On linear and semilinear elliptic problems with right hand side data integrable with respect to the distance to the boundary

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Dedicated to the memory of

Fuensanta Andreu

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Some Qualitative Properties for the Total Variation Flow

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$$\begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}\left(\dfrac{Du}{|Du|}\right) & \text{in } Q = (0,\infty)\times\Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & \text{on } S = (0,\infty)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } x\in\Omega, \end{cases}$$

$$u(t,x) = \varphi(x)$$

Motivated by the L. Rudin, S. Osher and E. Fatemi models for image restoration

20. **ANDREU, F., BALLESTER, C., CASELLES, V., MAZON, J. M.** *Minimizing Total Variation Flow.* Differential and Integral Equations **14** (2001), 321-360.
21. **ANDREU, F., BALLESTER, C., CASELLES, V., MAZON, J. M.** *The Dirichlet Problem for the Total Variation Flow* Journal of Functional Analysis. **180** (2001), 347-403.

$$u_0 \in L^1(\Omega) \text{ and } \varphi \in L^1(\partial\Omega)$$

Chiba (Japan, 1999): numerical experiences by Kobayashi, Giga,…

Finite extinction time, asymptotic profile, differences with p-Laplacian

$$u(t) = \overline{u_0} := \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx \qquad \forall t \geq T_0.$$

Remark 1. Theorem 4 could be compared with what happens in the study of the parabolic problem associated with the p-Laplacian operator. Consider the Dirichlet problem for the p-Laplacian:

$$P_D^p \begin{cases} \dfrac{\partial u}{\partial t} = \operatorname{div}(|Du|^{p-2}Du) & \text{in } Q = (0,\infty)\times\Omega \\ u(t,x) = 0 & \text{on } S = (0,\infty)\times\partial\Omega \\ u(0,x) = u_0(x) & \text{in } x \in \Omega, \end{cases}$$

with $1 < p < \infty$. The conditions on $\alpha(t)$ to generate a supersolution are

$$\alpha(t) \geq 0 \quad \text{and} \quad u_0(x) \leq \alpha(0) \quad \text{a.e. } x \in \Omega, \tag{3.5}$$

$$\alpha'(t) \geq 0, \tag{3.6}$$

and, in fact, those conditions are also sufficient for the total variation flow. Nevertheless, in the limit case $p = 1$, condition (3.6) can be substituted by the new one given in the above result (which is not the case of problem P_D^p).

Remark 4. It is well known (see [14, 15, 22]) that if $p > 2$ then there is *finite speed of propagation* (i.e., if $\operatorname{supp}(u_0) \subset B(0,r) \subset\subset \Omega$, then the solution of problem (P_D^p) satisfies that $\operatorname{supp}(u(t))$ is a compact set for any $t > 0$), but, if $1 < p \leq 2$ and $u_0 \geq 0$, $u_0 \neq 0$, then $u(t) > 0$ or $u(t) = 0$ in Ω for all $t > 0$ [15, 22]. Observe that (P_D) can be considered as the limit case $p = 1$ of problem (P_D^p) and the above result shows that there is no propagation of the support of the initial datum (or equivalently, there is an infinite waiting time). Finite time extinction of the solutions of (P_D^p) when $\frac{2N}{N+1} \leq p < 2$, $N \geq 2$ was proved in [8], and, for $1 < p < \frac{2N}{N+1}$, in [21] (see also [6, 25]). The same approach also proves the finite time extinction of solutions of (P_D) (see inequality (3.29) in the proof of Lemma 3).

Associated eigenvalue problem: first time in the literature, discontinuous solutions, tower solutions,…

THEOREM 3. *Let $u_0 \in L^\infty(\Omega)$ and let $u(t,x)$ be the unique solution of problem (P_D). Let $d(\Omega)$ be the smallest radius of a ball containing Ω. If $T^*(u_0) = \inf\{t > 0 : u(t) = 0\}$, then*

$$T^*(u_0) \leq \frac{d(\Omega)\,\|u_0\|_\infty}{N}. \tag{3.1}$$

Let

$$w(t,x) := \begin{cases} \dfrac{u(t,x)}{T^*(u_0) - t} & \text{if } 0 \leq t < T^*(u_0), \\ 0 & \text{if } t \geq T^*(u_0). \end{cases}$$

Then, there exists an increasing sequence $t_n \to T^*(u_0)$ and a solution $v^* \neq 0$ of the stationary problem

$$S_D \begin{cases} -\operatorname{div}\left(\dfrac{Dv}{|Dv|}\right) = v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$\lim_{n\to\infty} w(t_n) = v^* \quad \text{in } L^p(\Omega)$$

for all $1 \leq p < \infty$. Moreover v^* is a minimizer of $\Phi(\cdot) - \langle \cdot, v^* \rangle$ in $BV(\Omega) \cap L^2(\Omega)$.

Intelectual satisfactions, …

1. Introduction.

This talk concerns some works (in collaboration with J.M.Rakotoson) which started, many years ago, with an originally unpublished manuscript by H. Brezis (personal communication of him to the author),

I.D.D. 1

ne équation non linéaire avec conditions aux
mites dans L^1.
$\subset I\!R^N$ ouvert borné de frontière régulière Γ
$\delta(x)$ = distance de $x \in \Omega$ à Γ.
éliminaire linéaire

$$\Omega \subset I\!R^N, \; N \geqslant 2, \text{ a bounded open set}$$

$$\delta(x) = \operatorname{dist}(x, \partial\Omega)$$

Very weak solution

mme 1 Soit f mesurable sur Ω tel que $\delta f \in L^1(\Omega)$

- Soit $u_0 \in L^1(\Gamma)$.

lors il existe $u \in L^1(\Omega)$ unique vérifiant

$$- \int_\Omega u \, \Delta \varphi = \int_\Omega f \varphi \, dx - \int_\Gamma u_0 \frac{\partial \varphi}{\partial n}$$

$$\forall \varphi \in H^{2,\infty}(\Omega) \cap H_0^{1,\infty}(\Omega)$$

$H_0^{1,\infty}(\Omega)$ = fonctions lipsch. sur $\overline{\Omega}$, nulles sur Γ ;

[$\varphi \in H_0^{1,\infty}(\Omega)$, on a $|\varphi(x)| \leq \|\varphi\|_{Lip} \, \delta(x)$, de

te que $f\varphi \in L^1$]

e plus, il existe C, dépendant seulement de Ω

tel que

$$\text{2)} \quad \|u\|_{L^1(\Omega)} \leq C \left(\|\delta f\|_{L^1(\Omega)} + \|u_0\|_{L^1(\Gamma)} \right)$$

$$GD(\Omega) = \begin{cases} -\int_\Omega v\Delta\varphi\,dx = \int_\Omega f\varphi\,dx, \quad \forall\varphi \in V_1(\Omega), \\ \text{with } V_1(\Omega) = \left\{\varphi \in C^2(\overline{\Omega}),\ \varphi = 0 \text{ on } \partial\Omega\right\}, \end{cases}$$

$$|v|_{L^1(\Omega)} \leqslant c|f|_{L^1(\Omega,\,\mathrm{dist}\,(x,\partial\Omega))}.$$

Problemes

1) Est ce que $u \in W^{1,1}(\Omega)$?

H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited. *Advance in Diff. Eq.* **1**, (1996) 73-90.

L. Veron *Singularities of solutions of second order quasilinear equations*, Longman, Edinburgh Gate, Harlow,(1995) p 176-180.

X. Cabré, Y. Martel, Weak eigenfunctions for the linearization of extremal elliptic problems , *J. Funct. Anal.* **156**, (1998), 30-56.

P. Quittner, P. Souplet, *Superlinear Parabolic Problems*, Birkhäuser Basel (2007).

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On the differentiability of very weak solutions with right-hand side data integrable with respect to the distance to the boundary

J.I. Díaz [a,*], J.M. Rakotoson [b]

More recently

$$(u, L\varphi)_0 - (Vu, \varphi)_0 + \left(g(\cdot, u, \nabla u), \varphi\right)_0 = \mu(\varphi), \quad \forall \varphi \in C_c^2(\Omega).$$

J.I. Díaz, J.M. Rakotoson On very weak solutions of semilinear elliptic equations with right hand side data integrable with respect to the distance to the boundary, To appear.

2. On the differentiability for the linear case.

$$Lu = -\sum_{i,j=1}^{N} \partial_i (a_{ij}(x) \partial_j u) + \sum_{i=1}^{N} b^i(x) \partial_i u + c_0(x) u$$

$$a_{ij} \in C^{0,1}(\overline{\Omega}), \; b^i \in C^{0,1}(\overline{\Omega}), \qquad c_0 \in L^\infty(\Omega), \; c_0 \geqslant 0,$$

$$\sum_{i,j} a_{ij}(x) \xi_i \xi_j \geqslant \alpha |\xi|^2 \text{ for some } \alpha > 0, \qquad \forall \xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$$

$$c_0(x) - \frac{1}{2} \sum_{i=1}^{N} \partial_i b^i(x) \geqslant 0 \text{ a.e. in } \Omega.$$

$$L^* \varphi = -\sum_{i,j} \partial_j (a_{ij}(x) \partial_i \varphi) - \sum_{i=1}^{N} \partial_i (b^i \varphi) + c_0(x) \varphi$$

Theorem 1. *Let* $f \in L^1(\Omega, \delta)$ *and* $N' = \frac{N}{N-1}$.
Then there exists a unique function $v \in L^{N',\infty}(\Omega)$ *satisfying*

$$(DG_L(\Omega)) : \int_\Omega v L^* \varphi \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_0^1(\Omega).$$

Moreover, there exists a constant $c(\Omega, L) > 0$ *such that*

$$|v|_{L^{N',\infty}} \leqslant c(\Omega, L)|f|_{L^1(\Omega,\delta)}.$$

$$L^1(\Omega, \delta^\alpha) = \left\{ f : \Omega \to I\!R \text{ Lebesgue measurable} : \int_\Omega |f(x)|\delta(x)^\alpha \, dx \text{ is finite} \right\},$$

$$0 \leqslant \alpha \leqslant 1, \quad L^1(\Omega, \delta^1) = L^1(\Omega, \delta), \quad L^1(\Omega, \delta^0) = L^1(\Omega).$$

We used Lorentz's spaces:
George G. Lorentz (St. Petersburg, Russia 1910, Austin, Texas 2006).

R. Hunt, On L(p,q) spaces, *L'enseignement Math*. **12**, 1966 249-276

$1 < p < +\infty, \qquad q < +\infty$

$$L^{p,q}(\Omega) = \left\{ v : \Omega \to I\!R \text{ measurable } |v|^q_{L^{p,q}} = \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty \right\}$$

- the **decreasing rearrangement of a measurable function** u is given by

$$u_* : \Omega_* =]0, |\Omega|[\to I\!R, \quad u_*(s) = \inf\{t \in I\!R : |u > t| \leqslant s\}$$
$$u_*(0) = \operatorname*{ess\,sup}_{\Omega} u, \qquad u_*(|\Omega|) = \operatorname*{ess\,inf}_{\Omega} u;$$

- the **decreasing radial rearrangement of the function** u is defined, on the ball $\widetilde{\Omega}$ having the same measure as Ω, by

$$\underset{\sim}{u} : \widetilde{\Omega} \to I\!R, \quad \underset{\sim}{u}(x) = u_*(\alpha_N |x|^N).$$

$$|v|_{**}(t) = \frac{1}{t} \int_0^t |v|_*(s) ds \qquad \text{for } t \in \Omega_* =]0, |\Omega|[.$$

$$L^{p,\infty}(\Omega) = \left\{ v : \Omega \to I\!R \text{ measurable } |v|_{L^{p,\infty}} = \sup_{t \leqslant |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \right\}$$

$$W^1(\Omega, |\cdot|_{p,q}) = \left\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega) \right\}$$

$$W^2(\Omega, |\cdot|_{p,q}) = \left\{ v \in W^{2,1}(\Omega) : \partial_{ij} v \in L^{p,q}(\Omega) \text{ for } (i,j) \in \{1, \dots, N\}^2 \right\}$$

The above result contains the result given in

P. Quittner, P. Souplet, *Superlinear Parabolic Problems,* Birkhäuser Basel (2007).

since $L^p(\Omega, \delta) \subsetneq L^1(\Omega, \delta^{\frac{1}{p}})$

We also improve the result of Cabré and Martel

f is only in $L^1(\Omega, \delta)$ then the function v is in $L^{\frac{N}{N-1},\infty}(\Omega)$

Lemma 2.
Assume L is the self adjoint uniformly elliptic operator $L = -\sum_{i,j} \partial_i(a_{i,j}(\cdot)\partial_j)$.
Then there exists a function $\varphi_1 \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ *and* $\lambda_1 > 0$, $\forall p \in]1, +\infty[$ *satisfying*

$$\begin{cases} L\varphi_1 = \lambda_1 \varphi_1 \quad \text{in } \Omega, \\ \varphi_1 = 0. \end{cases}$$

Moreover, there are two constants $c_1 > 0$, $c_2 > 0$ *such that*

$$c_1 \delta(x) \leqslant \varphi_1(x) \leqslant c_2 \delta(x) \quad \forall x \in \Omega.$$

Theorem 2. *Under the same assumptions as for Lemma 2, the unique generalized function v given in Theorem 1 belongs to* $W^{1,q}(\Omega, \delta)$ *for* $1 \leqslant q < \dfrac{2N}{2N-1}$.

Theorem 3. *Let v be the unique solution of $(DG_L(\Omega))$ given in theorem 1. If $f \in L^1(\Omega, \delta^\alpha)$ for some $\alpha \in [0, 1[$ then*

$$|\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega).$$

Moreover, there exists $c(\Omega, L) > 0$

$$|\nabla v|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c(\Omega, L)|f|_{L^1(\Omega,\delta^\alpha)}$$

Theorem 4. *Let v be the unique solution of the generalized Dirichlet problem $(DG_L(\Omega))$, $f \geqslant 0$. Then $v \in W_0^{1,q}(\Omega)$ for some $q > 1$ if and only if there exists $\alpha \in [0, 1[$ such that $f \in L^1(\Omega, \delta^\alpha)$.*

Next, we want to analyze some specific case, namely when we "symmetrize" the equation.

Remark 2. *In general, when we consider the ball $\widetilde{\Omega}$ having the same measure $|\Omega|$ than Ω, then the distance to the boundary $\delta(x) = \delta_{\widetilde{\Omega}}(x)$ is given by*

$$\begin{cases} \delta_{\widetilde{\Omega}} = \alpha_N^{-\frac{1}{N}} |\Omega|^{\frac{1}{N}} - |x|, \\ x \in \widetilde{\Omega}. \end{cases}$$

if $\underset{\sim}{f} \in L^1(\widetilde{\Omega}, \delta_{\widetilde{\Omega}})$, $f \geqslant 0$ *then* $f \in L^1(\Omega)$

Some results on a ball : f is replaced by its increasing rearrangement $\widetilde{f}$

The aim of this section is to show that there are functions v whose data are in $L^1(\Omega, \delta)$ for which we have only the regularity $W^{1,1}(\Omega)$.

$$L^1(\Omega, \delta^{1^-}) = \bigcup_{0 \leqslant \alpha < 1} L^1(\Omega, \delta^\alpha),$$

a natural question concerns the global differentiability of v on the entire Ω when $f \in L^1(\Omega, \delta) \backslash L^1(\Omega, \delta^{1^-})$.

Ω is a ball

Proposition 1. *Assume that $L = -\Delta$. If $f \in L^1(\Omega, \delta |\operatorname{Ln} \delta|)$ then the function v solution of $(DG_L(\Omega))$ is in $W_0^{1,1}(\Omega)$.*
Moreover, there exists a constant $c > 0$:

$$|\nabla v|_{L^1} \leqslant c|f|_{L^1(\Omega, \delta |\operatorname{Ln} \delta|)}.$$

$$L^1(\Omega, \delta|\operatorname{Ln}\delta|) = \Big\{ f : \Omega \to I\!R \text{ Lebesgue measurable such that}$$

$$\int_\Omega |f|(x)\delta(x)|\operatorname{Ln}\delta(x)|\,dx < +\infty \Big\}.$$

$$L^1(\Omega, \delta^\alpha) \subsetneqq L^1(\Omega, \delta|\operatorname{Ln}\delta|) \subsetneqq L^1(\Omega, \delta). \quad \text{for } \alpha \in [0, 1[$$

We recall the following result which can be obtained by some direct integrations

Lemma 5.

Let $f \in L^1(\Omega,\delta)$, $f \geqslant 0$ *and let for* $n \in I\!N$, $T_n(f) = \min(f,n) \doteq f_n$. *Then the sequence* $(U_n)_{n\geqslant 0}$ *defined on* Ω *by*

$$U_n(x) = \frac{1}{N^2 \alpha_N^{\frac{2}{N}}} \int_{\alpha_N |x|^N}^{\alpha_N} \left[\sigma^{-2(1-\frac{1}{N})} \int_0^{\sigma} f_{n*}(t)dt\right] d\sigma,$$

is the unique solution of

$$\begin{cases} -\Delta U_n(x) = \underset{\sim}{f_n}(x) = f_{n*}(\alpha_N |x|^N), \quad x \in \Omega, \\ U_n = 0 \text{ on } \partial\Omega, \end{cases}$$

$U_n \in W_0^{1,q}(\Omega)$, $\quad \forall q < +\infty$.

Another lemma that shall explain the difference between the results when $f \in L^1(\Omega)$ and $f \in L^1(\Omega,\delta)$ is the following necessary and sufficient condition.

Lemma 6.

Under the same assumptions as in Lemma 5, we have $f \in L^1(\Omega)$ *if and only if* $\lim\limits_{n\to+\infty} U_n = U$ *is in* $L^1(\Omega)$.

And in this case (i.e. $f \in L^1(\Omega)$*), the function* U *is the unique solution of*

$$\begin{cases} -\Delta U = \underset{\sim}{f} \text{ in } \Omega, \\ U \in W_0^{1,q}(\Omega),\ 1 \leqslant q < \dfrac{N}{N-1}. \end{cases}$$

Theorem 5.
Let $h \in L^1(\Omega,\delta)$, $h \geqslant 0$. Then, the unique solution $\omega \in L^1(\Omega)$ of

$$\begin{cases} -\Delta\omega(x) = \widetilde{h}(x) = h_*(\alpha_N - \alpha_N|x|^N), & in\ \Omega, \\ \omega(x) = 0, & x \in \partial\Omega, \end{cases}$$

in the very weak sense given above belongs to $W_0^{1,1}(\Omega)$.
Moreover we have

$$|\omega|_{L^1(\Omega)} \leqslant \frac{1}{{\alpha_N}^{1+\frac{1}{N}} N} |h_*(\sigma)\sigma|_{L^1(\Omega_*)} \leqslant \frac{1}{{\alpha_N}^{\frac{1}{N}}} |h|_{L^1(\Omega,\delta)},$$

$$|\nabla\omega|_{L^1(\Omega)} \leqslant \frac{1}{N{\alpha_N}^{\frac{1}{N}}} |h_*(\sigma)\sigma|_{L^1(\Omega_*)} \leqslant |h|_{L^1(\Omega,\delta)}.$$

For this, we shall prove the following more general theorem which shows merely that for radial solution ω, one has the $W^{1,1}$-regularity.

Theorem 6. *Let f_0 be a given non negative measurable function on the interval Ω_* with $\sigma f_0(\sigma) \in L^1(\Omega_*)$.*
Then, $f \in L^1(\Omega, \delta_\Omega)$, with $f(x) = f_0(\alpha_N - \alpha_N|x|^N)$, and the unique generalized function $\omega \in L^1(\Omega)$ of $-\Delta\omega = f_0(\alpha_N - \alpha_N|x|^N)$ belongs to $W_0^{1,1}(\Omega)$.
Moreover we have

$$|\omega|_{L^1(\Omega)} \leqslant \frac{1}{{\alpha_N}^{1+\frac{1}{N}} N} |f_0(\sigma)\sigma|_{L^1(\Omega_*)} \leqslant \frac{1}{{\alpha_N}^{\frac{1}{N}}} |f|_{L^1(\Omega,\delta)},$$

$$|\nabla\omega|_{L^1(\Omega)} \leqslant \frac{1}{N{\alpha_N}^{\frac{1}{N}}} |f_0(\sigma)\sigma|_{L^1(\Omega_*)} \leqslant |f|_{L^1(\Omega,\delta)}.$$

As a complement for the Theorem 4, we can make precise the necessary and sufficient condition for radial solution as in the above theorem. This will allow us to construct easily some examples for the applications.

Lemma 7. *Let $q \in [1, N'[$. Then the function ω given in Theorem 6 is in $W_0^{1,q}(\Omega)$ if and only if we have*

$$\int_0^{\alpha_N} \sigma f_0(\sigma) \left(\int_\sigma^{\alpha_N} f_0(t)dt \right)^{q-1} d\sigma = \int_0^{\alpha_N} \left(\int_\sigma^{\alpha_N} f_0(t)dt \right)^q d\sigma \text{ is finite.}$$

We shall end this section by few examples of applications of the above results

Corollary 7.1.

Let Ω be the unit ball of $I\!R^N$ and $q \in \left[1, \dfrac{N}{N-1}\right[$ for $\gamma \in [1, 2[$, we consider

$$f(x) = \frac{1}{(1 - |x|^N)^\gamma}.$$

Then

$$f \in L^1(\Omega, \delta) \text{ and } f \notin L^1(\Omega).$$

Moreover

- *if $\gamma \in \left]1 + \dfrac{1}{q}, 2\right[$ then the function ω given in Theorem 5* **is not** *in $W_0^{1,q}(\Omega)$;*
- *if $\gamma \in \left[1, 1 + \dfrac{1}{q}\right[$ then the function $\omega \in W_0^{1,q}$ with $q \in \left[1, \min\left(\dfrac{1}{\gamma - 1}, \dfrac{N}{N-1}\right)\right[$.*

Corollary 7.2.

Let $g(\sigma) = \dfrac{4\alpha_{_N}}{\sigma\left|\operatorname{Ln}\frac{\sigma}{4\alpha_N}\right|^{\gamma}}$, *with* $\gamma > 1$ *we set* $f(\sigma) = -g'(\sigma),\ \sigma \in]0, \alpha_{_N}[$. *Then*

(1) $g \in L^1(]0, \alpha_{_N}[)$ *and* $g \notin L^q(]0, \alpha_{_N}[)$ *for all* $q > 1$.

(2) *Setting* $h(x) = f(\alpha_n - \alpha_N|x|^N),\ x \in \Omega,\ h \in L^1(\Omega, \delta|\operatorname{Ln}\delta|)$, *but* h *is not in* $L^1(\Omega, \delta^\alpha)$, *for any* $\alpha \in [0, 1[$.

(3) *The generalized function* $\omega \in L^1(\Omega)$ *solution of* $-\Delta\omega = h$ *belongs to* $W_0^{1,1}(\Omega)$ *but not to* $W_0^{1,q}(\Omega)$ *for* $q > 1$.

3. Proof of Theorem 1

Theorem 1. *Let* $f \in L^1(\Omega, \delta)$ *and* $N' = \dfrac{N}{N-1}$.
Then there exists a unique function $v \in L^{N',\infty}(\Omega)$ *satisfying*

$$(DG_L(\Omega)) : \int_\Omega v L^* \varphi \, dx = \int_\Omega f \varphi \, dx, \qquad \forall \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_0^1(\Omega).$$

Moreover, there exists a constant $c(\Omega, L) > 0$ *such that*

$$|v|_{L^{N',\infty}} \leqslant c(\Omega, L) |f|_{L^1(\Omega,\delta)}.$$

Proof

For $k \geqslant 1$, we define the usual truncation

$$T_k(\sigma) = \begin{cases} \sigma & if\ |\sigma| \leqslant k, \\ k\,\mathrm{sign}(\sigma) & otherwise, \end{cases} \quad \sigma \in \mathbb{R}$$

We set $f_k = T_k(f) \in L^1(\Omega, \delta) \cap L^\infty(\Omega)$ and $f_k \to f$ in $L^1(\Omega, \delta)$. By standard result there exists a unique function

$$v_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega), \quad \forall p \in [1, +\infty[: L v_k = f_k.$$

Next we want to show that v_k is a Cauchy sequence in $L^{N',\infty}(\Omega)$. For $n \geqslant 1$, $k \geqslant 1$, we set $v^{nk} = v_n - v_k$, $f^{nk} = f_n - f_k$. Then $Lv^{nk} = f^{nk}$ which implies that $\forall \varphi \in H^2(\Omega) \cap H_0^1(\Omega)$

$$\int_\Omega v^{nk} L^* \varphi \, dx = \int_\Omega f^{nk} \varphi \, dx. \tag{2}$$

For any E measurable in Ω, there exists a function $\varphi_E \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ such that

$$L^* \varphi_E = \chi_E \operatorname{sign}(v^{nk}). \tag{3}$$

From Sobolev embedding associated to Lorentz spaces

$$|\nabla \varphi_E|_\infty \leqslant c(\Omega) \Big[\operatorname*{Max}_{i,j} |\partial_{ij} \varphi_E|_{L^{N,1}} + |\varphi_E|_{H^1} \Big]$$

J.M. Rakotoson, Réarrangement relatif: un instrument d'estimation dans les problèmes aux limites, Springer-Verlag, Berlin, 2008.

We extend the Agmon-Douglis-Nirenberg theorem to Lorentz spaces and get

$$|\nabla \varphi_E|_\infty \leqslant c |\chi_E|_{L^{N,1}} \leqslant c |E|^{\frac{1}{N}}.$$

Since $\forall x \in \Omega$, we have

$$\left|\frac{\varphi_E(x))}{\delta(x)}\right| \leqslant c|\nabla \varphi_E|_\infty,$$

we get

$$\frac{|\varphi_E(x)|}{\delta(x)} \leqslant c|E|^{\frac{1}{N}}, \quad \forall x \in \Omega.$$

$$\int_E |v_n - v_k|\, dx = \int_\Omega f^{nk} \varphi_E\, dx.$$

$$\int_E |v_n - v_k|\, dx \leqslant c|E|^{\frac{1}{N}} \int_\Omega |f_n - f_k|(x)\delta(x)\, dx$$

for all E measurable set in Ω.

Using the Hardy-Littlewood inequality

$$\sup_{t \leqslant |\Omega|} \left[t^{1-\frac{1}{N}} |v_n - v_k|_{**}(t)\right] \leqslant c|f_n - f_k|_{L^1(\Omega,\delta)}.$$

Knowing that $L^{N',\infty}$ is the dual and associate space of $L^{N,1}$ we pass to the limit in relation that

$$\int_\Omega v_k L^*\psi\,dx = \int_\Omega f_k\psi\,dx, \quad \forall\psi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_0^1(\Omega)$$

as $k \longrightarrow +\infty$ to derive the result.

4. Some results for the semilinear case

Next, we shall need the following definitions :

- $$C_c(\Omega) = \Big\{ v : \Omega \to \mathbb{R},\ \text{continuous with compact support in } \Omega \Big\}$$

  is a Frechet space and we shall denote by

- $M(\Omega)$ its dual space. That is the set of all measure μ continuous on the Frechet space $C_c(\Omega)$.

- When we endow $C_c(\Omega)$ with the norm

$$|u|_\infty = \underset{x\in\Omega}{\mathrm{Max}}\ |u(x)|,\ \ u \in C_c(\Omega),$$

- then we shall denote by $M^1(\Omega)$ the dual of $\Big(C_c(\Omega), |\cdot|_\infty \Big)$.

- For $\mu \in M^1(\Omega)$, we denote its norm by

$$|\mu|_* = \sup \Big\{ < \mu, \varphi >,\ |\varphi|_\infty \leqslant 1,\ \varphi \in C_c(\Omega) \Big\}.$$

- We shall introduce the following weighted Radon measure : for $\alpha \in [0,1]$,

$$M^1(\Omega, \delta^\alpha) = \Big\{\mu \in M(\Omega) : \delta^\alpha \mu \in M^1(\Omega)\Big\}.$$

- We can define a norm on it by setting

$$||\mu||_{*,\delta^\alpha} = |\delta^\alpha \mu|_*, \quad \mu \in M^1(\Omega, \delta^\alpha).$$

Property 1.

1. $L^1(\Omega, \delta^\alpha) \subset\!\!> M^1(\Omega, \delta^\alpha)$.
2. $\forall\, \mu \in M^1(\Omega, \delta^\alpha)$, *there exists a sequence of* $(f_n)_n \subset C_c(\Omega)$ *such that* f_n *remains in a bounded set of* $L^1(\Omega, \delta^\alpha)$ *and* $f_n \rightharpoonup \mu$ *weakly in* $M(\Omega)$. *More precisely*

$$|f_n|_{L^1(\Omega,\delta^\alpha)} \leqslant ||\mu||_{*,\delta^\alpha}.$$

For convenience and clarity, we shall begin by the following linear equations with unbounded potential.

Theorem 4.
Let $V \in L^1_{loc}(\Omega)$, with $V \leqslant \lambda < \lambda_1$, λ_1 being the first eigenvalue of L or $V \in L^1(\Omega,\delta)$. Then, for any $f \in L^1(\Omega,\delta)$, there exists a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega,\delta)$, $1 \leqslant q < \dfrac{2N}{2N-1}$ satisfying $Vu \in L^1(\Omega,\delta)$, and $\displaystyle\int_\Omega uL\varphi dx - \int_\Omega Vu\varphi dx = \int_\Omega f\varphi dx \quad \forall \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_c(\Omega)$.

Moreover there are constants $c = c(\Omega, L, V, q) > 0$ such that

1. $|Vu|_{L^1(\Omega,\delta)} \leqslant c|f|_{L^1(\Omega,\delta)}$,
2. $|u|_{L^{N',\infty}(\Omega)} \leqslant c|f|_{L^1(\Omega,\delta)}$,
3. $\displaystyle\int_\Omega |\nabla u|^q \delta(x)dx \leqslant c|f|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\Big(1 + |f|^{N'}_{L^1(\Omega,\delta)}\Big)^{1-\frac{q}{2}}$.

If $f \in L^1(\Omega,\delta^\alpha)$ for some $\alpha \in [0,1[$ and $V \in L^{N,1}(\Omega,\delta^\alpha)$ then

$$u \in W^1_0\Big(\Omega, |\cdot|_{\frac{N}{N-1+\alpha},\infty}\Big) = \Big\{v \in W^{1,1}_0(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)\Big\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c|f|_{L^1(\Omega,\delta^\alpha)}.$$

As a consequence of the above Theorem 4, we may replace f by $\mu \in M^1(\Omega, \delta)$.

2.2. *The semilinear case.*

In this section, we shall add a perturbation $g(x,u,\nabla u)$ to the linear operator $Lu - Vu$. We shall start with the case where $f \in L^1(\Omega,\delta)$. We consider a non linear map:

H1./ $g : \Omega \times I\!R \times I\!R^N \longrightarrow I\!R$ which is a Caratheodory function, that is

$$x \longrightarrow g(x,s,\xi) \text{ is a measurable } \forall\, s \in R,\ \forall\, \xi \in I\!R^N$$

$$(s,\xi) \longrightarrow g(x,s,\xi) \text{ is continuous for a.e. } x.$$

H2./ $sg(x,s,\xi) \geqslant 0\ \forall\, \xi \in I\!R^N$, for a.e. x, and all $s \in I\!R$.

H3./ $\sup\limits_{\{\xi \in I\!R^N,\ |s| \leqslant t\}} |g(\cdot,s,\xi)|\delta(\cdot)$ is in $L^1_{loc}(\Omega)$ for all $t \geqslant 0$.

Theorem 6.
Under the same assumptions as for Theorem 4, and assumptions H1./ to H3./, there exists a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega,\delta)$, $1 \leqslant q < \dfrac{2N}{2N-1}$, *satisfying* $g(x,u,\nabla u) \in L^1(\Omega,\delta)$, $Vu \in L^1(\Omega,\delta)$, *and* $\forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_0(\Omega)$

$$\int_\Omega uL\varphi dx - \int_\Omega Vu\varphi + \int_\Omega \varphi g(x,u,\nabla u)dx = \int_\Omega f\varphi dx.$$

Moreover there are constants $c = c(\Omega, L, V, q) > 0$ such that

1. $|Vu|_{L^1(\Omega,\delta)} \leqslant c|f|_{L^1(\Omega,\delta)}$,
2. $|u|_{L^{N',\infty}(\Omega)} \leqslant c|f|_{L^1(\Omega,\delta)}$,
3. $\displaystyle\int_\Omega |\nabla u|^q \delta(x)dx \leqslant c|f|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\Big(1 + |f|^{N'}_{L^1(\Omega,\delta)}\Big)^{1-\frac{q}{2}}$.
4. *If $f \in L^1(\Omega, \delta^\alpha)$ for some $\alpha \in [0, 1[$ and $V \in L^{N,1}(\Omega, \delta^\alpha)$ then*

$$u \in W^1_0\Big(\Omega, |\cdot|_{\frac{N}{N-1+\alpha},\infty}\Big) = \Big\{v \in W^{1,1}_0(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)\Big\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c|f|_{L^1(\Omega,\delta^\alpha)}.$$

We can announce an analogous Theorem replacing f by $\mu \in M^1(\Omega, \delta)$ and replacing H3./ by

$H3_\mu./ : |g(x, s, \xi)| \leqslant a(x) + \beta|\xi|\delta(x)$ *for a.e* x, $\forall\, s \in I\!R$, $\forall\, \xi \in I\!R^N$, $\beta \geqslant 0$ *and* $a \in L^1(\Omega, \delta)$.

Remark 5. *In many applications the term involves a "natural growth" gradient of the form*

$$g(x,u,\nabla u) = h(u)|\nabla u|^2. \tag{27}$$

Although assumption $H3_\mu$./ fails for this special case we can adapt the conclusion of Theorem 7 to this choice of g. In particular, we can show that

$$g_\varepsilon(x,u,\nabla u) = \frac{h(u)}{1+\varepsilon(h(u))} \cdot \frac{|\nabla u|^2}{1+\varepsilon|\nabla u|^2},$$

has a solution u_ε which will converge in some sense to a solution with the growth (27).

2.3. Large solutions of semi-linear equations with $L^1(\Omega,\delta)$ as data.

$m > 1,$

$$|g(x,t,\xi)| \geqslant c_1|t|^m - c_2 \ \forall\, t \in I\!R,$$

$$(LSP)\begin{cases} -\Delta u + g(x,u,\nabla u) = f(x) & in\ \Omega, \\ u(x) \xrightarrow[x\to\partial\Omega]{} +\infty. \end{cases}$$

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3. Rearrangement comparison results.

Let us assume that $V \geqslant 0$ and that

$$g(x,s) = q(x)\beta(s) \text{ with } \beta \text{ a continuous nondecreasing}$$

$$\text{real function with } \beta(0) = 0 \text{ and } q \in L^1_{loc}(\Omega),\ q(x) \geqslant 0 \text{ a.e. } x \in \Omega.$$

$$SP(\widetilde{\Omega} : \widetilde{q}\beta, F) \equiv \begin{cases} -\Delta U - \underset{\sim}{V}(|x|)U + \widetilde{q}(|x|)\beta(U) = F(|x|) \text{ in } \widetilde{\Omega}, \\ U = 0 \text{ on } \partial\widetilde{\Omega}. \end{cases}$$

$$F(x) \geqslant \underset{\sim}{f}(x) \geqslant 0 \text{ a.e. } x \in \widetilde{\Omega},$$

Theorem 9. *Assume that* $V \geqslant 0$ *satisfies* $V \leqslant \lambda < \lambda_1$, λ_1 *the first Dirichlet eigenvalue, that* $g(x,u)$ *satisfies (42) and assume (43) for some* $F \in L^1(\widetilde{\Omega})$. *Then*

$$\int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{u}(|x|))\delta_{\widetilde{\Omega}}(x)\,dx \leqslant \int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{U}^q(|x|))\delta_{\widetilde{\Omega}}(x)\,dx \text{ for any } r \in (0,R). \tag{44}$$

In particular, if $q(x) \equiv 1$ *then*

$$||\beta(u)||_{L^1(\Omega,\delta)} \leqslant ||\beta(U^g)||_{L^1(\widetilde{\Omega},\delta_{\widetilde{\Omega}})}. \tag{45}$$

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Thanks for your attention

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