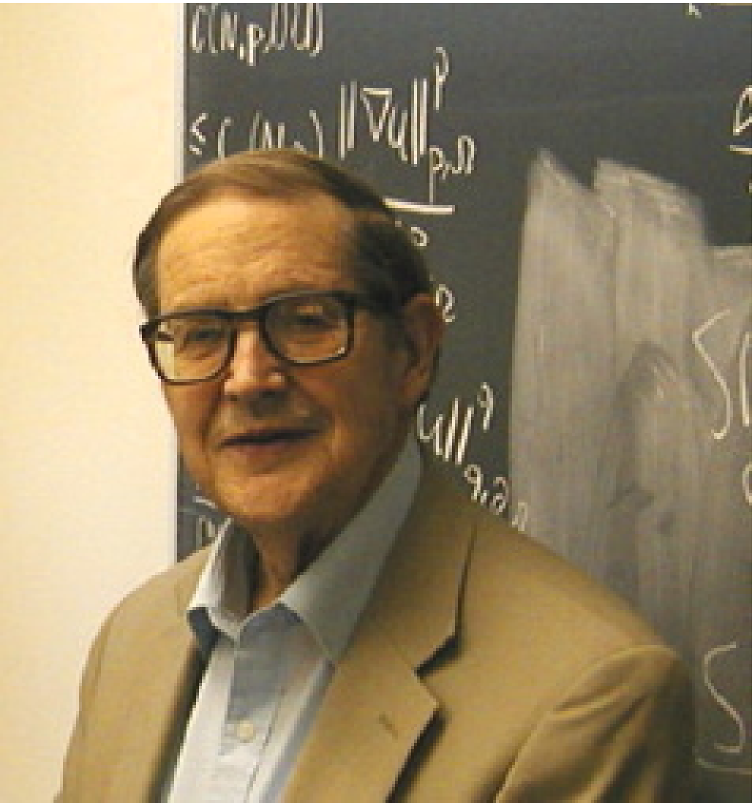
Beyond regular Lagrangians:
bifurcation branches of variational solutions to some singular elliptic equations and solutions with compact support of some Hamiltonian systems



J. I. Díaz

Universidad Complutense de Madrid

Introduction

We consider two different questions related to some problems arising in the Calculus of Variations and Analytical Dynamics in which the Lagrangian function $L(x,u,u_x)$ is not regular in the usual sense

$$(L_{pp}((x,u,p)>0)$$

but $L(x,u,u_x)$ is strictly convex in the sense of Weierstrass. In the first part, by a variant of a result by E. L. Lindelöf (1861), we study the number of solutions (bifurcating diagrams) of

$$\begin{cases} g'(u_x)_x = H(u)(g(u_x)-g'(u_x)u_x) \quad \text{in } (0,l), \\ u(0)=u_0,\ u(l)=\lambda, \end{cases}$$

as stationary points of the functional

$$J(u)=\int_0^l f(u(x))g(u_x(x))dx,$$

when it is assumed that $$H(r) = \frac{f'(r)}{f(r)}.$$

The result applies to some singular equations of the form

$$(|u_x|^{m-2}\ u_x)_x = \frac{1}{u}\ |u_x|^m$$

for any m>1.

In a second part, we apply the rigorous derivation of the Hamiltonian systems associated to such type of Lagrangians made by P. Pucci and J. Serrin (1994)

in order to study the existence of solutions with compact support to some Hamiltonian systems of the type

$$\begin{cases} \dot{p} = -g(q) \\ \dot{q} = -a(p) \end{cases}$$

with g and a continuous nondecreasing functions such that g(0)=a(0)=0.

2. Bifurcation branches of variational solutions to some singular elliptic equations

Consider the nonlinear boundary value problem depending on a parameter

$$\begin{cases} g'(u_x)_x = H(u)(g(u_x) - g'(u_x)u_x) & \text{in } (0,l), \\ u(0) = u_0, \ u(l) = \lambda, \end{cases}$$

A first example of functions g and H satisfyng the concrete assumptions we shall ask later leads to the singular equation

$$(|u_x|^{p-2} \ u_x)_x = \frac{1}{u} |u_x|^p$$

for any given p>1.

The main idea of our study starts with the analysis of the Euler-Lagrange equation for

$$J(u) = \int_0^l f(u(x))g(u_x(x))dx$$

on the set

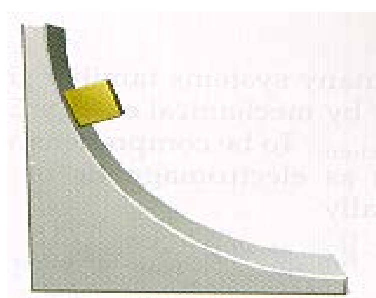
$$X = \{u \in C^1(0,l) \cap C^0[0,l],\ g(u_x) \in L^1(0,l) : u(0) = u_0,\ u(l) = \lambda\}$$

where f(u) is related to H(u) by the condition

$$H(r) = \frac{f'(r)}{f(r)}.$$

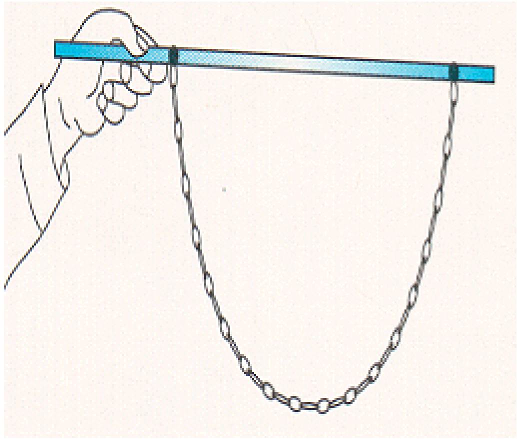
Our study is closed to some classical results of the **Calculus of Variations**: catenoids, mínimal surfaces of revolution and brachystochrones. In those cases the functional J corresponds to

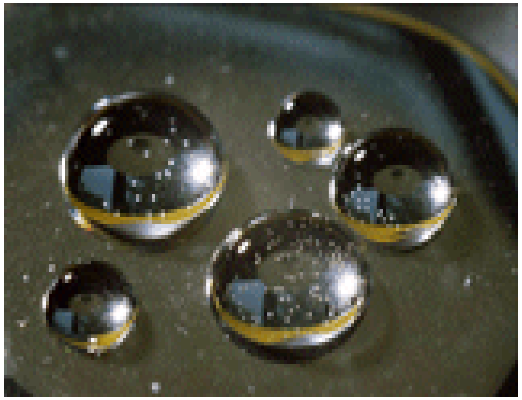
$g(s) = \sqrt{1+s^2}$ and $f(u) = u^{\alpha}$ with $\alpha = -1/2$ Brachystochrone:

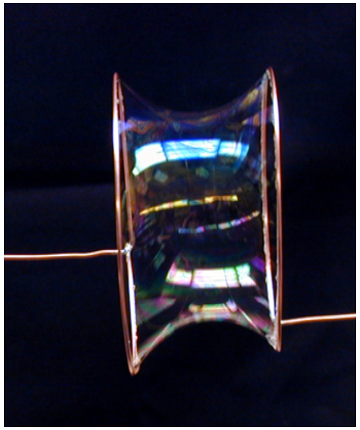


$$g(s) = \sqrt{1+s^2} \text{ and } f(u) = u^{\alpha} \text{ with } \qquad \alpha = 1$$

Catenoids and mínimal surfaces of revolution







M. Giaquinta and S. Hildebrandt, Calculus of Variations I, Springer-Verlag, Berlin, 1996,

H. Brezis and L. Boccardo (2001)

In the catenoid and mínimal surfaces of revolution a simple manipulation of the Euler-Lagrange equation leads to the singular equation

$$\left(\frac{u_x}{\sqrt{1+(u_x)^2}}\right)_x = \frac{1}{u\sqrt{1+(u_x)^2}}.$$

This equation also arises in the study "líquid bridges“ (experiences in spatial devices under microgravity for very sharp semiconductors).

Our results are closed to the Jacobi theory conjugated points

(C.G.J.Jacobi, Zur Theorie der Variations-Rechnung und der Theorie der Differential-Gleichungen. *Crelle's J. Reine Angew. Math*. **17**, 1837, 68-82).

Carl Gustav Jacob Jacobi

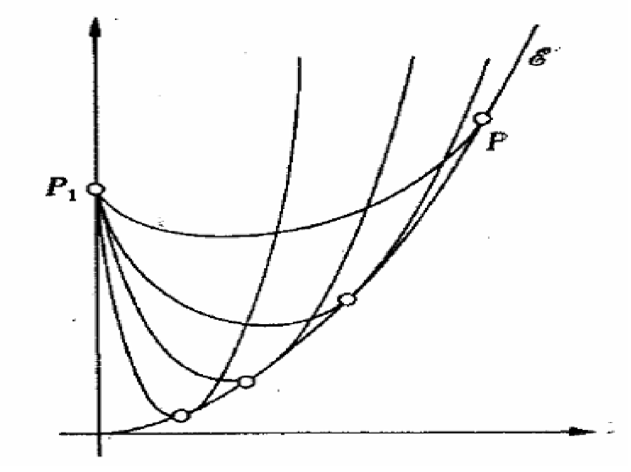
(1804-1851)

•It seems that the study of the number of catenoids (when one of the extreme points is arbitrary) is due to

E.L. Lindelof (*Leçons de calcul des variations*. Mallet-Bachhelier, Paris, 1861)

Ernst Leonard Lindelöf

(1870-1946)



- We shall avoid the *Legendre strict assummption* (which in our case would lead to the assumption $g''(s)>0 \ \forall s\in R$).

- Instead, we shall use the same approach than in

J.I.D., *Nonlinear Partial Differential Equations and Free Boundaries.* Research Notes in Mathematics nº106, Pitman, London, 1985,

when studying the sufficient conditions forthe occurrence of free boundaries for some second order quasilinear equations of a different type to the above singular equations.

- We shall show that in some cases there are exactly two solutions

- Our results lead to some bifurcation diagramms very similar to others arising in the study of some semilinear equations.

- Here, the bifurcating parametre de bifurcación is the second coordinate of P_2

Consider the problem

$$(P_\lambda)\begin{cases} g'(u_x)_x = H(u)(g(u_x) - g'(u_x)u_x) \\ u(0) = u_0,\ u(l) = \lambda \end{cases}$$

$$u_0, l > 0, \lambda \in [0, u_0)$$

under the structure conditions

$$g''(s) > 0,\ g'(s) > 0,\ g(-s) = g(s)\ \forall s > 0 \quad g(0) \geq 0, \tag{3}$$

$$sign(H(r)) = sign(g(s) - sg'(s)),\ \forall r, s > 0, \tag{4}$$

$$\begin{cases} \exists\ f : \mathbb{R} \to \mathbb{R}\ f \in C^1(0, +\infty) \text{ such that} \\ H(r) = \frac{f'(r)}{f(r)},\ \ f(r) > 0 \text{ si } r > 0. \end{cases} \quad (5)$$

Remark 1. Conditions (4) and (5) imply the strict monotonicity for function $f(r)$. So, if $sign(H(r)) = 1$ then necessarily $f'(r) > 0$ but if $sign(H(r)) = -1$ then $f'(r) < 0$ $\forall r > 0$.

Example 1. The study of catenoids and minimal surfaces of revolution corresponds to $g(s) = \sqrt{1+s^2}$. Then, $g'(s) = \frac{s}{\sqrt{1+s^2}}$ and $g(s) - sg'(s) = \frac{1}{\sqrt{1+s^2}} > 0$ and so $sign(H(r)) = 1$ (thus, for instance, $H(u) = \frac{\alpha}{u}$, and, thus, $f(r) = r^{\alpha}$, with $\alpha > 0$).

Example 2. If $g(s) = \frac{1}{2}s^2$ then $g(s) - sg'(s) = -\frac{1}{2}s^2$. More in general, if, $g(s) = \frac{1}{p}|s|^p$ with $p > 1$ we get that $g(s) - sg'(s) = -\frac{1}{p'}|s|^p$ (with $p' = \frac{p}{p-1}$) and so $sign(H(r)) = -1$ (this is the case, e.g., of $H(u) = \frac{-q}{u}$, and thus, $f(r) = r^{-q}$, con $q > 0$). Notice that the corresponding equation is $(|u_x|^{p-2} u_x)_x = \frac{q}{u}|u_x|^p$.

We shall need the auxiliary function

$$A(s) := sg'(s) - g(s).$$

Notice that $A'(s) = sg''(s)$ and so A is strictly increasing on $(0, +\infty)$ (respectively decreasing on $(-\infty, 0)$) due to (3). Moreover, $A(-s) = -sg'(-s) - g(-s) = A(s)$ if $s > 0$. So, in order to take the inverse function A we can connect both branches by the identity $\left(A^{-1}\right)_{-}(s) = -\left(A^{-1}\right)_{+}(-s)$ if $s < 0$. Notice that in Example 1 the two branches of A^{-1} are merely defined on $[-1, 0)$. In the Example 2 $\left(A^{-1}\right)_{+}(s) = (p's)^{1/p}$ which is defined $\forall s \geq 0$. In what follows we shall identify the positive branch $\left(A^{-1}\right)_{+}$ with A^{-1} if no confusion arises. We have

$$A(\mathbb{R}) = [r_A, R_A) \text{ with } -\infty < r_A < R_A \leq +\infty. \tag{6}$$

In Example 1 we have $r_A = -1$ and $R_A = 0$ and in Example 2 $r_A = 0$ and $R_A = +\infty$.

Theorem 1 *Assume (3),(4), (5). Suppose that we have one of the following two cases*

i) $g(s)=\sqrt{1+s^2}$ *y* $f(r)=r^{\alpha}$, *with* $\alpha>0$,

or

ii) $sign(g(s)-sg'(s))=-1$, $f(0)=+\infty$, $r_A=0$, $R_A=+\infty$ *and for any* $\lambda\geq 0$, $\tau, c>0$

$$\begin{cases} \int_{\lambda}^{\tau} \frac{ds}{A^{-1}(\frac{c}{f(s)})} = I_{\lambda}(c) < +\infty, I_{\lambda}(c) \text{ is decreasing in } c, \\ I_{\lambda}(c) \searrow 0 \text{ if } c \nearrow +\infty \text{ and } I_{\lambda}(c) \nearrow +\infty \text{ if } c \searrow 0. \end{cases} \tag{7}$$

Then, $\forall l\geq 0$ *there exists* $\lambda_0=\lambda_0(l)\geq 0$ *such that if* $\lambda\in[0,\lambda_0)$ *the problem* (P_{λ}) *do not have any solution, if* $\lambda=\lambda_0$ (P_{λ}) *has a unique solution and if* $\lambda\in(\lambda_0,u_0)$ *there are two solutions (with {u(x)=0}of zero measure). Any solution is a stationary point in* $X=\{u\in C^1(0,l)\cap C^0[0,l],\, g(u_x)\in L^1(0,l): u(0)=u_0,\, u(l)=\lambda\}$ *of the functional*

$$J(u)=\int_0^l f(u(x))g(u_x(x))dx.$$

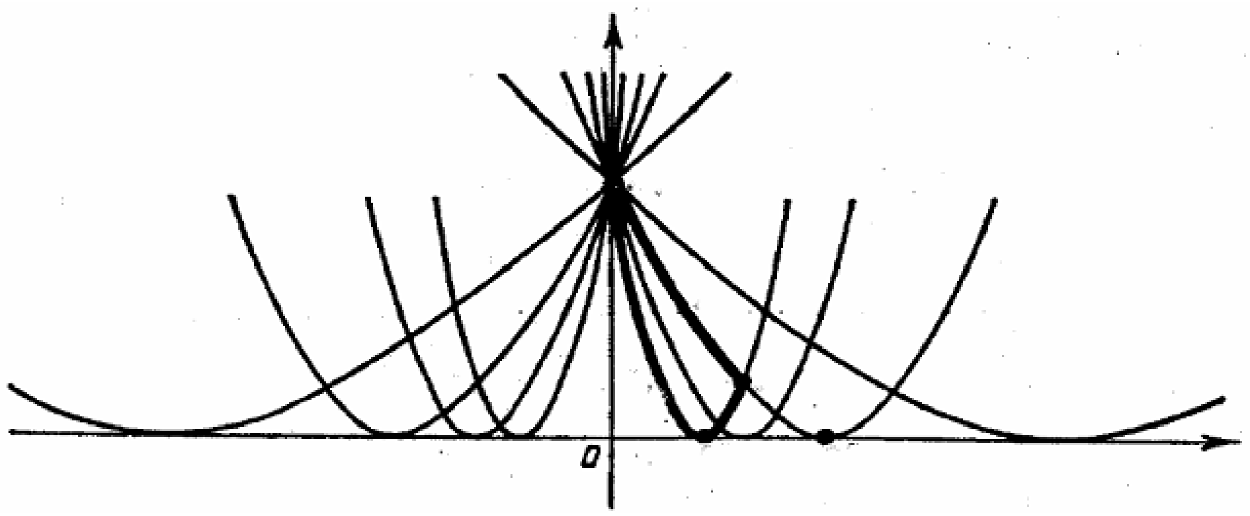
Moreover, in case *i) we have* $\lambda_0\in C^1([0,u_0))$, $\lambda_0(0)=0$ *and* $\lambda_0'(l)>0$ *if* $l>0$. *In case ii)* $\lambda_0(l)=0$ *for any* $l>0$.

Remark 2. In the case of Example 2, and if $f(r) = r^{-q}$, condition (7) holds if $0 < q < p$. It is possible to extends the conclusion of ii) when assumption (3) is replaced by

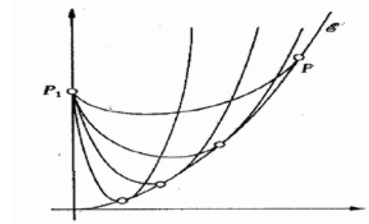
$$g''(s) > 0,\ g'(s) < 0,\ g(-s) = g(s)\ \forall s > 0 \text{ and } g(0) \geq 0. \tag{8}$$

Notice that in this case $sign(g(s) - sg'(s)) = 1$ and so conditions (4) and (5) lead to $f'(r) > 0$. An example of this situation is $g(s) = s^{-2}$ and $f(r) = r$ (quite similar to the "Newton problem ").

Comte-D. (2005, JEMS)



Proof. The proof of i) is a simple reformulation of well-know results (Lindelöf 1861) with $\lambda_0(l)$ given as the envolvent of an parametric family of curves.



For the proof of ii) we point out that the Euler-Lagrange equation for any stationary point of J becomes

$$(f(u)g'(u_x))_x = f'(u)g(u_x)$$

and that by assumption (5) it coincides with (P_λ). Since the Lagrangian in independent of x, $L(u,p) := f(u)g(p)$ there exists a constant c such that

$$u_x \frac{d}{dp} L(u, u_x) - L(u, u_x) = c,$$

i.e.

$$g'(u_x)u_x - g(u_x) = \frac{c}{f(u)},$$

and so, necessarily $c > 0$. On the other hand, any nonnegative solution must be convex since

$$u_{xx} = \frac{H(u)}{g''(u_x)}(g(u_x) - g'(u_x)u_x),$$

so, from (4), we deduce that necessarily $u_x < 0$ near $x = 0$. Thus

$$u_x = -A^{-1}\left(\frac{c}{f(u)}\right). \tag{9}$$

and then

$$\int_{u_0}^{u(x)} \frac{ds}{A^{-1}\left(\frac{c}{f(s)}\right)} = -x \tag{10}$$

once that $u_x(x) < 0$. Then, if we define the function (thanks to (7)) $\psi_c : [0, +\infty) \to [0, +\infty)$ by

$$\psi_c(\tau) = \int_0^\tau \frac{ds}{A^{-1}\left(\frac{c}{f(s)}\right)},$$

we get the identity

$$\psi_c(u(x)) - \psi_c(u_0) = -x. \tag{11}$$

But ψ_c is strictly increasing on $(0, +\infty)$ and so we can define its inverse function

$$\eta_c = (\psi_c)^{-1} \tag{12}$$

and thus

$$u(x) = \eta_c(\psi_c(u_0) - x). \tag{13}$$

The other boundary condition ($u(l) = \lambda$) requires

$$\lambda = \eta_c(\psi_c(u_0) - l)$$

or, equivalently

$$l = \psi_c(u_0) - \psi_c(\lambda) = \int_\lambda^{u_0} \frac{ds}{A^{-1}(\frac{c}{f(s)})}. \tag{14}$$

For any given l, u_0, condition (14) determines the constant $c = c(\lambda, l) > 0$ thanks to the assumption (7). Moreover, by construction, $u(x : c(\lambda, l))$ is strictly decreasing as function of x. If $\lambda = 0$ this function can not vanish except $x = l$.

Let us prove that if $\lambda \in (0, u_0)$ it is possible to construct a second solution. As before, there exists two values of the constant c, $c(0, \frac{l}{2})$ and $c(0, l)$, such that the corresponding solutions verify that

$$u(l : c(0, \tfrac{l}{2})) = u_0, \quad u(l : c(0, l)) = 0. \tag{15}$$

Using the continuity and the strict monotonicity of $u(x : c(\lambda, l))$ with respect to $c(\lambda, l)$, for any $c \in [c(0, l), c(0, \frac{l}{2})]$ there exists a unique $x_\lambda(c) \in (0, \frac{l}{2})$ such that $u(x_\lambda(c) : c) = \lambda$.

Extending by symmetry, with respect to x_0 the functions associated to any

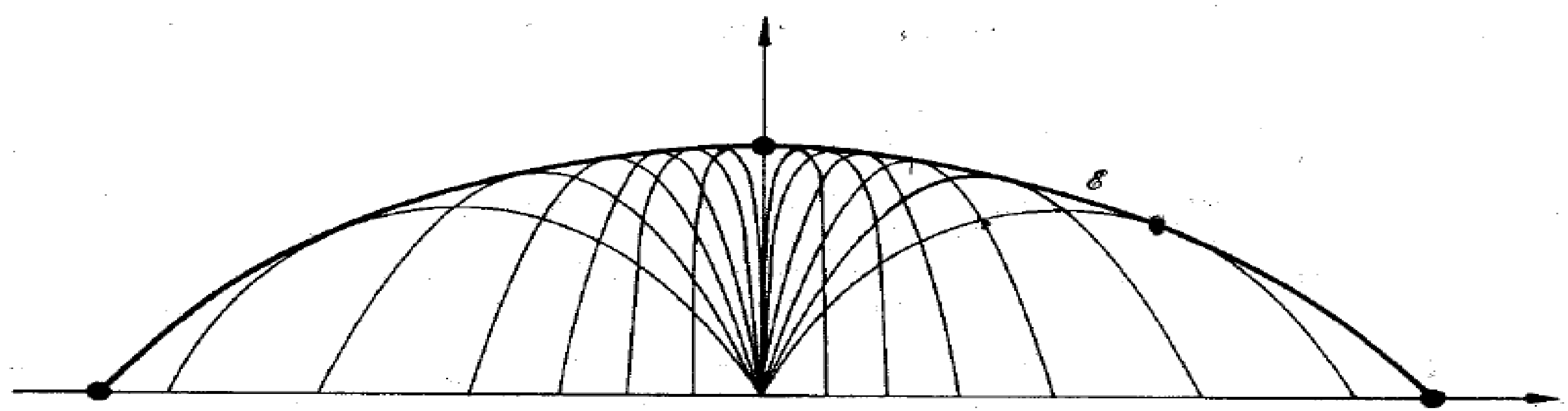
$$c \in [c(0, l), c(0, \tfrac{l}{2})]$$

we conclude that images points of $2x_0(c) - x_\lambda(c)$ cover the interval $[2\psi_{c(0,\frac{l}{2})}(u_0) - I_\lambda(c(0,\frac{l}{2})), 2\psi_{c(0,l)}(u_0) - I_\lambda(c(0,l))]$ which contents to l (thanks to (15)) and so there is a unique $c^* \in [c(0,l)), c(0,\frac{l}{2})]$ such that $u(l : c^*) = \lambda$ which provide the second solution.■

Remark 3. It is easy to see that the line $\lambda_0(l) = 0$ *for any* $l > 0$ is the envolvent of the family of curves $u(x) = \eta_c(\psi_c(u_0) - x))$ with $c \in (0, +\infty)$.

Remak 4. As in the classical case i) it can be proved that only the upper branch (maximal solutions) are minima of the functional J.

Remark 5. A similar result (after some suitable change of the assumptions) can be proved for concave solutions. The conclusion associated to pat (i) corresponds to the family of parabolas of Galileo.



3. Solutions with compact support of some Hamiltonian systems

Main motivation: Study of finite extinction time for (u, v) solution of the Hamiltonian system

$$\begin{cases} u_t - k_1 \Delta u = H_v(t, u, v) \\ v_t - k_2 \Delta v = -H_u(t, u, v) \end{cases}$$

(for simplicity) with $k_1, k_2 \geq 0$.

Remark. Several results in the literature on the blow up phenomenon for Hamoltonian systems (Ph. Souplet lecture) but lacke of results on extinction in finite time.

Super-simplified problem: consider the ordinary differential Hamiltonian system $(k_1 = k_2 = 0)$

$$\begin{cases} \dot{p} = H_q(t, p, q) \\ \dot{q} = -H_q(t, p, q) \end{cases}$$

or, to fix ideas,

$$\begin{cases} \dot{p} = -g(q) \\ \dot{q} = -a(p) \end{cases}$$

with g and a continuous increasing functions such that $g(0) = a(0) = 0$.

The Hamiltonian function is

$$H(p,q) = -A(p) + G(q)$$

where

$$G(s) = \int_0^s g(r)dr$$

and

$$A(s) = \int_0^s a(r)dr.$$

The Hamiltonian structure implies that any solution $(p(t), q(t))$ must satisfy that

$$H(p(t), q(t)) = -A(p_0) + G(q_0)$$

where

$$(p(0), q(0)) = (p_0, q_0).$$

Thus, if there is extinction in finite time we must have, necessarily, that

$$A(p_0) = G(q_0). \tag{H(0)}$$

Our main task is to identify the nature of the nonlinear terms g and a leading to this property.

We have

Theorem. *Assume H(0). Then, there is finite extinction time if and only if*

$$\int_0 \frac{ds}{\varphi^{-1}(G(s))} < \infty$$

where

$$\varphi = A^* \text{ (the Legendre transformed of } A).$$

Main idea of the proof (necessity). By taking $\widetilde{t} = T_e - t$ and $(\widetilde{p}(\widetilde{t}), \widetilde{q}(\widetilde{t})) = (p(t), q(t))$ we see that

$$\begin{cases} \dot{\widetilde{p}} = g(\widetilde{q}) \\ \dot{\widetilde{q}} = a(\widetilde{p}) \end{cases}$$

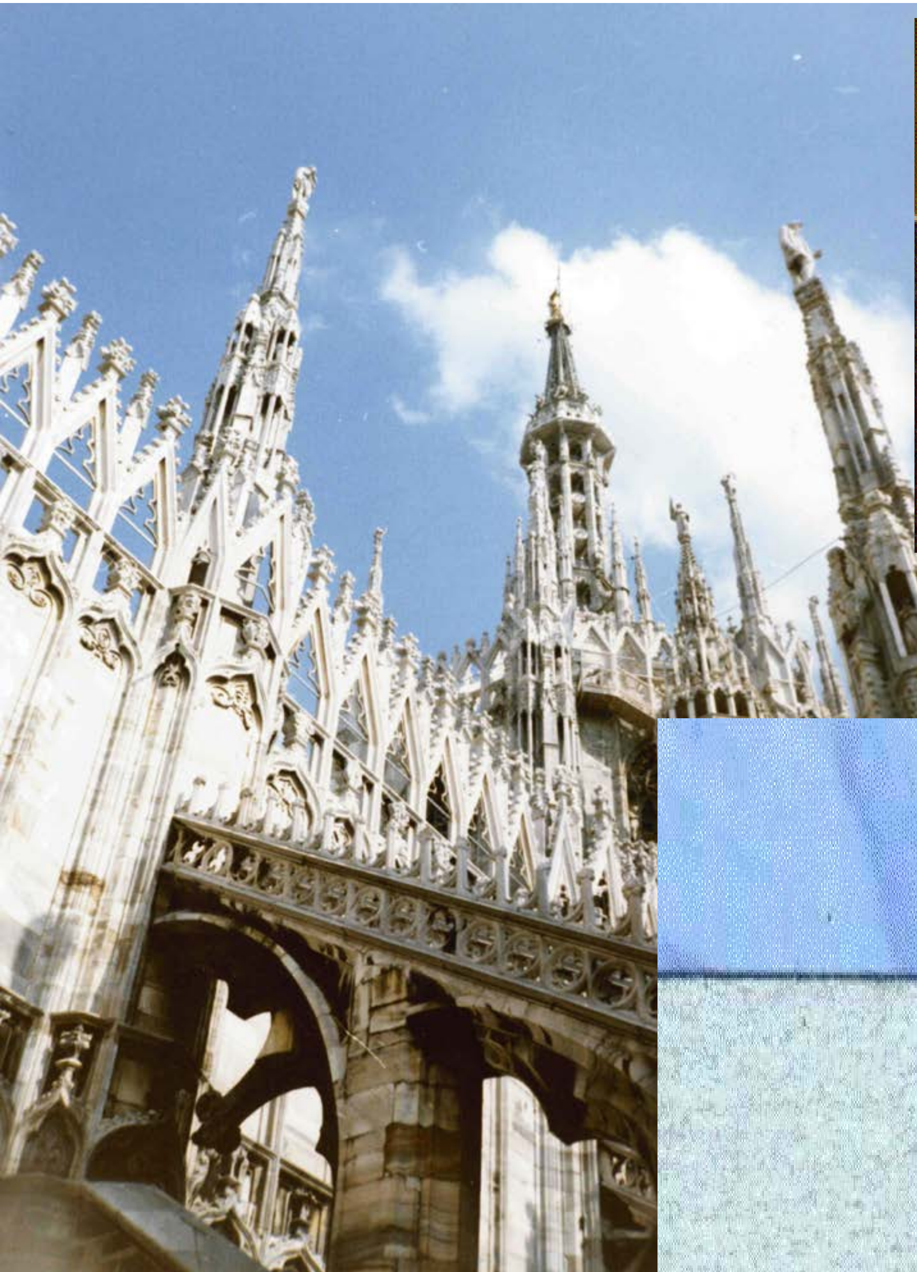
with

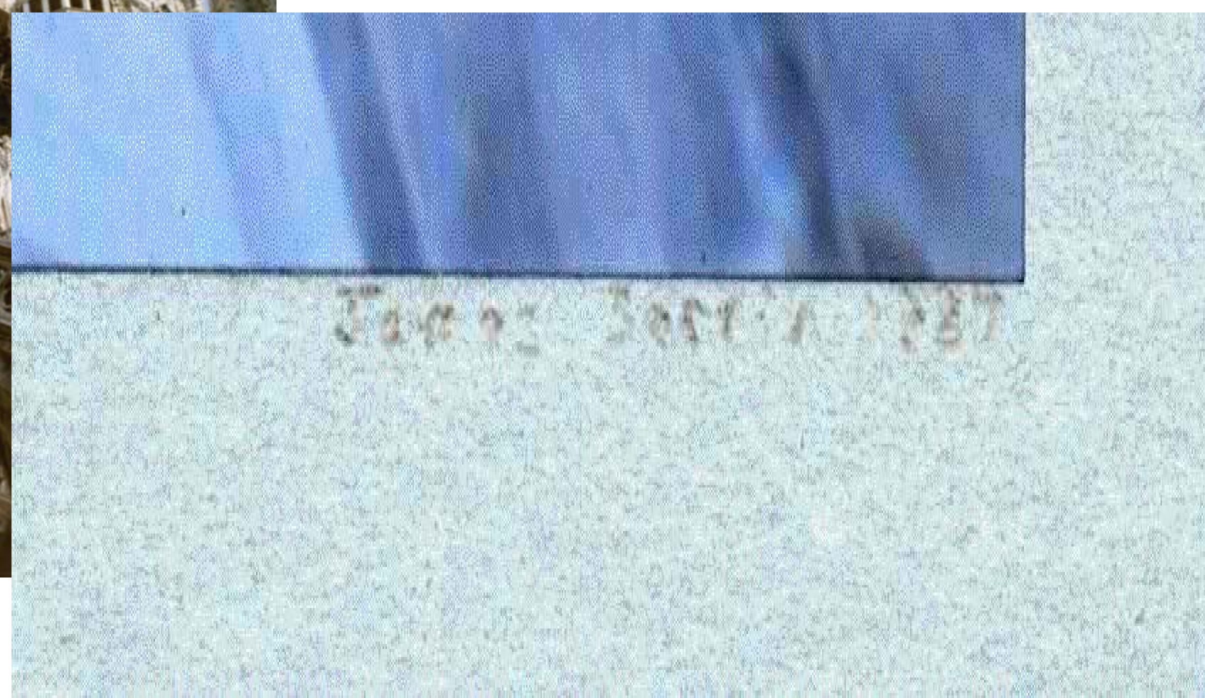
$$(\widetilde{p}(0), \widetilde{q}(0)) = (0, 0).$$

On the other hand, thanks to the Pucci-Serrin (1994) result, the problem can be equivalently formulated (Lagrangian formulation) as the Euler-Lagrange equation of the type

$$\frac{d}{dt}(Q(\dot{\widetilde{q}})) = g(\widetilde{q})$$

for a suitable function Q and, finally, it can be checked that the assumption coincides with the necessary (and sufficient) condition in order to have $\widetilde{q}(0) = \dot{\widetilde{q}}(0) = 0$ and $q(t) \neq 0$ for $t > 0$.■





Thanks for your attention