmetric functions

$$\mathbb{S}_k[\lambda(\mathrm{D}^2 u)] = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq \mathrm{N}} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \leq k \leq \mathrm{N}, \tag{12}$$

where $\lambda(\mathrm{D}^2 u) = \big(\lambda_1, \ldots, \lambda_{\mathrm{N}}\big)$ are the eigenvalues of $\mathrm{D}^2 u$. Note that the case $k = 1$ corresponds to the Laplacian operator while it is a fully nonlinear operator for the other choices of k. The case $k = \mathrm{N}$ corresponds to the Monge–Ampère operator. Some other properties for the k^{th} elementary symmetric function (12) are studied by the authors in [21, 22, 24, 25].

2. On the notion of solutions and the weak maximum principle

Many previous expositions on the nature of the solutions can be found in the literature, see for instance the survey [42]. Here, we shall offer a short presentation of it. Certainly in the class of $\mathcal{C}^2$ convex functions, the Monge–Ampère operator $\mathbb{S}_{\mathrm{N}}[\lambda(\mathrm{D}^2 u)]$ (see (12)) is elliptic because the cofactor matrix of $\mathrm{D}^2 u$ is positive definite. So that, as it is proved by several methods in [11, 12, 24, 30, 32, 36, 40, 41, 42, 43], there exists a $\mathcal{C}^2$ convex solution of the general boundary value problems as

$$\begin{cases} \det \mathrm{D}^2 u = \mathrm{H}(\mathrm{D}u, u, x), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{13}$$

under suitable assumptions on Ω, $\mathrm{H} > 0$ and φ. A main question arises now both in theory and in applications: what happens if $\mathrm{H} \geq 0$. Certainly, the ellipticity degeneracy occurs and in general the regularity $\mathcal{C}^2$ of solutions cannot be guaranteed. The so called "viscosity solutions" or the "generalized solutions" are suitable notions in order to remove the degeneracy of the operator. In fact, it can be proved that for a convex domain Ω both notions coincide (see [33]). A short description of all that is as follows. By a change of variable we get

$$|\mathrm{D}u(\mathrm{E})| = \int_{\mathrm{E}} \det \mathrm{D}^2 u \, dx = \int_{\mathrm{E}} \mathrm{H}(\mathrm{D}u, u, x) dx \tag{14}$$

for any Borel set $\mathrm{E} \subset \Omega$, where the left hand side makes sense merely when $u \in \mathcal{C}^1$ is convex. By the structure of the problem, u must be convex on Ω and consequently u is at least locally Lipschitz. While for locally Lipschitz functions the right hand side of (14) is well defined, slight but careful modifications are needed to give sense to the left hand side. The progress in this direction is achieved thanks to the notion of subgradients of a convex function u: given $\mathbf{p} \in \mathbb{R}^{\mathrm{N}}$, we say

$$\mathbf{p} \in \partial u(x) \quad \text{iff} \quad u(y) \geq u(x) + \langle \mathbf{p}, y - x \rangle, \quad \text{for all } y \in \Omega. \tag{15}$$

Thus, we can define the Radon measure

$$\mu_u(\mathrm{E}) \doteq |\partial u(\mathrm{E})| = \mathrm{meas}\{\mathbf{p} \in \mathbb{R}^{\mathrm{N}} : \ \mathbf{p} \in \partial u(x) \text{ for some } x \in \mathrm{E}\}. \tag{16}$$

Since the pioneering works by Aleksandrov [1] several authors have contributed to the study of the above measure (see, for instance, [42]). Then we arrive to

Definition 2.1. A convex function u on Ω is a "generalized solution" of (13) if

$$\mu_u(\mathrm{E}) = \int_{\mathrm{E}} \mathrm{H}(\mathrm{D}u, u, x) dx$$

for any Borel set $\mathrm{E} \subset \Omega$.

The continuity on $\overline{\Omega}$ is compatible with the usual realization of the Dirichlet boundary condition. Obviously, the condition $\mathrm{H} \geq 0$ cannot be removed. Certainly, the definition, as well as (16), can be extended to locally convex functions u on Ω, for which u can be constant on some subset of Ω.

This notion of generalized solution is specific of the equations governed by the Monge–Ampère operator, but other notion of solutions are available for other type of fully nonlinear equations with non divergence form. The most usual is the so called "viscosity solution" introduced by M.G. Crandall and P.L. Lions (see the users guide [16])

Definition 2.2. A convex function u on Ω is a viscosity solution of the inequality

$$\det \mathrm{D}^2 u \geq \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega \qquad \text{(subsolution)}$$

if for every $\mathcal{C}^2$ convex function Φ on Ω for which

$$(u - \Phi)(x_0) \geq (u - \Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2 \Phi(x_0) \geq \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), x_0\big).$$

Analogously, one defines the viscosity solution of the reverse inequality

$$\det \mathrm{D}^2 u \le \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega \qquad \text{(supersolution)}$$

as a convex function u on Ω such that for every $\mathcal{C}^2$ convex function Φ on Ω for which

$$(u-\Phi)(x_0) \le (u-\Phi)(x) \quad \text{locally at } x_0 \in \Omega$$

one has

$$\det \mathrm{D}^2\Phi(x_0) \le \mathrm{H}\big(\mathrm{D}\Phi(x_0), u(x_0), (x_0)\big).$$

Finally, when both properties hold we arrive to the notion of viscosity solution of

$$\det \mathrm{D}^2 u = \mathrm{H}(\mathrm{D}u, u, x) \quad \text{in } \Omega.$$

Note that the convexity condition on u and Φ are extra assumptions with respect to the usual notion of viscosity solution (see [16]). This is needed here because the Monge–Ampère operator is only degenerate elliptic on this class of functions. In fact, it can be seen that the convexity on Φ is only required for the correct definition of super solutions in viscosity sense. One proves the equivalence

u is a generalized solution of (13) if and only if u is a viscosity solution of (13),

provided that Ω is a convex domain and $\mathrm{H} \in \mathcal{C}(\mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \Omega)$ (see [33]).

With this intrinsic way of solve (13) one may study some complementary regularity results. In particular, we may get back the notion of classical solution by means of the following consistence result

Theorem 2.3 ([11]). *Let u be a strictly convex generalized solution of* (13) *in a convex domain $\Omega \subset \mathbb{R}^{\mathrm{N}}$, where $\mathrm{H} \in \mathcal{C}^{0,\alpha}(\mathbb{R}^{\mathrm{N}} \times \mathbb{R} \times \Omega)$ is positive. Then $u \in \mathcal{C}^{2,\alpha'}(\Omega) \cap \mathcal{C}^{1,1}(\overline{\Omega})$, for some $\alpha' \in]0,1[$, and u solves* (13) *in the classical sense.* □

We continue this section with the study of some comparison and existence results for the equation (7). In order to simplify the exposition, we only consider here the case $\lambda = 1$. All results of this section apply to the case of general increasing functions $f \in \mathcal{C}(\mathbb{R})$ satisfying $f(0) = 0$

$$\det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f(u-h) \quad \text{in } \Omega.$$

We begin by showing that the nature of the viscosity solution is intrinsic to the Maximum Principle.

Proposition 2.4 (Weak Maximum Principle I)**.** *Let $h_1, h_2 \in \mathcal{C}(\overline{\Omega})$. Let $u_2 \in \mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$ be a classical solution of*

$$-\det \mathrm{D}^2 u_2 + g\big(|\mathrm{D}u_2|\big) f(u_2 - h_2) \ge 0 \quad \textit{in } \Omega,$$

and let $u_1 \in \mathcal{C}(\overline{\Omega})$ be a convex viscosity solution of

$$-\det \mathrm{D}^2 u_1 + g\big(|\mathrm{D}u_1|\big) f(u_1 - h_1) \le 0 \quad \textit{in } \Omega.$$

Then one has

$$(u_1 - u_2)(x) \le \sup_{\partial\Omega} \big[u_1 - u_2\big]_+ + \sup_{\Omega} \big[h_1 - h_2\big]_+, \quad x \in \Omega.$$

Proof. By continuity there exists $x_0 \in \overline{\Omega}$ where $[u_1 - u_2]_+$ achieves the maximum value on $\overline{\Omega}$. We only consider the case $x_0 \in \Omega$ and $[u_1 - u_2]_+(x_0) > 0$, because otherwise the result follows. Then from the applications of the definition of viscosity solution for u_1 we can take $\Phi = u_2$ and so we deduce

$$\begin{aligned}0 \geq & - \det \mathrm{D}^2 u_2(x_0) + g\big(|\mathrm{D}u_2(x_0)|\big) f(u_1(x_0) - h_1(x_0)) \\ \geq & \, g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_1(x_0) - h_1(x_0)\big) - g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_2(x_0) - h_1(x_0)\big).\end{aligned}$$

Then, since f is increasing

$$(u_1 - u_2)(x_0) \leq \big(h_1 - h_2\big)(x_0) \leq \sup_{\partial\Omega} \big[u_1 - u_2\big]_+ + \sup_{\Omega} [h_1 - h_2]_+.$$

□

Remark 2.5. We note that the monotonicity on the zeroth order terms is the only assumption required on the structure of the equation and that our argument is strongly based on the notion of viscosity solution. An analogous estimate holds by changing the roles of u_1 and u_2 (but then we do not require the $\mathcal{C}^2$ function u_1 to be convex). Note also that we did not assume any convexity condition on the domain Ω. When Ω is convex these results can be extended to the class of the generalized solutions through the mentioned equivalence between such solution and the viscosity solutions. In [24] we extend Proposition 2.4 to non decreasing functions f. □

A very simple (and important fact) was used in our precedent arguments: if $u_1 \in \mathcal{C}^2$ and $u_2 - u_1 \in \mathcal{C}^2$ are convex functions on a ball $\mathbf{B}$ then

$$\det \mathrm{D}^2 u_2 \geq \det \mathrm{D}^2 u_1 \quad \text{in } \mathbf{B}.$$

This simple inequality can be extended to the case u_1 and $u_2 - u_1$ convex function on a ball $\mathbf{B}$, with $u_1 = u_2$ on $\partial\mathbf{B}$, by the "monotonicity formula"

$$\mu_{u_2}(\mathbf{B}) \leq \mu_{u_2}(\mathbf{B}) \tag{17}$$

(see [42]). So that, the Weak Maximum Principle can be extended to the class of generalized solutions

Theorem 2.6 (Weak Maximum Principle II)**.** *Let $h_1, h_2 \in \mathcal{C}(\overline{\Omega})$. Let $u_1, u_2 \in \mathcal{C}(\overline{\Omega})$ where u_1 is locally convex in Ω. Suppose*

$$- \det \mathrm{D}^2 u_1 + g\big(|\mathrm{D}u_1|\big) f\big(u_1 - h_1\big) \leq - \det \mathrm{D}^2 u_2 + g\big(|\mathrm{D}u_2|\big) f\big(u_2 - h_2\big) \quad \text{in } \Omega \tag{18}$$

in the generalized solution sense. Then

$$(u_1 - u_2)(x) \leq \sup_{\partial\Omega} [u_1 - u_2]_+ + \sup_{\Omega} [h_1 - h_2]_+, \quad x \in \Omega. \tag{19}$$

In particular,

$$|u_1 - u_2|(x) \leq \sup_{\partial\Omega} |u_1 - u_2| + \sup_{\Omega} |h_1 - h_2|, \quad x \in \Omega, \tag{20}$$

whenever the equality holds in (18).

Proof. As above, we only consider the case where the maximum of $[u_1 - u_2]_+$ on $\overline{\Omega}$ is achieved at some $x_0 \in \Omega$ with $[u_1 - u_2]_+(x_0) > 0$. Therefore, $\big(u_1 - u_2\big)(x) > 0$ and convex in a ball $\mathbf{B}_{\mathrm{R}}(x_0)$, R small. Let $\Omega^+ = \{u_1 > u_2\} \supseteq \mathbf{B}_{\mathrm{R}}(x_0)$. We construct $\widehat{u}_1(x) = u_1(x) + \gamma\big(|x - x_0|^2 - \mathrm{M}^2\big) - \delta$, where $\mathrm{M} > 0$ is large and $\gamma, \delta > 0$ such that $\widehat{u}_1 < u_1$ on $\partial\Omega^+$ and the set $\Omega^+_{\gamma,\delta} = \{\widehat{u}_1 > u_2\}$ is compactly contained in Ω and contains $\mathbf{B}_\varepsilon(x_0)$ for some ε small. By choosing γ, δ properly, we can assume that the diameter of $\Omega^+_{\gamma,\delta}$ is small so that u_1, and therefore $u_2 = (u_2 - u_1) + u_1$, are convex in it. Then (17) implies

$$\begin{aligned} 0 < (\gamma\varepsilon)^{\mathrm{N}}|\mathbf{B}_1(0)| &\leq \mu_{u_2}\big(\mathbf{B}_\varepsilon(x_0)\big) - \mu_{u_1}\big(\mathbf{B}_\varepsilon(x_0)\big) \\ &\leq \int_{\mathbf{B}_\varepsilon(x_0)} \big[g\big(|\mathrm{D}u_2|\big) f\big(u_2 - h_2\big) - g\big(|\mathrm{D}u_1|\big) f\big(u_1 - h_1\big)\big] dx. \end{aligned}$$

Since $g\big(|\mathrm{D}u_1(x_0)|\big) = g\big(|\mathrm{D}u_2(x_0)|\big) > 0$ (see Remark 2.7 below), by letting $\varepsilon \to 0$, the Lebesgue differentiation theorem implies

$$0 \leq g\big(|\mathrm{D}u_2(x_0)|\big) f\big(u_2(x_0) - h_2(x_0)\big) - g\big(|\mathrm{D}u_1(x_0)|\big) f\big(u_1(x_0) - h_1(x_0)\big),$$

whence

$$\big(u_1 - u_2\big)(x_0) < \big(h_1 - h_2\big)(x_0) \leq \sup_{\partial\Omega}\big[u_1 - u_2\big]_+ + \sup_{\Omega}\big[h_1 - h_2\big]_+$$

concludes the estimates. □

Remark 2.7. The above proof requires a simple fact, any convex function ψ in a convex open set $\mathcal{O} \subset \mathbb{R}^{\mathrm{N}}$ such that it achieves a local interior maximum at some $z_0 \in \mathcal{O}$ verifies $\mathrm{D}\psi(z_0) = \mathbf{0}$. Indeed, for any $\mathbf{p} \in \partial\psi(z_0)$ one has

$$\psi(x) \geq \psi(z_0) + \langle \mathbf{p}, x - z_0\rangle \geq \psi(x) + \langle \mathbf{p}, x - z_0\rangle \quad \text{with } x \text{ near } z_0,$$

thus

$$\langle \mathbf{p}, x - z_0\rangle \geq 0.$$

Then if $\tau > 0$ is small enough we may choose $x - z_0 = -\tau\mathbf{p} \in \mathcal{O}$ and deduce the contradiction

$$\tau|\mathbf{p}|^2 \leq 0.$$

□

A first consequence of the general theory for (13) and the Weak Maximum Principle is the following existence result

Theorem 2.8. *Let $\varphi \in \mathcal{C}(\partial\Omega)$ and assume the compatibility condition* (4). *Then there exists a unique locally convex function verifying*

$$\begin{cases} \det \mathrm{D}^2 u = g\big(|\mathrm{D}u|\big) f(u - h) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

in the generalized sense. In fact, one verifies

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}, \tag{21}$$

where U_φ is the harmonic function in Ω with $\mathrm{U}_\varphi = \varphi$ on $\partial\Omega$.

Proof. First we consider the generalized solution of the problem

$$\begin{cases} -\det \mathrm{D}^2 u + g\big(|\mathrm{D}u|\big)\big[f(u-h)\big]_+ = 0 & \text{in } \Omega. \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Since $\mathrm{H}\big(\mathrm{D}u, u, x\big) = g\big(|\mathrm{D}u|\big)\big[f(u-h)\big]_+ \geq 0$ we can apply well known results in the literature. In particular, from [43], it follows the existence and uniqueness of the solution u. The second point is to note that, by construction, the own locally convex function h verifies

$$-\det \mathrm{D}^2 h + g\big(|\mathrm{D}u|\big)\big[f(h-h)\big]_+ \leq 0 \quad \text{in } \Omega.$$

Therefore, by the Weak Maximum Principle and the assumption $h \leq \varphi$ on $\partial\Omega$ we get that

$$h \leq u \quad \text{in } \Omega,$$

whence

$$\big[f(u-h)\big]_+ = f(u-h)$$

concludes the existence. The uniqueness also follows from the Weak Maximum Principle. Finally, since u is locally convex, the arithmetic–geometric mean inequality lead to

$$0 \leq \det \mathrm{D}^2 u \leq \frac{1}{\mathrm{N}} \left(\Delta u\right)^{\mathrm{N}} \quad \text{in } \Omega,$$

whence the estimate

$$h(x) \leq u(x) \leq \mathrm{U}_\varphi(x), \quad x \in \overline{\Omega}$$

is completed by the weak maximum principe for harmonic functions. □

Remark 2.9. i) As it was pointed out in the Introduction, no sign assumption on h is required in Theorem 2.8. The simple structural assumption (4) implies that $h \leq u$ on $\overline{\Omega}$ and therefore the ellipticity, eventually degenerate, of the equation holds. Thus, the ellipticity holds once h behaves as a lower "obstacle" for the solution u. We note that these compatibility conditions are not required a priori in the Weak Maximum Principles because there we are working with functions whose existence is a priori assumed.
ii) Since u is locally convex on $\overline{\Omega}$, we can prove

$$\sup_{\Omega} |\mathrm{D}u| = \sup_{\partial\Omega} |\mathrm{D}u|,$$

(see [24]) then inequality (21) gives a priori bounds on $|\mathrm{D}u|$ on $\overline{\Omega}$, provided $h = \varphi$ on $\partial\Omega$ and $\mathrm{D}h$ is defined on $\partial\Omega$. The second derivative estimate is based on the inequality

$$\sup_{\Omega} |\mathrm{D}^2 u| \leq \mathrm{C}\left(1 + \sup_{\partial\Omega} |\mathrm{D}^2 u|\right) \tag{22}$$

for some constant C independent on u, as it will be proved in [24]. □

In the next section we prove a kind of Strong Maximum Principle which under suitable assumptions will avoid the appearance of the mentioned free boundary.

3. Flat regions in the stationary problem

In this section we focus the attention to a lower "obstacle" function h locally convex on $\overline{\Omega}$ having some region giving rise to the set

$$\mathrm{Flat}(h) = \bigcup_{\alpha} \mathrm{Flat}_{\alpha}(h)$$

where

$$\mathrm{Flat}_{\alpha}(h) = \{x \in \overline{\Omega} : h(x) = \langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha}, \text{ for some } \mathbf{p}_{\alpha} \in \mathbb{R}^{\mathrm{N}} \text{ and } a_{\alpha} \in \mathbb{R}\}. \tag{23}$$

Since

$$u(y) - \big(\langle \mathbf{p}_{\alpha}, y \rangle + a_{\alpha}\big) \geq u(x) - \big(\langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha}\big) + \langle \mathbf{p} - \mathbf{p}_{\alpha}, y - x \rangle,$$

thus

$$\mathbf{p} \in \partial u(x) \quad \Leftrightarrow \quad \mathbf{p} - \mathbf{p}_{\alpha} \in \partial \left(u(x) - \big(\langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha}\big)\right),$$

the equation (7) becomes

$$\det \mathrm{D}^2 u_{\alpha} = \lambda g\big(|\mathrm{D}u|\big) u_{\alpha}^{\frac{1}{\mathrm{p}}}, \quad x \in \mathrm{Flat}_{\alpha}(h), \tag{24}$$

for $u_{\alpha} = u - \big(\langle \mathbf{p}_{\alpha}, x \rangle + a_{\alpha}\big)$. Remember that $u_{\alpha} \geq 0$ in an open set $\mathcal{O} \subseteq \Omega$, if $u_h \geq 0$ on $\partial\mathcal{O}$. Assumption $g(|\mathbf{p}|) \geq 1$ leads us to study for the auxiliar problem

$$\begin{cases} \det \mathrm{D}^2 \mathrm{U} = \lambda \mathrm{U}^{\frac{1}{\mathrm{p}}} & \text{in } \mathbf{B}_{\mathrm{R}}(0), \\ \mathrm{U} \equiv \mathrm{M} > 0 & \text{on } \partial \mathbf{B}_{\mathrm{R}}(0), \end{cases} \tag{25}$$

for any $\mathrm{M} > 0$. From the uniqueness of solutions, it follows that U is radially symmetric, because by rotating it we would find another solutions. Moreover, by the comparison results U is nonnegative. Therefore, the solution U is governed by a nonnegative radial profile function $\mathrm{U}(x) = \widehat{\mathrm{U}}(|x|)$ for which some straightforward computations leads to

$$\det \mathrm{D}^2 \mathrm{U}(x) = \widehat{\mathrm{U}}''(r) \left(\frac{\widehat{\mathrm{U}}'(r)}{r} \right)^{\mathrm{N}-1} = \frac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\big(\widehat{\mathrm{U}}'(r)\big)^{\mathrm{N}} \right]'. \tag{26}$$

Remark 3.1. For $\mathrm{N} = 1$, equation (25) becomes

$$\widehat{\mathrm{U}}''(r) = \lambda \widehat{\mathrm{U}}^{\frac{1}{\mathrm{p}}}$$

whose annulation set was studied in [26]. Note that for $\mathrm{N} > 1$ equation (26) does not coincide with the $(\mathrm{N}-1)$–Laplacian considered in [26]. □

We start by considering the initial value problem

$$\begin{cases} \dfrac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\big(\mathrm{U}'(r)\big)^{\mathrm{N}} \right]' = \lambda \big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{p}}}, \quad \lambda > 0 \\ \mathrm{U}(0) = \mathrm{U}'(0) = 0. \end{cases} \tag{27}$$

Obviously, $\mathrm{U}(r) \equiv 0$ is always a solution, but we are interested in the existence of nontrivial and non–negative solutions. It will be useful the following result

Lemma 3.2. *Assume* $\mathrm{Np} > 1$. *Consider the function*

$$\mathrm{U}(r) = \mathrm{C} r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0, \tag{28}$$

where C *is a positive constant. Let*

$$\mathrm{C}_{\mathrm{p,N}} = \left(\frac{(\mathrm{Np}-1)^{\mathrm{N}+1}}{(2\mathrm{Np})^{\mathrm{N}}(\mathrm{Np}+1)} \right)^{\frac{\mathrm{p}}{\mathrm{Np}-1}}. \tag{29}$$

Then,

$$-\frac{r^{1-\mathrm{N}}}{\mathrm{N}} \left[\big(\mathrm{U}'(r)\big)^{\mathrm{N}} \right] + \lambda \big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{p}}} = \left[\lambda - \left(\frac{\mathrm{C}}{\mathrm{C}_{\mathrm{p,N}}} \right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}} \right] \mathrm{C}^{\frac{1}{\mathrm{p}}} r^{\frac{2\mathrm{N}}{\mathrm{Np}-1}}. \tag{30}$$

Therefore,

(i) *if* $\mathrm{C} < \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}}$ *the function* $\mathrm{U}(r)$ *is a supersolution of the equation* (27),

(ii) *if* $\mathrm{C} = \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}}$ *the function* $\mathrm{U}(r)$ *is the solution of the equation* (27),

(iii) *if* $\mathrm{C} > \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}}$ *the function* $\mathrm{U}(r)$ *is a subsolution of the equation* (27).

Proof. The conclusions are immediate by some straightforward computations on the function $\mathrm{U}(r)$. □

Since $\mathrm{Np} > 1$, the function

$$\mathrm{U}(r) = \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C}_{\mathrm{p,N}} r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0, \tag{31}$$

enables us to construct functions vanishing in a ball $\mathbf{B}_\tau(0)$

$$v_\tau(x) \doteq \mathrm{U}\big([|x| - \tau]_+\big), \quad x \in \mathbb{R}^{\mathrm{N}}, \tag{32}$$

which solves

$$-\det \mathrm{D}^2 v_\tau(x) + \lambda \big(v_\tau(x)\big)^{\frac{1}{\mathrm{p}}} = 0, \quad x \in \mathbb{R}^{\mathrm{N}}.$$

Moreover, given $\mathrm{M} > 0$, it verifies

$$v_\tau(x) = \mathrm{M}, \quad |x| = \mathrm{R}$$

once we take

$$\tau = \mathrm{R} - \mathrm{U}^{-1}(\mathrm{M}) = \left(\frac{\mathrm{M}}{\mathrm{C}_{\mathrm{p,N}}} \right)^{\frac{\mathrm{Np}-1}{2\mathrm{Np}}} \left[\lambda_*^{-\frac{1}{2\mathrm{N}}} - \lambda^{-\frac{1}{2\mathrm{N}}} \right]$$

with

$$\lambda \geq \lambda_* \doteq \frac{1}{\mathrm{R}^{2\mathrm{N}}} \left(\frac{\mathrm{M}}{\mathrm{C}_{\mathrm{p,N}}} \right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}}. \tag{33}$$

Now for the solution of (5) we may localize a core of the flat region $\mathrm{Flat}(u)$ inside the flat subregion $\mathrm{Flat}_\alpha(h)$ of the "obstacle".

Theorem 3.3. *Let h be locally convex on $\overline{\Omega}$. Let us assume that there exists $\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_\alpha(h)$ with*

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \mathrm{M} \leq \max_{\overline{\Omega}}(u-h), \quad x \in \partial \mathbf{B}_{\mathrm{R}}(x_0), \tag{34}$$

where u is a generalized solution of (7)*, for some* $\mathrm{M} > 0$*. Then, if* $\mathrm{Np} > 1$ *and*

$$\lambda \geq \lambda^* \doteq \frac{1}{\mathrm{R}^{2\mathrm{N}}}\left(\frac{\mathrm{M}}{\mathrm{C_{p,N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}},$$

one verifies

$$0 \leq u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \leq \lambda^{\frac{\mathrm{p}}{\mathrm{Np}-1}} \mathrm{C_{p,N}}\big(\big[|x-x_0| - \tau\big]_+\big)^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0), \tag{35}$$

where

$$\tau = \left(\frac{\mathrm{M}}{\mathrm{C_{p,N}}}\right)^{\frac{\mathrm{Np}-1}{2\mathrm{Np}}} \left[\lambda_*^{-\frac{1}{2\mathrm{N}}} - \lambda^{-\frac{1}{2\mathrm{N}}}\right], \tag{36}$$

once we assume that

$$\left(\frac{\mathrm{M}}{\mathrm{C_{p,N}}}\right)^{\frac{\mathrm{Np}-1}{2\mathrm{Np}}} \lambda^{-\frac{1}{2\mathrm{N}}} < \mathrm{R} \leq \mathrm{dist}(x_0, \partial\Omega). \tag{37}$$

In particular, the function u is flat on $\overline{\mathbf{B}}_\tau(x_0)$. More precisely,

$$u(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha \quad \textit{for any } x \in \overline{\mathbf{B}}_\tau(x_0).$$

Proof. The result is a direct consequence of previous arguments. Indeed, for simplicity we can assume $x_0 = 0$. Since $g(|\mathbf{p}|) \geq 1$, by the comparison results we get that

$$0 \leq u_\alpha(x) \leq v_\tau(x), \quad x \in \mathbf{B}_{\mathrm{R}}(0)$$

(see (24) and (32)) and so the conclusions hold. □

Remark 3.4. We have proved that under the above assumptions the flat region of u is a non–empty set. Obviously, $\mathrm{Flat}(h) \subset \mathrm{Flat}(u)$ whenever (34) fails, even if $\mathrm{Np} > 1$. We shall examine the optimality of (35) in [24] following different strategies carry out in [26] for other free boundary problems. □

Remark 3.5. We point out that the above result applies to the case in which $\varphi \equiv 1$ and $h \equiv 0$ (the so called "dead core" problem) as well as to cases in which u is flat only near $\partial\Omega$ (take for instance, $h(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha$ in Ω and $\varphi \equiv h$ on $\partial\Omega$). □

Theorem 3.3 gives some estimates on the localization of the points inside $\mathrm{Flat}(h)$ where u becomes flat too. The following result shows that if h decays in a suitable way at the boundary points of $\mathrm{Flat}(h)$ then the solution u becomes also flat in those points of the boundary of $\mathrm{Flat}(h)$. In this result the parameter λ is irrelevant, therefore with no loss of generality we shall assume that $\lambda = 1$.

Theorem 3.6. *Let us assume* $\mathrm{Np} > 1$. *Let* $x_0 \in \partial\mathrm{Flat}_\alpha(h)$ *such that*

$$h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_\mathrm{R}(x_0) \cap \big(\mathbb{R}^\mathrm{N} \setminus \mathrm{Flat}(h)\big), \tag{38}$$

and

$$0 \le \max_{|x-x_0|=\mathrm{R}} \big\{u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big\} \le \mathrm{C}\mathrm{R}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \tag{39}$$

for some suitable positive constants K *and* C *(see* (41) below*) and* u *is a generalized solution of* (7). *Then*

$$u(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha. \tag{40}$$

Proof. Define the function

$$\mathrm{V}(x) = u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big),$$

which by construction is nonnegative in $\partial\mathbf{B}_\mathrm{R}(x_0)$ (see (39)). In fact, the Weak Maximum Principle implies that V is non negative on $\overline{\mathbf{B}}_\mathrm{R}(x_0)$. Then

$$\begin{aligned}
-\big(\det \mathrm{D}^2\mathrm{V}(x)\big)^{\frac{1}{\mathrm{N}}} + \big(\big(\mathrm{V}(x)\big)^{\frac{1}{\mathrm{Np}}} &= -\big(\det \mathrm{D}^2 u(x)\big)^{\frac{1}{\mathrm{N}}} + \big(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{1}{\mathrm{Np}}} \\
&= -\big(u(x) - h(x)\big)^{\frac{1}{\mathrm{Np}}} + \big(u(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{1}{\mathrm{Np}}} \\
&\le \big(h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)\big)^{\frac{1}{\mathrm{Np}}} \\
&\le \mathrm{K}^{\frac{1}{\mathrm{Np}}}|x - x_0|^{\frac{2\mathrm{p}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_\mathrm{R}(x_0),
\end{aligned}$$

where we have used a kind of Minkovsky inequality (see appendix)

$$(a+b)^{\frac{1}{\mathrm{Np}}} \le a^{\frac{1}{\mathrm{Np}}} + b^{\frac{1}{\mathrm{Np}}}, \ a, b \ge 0, \quad \text{where } \mathrm{Np} > 1,$$

as well as (38). On the other hand, since the function

$$\mathrm{U}_\mathrm{p}(r) = \mathrm{C}_{\mathrm{p},\mathrm{N}} r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \ge 0,$$

verifies

$$\left(\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\mathrm{U}_\mathrm{p}'(r)\big)^\mathrm{N}\right]\right)^{\frac{1}{\mathrm{N}}} = \big(\mathrm{U}_\mathrm{p}(r)\big)^{\frac{1}{\mathrm{Np}}}$$

(see (30)), we have

$$-\left(\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\mathrm{U}'(r)\big)^\mathrm{N}\right]'\right)^{\frac{1}{\mathrm{N}}} + \big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{Np}}} = \left[1 - \left(\frac{\mathrm{C}}{\mathrm{C}_{\mathrm{p},\mathrm{N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{N}^2\mathrm{p}}}\right]\mathrm{C}^{\frac{1}{\mathrm{N}}} r^{\frac{2\mathrm{p}}{\mathrm{Np}-1}}$$

for $\mathrm{U}(r) = \mathrm{C}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}$. Hence, if we take $\mathrm{C} < \mathrm{C}_{\mathrm{p},\mathrm{N}}$ and then K such that

$$\mathrm{K}^{\frac{1}{\mathrm{Np}}} \le \mathrm{C}^{\frac{1}{\mathrm{N}}}\left[1 - \left(\frac{\mathrm{C}}{\mathrm{C}_{\mathrm{p},\mathrm{N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{N}^2\mathrm{p}}}\right] \tag{41}$$

we obtain

$$-\big(\det \mathrm{D}^2\mathrm{V}(x)\big)^{\frac{1}{\mathrm{N}}} + \big(\mathrm{V}(x)\big)^{\frac{1}{\mathrm{Np}}} \leq -\big(\det \mathrm{D}^2\mathrm{U}(|x|)\big)^{\frac{1}{\mathrm{N}}} + \big(\mathrm{U}(|x|\big)^{\frac{1}{\mathrm{Np}}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0).$$

Finally, by choosing R satisfying (39) one has

$$\mathrm{V}(x) \leq \mathrm{U}(|x|), \quad x \in \partial\mathbf{B}_{\mathrm{R}}(x_0),$$

whence the comparison principle concludes

$$0 \leq \mathrm{V}(x) \leq \mathrm{C}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),$$

and so $u(x_0) = \big(\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha\big)$. □

Remark 3.7. The assumption (39) is satisfied if we know that the ball $\mathbf{B}_{\mathrm{R}}(x_0)$ where (38) holds is assumed large enough. The above result is motivated by [26, Theorem 2.5]. By adapting the reasoning used in previous results of the literature (see [2, 3, 27]) it can be shown that the decay of $h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ near the boundary point x_0 is optimal in the sense that if

$$h(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) > \mathrm{C}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \quad \text{in a neighbourhood of } x_0$$

then it can be shown that

$$u(x_0) - \big(\langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha)\big) > \mathrm{C}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}} \quad \text{for } x \text{ near } x_0.$$

This type of results gives very rich information on the non–degeneracy behavior of the solution near the free boundary. This is very useful to the study of the continuous dependence of the free boundary with respect to the data h and φ (see [27]). □

Now we examine the case in which the solution cannot be flat (*i.e.* the free boundary cannot appear) independent on "size" of Ω, obviously it requires the condition

$$\mathrm{Np} \leq 1.$$

This will be proved by a version of the Strong Maximum Principle. We shall follow the classical reasoning by E. Hopf (see *e.g.* [30]). Again, since the parameter λ is also irrelevant, in this result, with no loss of generality, we assume here $\lambda = 1$. So, we begin with

Lemma 3.8 (Hopf boundary point lemma)**.** *Assume* $\mathrm{Np} \leq 1$. *Let u be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + u^{\frac{1}{\mathrm{p}}} \geq 0 \quad \textit{in } \Omega.$$

Let $x_0 \in \partial\Omega$ be such that $u(x_0) \doteq \liminf_{\substack{x\to x_0\\ x\in\Omega}} u(x)$ *and*

$$\begin{cases} \textit{i)} & u \textit{ achieves a strict minimum on } \Omega\cup\{x_0\},\\ \textit{ii)} & \exists\ \mathbf{B}_{\mathrm{R}}(x_0 - \mathrm{R}\mathbf{n}(x_0)) \subset \Omega, \quad (\ \partial\Omega \textit{ satisfies an interior sphere condition at } x_0).\end{cases}$$

Then

$$\liminf_{\tau\to 0} \frac{u(x_0 - \tau\mathbf{n})}{\tau} \geq C > 0, \tag{42}$$

where $\mathbf{n}$ stands for the outer normal unit vector of $\partial\Omega$ at x_0 and C is a positive constant depending only on the geometry of $\partial\Omega$ at x_0.

Proof. Let $y = x_0 - \mathrm{R}\mathbf{n}(x_0)$ and $\mathbf{B}_\mathrm{R} \doteq \mathbf{B}_\mathrm{R}(y)$. As it was pointed out before, equation (7) leads to the study of the differential equation

$$\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = \big(\Phi(r)\big)^{\frac{1}{\mathrm{p}}}, \quad r > 0,$$

for radially symmetric solutions. We consider now the classical solution of the two point boundary problem

$$\begin{cases} \dfrac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = \big(\Phi(r)\big)^{\frac{1}{\mathrm{p}}}, \quad 0 < r < \dfrac{\mathrm{R}}{2}, \\ \Phi(0) = 0, \quad \Phi\left(\dfrac{\mathrm{R}}{2}\right) = \Phi_1 > 0. \end{cases} \tag{43}$$

The existence of solution follows from standard arguments and the uniqueness of solution can be proved as in Theorem 2.6, whence

$$\Phi'(0) \geq 0 \quad \Rightarrow \quad \Phi'(r) > 0 \qquad \Rightarrow \quad \Phi''(r) > 0.$$

Then

$$0 \leq \Phi(r) \leq \Phi_1, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

We note that the singularity at $r = 0$ must be removed by the condition

$$\lim_{r\to 0} \frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\Phi'(r)\big)^{\mathrm{N}}\right]' = 0. \tag{44}$$

Let r_0 be the largest r for which $\Phi(r) = 0$. We want to prove that $r_0 = 0$ by proving that $r_0 > 0$ leads to a contradiction. In order to do that we multiply (43) by $r^{\mathrm{N}-1}\Phi'(r)$ and get

$$\left[\big(\Phi'(r)\big)^{\mathrm{N}+1}\right]' = (\mathrm{N}+1)\big(\Phi(r)\big)^{\frac{1}{\mathrm{p}}}\Phi'(r)r^{\mathrm{N}-1}, \quad 0 < r < \frac{\mathrm{R}}{2}.$$

Next, since $\Phi'(r_0) = 0 = \Phi(r_0)$, an integration between r_0 and r leads to

$$\begin{aligned} \big(\Phi'(r)\big)^{\mathrm{N}+1} &= \frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1}\big(\Phi(r)\big)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-1} - \frac{\mathrm{p}(\mathrm{N}+1)(\mathrm{N}-1)}{\mathrm{p}+1}\int_{r_0}^{r} \big(\Phi(s)\big)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-2} ds \\ &\leq \frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1}\big(\Phi(r)\big)^{\frac{\mathrm{p}+1}{\mathrm{p}}} r^{\mathrm{N}-1}, \quad r_0 < r < \frac{\mathrm{R}}{2}. \end{aligned}$$

Because $\mathrm{Np} \leq 1$, a new integration between r_0 and $\dfrac{\mathrm{R}}{2}$ yields the conjectured contradiction because

$$\infty = \int_0^{\Phi_1} \frac{ds}{s^{\frac{\mathrm{p}+1}{\mathrm{p}(\mathrm{N}+1)}}} = \int_{r_0}^{\frac{\mathrm{R}}{2}} \frac{\Phi'(r)}{\big(\Phi(r)\big)^{\frac{\mathrm{p}+1}{\mathrm{p}(\mathrm{N}+1)}}} dr \leq \left(\frac{\mathrm{p}(\mathrm{N}+1)}{\mathrm{p}+1}\right)^{\frac{1}{\mathrm{N}+1}} \int_{r_0}^{\frac{\mathrm{R}}{2}} r^{\frac{\mathrm{N}-1}{\mathrm{N}+1}} dr < \infty.$$

So that, we have proved $\Phi'(0) > 0$ and also

$$0 < \Phi(r) < \Phi_1, \ \Phi'(r) > 0, \quad 0 < r < \frac{\mathrm{R}}{2},$$

as well as $\Phi''(0) = 0$ (see (44)). Hence, straightforward computations on the $\mathcal{C}^2$ convex function $w(x) = \Phi(\mathrm{R} - |x - y|)$, defined in the annulus $\mathcal{O} \doteq \mathbf{B}_{\mathrm{R}} \setminus \overline{\mathbf{B}}_{\frac{\mathrm{R}}{2}}$, prove

$$\begin{cases} \det \mathrm{D}^2 w(x) = \big(w(x)\big)^{\frac{1}{\mathrm{p}}}, \quad x \in \mathcal{O}, \\ w(x) = \Phi_1, \quad x \in \partial \mathbf{B}_{\frac{\mathrm{R}}{2}}, \\ w(x) = 0, \quad x \in \partial \mathbf{B}_{\mathrm{R}}. \end{cases}$$

Moreover, by construction

$$u(x) > 0, \quad x \in \partial \mathbf{B}_{\frac{\mathrm{R}}{2}} \quad \Rightarrow \quad u(x) \geq w(x), \quad x \in \partial \mathbf{B}_{\mathrm{R}},$$

for Φ_1 small enough. Then the Weak Maximum Principle of Proposition 2.4 implies

$$(u - w)(x) \geq 0, \quad x \in \overline{\mathcal{O}}.$$

that leads to

$$\frac{u(x_0 - \tau \mathbf{n})}{\tau} \geq \frac{\Phi(\mathrm{R} - \mathrm{R}(1 - \tau))}{\tau}, \qquad (\tau \ll 1)$$

whence

$$\liminf_{\tau \to 0} \frac{u(x_0 - \tau \mathbf{n})}{\tau} \geq \Phi'(0) > 0.$$

□

Remark 3.9. In fact, above result implies

$$\liminf_{\substack{x \to x_0 \\ x \in \Omega}} \frac{u(x)}{|x - x_0|} \geq \Phi'(0) > 0.$$

□

Our main result proving the absence of the free boundary is the following

Theorem 3.10 (Hopf's Strong Maximum Principle)**.** *Assume* $\mathrm{Np} \leq 1$. *Let u be a nonnegative viscosity solution of*

$$-\det \mathrm{D}^2 u + u^{\frac{1}{\mathrm{p}}} \geq 0 \quad \textit{in } \Omega.$$

Then u cannot vanish at some $x_0 \in \Omega$ unless u is constant in a neighborhood of x_0.

Proof. Assume that u is non–constant and achieves the minimum value $u(x_0) = 0$ on some ball $\mathbf{B} \subset \Omega$. Then we consider the semi-concave approximation of u, *i.e.*

$$u^{\varepsilon}(x) \doteq \inf_{y \in \Omega} \left\{ u(y) + \frac{|x - y|^2}{2\varepsilon^2} \right\}, \quad x \in \mathbf{B}_{\varepsilon} \qquad (\varepsilon > 0), \tag{45}$$

where $\mathbf{B}_{\varepsilon} \doteq \{x \in \mathbf{B} : \ \mathrm{dist}(x, \partial \mathbf{B}) > \varepsilon\sqrt{1 + 4 \sup_{\mathbf{B}} |u|}\}$. For ε small enough we can assume $x_0 \in \mathbf{B}_{\varepsilon}$. Then u^{ε} achieves the minimum value in $\mathbf{B}_{\varepsilon}$, with $u(x_0) = u^{\varepsilon}(x_0) = 0$. Moreover, u^{ε} satisfies

$$-\det \mathrm{D}^2 u_{\varepsilon} + u_{\varepsilon}^{\frac{1}{\mathrm{p}}} \geq 0 \quad \text{on } \mathbf{B}_{\varepsilon}. \tag{46}$$

(see, for instance [43, Proposition 2.3] or [6, 16] for general fully nonlinear equations). By classic arguments, if we denote

$$\mathbf{B}_{\varepsilon}^{+} \doteq \{x \in \mathbf{B}_{\varepsilon} : \ u^{\varepsilon}(x) > 0\},$$

there exists the largest ball $\mathbf{B}_{\mathrm{R}}(y) \subset \mathbf{B}_{\varepsilon}^{+}$ (see [30]). Certainly there exists some $z_0 \in \partial\mathbf{B}_{\mathrm{R}}(y) \cap \mathbf{B}_{\varepsilon}$ for which $u^{\varepsilon}(z_0) = 0$ is a local minimum. Then, Lemma 3.8 implies

$$\mathrm{D}u^{\varepsilon}(z_0) \neq \mathbf{0}$$

contrary to

$$\mathrm{D}u^{\varepsilon}(z_0) = \mathbf{0}, \tag{47}$$

as we shall prove in Lemma 3.13 below. Therefore, u^{ε} is constant on $\mathbf{B} \subset \Omega$, *i.e.*

$$u^{\varepsilon}(y) = u^{\varepsilon}(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

Finally, for every $y \in \mathbf{B}$ we denote by $\widehat{y}$ the point of Ω such that

$$u^{\varepsilon}(y) = u(\widehat{y}) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2$$

whence

$$u(x_0) = u^{\varepsilon}(x_0) = u^{\varepsilon}(y) = u(y) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2 \geq u(x_0) + \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2 \geq u(x_0) \quad \Rightarrow \quad \widehat{y} = y.$$

So that, one concludes

$$u(y) = u^{\varepsilon}(y) = u^{\varepsilon}(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

□

Corollary 3.11. *Assume* $\mathrm{Np} \leq 1$. *Let u be a generalized solution u of* (7). *Then if $u(x_0) > h(x_0)$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point x_0 of a ball $\overline{\mathbf{B}} \subseteq \overline{\Omega}$ then $u > h$ on $\overline{\mathbf{B}}$, consequently the equation* (7) *is elliptic in $\overline{\mathbf{B}}$. In particular, if $\varphi(x_0) > h(x_0)$ at some $x_0 \in \partial\Omega$ or $\det \mathrm{D}^2 h(x_0) > 0$ at some point $x_0 \in \Omega$ the problem* (5) *is elliptic non degenerate in path-connected open sets Ω, provided the compatibility condition* (4) *holds.*

Proof. From Theorem 3.10, both cases imply $u > h$ on $\overline{\mathbf{B}}$. Finally, a continuity argument concludes the proof. □

Remark 3.12. Straightforward computations enable us to extend Lemma 3.8, Theorem 3.10 and Corollary 3.11 to the general case $g(|\mathbf{p}|) \geq 1$, since we know that $u \in \mathrm{W}^{1,\infty}(\Omega)$ (see the comments of Remark 2.9). □

We end this section by proving the property (47) used in the proof of Theorem 3.10

Lemma 3.13. *Let ψ be a function achieving a local minimum at some $z_0 \in \mathcal{O}$. Assume that there exists a function $\widehat{\psi}$ defined in $\mathcal{O}$ such that $\widehat{\psi}(z_0) = 0$, $\Psi = \psi + \widehat{\psi}$ is concave on $\mathcal{O}$ and*

$$\widehat{\psi}(x) \geq -\mathrm{K}|x - z_0|^2, \quad x \in \mathcal{O}, \ \textit{with } |x - z_0| \textit{ small},$$

for some constant $\mathrm{K} > 0$. *Then the function ψ is differentiable at z_0 and* $\mathrm{D}\psi(z_0) = \mathbf{0}$.

Proof. By simplicity we can take $z_0 = 0 \in \mathcal{O}$. By applying the convex separation theorem there exists $\mathbf{p} \in \mathbb{R}^{\mathrm{N}}$ such that

$$\Psi(x) \leq \Psi(0) + \langle \mathbf{p}, x \rangle = \psi(0) + \langle \mathbf{p}, x \rangle, \quad x \in \mathcal{O}, \text{ with } |x| \text{ small.}$$

Then we have

$$\begin{aligned} \psi(x) &= \Psi(x) - \widehat{\psi}(x) \leq \psi(0) + \langle \mathbf{p}, x \rangle + \mathrm{K}|x|^2 \\ &\leq \psi(x) + \langle \mathbf{p}, x \rangle + \mathrm{K}|x|^2, \quad x \in \mathcal{O}, \text{ with } |x| \text{ small} \end{aligned} \tag{48}$$

whence

$$-\langle \mathbf{p}, x \rangle \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O}, \text{with } |x| \text{ small.}$$

For $\tau > 0$ small enough we can choose $x = -\tau \mathbf{p} \in \mathcal{O}$ and $\tau \mathrm{K} < 1$, for which

$$\tau |\mathbf{p}|^2 \leq \mathrm{K} \tau^2 |\mathbf{p}|^2.$$

Therefore $\mathbf{p} = 0$. Finally, (48) leads to

$$0 \leq \psi(x) - \psi(0) \leq \mathrm{K}|x|^2, \quad x \in \mathcal{O}, \text{ with } |x| \text{ small,}$$

and the result follows. □

Remark 3.14. The result is immediate if ψ is concave, in this case we can choose $\widehat{\psi} \equiv 0$. The convex version follows by changing ψ and $\widehat{\psi}$ by $-\psi$ and $-\widehat{\psi}$, respectively (see Remark 2.7 above). □

Note that since the function u^{ε} defined in (45) is semi concave, the property (47) holds

4. The evolution problem. Study of the associated free boundary and the global flatness in finite time

We start by considering the existence of solution of (2) by means of the accretivity of the operator. The definition of the operator uses odd increasing functions $f \in \mathcal{C}(\mathbb{R})$, such that $f(0) = 0$. Then, we say $u \in \mathrm{D}(\mathcal{A})$ if $u \in \mathcal{C}(\overline{\Omega})$ is a locally convex function on $\overline{\Omega}$ prescribing $\varphi \in \mathcal{C}(\partial\Omega)$ on $\partial\Omega$ and there exists a nonpositive continuous function v in Ω such that u is a generalized solution of

$$\begin{cases} \dfrac{f^{-1}\big(-\det \mathrm{D}^2 u\big)}{g\big(|\mathrm{D}u|\big)} = v & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

or equivalently

$$\begin{cases} \det \mathrm{D}^2 u = f(\big(-g\big(|\mathrm{D}u|\big)v\big) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases}$$

for a more precise sense. Then we denote by $\mathcal{A}u$ the set of all such $v \in \mathcal{C}(\overline{\Omega})$.

Theorem 4.1. *The operator $\mathcal{A}$ is T-accretive on the Banach space $\mathbb{X} = \mathcal{C}(\overline{\Omega})$ equipped with the supreme norm. In particular,*

$$\begin{aligned} \|[u_1 - u_2]_+\| &\leq \sup_{\Omega} \big\|\big[u_1 - u_2 + \varepsilon\big(\mathcal{A}u_1 - \mathcal{A}u_2\big)\big]_+\big\|, \\ \big\|u_1 - u_2\big\| &\leq \sup_{\Omega} \big\|u_1 - u_2 + \varepsilon\big(\mathcal{A}u_1 - \mathcal{A}u_2\big)\big\|, \end{aligned} \tag{49}$$

for $\varepsilon > 0,\ u_i \in \mathrm{D}(\mathcal{A})$.

Proof. It is a mere application of Theorem 2.6. □

Remark 4.2. We recall that the accretiveness of the operator implies that the resolvent $\mathcal{J}_\varepsilon = (\mathrm{I} + \varepsilon\mathcal{A})^{-1}$ is a contraction on $\mathbb{X}$. □

Certainly, one has

$$\mathrm{D}(\mathcal{A}) \subset \widehat{\mathbb{X}}_\varphi \doteq \{w \in \mathcal{C}(\overline{\Omega}) :\ w \text{ locally convex on } \overline{\Omega} \text{ and } w = \varphi \text{ on } \partial\Omega\}.$$

In fact, we have

Corollary 4.3. *The operator $\mathcal{A}$ satisfies $\overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi$ as well as the range condition*

$$R(\mathrm{I} + \varepsilon\mathcal{A}) \supset \overline{\mathrm{D}(\mathcal{A})}, \quad \varepsilon > 0.$$

Proof. By well-known results (see, *e.g.* expression (2.1) of [42]), any w locally convex function can be approximated uniformly by a sequence of smooth locally convex functions $w_n \in \mathcal{C}(\overline{\Omega})$ such that $w_n = \varphi$ on $\partial\Omega$. Then we can assume that $\det \mathrm{D}^2 w_n \in \mathcal{C}(\overline{\Omega})$, and so $w_n \in \mathrm{D}(\mathcal{A})$ (note that we merely have that $\det \mathrm{D}^2 w_n \rightharpoonup \det \mathrm{D}^2 w$ weakly in the sense of measures). Therefore

$$\overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi.$$

On the other hand, for each $h \in \overline{\mathrm{D}(\mathcal{A})}$ and $\varepsilon > 0$ by means of a simple adaptation of the proof of Theorem 2.8 one proves that there exists a unique solution of

$$\begin{cases} \det \mathrm{D}^2 u = f\left(g\big(|\mathrm{D}u|\big)\dfrac{u-h}{\varepsilon}\right), & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases}$$

thus

$$\big(\mathrm{I} + \varepsilon\mathcal{A}\big)u = h.$$

□

Crandall–Liggett generation theorem (see [15]) and Corollary 4.3 enables us to show that $\mathcal{A}$ generates a nonlinear semigroup of contractions $\{S(t)\}_{t\geq 0}$ on $\mathbb{X}$ and

$$S(t)u_0 = \lim_{\substack{n\to\infty \\ \varepsilon n \to t}} (\mathrm{I} + \varepsilon\mathcal{A})^{-1} u_0, \quad \text{for any } u_0 \in \overline{\mathrm{D}(\mathcal{A})} = \widehat{\mathbb{X}}_\varphi, \tag{50}$$

uniformly for t in bounded subsets of $]0,\infty[$. Furthermore, the mapping $t \mapsto S(t)u_0$ is continuous from $[0,\infty[$ into $\mathbb{X}$. In general the semigroup generated by such accretive operators $\mathcal{A}$ can be regarded as the so called "mild solution" of the Cauchy problem

$$\left\{\begin{array}{l} u_t + \mathcal{A}u = 0 \quad t>0, \\ u(0) = u_0, \end{array}\right. \tag{51}$$

(see [15]). A different characterization is possible.

Proposition 4.4. *Assume* $u_0 \in \overline{\mathrm{D}(\mathcal{A})} \subset \mathbb{X}$. *Then*

$$u(x,t) = S(t)u_0(x), \quad x \in \Omega,\ 0 < t < \mathrm{T} < \infty, \tag{52}$$

satisfies (2) *in the viscosity sense.*

Proof. Assume $\Phi \in \mathcal{C}^2(\Omega\times]0,\mathrm{T}[)$ such that $\Phi(\cdot,t)$ is convex on Ω, for all $t \in [0,\mathrm{T}]$ and $u-\Phi$ attains a strict local maximum at $(x_0,t_0) \in \Omega\times]0,\mathrm{T}[)$. For each $\varepsilon > 0$, consider the step function $u_\varepsilon(t) \in \mathrm{D}(\mathcal{A})$ solving

$$\left\{\begin{array}{l} \det \mathrm{D}_x^2 u_\varepsilon(t+\varepsilon) = f\left(g\big(|\mathrm{D}_x u_\varepsilon(t+\varepsilon)|\big)\dfrac{u_\varepsilon(t+\varepsilon)-u_\varepsilon(t)}{\varepsilon}\right) \quad \text{in } \Omega, \quad t>0, \\ u_\varepsilon(t) = u_0 \quad \text{if } 0 \le t \le \varepsilon. \end{array}\right.$$

We may assume $t_0 \neq k\varepsilon$ by appropriate choice of ε. Since $u_\varepsilon(t) \to S(t)u_0$ uniformly on $[0,\mathrm{T}]$ in $\mathbb{X}$ as $\varepsilon \to 0$, $u_\varepsilon(x,t+\varepsilon) - \Phi(x,t)$ has a local maximum at some point $(x_\varepsilon,t_\varepsilon)$, such that $(x_\varepsilon,t_\varepsilon) \in \Omega\times]0,\mathrm{T}[$, $x_\varepsilon \to x_0$, $t_\varepsilon \to t_0$, as $\varepsilon \to 0$. Hence,

$$\det \mathrm{D}_x^2 u_\varepsilon(t_\varepsilon+\varepsilon) - \det \mathrm{D}_x^2\Phi \le 0 \quad \text{at } x_\varepsilon,$$

according to the definition of $\mathcal{A}$ (note $u_\varepsilon(\cdot+\varepsilon) \in \mathrm{D}(\mathcal{A})$). Moreover, if ε is small enough, we have

$$\frac{u_\varepsilon(x_\varepsilon,t_\varepsilon+\varepsilon)-u_\varepsilon(x_\varepsilon,t_\varepsilon)}{\varepsilon} \ge \frac{\Phi(x_\varepsilon,t_\varepsilon)-\Phi(x_\varepsilon,t_\varepsilon-\varepsilon)}{\varepsilon},$$

whence

$$\det \mathrm{D}^2\Phi(x_\varepsilon,t_\varepsilon) \ge f\left(g\big(|\mathrm{D}_x\Phi(x_\varepsilon,t_\varepsilon-\varepsilon)|\big)\frac{\Phi(x_\varepsilon,t_\varepsilon-\varepsilon)-\Phi(x_\varepsilon,t_\varepsilon)}{\varepsilon}\right).$$

If we let $\varepsilon \to 0$, then, since $(x_\varepsilon,t_\varepsilon) \to (x_0,t_0)$, we obtain

$$-\det \mathrm{D}_x^2\Phi(x_0,t_0) + f\big(g\big(|\mathrm{D}_x\Phi(x_0,t_0)|\big)\Phi_t(x_0,t_0)\big) \le 0.$$

The opposite inequality has an analogous proof should $u-\Phi$ attain a local minimum at (x_0,t_0). ☐

Remark 4.5. Since $u(t) \in \overline{\mathrm{D}(\mathcal{A})}$ the property

$$0 \le \det \mathrm{D}_x^2 u(t),$$

holds in the generalized sense *a.e.* $t > 0$. Note that a priori we merely know that $S(t)\big(\overline{\mathrm{D}(\mathcal{A})}\big) \subset \overline{\mathrm{D}(\mathcal{A})}$ and so the time derivative u_t must be understood in a large sense. Nevertheless, it is possible to apply different regularity results according f see, for instance, [19] and its references. In any case, at least u_t is a nonnegative measure and $u(\cdot,t)$ is a locally convex function. ☐

As it was pointed out in the introduction our study on the free boundary uses the power like nonlinearity, $f(t) = t^{\mathrm{p}}$ and its inverse function $f^{-1}(t) = t^{\frac{1}{\mathrm{p}}}$ which must be understood as the restriction to $\overline{\mathbb{R}}_+$ of the odd functions $f(t) = |t|^{\mathrm{p}-1}t$ and $f^{-1}(t) = |t|^{\frac{1-\mathrm{p}}{\mathrm{p}}}t$, respectively. So that, from the viscosity notion point of view we may rewrite (51) in the usual form

$$\begin{cases} u_t = \dfrac{\big(\det \mathrm{D}^2 u\big)^{\mathrm{p}}}{g(|\mathrm{D}u|)} & \text{in } \Omega \times \mathbb{R}_+, \\ u(x,t) = \varphi(x), & (x,t) \in \partial\Omega \times \mathbb{R}_+, \\ u(x,0) = u_0(x). & x \in \Omega. \end{cases}$$

(see (2) above).

Our results on the free boundary begin by studying how a possible region of flatness of the initial datum u_0 shrinks when t increases. We start by considering the interior points of $\mathrm{Flat}(u_0)$. As in Section 3, for $u_0 \in \overline{\Omega}$ we denote

$$\mathrm{Flat}(u_0) = \bigcup_{\alpha} \mathrm{Flat}_\alpha(u_0)$$

where

$$\mathrm{Flat}_\alpha(u_0) = \{x \in \overline{\Omega} : u_0(x) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha, \text{ for some } \mathbf{p}_\alpha \in \mathbb{R}^{\mathrm{N}} \text{ and } a_\alpha \in \mathbb{R}\}.$$

Theorem 4.6. *Let* $\mathrm{Np} > 1$ *and*

$$\mathbf{B}_{\mathrm{R}}(x_0) \subset \mathrm{Flat}_\alpha(u_0) \tag{53}$$

for some $\mathrm{R} > 0$. *Then there exists* $t^* = t^*(u_0) > 0$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha, \quad 0 \leq t < t^*,$$

where u is the viscosity solution of (2).

Proof. We need a suitable local separable supersolution $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$. The time function $\eta(t)$ is given by

$$\eta'(t) = \delta\big(\eta(t)\big)^{\mathrm{Np}}, \quad t > 0, \quad \text{for some } \delta > 0, \tag{54}$$

whose solution is

$$\eta(t) = \left[\frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}} - \frac{\delta}{\mathrm{Np}-1}t\right]^{-\frac{1}{\mathrm{Np}-1}}. \tag{55}$$

Note that $\eta(t)$ blows up at

$$t^*\big(\delta, \eta(0)\big) \doteq \frac{\mathrm{Np}-1}{\delta}\frac{1}{\big(\eta(0)\big)^{\mathrm{Np}-1}}.$$

The spatial dependence is given by the function

$$\mathrm{U}(r) = \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \geq 0,$$

(see (31)). In [25] it is proved the regularity

$$\mathrm{D}u \in \mathrm{L}^{\infty}\big(0,\infty : \mathrm{L}^{\infty}(\Omega)\big).$$

Then the convexity of the solution enables us to choose $\eta(0)$, δ and R such that

$$\max_{\mathbf{B}_{\mathrm{R}}(x_0)\times\overline{\mathbb{R}}_+} u \leq \eta(0)\delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}\mathrm{R}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}. \tag{56}$$

So that, we consider now the function

$$\mathrm{V}(x,t) = u(x,t) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$$

for which

$$\mathrm{V}(x,0) = 0, \quad x \in \mathbf{B}_{\mathrm{R}}(x_0),$$

(see (53)) and

$$\mathrm{V}(x,t) \leq \overline{\mathrm{U}}(x,t), \quad (x,t) \in \partial\mathbf{B}_{\mathrm{R}}(x_0) \times [0, t^*\big(\delta, \eta(0)\big[$$

hold (see (56)). On the other hand, we have that

$$\frac{1}{\delta} \geq g\big(|\mathrm{D}_x\mathrm{V}(x,t)|\big) \geq 1, \quad (x,t) \in \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[$$

for a suitable choice of δ. Therefore, one has

$$-\frac{\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\mathrm{V}|\big)} \leq -\delta\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[,$$

and

$$\mathrm{V}_t - \frac{\big(\det \mathrm{D}_x^2\mathrm{V}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\mathrm{V}(x,t)|\big)} = 0 \leq \overline{\mathrm{U}}_t - \frac{\big(\det \mathrm{D}_x^2\overline{\mathrm{U}}\big)^{\mathrm{p}}}{g\big(|\mathrm{D}_x\overline{\mathrm{U}}(x,t)|\big)} \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[.$$

Thus, by the comparison principle

$$0 \leq \mathrm{V}(x,t) \leq \overline{\mathrm{U}}(x,t), \quad (x,t) \in \mathbf{B}_{\mathrm{R}}(x_0) \times \big[0, t^*\big(\delta, \eta(0)\big)\big[$$

which concludes the proof. □

Remark 4.7. It is easy to see that the above argument gives a simple estimate on the shrinking of the free boundary

$$\mathcal{F}_\alpha(t) = \partial\{(x,t) :\ u(x,t) = \langle \mathbf{p}_\alpha, x\rangle + a_\alpha\} \tag{57}$$

from the rest. Essentially,

$$\limsup_{t\to 0} \operatorname{dist}\big(\mathcal{F}_\alpha(t), \mathcal{F}_\alpha(0)\big)t^{-\frac{1}{\mathrm{Np}-1}} \leq \mathrm{C},$$

for some positive constant C. □

The next result shows that if $u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ arrives to some points of the boundary of its support flat enough, let us say (38), then there exists a "finite waiting time" for those points.

Theorem 4.8. *Let* $\mathrm{Np} > 1$ *and let* $x_0 \in \Omega$ *be such that*

$$u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad x \in \mathbf{B}_\mathrm{R}(x_0), \tag{58}$$

for suitable positive constants K *and* R. *Then, there exists* $\widetilde{t} = \widetilde{t}(x_0)$ *such that*

$$u(x_0, t) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha \quad \text{if } 0 \le t < \widetilde{t},$$

where u *is the viscosity solution of* (2).

Proof. As in the above proof we use a local separable supersolution $\overline{\mathrm{U}}(x,t) = \mathrm{U}(|x|)\eta(t)$, where $\eta(t)$ was given in (54) and

$$\mathrm{U}(r) = \mathrm{C}r^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}, \quad r \ge 0,$$

for $\mathrm{C} > 0$. Then

$$-\frac{r^{1-\mathrm{N}}}{\mathrm{N}}\left[\big(\mathrm{U}'(r)\big)^{\mathrm{N}}\right] + \delta\big(\mathrm{U}(r)\big)^{\frac{1}{\mathrm{p}}} = \mathrm{C}^{\frac{1}{\mathrm{p}}}\left[1 - \left(\frac{\mathrm{C}}{\delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}}\right)^{\frac{\mathrm{Np}-1}{\mathrm{p}}}\right] r^{\frac{2\mathrm{N}}{\mathrm{Np}-1}} > 0$$

for $\mathrm{C} \in \big]0, \delta^{\frac{\mathrm{p}}{\mathrm{Np}-1}}\mathrm{C}_{\mathrm{p,N}}\big[$ (see (30)). Then the reasonings are similar to those of the proof of Theorem 4.6 because now (58) provides the inequality

$$\mathrm{V}(x,0) \le \overline{\mathrm{U}}(x,0)$$

before derived from (53). □

Remark 4.9. A similar waiting time result was obtained by Choop, Evans and Ishii [14] for the special case $\mathrm{p} = 1$ and $\mathrm{N} = 2$ under a global formulation on the assumption on the initial datum. Essentially, their assumption is $u_0 \in \mathcal{C}^4(\Omega)$. Note that for this special case $\mathrm{p} = 1$ and $\mathrm{N} = 2$ the condition (58) becomes

$$u_0(x) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big) \le \mathrm{K}|x - x_0|^4, \quad x \in \mathbf{B}_\mathrm{R}(x_0). \tag{59}$$

In particular, any C^4 partially flat function satisfies (59) at all points of the boundary of $\mathcal{F}_\alpha(0)$ (see (57)). So, our result can be regarded as a local and generalized version of the result of [14]. □

We continue our study on the evolution of the free boundary by showing that, in the case of a bounded domain Ω, in most of the cases $\mathcal{F}_\alpha(t)$ is shrinking.

Theorem 4.10. *Let* $\mathrm{Np} > 1$ *and assume* x_0 *such that*

$$u_0(x_0) = \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha.$$

Then there exists $\widehat{t} > 0$ *such that*

$$u(x_0, t) > \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha, \quad t > \widehat{t}, \tag{60}$$

where u *is the viscosity solution of* (2).

As a matter of fact, it is enough to show that

$$u(x_0,\widehat{t}) > \langle \mathbf{p}_\alpha, x_0\rangle + a_\alpha, \tag{61}$$

because since $u_t \geq 0$ we get (60) for any $t > \widehat{t}$. In order to prove Theorem 4.10 we shall use other suitable supersolution based on the self–similar solution of the Cauchy problem associated to the case $g(s) \equiv \delta$, for suitable $\delta \geq 1$. We start by pointing out that by arguing by adimensionalization we get:

Lemma 4.11. *Let $u(x,t)$ be a viscosity solution of*

$$u_t = \big(\det \mathrm{D}_x^2 u\big)^p \quad \textit{in } \mathbb{R}^{\mathrm{N}} \times \mathbb{R}_+. \tag{62}$$

Then the change of scale $x' = \mathrm{L}x,\ t' = \mathrm{T}t$ allows to define

$$u'(x',t') = \mathrm{L}^{\frac{2\mathrm{Np}}{\mathrm{Np}-1}}\mathrm{T}^{-\frac{1}{\mathrm{Np}-1}}u(x,t)$$

which is also a viscosity solution of (62). □

A more deep conclusion on the self–similar solution of (62) is the following

Theorem 4.12. *Assume* $\mathrm{Np} > 1$. *Then, there exists a family of convex compactly supported similarity solutions of* (62) *given by*

$$u(r,t;\sigma,\beta) \doteq t^{-\sigma}\Lambda(\eta), \quad \eta \doteq \frac{r}{t^\beta} \quad \textit{for suitable } \sigma,\ \beta > 0.$$

The proof of Theorem 4.12 requires the analysis of the correspondent phase–plane system

$$\begin{cases} \dfrac{dq}{d\eta} = -\left[\dfrac{\mathrm{N}^{\mathrm{p}}\beta}{\eta}\Big[\dfrac{\sigma}{\beta}\Lambda(\eta) + \eta q^{\frac{1}{\mathrm{N}}}\Big]\right]^{\frac{1}{\mathrm{p}}}, \\ \dfrac{d\Lambda}{d\eta} = q^{\frac{1}{\mathrm{N}}}, \end{cases}$$

where $q = (\Lambda')^{\mathrm{N}}\big(\mathrm{sign}\ \Lambda'\big)$. By simplicity, we do not present here the details but send the reader to [25]. In any case, we can indicate that the proof is a non–difficult variation of some results in the literature (see, for instance, Bernis, Hulshof and Vázquez [7] and Igbida [35]). See also Daskalopoulus and Lee [18] for the case of the "focusing problem" associated to (62).

Proof of Theorem 4.10. As in the proof of Theorem 4.6 we can assume that the solution u has bounded gradient

$$\mathrm{D}u \in \mathrm{L}^\infty\big(\Omega\times]0,\widehat{t}[\big)$$

for any given $\widehat{t}$. So that $\mathrm{V}(x,t) = u(x,t) - \big(\langle \mathbf{p}_\alpha, x\rangle + a_\alpha\big)$ verifies

$$\mathrm{V}_t(x,t) - \delta\big(\det \mathrm{D}_x^2\mathrm{V}(x,t)\big)^{\mathrm{p}} \geq 0,$$

for a suitable $\delta > 0$, that we suppose here $\delta = 1$ to simplify the notation, otherwise the needed modifications are simple. Let $x_0 \in \Omega$ such that

$$u_0(x_0) = \langle \mathbf{p}_\alpha, x_0 \rangle + a_\alpha.$$

Then we consider σ and β for which

$$u\big(|x - x_0|, t; \sigma, \beta\big) \leq \mathrm{V}(x,t) \quad \text{for any } t \geq 0 \text{ and } x \in \mathbf{B}_{\mathrm{R}}(x_0)$$

for some $\mathrm{R} > 0$ (see Theorem 4.12). Since

$$u\big(r_0, t; \sigma, \beta\big) > 0$$

for any $r_0 > 0$ once that t is large enough, we conclude the result. □

Remark 4.13. Other properties of the solution of (51) can be obtained from the family of self-similar solution given in Theorem 4.12 (see [25] for details). □

Our last goal is the study of the asymptotic behavior of u as $t \to \infty$ from a peculiar point of view. We start by proving that if $\mathrm{Np} \geq 1$ then the stabilization to a stationary solution requires infinite time. From now on, any locally convex function on $\overline{\Omega}$ such that $\det \mathrm{D}^2 h = 0$, *a.e.* in Ω will be called *flat convex function.* By simplicity we assume $g\big(|\mathbf{p}|\big) \equiv 1$.

Theorem 4.14. *Assume that* $\mathrm{Np} \geq 1$. *Let* $\overline{h}$ *a flat convex function on* $\overline{\Omega}$ *such that* $\varphi \leq \overline{h}$ *on* $\partial\Omega$. *Then for each* $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ *such that* $u_0 \leq \overline{h}$ *on* $\overline{\Omega}$ *and* $u_0 < \overline{h}$ *in some set* $\Omega' \subset \Omega$ *with positive measure, there exists a positive constant* $\mathrm{C}_{\Omega'}$ *such that*

$$\begin{cases} \liminf\limits_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big) t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} > 1, \\ \liminf\limits_{t\to\infty} \big(\overline{h}(x) - u(x,t)\big) e^t \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega', \tag{63}$$

where u *is the viscosity solution of* (2). *Analogously, let* $\underline{h}$ *be a flat convex function on* $\overline{\Omega}$ *such that* $\varphi \geq \underline{h}$ *on* $\partial\Omega$ *verifying* $u_0 \geq \underline{h}$ *on* $\overline{\Omega}$ *and* $u_0 > \overline{h}$ *in some set* $\Omega' \subset \Omega$ *with positive measure, there exists a positive constant* $\mathrm{C}_{\Omega'}$ *such that*

$$\begin{cases} \liminf\limits_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big) t^{\frac{1}{\mathrm{Np}-1}} \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} > 1, \\ \liminf\limits_{t\to\infty} \big(u(x,t) - \underline{h}(x)\big) e^t \geq \mathrm{C}_{\Omega'} & \text{if } \mathrm{Np} = 1, \end{cases} \quad x \in \Omega'. \tag{64}$$

Proof. If h be a flat such that $\varphi \leq \overline{h}$ on $\partial\Omega$. Then, one has

$$\overline{h}_t = \big(\det \mathrm{D}_x^2 \overline{h}\big)^{\mathrm{p}}$$

whence $\underline{u}(x,t) = u(x,t) - \overline{h}(x)$ verifies

$$\begin{cases} (\underline{u})_t = \big(\det \mathrm{D}_x^2 \underline{u}\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ \underline{u} \leq 0, \ \underline{u} \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases} \tag{65}$$

in the viscosity sense and

$$\underline{u}(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Here the key idea is to consider the auxiliar problem

$$\begin{cases} \phi'(t) + \dfrac{2}{\mathrm{m}}\big(k\phi(t)\big)^{\mathrm{Np}} = 0, \quad t \geq 0, \\ \phi(0) = 1, \quad \phi(\infty) = 0, \end{cases} \tag{66}$$

whose solution is

$$\phi(t) = \begin{cases} \left[1 + \dfrac{2}{\mathrm{m}} \dfrac{k^{\mathrm{Np}}}{\mathrm{Np}-1} t\right]^{-\frac{1}{\mathrm{Np}-1}}, & \text{if } \mathrm{Np} > 1, \\ e^{-\frac{2k}{\mathrm{m}}t}, & \text{if } \mathrm{Np} = 1, \end{cases}$$

where k is a positive constant to be choosen and m is a positive constant such that $\overline{h} - u_0 \geq \mathrm{m}$ in some $\mathbf{B}_{2\mathrm{R}} \subset \Omega$. Let $\psi(x_1) \in \mathcal{C}^2$ a non positive function such that

$$\begin{cases} \psi(x_1) = 0, \quad x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ -\mathrm{m} < \psi(x_1) < -\dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, \quad x \in \mathbf{B}_{\mathrm{R}}, \\ -\dfrac{\mathrm{m}}{2} < \psi(x_1) < 0 \quad \text{and} \quad \psi''(x_1) \leq 0, \quad x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}. \end{cases}$$

Then the function

$$W(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} W(x,t) < 0, & (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \overline{\mathbb{R}}_+, \\ W(x,t) = 0, & (x,t) \in \partial\Omega \times \overline{\mathbb{R}}_+, \\ W(x,0) = \phi(0)\psi(x_1) \geq -\mathrm{m} > \big(u_0 - h\big)(x), & x \in \mathbf{B}_{2\mathrm{R}}, \\ W(x,0) = \phi(0)\psi(x_1) = 0 \geq \big(u_0 - h\big)(x), & x \in \overline{\Omega} \setminus \overline{\mathbf{B}}_{2\mathrm{R}}. \end{cases}$$

because $\phi(0) = 1$. Moreover, from (66) we get

$$W_t(x,t) + \big(-\det \mathrm{D}_x^2 W(x,t)\big)^{\mathrm{p}} \doteq r(x,t)$$

where

$$r(x,t) \geq \begin{cases} \big(k\phi(t)\big)^{\mathrm{Np}} + \Big(-\big(\phi(t)\big)^{\mathrm{Np}}\Big)\big(\psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \mathbf{B}_{\mathrm{R}}, \\ \phi'(t)\psi(x_1) + \big(\phi(t)\big)^{\mathrm{Np}}\big(-\psi''(x_1)\big)^{\mathrm{p}} \geq 0, & x \in \Omega \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

for $t > 0$, provided k is large. Then, from (65), comparison results lead to

$$u(x,t) - \overline{h}(x) \leq W(x,t) \leq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$\overline{h}(x) - u(x,t) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+. \tag{67}$$

We may repeat the reasoning with a flat function $\underline{h}$ such that $\varphi \geq \underline{h}$ on $\partial\Omega$. So, the function $\overline{u}(x,t) = u(x,t) - \underline{h}(x)$ verifies

$$\begin{cases} (\overline{u})_t = \big(\det \mathrm{D}^2_x \overline{u}\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ \overline{u} \geq 0,\ \overline{u} \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases}$$

in the viscosity sense and

$$\overline{u}(x,t) \geq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

Now, we consider a non negative function $\psi(x_1) \in \mathcal{C}^2$ such that

$$\begin{cases} \psi(x_1) = 0, \quad x \notin \overline{\mathbf{B}}_{2\mathrm{R}}, \\ \dfrac{\mathrm{m}}{2} < \psi(x_1) < \mathrm{m} \quad \text{and} \quad \psi''(x_1) \leq 0, \quad x \in \mathbf{B}_{\mathrm{R}}, \\ 0 < \psi(x_1) < \dfrac{\mathrm{m}}{2} \quad \text{and} \quad \psi''(x_1) \geq 0, \quad x \in \mathbf{B}_{2\mathrm{R}} \setminus \overline{\mathbf{B}}_{\mathrm{R}}, \end{cases}$$

where m is a positive constant such that $u_0 - \underline{h} \geq \mathrm{m}$ in some $\mathbf{B}_{2\mathrm{R}} \subset \Omega$. Arguing as above one proves that the function

$$w(x,t) = \phi(t)\psi(x_1), \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+,$$

verifies

$$\begin{cases} w_t(x,t) + \big(-\det \mathrm{D}^2_x w(x,t)\big)^{\mathrm{p}} \leq 0 & \text{in } \Omega \times \mathbb{R}_+, \\ w \leq \overline{u} & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases}$$

provided k is large. Then, we obtain

$$u(x,t) - \underline{h}(x) \geq w(x,t) \geq 0, \quad (x,t) \in \Omega \times \mathbb{R}_+.$$

In particular,

$$u(x,t) - \underline{h}(x) \geq \frac{\mathrm{m}}{2}\phi(t) > 0, \quad (x,t) \in \mathbf{B}_{2\mathrm{R}} \times \mathbb{R}_+. \tag{68}$$

□

Note that, in particular, Theorem 4.14 implies a kind of non flattened global retention property:

$$\begin{cases} u_0(x) < \overline{h}(x),\ x \in \Omega' \subset \Omega & \Rightarrow \quad u(x,t) < \overline{h}(x),\ x \in \Omega' \text{ for all } t \geq 0 \\ \underline{h}(x) < u_0(x),\ x \in \Omega' \subset \Omega & \Rightarrow \quad \underline{h}(x) < u(x,t),\ x \in \Omega' \text{ for all } t \geq 0 \end{cases} \tag{69}$$

holds. Clearly, the second retention property also follows from $u_t \geq 0$.

Our final result in this paper shows that when $\mathrm{Np} < 1$ the asymptotic behavior is very fast. It is the property of "finite global flattened time" which we prove by means of some ideas used by first time in [23]. Again, for simplicity we assume $g\big(|\mathbf{p}|\big) \equiv 1$.

Theorem 4.15. *Let $h(x) = \langle \mathbf{p}, x\rangle + a$ on $\overline{\Omega}$ and suppose $\varphi = h$ in the definition of the operator $\mathcal{A}$. Assume* $\mathrm{Np} < 1$. *Then for each $u_0 \in \overline{\mathrm{D}(\mathcal{A})}$ such that $u_0 \leq h$ on $\overline{\Omega}$ there exists a time T_0, depending on $h - u_0$, such that*

$$u(x,t) = \langle \mathbf{p}, x\rangle + a, \quad x \in \overline{\Omega}, \quad t \geq \mathrm{T}_0.$$

where u is the viscosity solution of (2)*.*

Proof. Let us denote $u_h(x,t) = u(x,t) - h(x)$. As in the proof of Theorem 4.14 one verifies (65), thus

$$\begin{cases} (u_h)_t = \big(\det \mathrm{D}^2_x u_h\big)^{\mathrm{p}} & \text{in } \Omega \times \mathbb{R}_+, \\ u_h \leq 0, \ u_h \not\equiv 0 & \text{on } \big(\partial\Omega \times \mathbb{R}_+\big) \cup \big(\overline{\Omega} \times \{0\}\big), \end{cases}$$

in the viscosity sense, whence

$$u_h(x,t) \leq 0, \quad (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

In fact, if $u_0 = h$ one derives the coincidence

$$u_h(x,t) = 0 \quad \text{for any } (x,t) \in \overline{\Omega} \times \overline{\mathbb{R}}_+.$$

So that, suppose

$$u_0 \leq h, \ u_0 \not\equiv h.$$

It is clear that the "finite flattened time property" is strongly based on the initial value problem

$$\begin{cases} \mathrm{m}\Theta'(t) = \big(2\Theta(t)\big)^{\mathrm{Np}}, \quad t \geq 0, \\ \Theta(0) = 0 \end{cases}$$

whose solution is

$$\Theta(t) = \left(\frac{2^{\mathrm{Np}}(1-\mathrm{Np})}{\mathrm{m}}\right)^{\frac{1}{1-\mathrm{Np}}} t^{\frac{1}{1-\mathrm{Np}}},$$

provided $\mathrm{Np} < 1$ and m is a positive constant. Then, for each $\mathrm{T}_0 > 0$ the profile function

$$\mathcal{T}(t) = \begin{cases} \Theta\big(\mathrm{T}_0 - t\big), & \text{if } 0 < t \leq \mathrm{T}_0, \\ 0, & \text{otherwise,} \end{cases}$$

satisfies

$$\mathcal{T}'(t)\mathrm{m} + \big(2\mathcal{T}(t)\big)^{\mathrm{Np}} = 0, \quad t > 0. \tag{70}$$

On the other hand, for $\mathrm{R} > 0$ large, we consider the function

$$\zeta(x) = 2^{\mathrm{N}-1}\big(x_1^2 - \mathrm{R}^2\big) \leq 0, \quad x \in \overline{\Omega}$$

which verifies

$$\begin{cases} -m < \zeta(x) < -\mathrm{M} < 0, \quad x \in \overline{\Omega}, \quad -\mathrm{m} \doteq \min\limits_{x\in\overline{\Omega}} \zeta, \ -\mathrm{M} \doteq \max\limits_{x\in\overline{\Omega}} \zeta \\ \det \mathrm{D}^2\zeta(x) \equiv 2^{\mathrm{N}}, \quad x \in \overline{\Omega}. \end{cases}$$

It enables us to define

$$V(x,t)=\mathcal{T}(t)\zeta(x),\quad (x,t)\in\overline{\Omega}\times\overline{\mathbb{R}}_+,$$

for which $v(x,t)\leq 0,\ (x,t)\in\partial\Omega\times\mathbb{R}_+$ and $V(x,0)\leq -\Theta\big(\mathrm{T}_0\big)\mathrm{M},\ x\in\overline{\Omega}$, whence

$$v(x,0)\leq\big(u_0-h\big)(x),\quad x\in\overline{\Omega},$$

provided $\mathrm{T}_0=\Theta^{-1}\big(\|h-u_0\|_\infty\mathrm{M}^{-1}\big)$. Moreover, for each $(x,t)\in\Omega\times\mathbb{R}_+$ one has

$$v_t(x,t)+\big(-\det\mathrm{D}_x^2v(x,t)\big)^{\mathrm{p}}\leq-\mathcal{T}'(t)\mathrm{m}+f_{\mathrm{p}}^{-1}\Big(-2\big(\mathcal{T}(t)\big)^{\mathrm{N}}\Big)=0$$

(see (70)). Thus

$$v_t(x,t)-\big(\det\mathrm{D}_x^2v(x,t)\big)^{\mathrm{p}}\leq 0,\quad (x,t)\in\Omega\times[0,\mathrm{T}].$$

This function V can be considered as an eventual test function for the viscosity solution u_h (see (65)), then, arguing as in the proof of Theorem 2.4, we deduce

$$v(x,t)\leq u_h(x,t)\leq 0,\quad (x,t)\in\overline{\Omega}\times[0,\mathrm{T}[,$$

whence the finite global flattened time property holds. □

Remark 4.16. In fact, the condition $\mathrm{Np}<1$ is also necessary for the property of "finite global flattened time" as we have shown in Theorem 4.14. The general case $g\big(|\mathbf{p}|\big)\geq 1$ is studied in [25]. □

Remark 4.17. The study of the problems (2) and (5) can be complemented with some other qualitative studies such other symmetrization properties (the "comparison of the rearrangements"). That was carried out in [9] in the case of some related stationary problems. The consideration of the parabolic problem is the main goal of the paper [8]. □

Acknowledgement The authors wish acknowledge to the editors for their suggestions in order to improve the presentation of this contribution.

Appendix

A kind of Mikovsky inequality Given the positive numbers $a_1,\ldots,a_n$ and $b_1,\ldots,b_n$ one has the pondered arithmetic mean

$$\frac{1}{p}\sum_{i=1}^{n}\frac{a_i^p}{\displaystyle\sum_{i=1}a_i^p}+\frac{1}{p'}\sum_{i=1}^{n}\frac{b_i^{p'}}{\displaystyle\sum_{i=1}b_i^{p'}}=1,\quad p>1,$$

then the Arithmetic-Geometric inequality gives

$$\sum_{i=1}^{n}\frac{a_i}{\Big(\displaystyle\sum_{i=1}a_i^p\Big)^{\frac{1}{p}}}\frac{b_i}{\Big(\displaystyle\sum_{i=1}b_i^{p'}\Big)^{\frac{1}{p'}}}\leq 1$$

thus

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1} b_i^{p'}\right)^{\frac{1}{p'}} \qquad \text{Hölder inequality (1889)}$$

The opposite inequality is valid if p or p' are negative. In both cases, equality holds is a_i^p and $b_i^{p'}$ are proportional. Next,

$$\begin{aligned}\sum_{i=1}^{n}(a_i+b_i)^p &= \sum_{i=1}^{n} a_i(a_i+b_i)^{p-1} \sum_{i=1}^{n} b_i(a_i+b_i)^{p-1} \\ &\leq \left(\left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}}\right) \left(\sum_{i=1}^{n}(a_i+b_i)^{(p-1)p'}\right)^{\frac{1}{p'}}\end{aligned}$$

implies

$$\left(\sum_{i=1}^{n}(a_i+b_i)^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} b_i^p\right)^{\frac{1}{p}} \qquad \text{Minkosky inequality (1889)}$$

If $p < 1;\ p \neq 0$ the reverse inequality is valid. In both cases, equality holds is a_i and b_i are proportional. In particular, if $n = 2$ and $a_1 = a^{\frac{1}{p}},\ b_2 = b^{\frac{1}{p}},\ a_2 = b_1 = 0$ one has

$$(a+b)^{\frac{1}{p}} \leq a^{\frac{1}{p}} + b^{\frac{1}{p}},$$

whence

$$\left(\sum_{i=1}^{n} a_i\right)^{\frac{1}{p}} \leq \sum_{i=1}^{n} a_i^{\frac{1}{p}}$$

If $p < 1;\ p \neq 0$ the reverse inequality is valid.

References

[1] Aleksandrov, A.D.: Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected to it, *Uzen. Zap. Leningrad. Gos. Univ.*, **37** (1939), 3–35. (Russian)

[2] Álvarez, L.: On the behavior of the free boundary of some nonhomogeneous elliptic problems, *Appl. Anal.*, **36** (1990), 131–144.

[3] Álvarez, L., Díaz, J.I.: On the retention of the interfaces in some elliptic and parabolic nonlinear problems, *Discrete Contin. Dyn. Syst.*, **25**(1) (2009), 1–17.

[4] Ambrosio, L.: Lecture Notes on Optimal Transport Problems, *Mathematical Aspects of Evolving Interfaces*, Springer Verlag, Berlin, Lecture Notes in Mathematics (1812), (2003), 1–52.

[5] Ampère, A.M.: Mémoire contenant l'application de la théorie, *J. l'École Polytechnique*, 1820.

[6] Barles, G., Busca, J.: Existence and comparison results for fully nonlinear degenerate elliptic equations without zeroth-order term, *Comm. in P.D.E.*, **26** (11&12) (2001), 2323–2337.

[7] Bernis, F., Hulshof, J., Vázquez, J.L.: A very singular solution for the dual porous media equation and the asymptotic behaviour of general soliutions, *J. reine und angew. Math.*, **435** (1993), 1–31.

[8] Brandolini, B., Díaz, J.I.: *work in progress.*

[9] Brandolini, B., Trombetti, C.: Comparison results for Hessian equations via symmetrization, *J. Eur. Math. Soc. (JEMS)*, **9**(3) (2007), 561–575.

[10] Budd, C., Galaktionov, V.: On self-similar blow-up in evolution equations of Monge-Ampère type, *IMA J. Appl. Math.*, (2011), doi: 10.1093/imamat/hxr053.

[11] Caffarelli, L.: Some regularity properties of solutions of the Monqe-Ampère equation, *Comm. Pure Appl. Math.* **44** (1991), 965–969.

[12] Caffarelli, L., Nirenberg, L., Spruck, J.: Nonlinear second-order elliptic equations. V. The Dirichlet problem for Weingarten hypersurfaces. *Comm. Pure Appl. Math.*, **42** (1988), 47-70.

[13] Caffarelli, L., Salsa, S.: A Geometric Approach to Free Boundary Problemas, *American Mathematical Society*, 2005.

[14] Chopp, D., Evans, L.C., Ishii, H.: Waiting time effects for Gauss curvature flows, *Indiana Univ. Math. J.*, **48** (1999), 311–334.

[15] Crandall, M.G., Liggett, T.M.: Generation of semigroups of nonlinear transformations on general Banach spaces, *Amer. J. Math.*, **93** (1971), 265-298.

[16] Crandall, M.G., Ishii, H., Lions, P.-L.: Users guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc.*, **27** (1992), 1-67.

[17] Daskalopoulos, P., Hamilton, R.: The free boundary in the Gauss curvature flow with flat slides. *J. reine angew. Math.*, **510** (1999), 187–227.

[18] Daskalopoulos, P., Lee, K.: Free-boundary regularity on the focusing problem for the Gauss Curvature Flow with flat sides*Math. Z.*, **237** (2001), 847–874.

[19] Daskalopoulos, P., Savin, O.: $\mathcal{C}^{1,\alpha}$ regularity of solutions to parabolic Monge–Ampère equations, *to appear.*

[20] Díaz, G.: Some properties of second order of degenerate second order P.D.E. in non-divergence form, *Appl. Anal.*, **20** (1985), 309-336.

[21] Díaz, G.: The influence of the geometry in the large solution of Hessian equations perturbated with a superlinear zeroth order term, *work in progress.*

[22] Díaz, G.: The Liouville Theorem on Hessian equations perturbated with a superlinear zeroth order term, *work in progress.*

[23] Díaz, G., Díaz, J.I.: Finite extinction time for a class of non–linear parabolic equations, *Comm. in Partial Equations*, **4**(11) (1979), 1213–1231

[24] Díaz, G., Díaz. J.I.: On some free boundary problems arising in fully nonlinear equation involving hessian functions. I The stationary equation, *work in progress.*

[25] Díaz, G., Díaz. J.I.: On some free boundary problems arising in fully nonlinear equation involving Hessian functions. II The evolution equation, *work in progress.*

[26] Díaz, J.I.: *Nonlinear Partial Differential Equations and Free Boundaries, Vol. 1 Elliptic Equations*, Res. Notes Math, **106**. Pitman, 1985.

[27] Díaz, J.I., Mingazzini, T., Ramos, A. M.: On an optimal control problem involving the location of a free boundary, *Proceedings of the XII Congreso de Ecuaciones Diferenciales y Aplicaciones /Congreso de Matemática Aplicada (Palma de Mallorca), Spain, Septembre*, (2011), 5-9.

[28] Fiery, W.J.: Shapes of worn stones, *Mathematika*, **21** (1974), 1–11.

[29] Gangbo, W., Mccann, R.J.: The geometry of optimal transportation, *Acta Math.*, **177** (1996), 113–161.

[30] Gilbarg, D., Trudinger, N.S.: *Elliptic Partial Differential Equations of Second Order* Springer–Verlag, Berlin,(1983).

[31] Goursat, E.: Leçons sur l'Integration des Équations aux Derivées Partielles du Second Order à Deux Variables Indepéndantes, Herman, Paris, 1896.

[32] Guan, P., Trudinger, N.S., Wang, X.: On the Dirichlet problem for degenerate Monge–Ampère equations, *Acta Math.*, **182** (1999), 87-104.

[33] Gutiérrez, C.E.: *The Monge–Ampère equation*, Birkhauser, Boston, MA, 2001.

[34] Hamilton, R.: Worn stones with at sides; in a tribute to Ilya Bakelman, *Discourses Math. Appl.*, **3** (1993), 69–78.

[35] Igvida, N.: Solutions auto–similaires pour une equation de Barenblatt, *Rev. Mat. Apl.*, **17**, (1991), 21–30.

[36] Lions, P.L.: Sur les equations de Monge-Ampère I, II, Manuscripta Math., **41** (1983), 1–44; *Arch. Rational Mech. Anal.*, **89** (1985), 93–122.

[37] Monge, G.: Sur le calcul intégral des équations aux differences partielles, *Mémoires de l'Académie des Sciences*, (1784).

[38] Nirenberg, L.: Monge–Ampère equations and some associated problems in Geometry, in *Proccedings of the International Congress of Mathematics*, Vancouver 1974.

[39] Pucci, P., Serrin J.: *The Maximum Principle*, Birkhäuser, Basel, 2007.

[40] Talenti, G.: Some estimates of solutions to Monge–Ampère type equations in dimension two, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (4) **VIII**(2) (1981), 183–230.

[41] Trudinger, N.S.: The Dirichlet problem for the prescribed curvature equations. *Arch. Ration. Mech. Anal.*, **111** (1990), 153-179.

[42] Trudinger, N,S., Wang, X.-J.: The Monge-Ampère equation and its geometric applications, in *Handbook of Geometric Analysis*, Vol. I, International Press (2008), 467–524.

[43] Urbas, J.I.E.: On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations. *Indiana Univ. Math. J.*, **39** (1990) 355–382.

[44] Villani, C.: *Optimal transport: Old and New*, Springer Verlag (Grundlehren der mathematischen Wissenschaften), 2008.

[45] Vázquez, J.L.: A strong Maximum Principle for some quasilinear elliptic equations, *Appl Math Optim.*, **12** (1984), 191–202.