Estabilización para tiempos grandes hacia ondas viajeras en ecuaciones cuasilineales de tipo Fisher-KPP

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I - Introduction

Usual result on stabilization of solutions, as $t \to +\infty$, for evolution boundary value problems (specially in bounded domains) $\Omega \subset \mathbf{R}^N$, $N \geq 1$ and in absence of blow up: let $Q_\infty = \Omega \times (0, +\infty)$, $\Sigma_\infty = \partial\Omega \times (0, +\infty)$

$$\left\{ \begin{array}{lcll} u_t + Au & = & f(x,t) & \text{in } Q_\infty, \\ Bu & = & g(x,t) & \text{on } \Sigma_\infty, \\ u(x,0) & = & u_0(x) & \text{on } \Omega. \end{array} \right.$$

Here, Au denotes a nonlinear operator and Bu denotes a boundary operator (we assume, for simplicity, that A and B are *autonomous operators*).

In the study of the s it is usually assumed that $f(x,t) \to f_\infty(x)$ and $g(x,t) \to g_\infty(x)$ as $t \to +\infty$, in some functional spaces then $u(x,t) \to u_\infty(x)$, as $t \to +\infty$, in some functional space, with $u_\infty(x)$ solution of

$$\left\{ \begin{array}{lcll} Au_\infty & = & f_\infty(x) & \text{in } \Omega, \\ Bu_\infty & = & g_\infty(x) & \text{on } \partial\Omega. \end{array} \right.$$

When Ω is unbounded (e.g. $N = 1$ and $\Omega = (-\infty, +\infty)$) and $f(x,t) \equiv 0$ we see that, some times, any solution of $Au_\infty = 0$ must be a constant:

if Au is a pure diffusion operator then **any** constant $u_\infty = C$,
if Au is a diffusion-reaction operator then **only some** constants $u_\infty = C$,

Examples:

1. $Au = -u_{xx}$ or $Au = -\varphi(u)_{xx}$
2. $Au = -u_{xx} - u(1-u)$ or $Au = -\varphi(u)_{xx} - u + \varphi(u)$ with $\varphi(0)=0$, $\varphi(1)=1$ then the only constants solutions are $u_\infty(x) \equiv 1$ and $u_\infty(x) \equiv 0$.

The *Fisher equation* or *logistic equation*

$$u_t = u_{xx} + u(1-u)$$

R.A. Fisher, *The wave of advance of advantageous genes*, Annals of Eugenics **7** (1937), 355–369.

The *KPP equation* $\quad u_t = u_{xx} + c(u),$

with c differentiable for $0 \le u \le 1$,

$$c(0) = 0, \quad c(u) > 0 \quad \text{for } 0 < u < 1, \quad c(1) = 0$$

A. Kolmogorov - I. Petrovsky - N. Piscunov, Étude de l'équation de la diffusion avec croissance de la quantité de matiere et son application a un probleme biologique. Bulletin Univ. Moscou, Ser. Internationale, Math., Mec., 1, 1937, 1--25.

$$u_t = (u^{m-1}u_x)_x + u^p(1-u^q) \text{ with } m,\ p \text{ and } q \text{ positive parameters.}$$

J.D. Murray, *Lectures on Nonlinear-Differential-Equation Models in Biology*, Clarendon Press, Oxford, 1977.

J.D. Murray, *Mathematical Biology*, Springer-Verlag, Berlin, 1989.

Very special **transient patterns:**

if, *typically*, Au is a pure diffusion operator:

symmetric (fundamental, *Barenblatt type*) **solutions** $U_q(x,t)$

if, *typically*, Au is a diffusion-reaction operator

travelling waves solutions $U_q(x,t)$

here q is a parameter.

Note that "very special **transient patterns**" $\Longrightarrow$ "very special **initial data** $U_q(x,0)$"

Natural question: asymptotic behaviour of solutions with more general initial data $u_0(x)$???

Natural answer : $\|u(t,.) - U_q(t,.)\| \to 0$ as $t \to \infty$ "in some suitable sense".

Far to be trivial to prove !!!

For instance: how is selected the parameter q in terms of $u_0(x)$ among all the possible parameters ???

Some other references

Ya. I. Kanel, The behaviour of solutions of the Cauchy problem when the time tends to infinity in the case of quasilinear equations arising in the theory of combustion. Soviet Math. Dokl. 1, 1960, 533--536.

A.V. Volpert - Vi.A. Volpert - Vl.A. Volpert, Travelling wave solutions of parabolic systems. American Mathematical Society, Providence, Rhode Island, 1994, Translation of Mathematical Monographs, vol. 140.

B.H. Gilding - R. Kersner, *Travelling waves in Nonlinear Reaction-Convection-Diffusion*. Birkhäuser Verlag, Basel, 2004

G.S. Medvedev - K. Ono - P.J. Holmes, Travelling wave solutions of the degenerate Kolmogorov-Petrovski-Piskunov equation. *European Journal of Applied Mathematics*, **14**, 2003, 343--367.

Z. Biro, *On the stability of the travelling waves / Stability of travelling waves for degenerate parabolic reaction-diffusion equations of KPP-type,* *Advanced Nonlinear Studies* **2** (2002), 357-371.

Convergence for general initial data:

S. Kamin - P. Rosenau, Convergence to the Travelling Wave Solution for a Nonlinear Reaction-Diffusion Equation, *Rendiconti Mat. Acc. Lincei* 2004,

2. On the paper D-Kamin-Rosenau.

We consider the Cauchy problem for

$$u_t = \varphi(u)_{xx} + \psi(u) \ , \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

$\psi(s)$ vanishing only for $s = 0$ and $s = 1$.

Here we shall assume, mainly, that

$$\psi(u) = u - \varphi(u)$$

two constants stationary states $\overline{u}_\infty(x) \equiv 1$ and $\underline{u}_\infty(x) \equiv 0$

prove that under some conditions the solution of the Cauchy problem

converges, as $t \to \infty$, to a travelling wave linking, from $-\infty$ to $+\infty$, these constant

values, 1 and 0

the special case $\varphi(u) = u^m$ was studied in S. Kamin and P. Rosenau,

hypothesis on the function $\varphi(u)$

$$(H_1) \quad \varphi \in C^0(\mathbb{R}) \cap C^1[0,1] \cap C^2(0,1),\ \varphi'(s) > 0 \text{ for } s \in (0,1),\ \varphi(0){=}0\ ,\ \varphi(1){=}1\ .$$

It is obvious that if $\varphi(s)$ satisfies (H_1) then $\varphi'(0) \geq 0.$

If $\varphi'(0) = 0$ then the equation degenerates at the points $(x,t) \in \mathbb{R} \times \mathbb{R}^+$ where $u = 0$ and a notion of weak solution should be defined.

We shall also use the additional assumption:

$$(H_2): \quad \varphi(s) < s \quad \text{for all} \quad s \in (0,1)\ .$$

Cauchy problem with the initial data

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}\ .$$

$u_0 \in C^0(\mathbb{R})$, $0 \leq u_0(x) \leq 1$ a.e. $x \in \mathbb{R}$.

Let $S = \mathbb{R} \times \mathbb{R}^+$ and, for a given $T > 0$, $S_T = \mathbb{R} \times [0, T]$.

Definition By a solution of the Cauchy problem we mean a nonnegative function u such that $u \in C(S_T)$, for any $T > 0$, which satisfies the identity

$$\iint_{S_T} [\zeta_t u + \zeta_{xx}\varphi(u) + \zeta(u - \varphi(u))]dxdt+$$
$$+\int_{\mathbb{R}} \zeta(0,x)u_0(x)dx = \int_{\mathbb{R}} \zeta(T,x)u(T,x)dx$$

for any $\zeta \in C^{2,1}(S_T)$ which vanishes for large $|x|$.

We consider the set of travelling waves solutions with velocity equal to 1.

Definition Function $U(t,x) = f(x-t)$ is called a $(1,0)$-travelling wave *(in short* $(1,0)$-(TW)) if U is a solution and $f(\eta)$ links the constant values, 1 and 0 in the sense that $f(\eta) \to 1$ as $\eta \to -\infty$ and $f(\eta) \to 0$ as $\eta \to \infty$.

In fact, we shall see below that under assumption H_2 any $(1,0)$-(TW) is monotone decreasing.

In the book

B.H. Gilding - R. Kersner, *Travelling waves in Nonlinear Reaction-Convection-Diffusion*. Birkhäuser Verlag, Basel, 2004,

the detailed study of TW solutions is performed

Nevertheless we present the proof of existence of $(1,0)$-TW which is quite simple

To do that, let

$$J(y) = \int_{\frac{1}{2}}^{y} \frac{ds}{s - \varphi^{-1}(s)}.$$

Note that by H_2 we have $\varphi^{-1}(s) > s$ for $s \neq 0$, $s \neq 1$, therefore $J(y)$ is a decreasing function of y. Moreover, the integral may diverge at the points where $s = \varphi^{-1}(s)$,

that means for $s = 0$ and for $s = 1$. Let $\eta_0 = J(0)$

We have

$$0 < \eta_0 \leq \infty \ .$$

Lemma *Suppose hypothesis $H_1 - H_2$ are satisfied. Then there exist a $(1,0)$-TW $U(t,x) = f(x-t)$ where $f \in C(\mathbb{R})$ is given by*

$$\eta = J[\varphi(f(\eta))] \quad \textit{for} \quad \eta \in (\eta_1, \eta_2)$$

$$f(\eta) = 0 \quad \textit{for} \quad \eta \geq \eta_0$$

Any other $(1,0)$-TW *with velocity equal 1 is given by* $U(t,x) = f(x - x_0 - t)$ *where* x_0 *is an arbitrary point of* $\mathbb{R}$.

We say that the $(1,0)$-TW *has a sharp front at* $\eta = \eta_0$ if $\eta_0 < \infty$.

Clearly there are several possibilities for the behavior of TW solution. It may have a sharp front or not depending on the behaviour of $\varphi(s)$ near $s = 0$.

Examples 1) $\varphi(s) = s^m$, $m > 1$ a sharp front arises at $f = 0$. 2) $\varphi(s) = \frac{1}{3}s + \frac{2}{3}s^2$, $f(\eta) > 0$ for all η.

Remark The $(1,0)$-TW constructed above is propagating to the right. Changing x to $-x$ leads to a $(0,1)$-TW propagating to the left.

In order to formulate the main result we have to add the next hypothesis about the *weighted global integrability* of the profile of the travelling

$$(H_3) : I = \int_{-\infty}^{\infty} f(\eta) e^{\eta} d\eta < \infty$$

Remark It is clear that if $\eta_0 < \infty$ then $I < \infty$ but assumption is satisfied for a very large class of profiles having $\eta_0 = \infty$.

So, for instance, if we suppose that for some $\delta > 0$

$$\varphi(s) = \alpha s \quad \text{for all} \quad s \in [0, \delta] \ .$$

Then $I < \infty$ if $\alpha < \frac{1}{2}$ but $I = \infty$ if $\alpha \geq \frac{1}{2}$

The unique determination of a *weighted global integrable* $(1,0)$-TW travelling wave can be attaint by associating a special point $x_0 \in \mathbb{R}$ to it.

for any $x \in \mathbb{R}$ we can define the function

$$I(x) = \int_{-\infty}^{\infty} f(\eta - x) e^{\eta} d\eta$$

transforming increasingly $\mathbb{R}$ onto $(0, +\infty)$. Thus, given $q > 0$ there exists a unique $x_0 = x_0(q)$ such that

$$\int_{-\infty}^{\infty} f(\eta - x_0) e^{\eta} d\eta = q \ .$$

In what follows, given $q > 0$, we shall denote by $U_q(t,x)$ the *weighted global integrable* $(1,0)$-TW travelling wave

$$U_a(t,x) = f(x - t - x_0)$$

Our main result for the Cauchy problem is the following:

Theorem *Suppose that the assumptions $(H_1)-(H_2)$ and (H_3) are satisfied. Suppose also that $0 \le u_0(x) \le 1$, $u_0 \in C(\mathbb{R})$, $u_0(x) \not\equiv 0$,*

$$\int_{-\infty}^{\infty} u_0(x) e^x dx = q < \infty$$

Then

$$\int_{-\infty}^{\infty} |u(t,x) - U_q(t,x)| e^{x-t} dx \to 0$$

as $t \to \infty$

For the proof of this theorem we shall use next properties of the solutions

(P_1) There exists a bounded weak solution of the Cauchy problem

(P_2) *Comparison principle*: if u_1 and u_2 are weak solutions and $u_1(0,x) \le u_2(0,x)$ then $u_1(t,x) \le u_2(t,x)$ all $x \in \mathbb{R}$, $t \in \mathbb{R}^+$.

(P_3) This solution is classical at the points where $0 < u(t,x) < 1$.

(P_4) If $u(t_0, x_0) > 0$ then $u(t, x_0) > 0$ for all $t > t_0$.

(P_5) For every x_0 there exists $T = T(x_0)$ such that $u(t, x_0) > 0$ for every $t \geq T(x_0)$.

Notice that the comparison principle implies the uniqueness of bounded weak solution. In fact we shall use some peculiar form of construction of the bounded weak solution

$(P_6 = P_1')$ There exists a bounded weak obtained as the limit of the classical solutions u_ϵ, $0 < u_\epsilon(t,x) < 1$, $u_\epsilon \to u$ uniformly on any bounded set.

$(P_7 = P_1'')$ If $u_0^\epsilon \to u_0$ uniformly on any compact set of $\mathbb{R}$, then $u^\epsilon(t,x) \to u(t,x)$ uniformly on any $K \subset \mathbb{R}^+ \cap \mathbb{R}$

the equation can be rewriten in the form

$$e^{dt}(e^{-dt}u)_t = e^{-bx}(e^{bx}\varphi(u))_{xx} \qquad \text{for some } d \text{ and } b.$$

(P_8) Every sequence of uniformly bounded solutions is equicontinuous on every compact set of S_T for any $T > 0$.

Ph. Benilan, J.I. Díaz, Pointwise gradient estimates of solutions of onedimensional nonlinear parabolic problems, *J. Evolution Equations*, **3** (2004) 557-602.

Theorem *For a given $T>0$ let $Q=\mathbb{R}\times(0,T)$. Suppose that $0\leq u(x,t)\leq 1$, $\psi(1)\leq 0\leq\psi(0)$. Then*

$$\varphi(u)_x^2\leq\frac{\max(u,1-u)}{t}\{2T\sup_{[0,1]}\Phi^+ + 2T\int_0^1(\Phi'(s))^-ds\}. \qquad \Phi(u)=\psi(u)\varphi'(u)$$

In particular, if $\psi(u)=u-\varphi(u)$ and we assume (H_2) then

$$|\varphi(u(x,t))_x|\leq\sqrt{\frac{2T\Lambda}{t}} \text{ for any } (x,t)\in\mathbb{R}\times(0,T)$$

with

$$\Lambda=\int_0^1\varphi'(s)^2ds.$$

To prove Theorem we need several lemmas,

Lemma *Suppose that* $u_0(x)$ *has compact support at the right and* $u_0(x) \leq 1-\delta$, $\delta > 0$. *Then for some* $q^* > 0$ $\quad u(t,x) \leq U_{q^*}(t,x) \quad$ *and for all* $t \geq 0$

$$\int_{-\infty}^{\infty} u(t,x)e^{x-t}dx \leq C < \infty.$$

Lemma *(weighted conservation law). For all* $t > 0$

$$\int_{-\infty}^{\infty} u(t,x)e^{x-t}dx = \int_{-\infty}^{\infty} u_0(x)e^{x}dx \ .$$

Remark

The presence of the weight e^{x-t} in the above conservation law is essential.

Notice, for instance, that in fact

$$\int_{-\infty}^{\infty} u(t,x)dx > \int_{-\infty}^{\infty} u_0(x)dx$$

Lemma *(weighted contraction principle) Let u and v be weak solutions*

$$\int_{-\infty}^{\infty} u_0(x)e^x dx < \infty\ , \quad \int_{-\infty}^{\infty} v_0(x)e^x dx < \infty\ .$$

Then

$$\int_{-\infty}^{\infty} |u(T,x) - v(T,x)|e^{x-T} dx \leq \int_{-\infty}^{\infty} |u(\tau,x) - v(\tau,x)|e^{x-\tau} dx$$

for $0 \leq \tau \leq T$.

Multiply the difference of two equations by $e^{x-t}\eta_\ell(x)p[\varphi(u) - \varphi(v)]$, where η_ℓ is the cut function.

Then integrate by parts and let $p(s)$ tend the $\mathrm{sign}_+ s$. Passing to the limit as $\ell \to \infty$ we obtain

Proof of Theorem Assume first that $u_0(x)$ satisfies the assumptions of Lemma Let

$$u_h(t,x) = u(t+h, x+h), h > 0$$

and instead of study the behaviour of $u(t,x)$ as $t \to \infty$ we consider the behaviour of the sequence $\{u_h\}$ on bounded sets of S_T as $h \to \infty$. Such shifthing transformation plays here the same role as a scaling transformation for the proof of attractivity properties of self-similar solutions

S. Kamin, L.A. Peletier and J. L. Vázquez, Barenblatt...

Note that $U_q(t,x)$ is invariant with respect to the shifting for any q that means that $U_q(t+h, x+h) = U_q(t,x)$.

It follows

$$\int_{-\infty}^{\infty} u_h(t,x) e^{x-t} dx = \int_{-\infty}^{\infty} u(t+h, y) e^{y-t-h} dy = q \ .$$

and

$$u_h(t,x) = u(t+h, x+h) \leq U_{q^*}(t+h, x+h) = U_{q^*}(t,x)$$

Sequence $\{u_h(t,x)\}$ is uniformly bounded, and thus,by (P_8), is equicontinuous on any bounded set in $\mathbb{R}^+ \times \mathbb{R}$

Therefore there exists a subsequence $h_i \to \infty$ such that

$$u_{h_i}(t,x) \longrightarrow w(t,x) \ ,$$

and the convergence is uniform on any bounded set. The limit function w is defined for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ and is a weak solution It follows $w(t,x) \leq U_{q^*}(t,x)$. and

$$\int |u_{h_i}(t,x) - w(t,x)| e^{x-t} dx \longrightarrow 0 \text{ as } h_i \longrightarrow \infty$$

$$\int w(t,x) e^{x-t} dx = q \ .$$

Let $\hat{q}$ be some fixed number. Define

$$I^h(t;\hat{q}) = \int |u_h(t,x) - U_{\hat{q}}(t,x)| e^{x-t} dx \ .$$

Obviously

$$I^h(t;\hat{q}) = \int |u(t+h,x+h) - U_{\hat{q}}(t+h,x+h)|e^{x-t}dx$$
$$= \int |u(t+h,y) - U_q(t+h,y)|e^{y-(t+h)}dy = I^0(t+h);\hat{q})$$

By the contraction principle $I^0(t+h;\hat{q})$ is a nonincreasing function of t and h, therefore, there exists

$$\lim_{h\to\infty} I^h(t;\hat{q}) = I^\infty(\hat{q}) \geq 0 \quad \text{for all} \quad t$$

It follows

$$\int_{-\infty}^{\infty} |w(t,x) - U_{\hat{q}}(t,x)|e^{x-t}dx = I^\infty(\hat{q}) \ .$$

Lemma *Suppose that*

$$w(0,x_1) = U_{\tilde{q}}(0,x_1) \in (0,1)$$

for some $\tilde{q}$ *and some* x_1. *Then*

$$\frac{\partial w}{\partial x}(0,x_1) = \frac{\partial U_{\tilde{q}}}{\partial x}(0,x_1)$$

In order to proceed with the proof of Theorem

we suppose that for all q

$$w(0,x) \neq U_{\tilde{q}}(0,x)$$

Then, for any point $x \in \mathbb{R}$ such that $w(0,x) \in (0,1)$ there exists some $\overline{q} = \overline{q}(x)$ such that $w(0,x) = U_{\overline{q}}(0,x)$. By Lemma we have $\frac{\partial w}{\partial x}(0,x) = \frac{\partial U_{\overline{q}}}{\partial x}(0,x)$

This means that $w(0,x)$ is the envelope of the set of curves $U_{\overline{q}}(0,x)$ and this is impossible. Hence $w(0,x) \neq U_{\tilde{q}}(0,x)$ is wrong.

Therefore, we finally proved that for some $\tilde{q}$

$$w(0,x) \equiv U_{\tilde{q}}(0,x)\ .$$

we obtain that $\tilde{q} = q$, therefore

$$w(t,x) = U_q(t,x)\ .$$

Hence $\lim_{h_i \to \infty} u_{h_i}$ does not depend on the subsequence and the whole sequence u_h converges to U_q. This convergence is in the "weighted" L^1 norm and, as follows from the proof, is uniform on any compact set. Thus we have

$$\int |u(t+h,x+h) - U_q(t+h,x+h)|e^{x-t}dx$$
$$= \int |u(\tau,y) - U_q(\tau,y)|e^{y-\tau}dy \to 0\ ,$$

as $\tau \to \infty$ and

$$|u(\tau, y) - U_q(\tau, y)| \to 0 \ ,$$

uniformly on every set

$$\alpha < y - \tau < \beta \ ,$$

for any fixed α, β.

Thus Theorem is proved for the case where $u_0(x)$ satisfies the assumptions of Lemma Now suppose that $0 \leq u_0(x) \leq 1$. Let $u_0^\epsilon(x)$ be a sequence of functions, each compactly supported from the right, and

$$\int |u_0^\epsilon(x) - u_0(x)| e^x dx \leq \varepsilon \ .$$

The sequence u_0^ϵ may be chosen such that

$$\int u_0^\epsilon(x) e^x dx = q \ .$$

Let $u^\epsilon(t, x)$ be the solution with initial data

$$u^\epsilon(0, x) = u_0^\epsilon(x).$$

By the contraction principle

$$\int |u^\epsilon(t,x) - u(t,x)| e^{x-t} dx \leq \varepsilon \ .$$

Moreover, as we proved above,

$$\int |u^\epsilon(t,x) - U_q(t,x)| e^{x-t} dx \to 0 \ ,$$

as $t \to \infty$. Because ε is arbitrarily small Theorem holds.

Other results:

a) Case of $u_0(x)$ with compact support

b) Case of other reaction terms

$$\psi(u) \neq u - \varphi(u)$$

We impose the following assumptions on $\psi(u)$:

$$\psi(s) \in C^1[0,\infty),\ \psi(0) = \psi(1) = 0,\ \psi(s) > 0 \ \text{ for } \ s \in (0,1),$$
$$\psi(s) < 0 \ \text{ for } \ s > 1\ ,\ \psi'(0) > 0,\ \psi'(1) < 0\ .$$

$$\int_0^\delta \frac{\varphi'(s)}{s} ds < \infty$$

Definition $(1,0) - TW$ solution of the equation is a weak solution of the form $U(t,x) = f(x - ct)$ such that $f(-\infty) = 1$, $f(+\infty) = 0$ and $f(s)$ is monotone decreasing.

there exists c^* such that there exists a unique TW solution for any $c \geq c^*$ and there does not exist any TW solution for $c < c^*$. Moreover solution U with the minimal speed has a sharp front, which means that $f(s) = 0$ for $s > s_0$ with some s_0.

Theorem *Suppose* $u_0 \not\equiv 0$ *and* $u_0(x) = 0$ *for* *all* $x \geq \tilde{x}$, *where* $\tilde{x}$ *is some fixed number. Then*

$$u(t,x) \to 1 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad 0 \leq x \leq ct \quad \text{for} \quad c < c^*$$

and

$$u(t,x) \to 0 \quad \text{as} \quad t \to \infty \quad \text{uniformly in} \quad x \geq ct \quad \text{for} \quad c > c^*\ .$$

Gracias por vuestra atención