Stabilization beyond the distributions.

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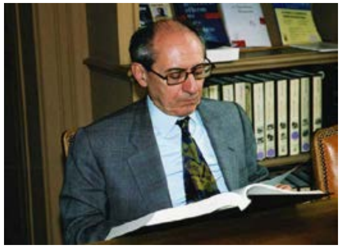

joint work with
E. Sánchez Palencia,
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1. Introduction

•The fundamental role of the *Distributions Theory,* offering the correct framework in which most of the models of the Mathematical-Physics must be formulated is well-known in our days

•Nevertheless, there is a large amount of singular problems, (arising in many different contexts as, e. g. in *thin shell theory)* which lead to formulations beyod the distributions.

•The main goal of this talk is to present some results showing that, even in this case, it is possible to show the stabilisation, as t goes to infinity, in a context more general than the space of distributions.

•In fact, this philosophy has many commoun points with a series of papers dealing with some singular stationary problems

N. Meunier, J. Sanchez-Hubert and E. Sanchez Palencia (2001)

Y. V. Egorov, N. Meunier and E. Sanchez-Palencia, Rigorus and heuristic treatment of certain sensitve singular perturbations (2007)

J. I. Díaz, E. Sánchez-Palencia, On slender shells and related problems suggested by Torroja's structures, *Asymptotic Analysis*, **52**, 2007, 259-297.

In this talk I will report some recent results obtained in collaboration with E. Sanchez Palencia.

We consider the transient displacements of a thin shell under a viscoelastic constitutive law. Then, according G. Duvaut, and J.L. Lions (1972) (Section 6, Chapter 3) and J. Sanchez-Hubert and E. Sanchez Palencia (1997) we arrive to a formulation of the type

$$\frac{\partial^2 \mathbf{U}}{\partial t^2} + c(\frac{\partial \mathbf{U}}{\partial t}, \mathbf{v}) + a(\mathbf{U}, \mathbf{v}) + \varepsilon^2 b(\mathbf{U}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$$

for suitable test functions **v** in the energy space V (a certain Sobolev space) and for some bilinear forms a,b, and c.

The so called ***quasi-static problem*** corresponds to

$$c(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) + \varepsilon^2 b(\mathbf{u}, \mathbf{v}) = \langle \mathbf{f}, \mathbf{v} \rangle$$

and it provides a reasonable good approximation for the asymptotic time since it was shown in G. Duvaut, and J.L. Lions (1972) that (under suitable conditions: see Corollary 6.1, Chapter 3)

$$\begin{cases} \|\mathbf{u}(t,.) - \mathbf{U}(t,.)\|_V \le Ce^{-\gamma t} & \text{and} \\ e^{\gamma t}(\frac{\partial \mathbf{u}}{\partial t} - \frac{\partial \mathbf{U}}{\partial t}) \in L^2(0,+\infty : V), & \text{for some } C, \gamma > 0. \end{cases}$$

The main goal of this talk is to analyze the stabilization of solutions of (1.1), as $t \to +\infty$, to the solutions $\mathbf{u}_\infty^\epsilon$ of the stationary problem

$$a(\mathbf{u}_\infty^\epsilon, \mathbf{v}) + \varepsilon^2 b(\mathbf{u}_\infty^\epsilon, \mathbf{v}) = \langle \mathbf{f}_\infty, \mathbf{v} \rangle. \tag{1.2}$$

Moreover, as shown in a series of papers (N. Meunier, J. Sanchez-Hubert and E. Sanchez Palencia (2001), Y. V. Egorv,N. Meunier and E. Sanchez-Palencia (2007)) the solutions $\mathbf{u}_\infty^\epsilon$ of (1.2) converges, when $\epsilon \to 0$ (in a functional space which is not included in the distributions space $\mathcal{D}'$), to a solution $\mathbf{u}_\infty$ of

$$a(\mathbf{u}_\infty, \mathbf{v}) = \langle \mathbf{f}_\infty, \mathbf{v} \rangle. \tag{1.3}$$

As a matter of fact, we shall not work directly with the displacements $\mathbf{u}$ since it can be shown that the singular perturbation problem can be reduced to a formulation on its trace on a part of the boundary Γ (here assumed as a one-dimensional compact manifold without boundary). The bilinear forms a, b, c and the energy space V must be also adapted to this trace formulation. We shall recall this, in Section2, for the simpler case of a thin stationary shell (following Egorv-Meunier-Sanchez-Palencia (2007)).

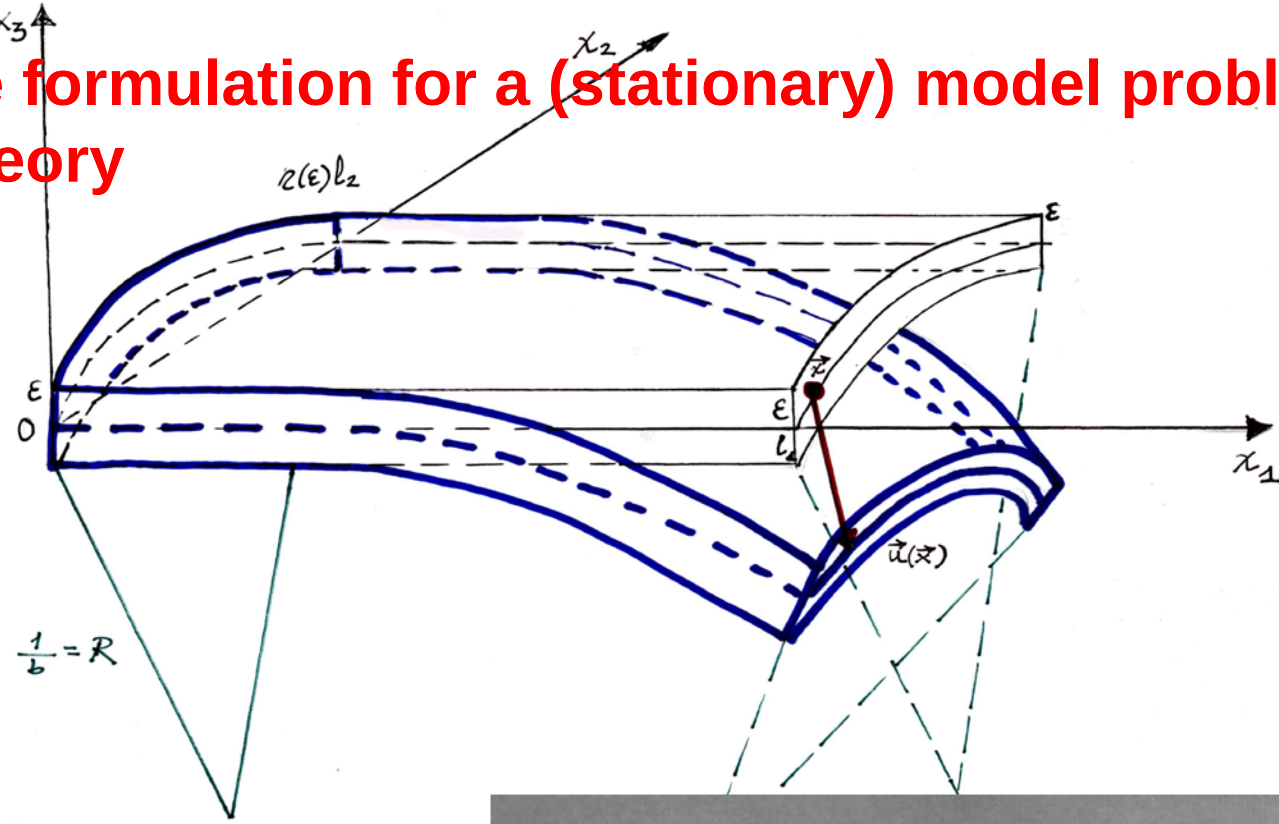
Before to consider the *quasi-static problem* (1.1) it seems instructive to start (in Section 3) by a more academic formulation but for which we can prove directly the stabilization of solutions beyond $\mathcal{D}'$.

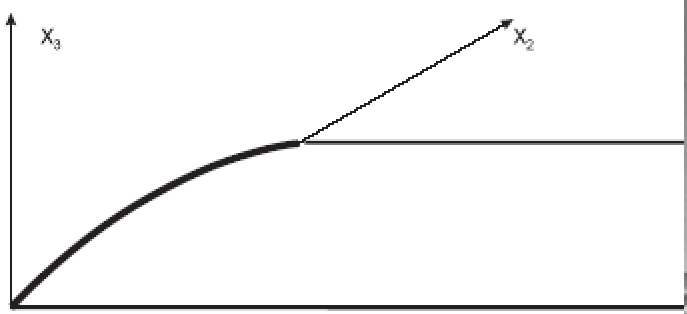
$$(EP)\begin{cases} \frac{du}{dt}(t) + Au = f(t) \quad t > 0,\ in\ V' \\ u(0) = 0 \end{cases}$$

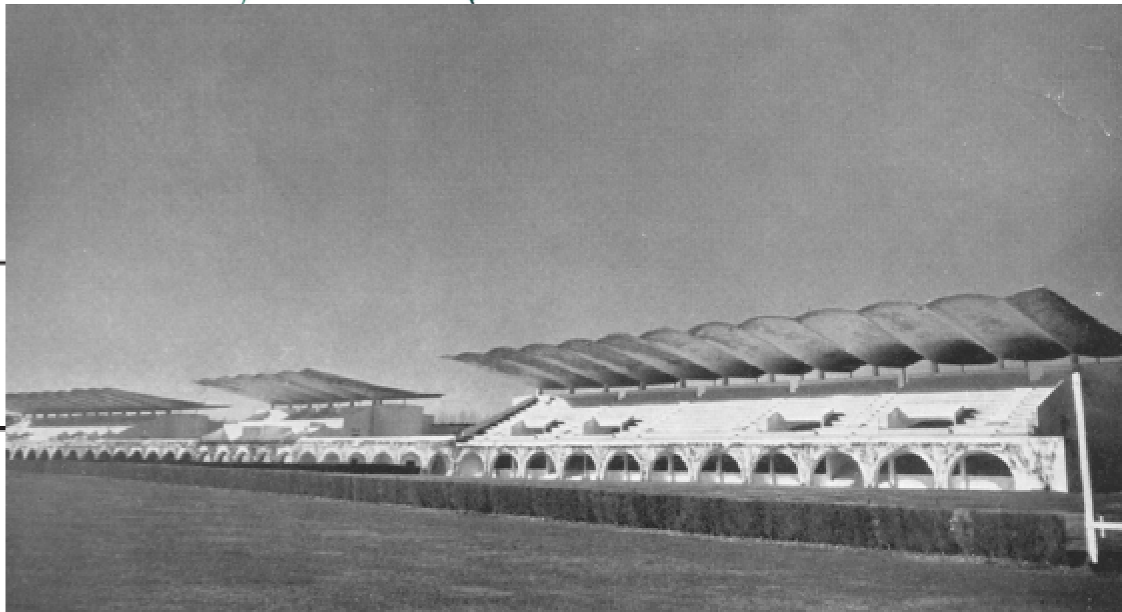
where $V = H^m(\Gamma)$ (and so $V' = H^{-m}(\Gamma)$) with Γ a one-dimensional compact manifold without boundary (for instance a circle), $m \in \mathbb{R}^+$, $A = S^*S$ and S is a suitable smoothing operator, i.e. verifying that $S(H^s(\Gamma)) \subset H^r(\Gamma))$ for any $s, r \in \mathbb{R}^+$ (in particular $S(\mathcal{D}'(\Gamma)) \subset C^\infty(\Gamma))$. Here S^* denotes the adjoint of S.

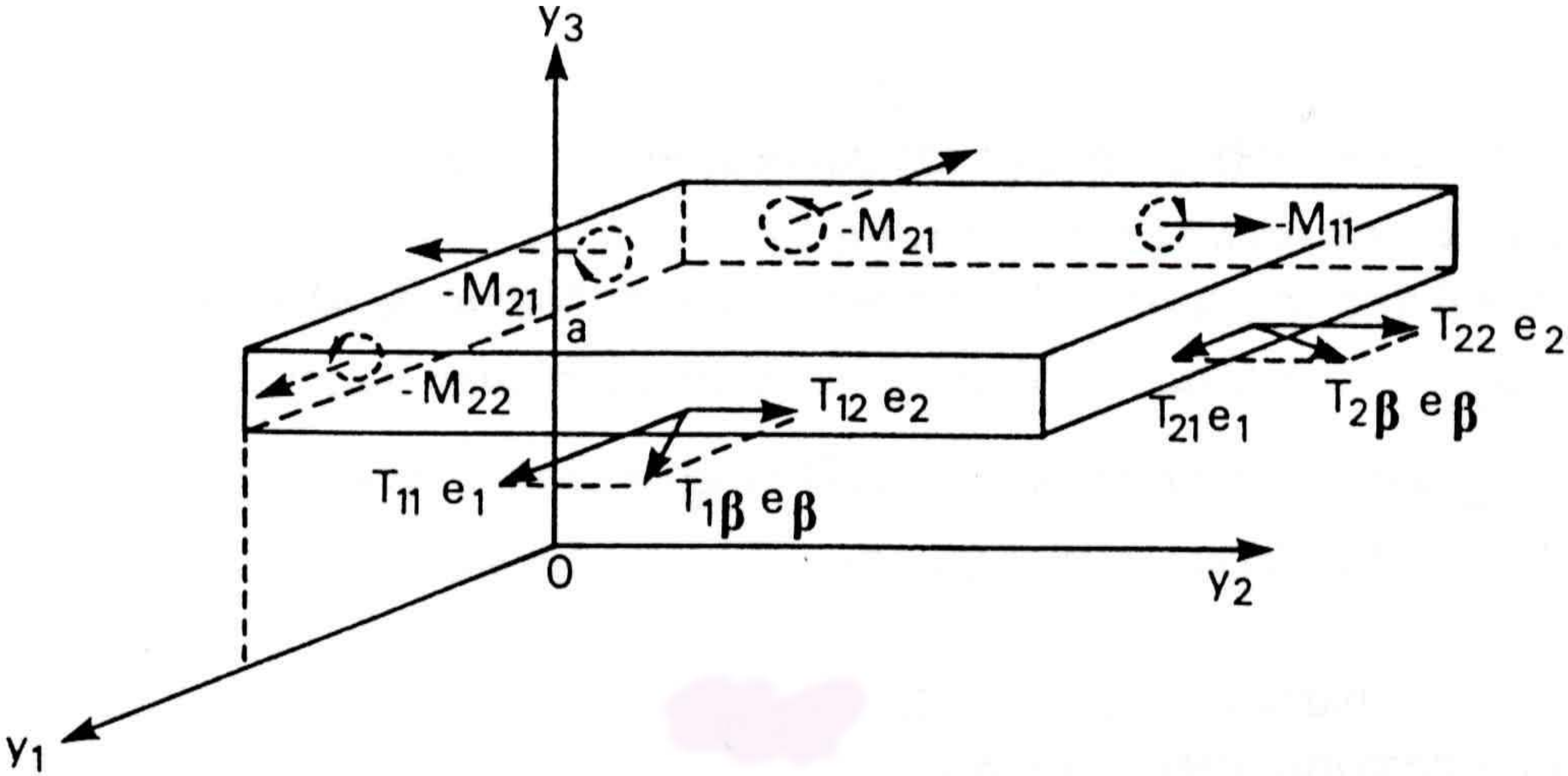
Finally, in Section 4 we shall consider the stabilization for the trace formulation of the *quasi-static problem* (1.1).

2. Trace formulation for a (stationary) model problem in shell theory







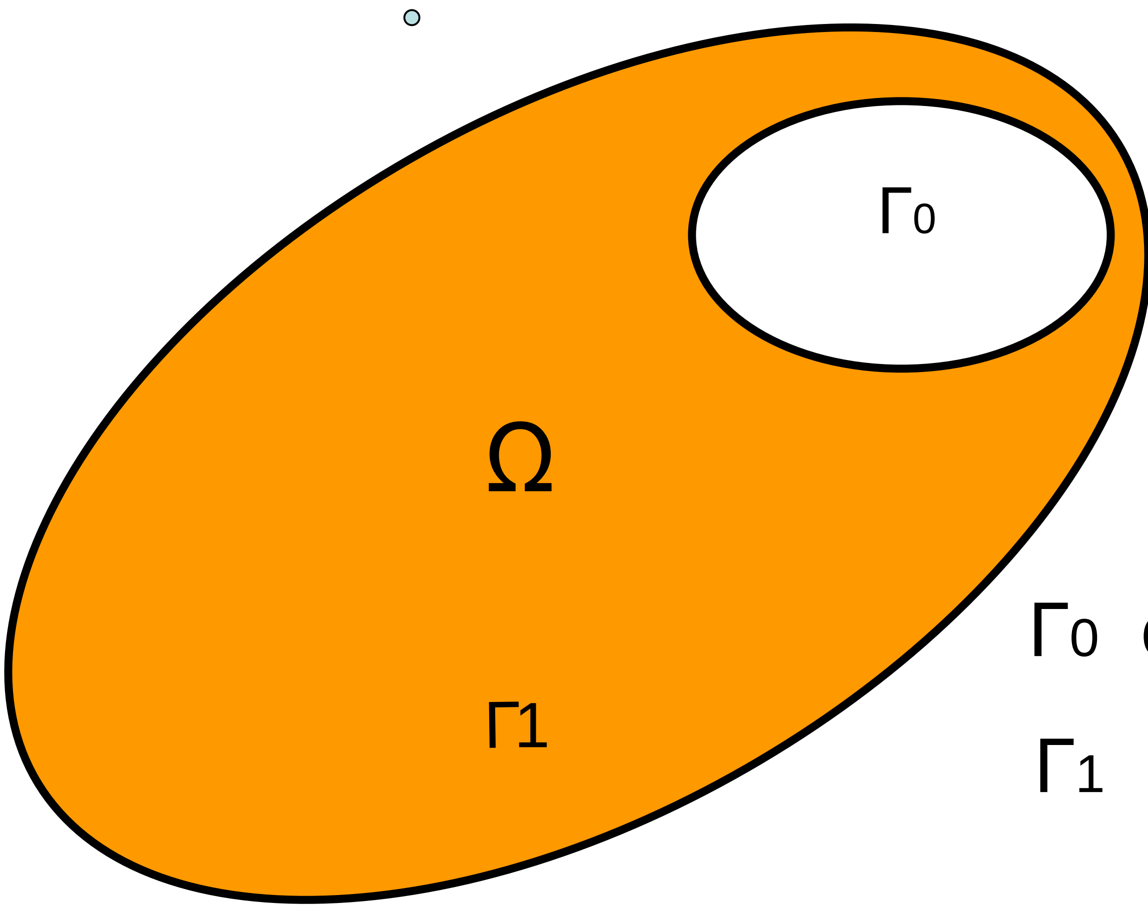


Uncoupling phenomenon (due to the symmetry with respect to $x_3 = 0$):

the total energy is the addition of membrain enery and flection energy

The elliptic case $V \subset H \subset V'$

$$V' \text{ very small } \Longrightarrow V \text{ very large}$$



Γ_0 classical BC

Γ_1 non-classical BC

$$a(u,v) = \int_\Omega \triangle u \, \triangle v \, \mathrm{d}x, \tag{4.1}$$

$$b(u,v) = \int_\Omega \sum_{\alpha,\beta=1}^{2} \partial_{\alpha\beta} u \, \partial_{\alpha\beta} v \, \mathrm{d}x. \tag{4.2}$$

We consider the following variational problem

$$\begin{cases} \text{Find } u^\varepsilon \in V \text{ such that, } \forall v \in V \\ a(u^\varepsilon, v) + \varepsilon^2 b(u^\varepsilon, v) = \langle f, v \rangle, \end{cases} \tag{4.3}$$

where the space V is the "energy space" with the essential boundary conditions on Γ_0

$$V = \{v \in H^2(\Omega);\ v_{|\Gamma_0} = \frac{\partial v}{\partial n}_{|\Gamma_0} = 0\}, \tag{4.4}$$

The classical formulation of problem (4.3) is:

$$(1+\varepsilon^2)\triangle^2 u^\varepsilon = f \text{ on } \Omega, \tag{4.5}$$

with the principal (or essential) boundary conditions, that is to say the boundary conditions that are included in the definition of V:

$$u = \frac{\partial u}{\partial n} = 0, \text{ on } \Gamma_0 \tag{4.6}$$

and the natural boundary conditions on Γ_1:

$$\begin{cases} \triangle u^\varepsilon + \varepsilon^2 \frac{\partial^2 u^\varepsilon}{\partial n^2} + L(\varepsilon, x, D)u^\varepsilon = 0 \text{ on } \Gamma_1 \\ -(1+\varepsilon^2)\frac{\partial}{\partial n}\triangle u^\varepsilon - \varepsilon^2 \frac{\partial^3 u^\varepsilon}{\partial n \partial t^2} + M(\varepsilon, x, D)u^\varepsilon = 0 \text{ on } \Gamma_1, \end{cases} \tag{4.7}$$

where L and M are differential operators of orders less than two and three respectively.

The Shapiro- Lopatinskii (SL) condition:

* Agmon-Douglis-Nirenberg (1965)

* J. Sanchez-Hubert and E. Sanchez Palencia: *Coques élastiques minces. Propriétés asymptotiques*, Masson, Paris 1997.

Proposition 4.1. *The boundary value problem (**??**) satisfies the SL condition for $\varepsilon > 0$ on Γ_0 and Γ_1.*

*When $\varepsilon = 0$, the boundary value problem (**??**) satisfy the SL condition on Γ_0 but it does not on Γ_1.*

The elements of the formal asymptotics

$\left\{ \begin{array}{l} \text{Minimize in } V, \\ a(u^\varepsilon, u^\varepsilon) + \varepsilon^2 b(u^\varepsilon, u^\varepsilon) - 2\langle f, u^\varepsilon \rangle. \end{array} \right.$ The natural trend consists in avoiding the a-energy which occurs with the factor 1 and leaving the b-energy which has a factor ε^2.

Clearly, this is not possible when (**??**) is satisfied since the kernel reduces to zero function. Nevertheless, in our case, we saw ((4.11) and (4.13)) that (**??**) follows from the uniqueness theorem for the Cauchy problem. This uniqueness is classical, but the solution u is unstable in the sense that there can be "large u" in the V norm (or in other spaces) for "small f" in the V' norm (or in other spaces). It then appears that the same reasoning yields that for small values of ε, the solution u^ε will be precisely among elements with small $a(u^\varepsilon, u^\varepsilon)$, that is to say with small $\triangle u^\varepsilon$ in L^2.

Let us now build such functions $u^\varepsilon \in V$ such that $\|\triangle u^\varepsilon\|_{L^2}$ is very small. The main idea is to consider functions in a larger space than V such that $\triangle v = 0$ The functions of this bigger space will not satisfy the two boundary conditions on Γ_0. Then we shall modify them in a narrow boundary layer along Γ_0 in order to satisfy the two boundary conditions with small value of a-energy.

More precisely, let us consider the vector space:

$$G^0 = \{v \in C^\infty(\overline{\Omega}),\ \triangle v = 0 \text{ on } \Omega,\ v = 0 \text{ on } \Gamma_0\}.$$

Obviously, as the Dirichlet problem for the laplacian on Ω is well posed in C^∞, the space G^0 is isomorphic with the space of traces on Γ_1:

$$\{w \in C^\infty(\Gamma_1)\}$$

the isomorphism is obtained by solving the Dirichlet problem:

$$\begin{cases} \triangle \tilde{w} = 0 \text{ on } \Omega, \\ \tilde{w} = 0 \text{ on } \Gamma_0, \\ \tilde{w} = w \text{ on } \Gamma_1. \end{cases}$$

In the sequel, we shall consider indifferently the functions $\tilde{w}$ on $\overline{\Omega}$ or their traces w on $\Gamma_{\mathbf{1}}$.

Once the layer is constructed, we compute the a-energy of it, as well as the $\varepsilon^2 b$-energy of the (modified) $\tilde{w}$ function, in order to consider the variational problem (4.3) in the restricted space.

The boundary layer on Γ_0

Let $\tilde{w}$ be in G^0 with fixed $\varepsilon > 0$. Our aim is to build a modified function $\tilde{w}^a$ of $\tilde{w}$ in a narrow boundary layer of Γ_0 in order to satisfy the supplementary boundary condition $\frac{\partial \tilde{w}}{\partial n} = 0$ on Γ_0 together with equation (4.5).

After using a partition of unity (index j) and a diffeomorphism to work in the half-plane, we use high frequency approximation analogous to those used in the construction of a parametrix, and tangential Fourier transform, $y_1 \to \xi_1$, of $\tilde{w}_j$.

The local structure of the Fourier transform of $\tilde{w}_j$ close to Γ_0.

$$\Big(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\Big)\tilde{w}_j + T(y, D)\tilde{w}_j = 0 \text{ on } \mathbf{R} \times (0, t),$$

for some $t > 0$, T being a differential operator of order less than two.

Now, according to the general trends of our approximation,

$$\left(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\right)\tilde{w}_j = 0 \text{ on } \mathbf{R}\times(0,t).$$

and taking the tangential Fourier transform:

$$\mathcal{F}(\tilde{w}_j)(\xi_1, y_2) = \lambda e^{|\xi_1|y_2} + \mu e^{-|\xi_1|y_2}.$$

And as it vanishes on Γ_0 $(y_2 = 0)$,

$$\mathcal{F}(\tilde{w}_j)(\xi_1, y_2) = 2\lambda \sinh(|\xi_1|y_2).$$

Writing it in terms of $\mathcal{F}\left(\frac{\partial \tilde{w}_j}{\partial y_2}\right)(\xi_1, 0) = 2\lambda|\xi_1|$, we obtain

$$\mathcal{F}\left(\tilde{w}_j\right)(\xi_1, y_2) = \mathcal{F}\left(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\right)\frac{\sinh(|\xi_1|y_2)}{|\xi_1|}.$$

We now proceed to the modification of $\tilde{w}_j$ in $\tilde{w}_j^a$ in a narrow boundary layer of Γ_0 in order to satisfy (always within our approximation) the equation coming from (4.5) for small ε. We obtain

$$\Big(\frac{\partial^2}{\partial y_1^2}+\frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\tilde{w}_j^{\boldsymbol{a}}=0 \text{ on } \mathbf{R}\times(0,t),$$

hence the tangential Fourier transform reads

$$\Big(-|\xi_1|^2+\frac{\partial^2}{\partial y_2^2}\Big)^{(2)}\mathcal{F}(\tilde{w}_j^a)=0.$$

Consequently, $\mathcal{F}(\tilde{w}_j^a)$ should take the form

$$\mathcal{F}(\tilde{w}_j^a)(\xi_1,y_2)=(\alpha+\gamma y_2)e^{|\xi_1|y_2}+(\beta+\delta y_2)e^{-|\xi_1|y_2}.$$

...

Imposing that $\mathcal{F}(\tilde{w}_j^a)(\xi_1,0)=\mathcal{F}\Big(\frac{\partial\tilde{w}_j^a}{\partial y_2}\Big)(\xi_1,0)=0$, we obtain

$$\mathcal{F}(\tilde{w}_j^a)(\xi_1,y_2)=\mathcal{F}\Big(\frac{\partial\tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\Big(\frac{\sinh(|\xi_1|y_2)}{|\xi_1|}-y_2e^{-|\xi_1|y_2}\Big).$$

This amounts to saying that the modification of the function $\tilde{w}_j$ consists in adding to it the inverse Fourier transform of

$$\mathcal{F}\Big(\frac{\partial \tilde{w}_j}{\partial y_2}_{|y_2=0}\Big)\Big(-y_2 e^{-|\xi_1| y_2}\Big).$$

Hence summing over j and defining on Γ_0 the family (with parameter y_2) of pseudo-differential smoothing operators $\delta\sigma(\varepsilon, D_1, y_2)$ with symbol:

$$\delta\sigma(\varepsilon, \xi_1, y_2) = -y_2 e^{-|\xi_1| y_2} h(\varepsilon, \xi, y_2),$$

we see that the modification of the function $\tilde{w}$:

$$\delta\tilde{w} = \tilde{w}^a - \tilde{w}$$

is precisely the action of $\delta\sigma(\varepsilon, D_1, y_2)$ on $\frac{\partial \tilde{w}_j}{\partial y_2}(y_1, 0)$:

$$\delta\tilde{w} = \delta\sigma(\varepsilon, D_1, y_2)\frac{\partial \tilde{w}_j}{\partial y_2}(y_1, 0).$$

Let $\tilde{v}$ and $\tilde{w}$ be two elements in G^0 and $\tilde{v}^a, \tilde{w}^a$ the corresponding elements modified in the boundary layer. As the given $\tilde{v}$ and $\tilde{w}$ are harmonic in Ω, the a-form is only concerned with the modification terms $\delta\tilde{v}$ and $\delta\tilde{w}$. Then, within our approximation, we have:

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} dy_1 \int_0^{+\infty} \triangle(\delta\tilde{v})\overline{\triangle(\delta\tilde{w})}\ \mathrm{d}y_2.$$

where the integral in $\mathrm{d}y_2$ is only effective in the narrow layer. Using the partition of the unity θ_j and denoting as before by $\delta w_j(\cdot, y_2)$ the extension with value 0 to $\mathbb{R}$ of $\theta_j(\cdot, y_2)\delta w(\cdot, y_2)$, we have

$$a(\tilde{v}^a, \tilde{w}^a) = \Sigma_{j,k} \int_{\Gamma_0} dy_1 \int_0^{+\infty} \triangle(\delta\tilde{v}_j)\overline{\triangle(\delta\tilde{w}_k)}\ \mathrm{d}y_2.$$

We now simplify this last expression using a sesquilinear form involving pseudo-differential operators.

Indeed, denoting by $P(\frac{\partial}{\partial y_1})$ the pseudo-differential operator with symbol

$$P(\xi_1) = (2|\xi_1|)^{1/2} h(\varepsilon, \xi, y_2), \tag{5.25}$$

and summing over j and k, we obtain that

$$a(\tilde{v}^a, \tilde{w}^a) = \int_{\Gamma_0} P(\frac{\partial}{\partial s})\frac{\partial \tilde{v}}{\partial n}_{|\Gamma_0} \overline{P(\frac{\partial}{\partial s})\frac{\partial \tilde{w}}{\partial n}_{|\Gamma_0}}\ \mathrm{d}s. \tag{5.26}$$

Taking account of the perturbation term $\varepsilon^2 b$.

We shall consider the minimization problem (??) on G^0 instead of on V. The a-energy should be computed using the previous formula. This modified problem obviously involves the a-energy and the $\varepsilon^2 b$-energy. A natural space for handling it should be the completion G of G^0 with the norm:

$$\|v\|_G^2 = \int_{\Gamma_0} \left| P(\frac{\partial}{\partial s}) \frac{\partial v}{\partial n}_{|\Gamma_0} \right|^2 \mathrm{d}s + b(v,v).$$

G is the space of the harmonic functions of $H^2(\Omega)$ vanishing on Γ_0 It may be identified with the space of traces $H^{3/2}(\Gamma_1)$. But we shall do in other approximations to compute the $\varepsilon^2 b$ form and an equivalent formulation directly on the space of traces $H^{3/2}(\Gamma_1)$.

We recall that the elements of G^0 (and then of G) are denoted by $\tilde{w}$ and the associated traces on Γ_1 are denoted by w.

We shall use an analogous approximation of the problem in the vicinity of Γ_1.

Then, defining the pseudo-differential operator $Q(\frac{\partial}{\partial s})$ of order 3/2 with principal symbol

$$\sqrt{2}|\xi_1|^{3/2}, \tag{5.32}$$

or equivalently as previously:

$$\sqrt{2}(1+|\xi_1|^2)^{3/4}, \tag{5.33}$$

we have (always within our approximation):

$$b(\tilde{v},\tilde{w}) = \int_{\Gamma_1} Q(\frac{\partial}{\partial s})v\ \overline{Q(\frac{\partial}{\partial s})w}\ \mathrm{d}s. \tag{5.34}$$

We observe that the operator Q is only concerned with the trace on Γ_1, so that we may either write $\tilde{v}$, $\tilde{w}$ or v, w in (5.34).

The formal asymptotic problem becomes:

$$\begin{cases} \text{Minimize in } G \text{ the functional} \\ \int_{\Gamma_0} \left|P(\frac{\partial \tilde{v}}{\partial n})\right|^2 \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} \left|Q(\tilde{v})\right|^2 \mathrm{d}s - 2\int_\Omega f\tilde{v}\ \mathrm{d}x. \end{cases}$$

The associated Lax-Milgram form is

$$\begin{cases} \text{Find } \tilde{v}^\varepsilon \in G \text{ such that } \forall \tilde{w} \in G \\ \int_{\Gamma_0} P(\frac{\partial \tilde{v}^\varepsilon}{\partial n})\overline{P(\frac{\partial \tilde{w}}{\partial n})}\ \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\tilde{v}^\varepsilon)\ \overline{Q(\tilde{w})}\ \mathrm{d}s = \langle f, w\rangle. \end{cases}$$

Reduction to a problem on Γ_1

In order to exhibit more clearly the unusual character of the problem, we shall now write (5.36) under another equivalent form involving only the traces on Γ_1 . This new form will be in the framework of Section 2. Let us define $\mathcal{R}_0$ as follows. For a given $w \in C^\infty(\Gamma_1)$ we solve (5.5) and we take the trace of $\frac{\partial \tilde{w}}{\partial n}$ on Γ_0, then

$$\frac{\partial \tilde{w}}{\partial n}_{|\Gamma_0} = \mathcal{R}_0 w. \tag{5.46}$$

Using the regularity properties of the solution of (5.5), it follows that $\mathcal{R}_0 w$ is in $C^\infty(\Gamma_0)$, so that $\mathcal{R}_0$ is a smoothing operator.

Then, (5.36) may be written as a problem for the traces on Γ_1:

$$\begin{cases} \text{Find } v^\varepsilon \in H^{3/2}(\Gamma_1) \text{ such that } \forall w \in H^{3/2}(\Gamma_1) \\ \int_{\Gamma_0} P(\frac{\partial}{\partial s})\mathcal{R}_0 v^\varepsilon \overline{P(\frac{\partial}{\partial s})\mathcal{R}_0 w} \, \mathrm{d}s + \varepsilon^2 \int_{\Gamma_1} Q(\frac{\partial}{\partial s}) v^\varepsilon \, \overline{Q(\frac{\partial}{\partial s}) w} \, \mathrm{d}s = \int_\Omega F \tilde{w} \, \mathrm{d}x, \end{cases} \tag{5.47}$$

where the configuration space is obviously $H^{3/2}(\Gamma_1)$. The left hand side with $\varepsilon > 0$ is continuous and coercive.

Here $F \in H^{-3/2}(\Gamma_1)$ is defined from $f \in V'$ by

$$\langle F, w \rangle_{H^{-3/2}(\Gamma_1), H^{3/2}(\Gamma_1)} = \langle f, \tilde{w} \rangle. \tag{5.48}$$

This problem may be written:

$$\left(\mathcal{R}_0^* P^*(\frac{\partial}{\partial s}) P(\frac{\partial}{\partial s}) \mathcal{R}_0 + \varepsilon^2 Q^*(\frac{\partial}{\partial s}) Q(\frac{\partial}{\partial s})\right) \tilde{v}^\varepsilon = F. \qquad (5.49)$$

We then, define the new operators (but we use the same notations)

$$\begin{aligned} \mathcal{A} &= \mathcal{R}_0^* P^* P \mathcal{R}_0 \ \in \mathcal{L}(H^s(\Gamma_1), H^r(\Gamma_0)), \forall s, r \in \mathbf{R}, & (5.50) \\ \mathcal{B} &= Q^* Q \ \in \mathcal{L}(H^{3/2}(\Gamma_1), H^{-3/2}(\Gamma_1)). & (5.51) \end{aligned}$$

Obviously, $\mathcal{B}$ is an elliptic pseudo-differential operator of order 3, whereas $\mathcal{A}$ is a smoothing (non local) operator. The problem becomes

$$\left(\mathcal{A} + \varepsilon^2 \mathcal{B}\right) v^\varepsilon = F, \text{ in } H^{-3/2}(\Gamma_1). \qquad (5.52)$$

This problem is in the general framework of the class of very simple sensitive problems previously considered.

Proposition 5.1. *Let $F \in H^{-3/2}(\Gamma_1)$ and $F \notin C^\infty(\Gamma_1)$, then the problem (5.55) has no u^0 solution in $\mathcal{D}'(\Gamma_1)$.*

Proof. If $u^0 \in \mathcal{D}'(\Gamma_0)$ was a solution of (5.55), as Γ_1 is compact, u^0 should be in some H^s, then recalling (5.50), we should have $\mathcal{A}u^0 \in C^\infty(\Gamma_0)$, which is not possible. □

Example

$$\hat{u}^\lambda(\xi) = \begin{cases} \cosh \xi & \text{for} \quad |\xi| < \lambda \\ 0 & \text{for} \quad |\xi| > \lambda \end{cases}$$

so that as $\lambda \to \infty$

$$\hat{u}^\lambda(\xi) \to \hat{u}(\xi) \quad \text{in} \quad D'(\mathbb{R}_\xi).$$

Consequently the inverse Fourier transforms satisfy

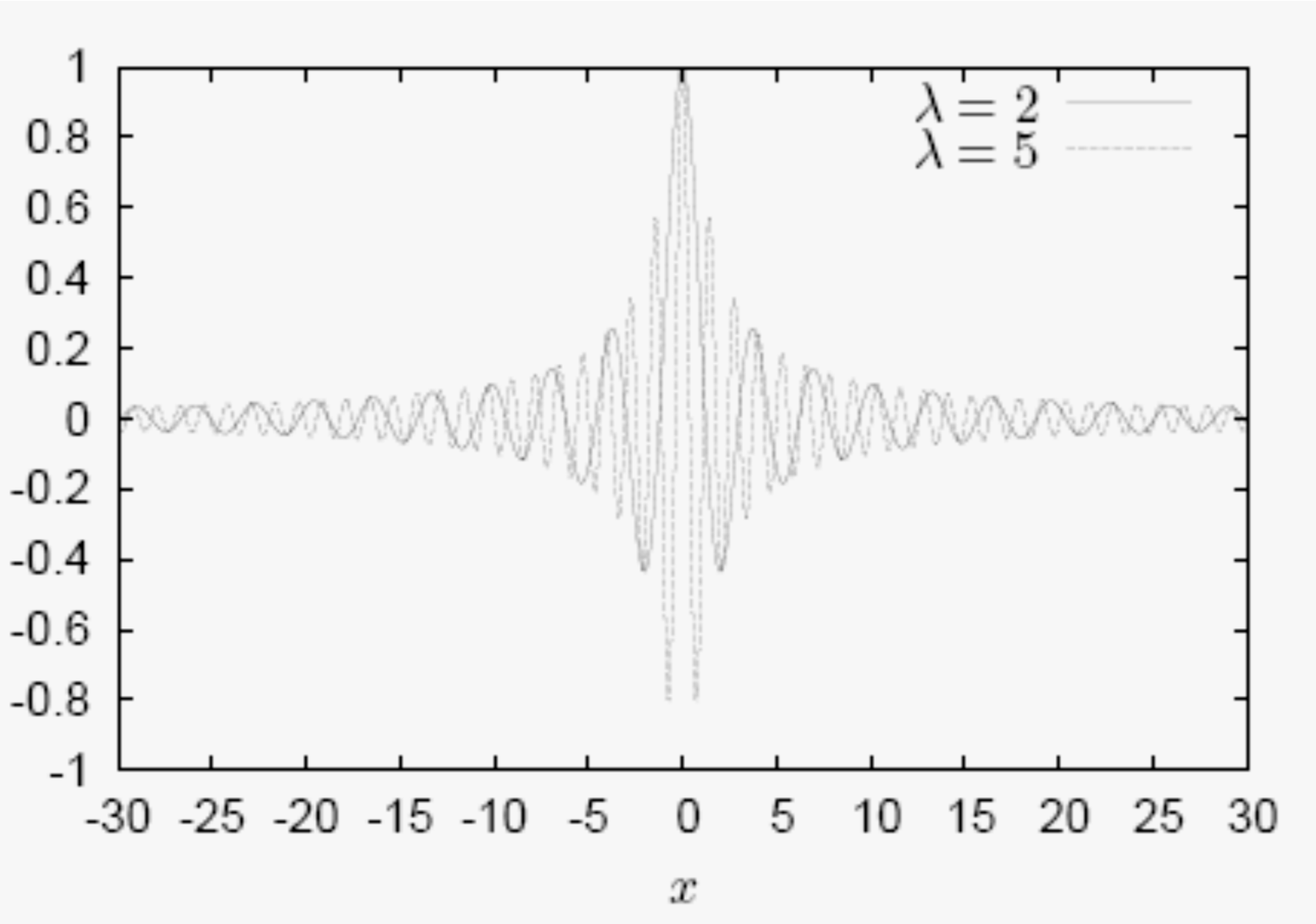
$$u(x)^\lambda \to u(x) \quad \text{in} \quad Z'(\mathbb{R}_x)$$

$$u^\lambda(x) \simeq \frac{e^\lambda}{2\pi}\psi^\lambda(x)$$

with

$$\psi^\lambda(x) = \frac{1}{(x^2+1)}[\cos(\lambda x) + x\sin(\lambda x)].$$

The function ψλ



3. A special sensitive evolution problem

In this Section, for the sake of simplicity in the exposition, we shall assume that $\Gamma = S^1 = \mathbb{T}$ is the unit circle . So that, we can assume that any function defined on $\mathbb{T}$ is 2π-periodic in x.

The sequence of functions e^{ikx} can be used to define any function v defined on $\mathbb{T}$ and the Fourier series of v

$$v = \sum_{-\infty}^{+\infty} v_k w_k.$$

with

$$w_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

As, w_k is an orthonormal basis in $L^2(\mathbb{T})$, we have

$$v \in L^2(\mathbb{T}) \text{ iff } \|v\|_{L^2}^2 = \sum_{k=-\infty}^{+\infty} |v_k|^2 < \infty.$$

Differentiating with respect to x amounts to multiply any v_k by ik, so the following equivalence holds

$$v \in H^1(\mathbb{T}) \text{ iff } \|v\|_{H^1}^2 = \sum_{k=-\infty}^{+\infty} |v_k|^2 (1+k^2) < \infty$$

and by duality

$$v \in H^{-1}(\mathbb{T}) \text{ iff } \|v\|_{H^{-1}}^2 = \sum_{k=-\infty}^{+\infty} \frac{|v_k|^2}{(1+k^2)} < \infty$$

For $m > 1$, $m \in \mathbb{N}$ we can use over the space $H^m(\mathbb{T})$ the norm (equivalent to the one obtained from the condition $u \in H^m(\mathbb{T})$ iff $u_x \in H^{m-1}(\Gamma)$)

$$\|v\|_{H^m}^2 = \sum_{k=-\infty}^{+\infty} |v_k|^2 (1+k^{2m})$$

and so, by duality

$$\|v\|_{H^{-m}}^2 = \sum_{k=-\infty}^{+\infty} \frac{|v_k|^2}{(1+k^{2m})}$$

Moreover, by interpolation the above equivalence can be extended to any $m \in \mathbb{R}$.

We can define

$$H^\infty(\mathbb{T}) = \bigcap_{p\in\mathbb{N}} H^p(\mathbb{T})$$

and we have that

$$\mathcal{D}(\mathbb{T}) \subset H^\infty(\mathbb{T}) \subset C^\infty(\mathbb{T})$$

So, by duality

$$H^{-\infty}(\mathbb{T}) := (H^\infty(\mathbb{T}))' \subset \mathcal{D}'(\mathbb{T}).$$

We consider now S a linear *smoothing operator*, $S : \mathcal{D}'(\mathbb{T}) \to \mathcal{D}'(\mathbb{T})$. For instance we can define

$$S(\sum_{k=-\infty}^{+\infty} v_k w_k) = \sum_{k=-\infty}^{+\infty} v_k e^{-\frac{|k|}{2}} w_k. \tag{2.4}$$

Then we get that $S(H^s(\mathbb{T})) \subset H^r(\mathbb{T}))$ for any $s, r \in \mathbb{R}^+$ (in particular $S(\mathcal{D}'(\mathbb{T})) \subset C^\infty(\mathbb{T})$).

More in general, let S be a (linear) smoothing operator, i.e. $S(H^s(\mathbb{T})) \subset H^r(\mathbb{T}))$ for any $s, r \in \mathbb{R}^+$.We assume that S is injective

$$v \in V,\, Sv = 0 \implies v = 0.$$

We then define on V the Hermitian form

$$a(u,v) = \int_{\mathbb{T}} Su\, \overline{Sv} dx$$

and the operator A, of $\mathcal{L}(V,V')$ given by

$$\langle Au, v\rangle = a(u,v) \quad \forall u,v \in V,$$

Notice that A can be written in the form

$$A = S^* S,$$

where we understand that S and its adjoint S^* are considered as

$$S \in \mathcal{L}(H^m, H^0),\ S^* \in \mathcal{L}(H^0, H^{-m}).$$

Lemma 1 *The operator $A \in L(V,V')$ is injective.*
Proof. Let $v \in V$ be such that $Av = 0$. Then

$$0 = \langle Av, v\rangle = a(v,v) = \|Sv\|_0^2\,,$$

and from the injectivity of S we obtain that $v = 0$.

Now, we follows an idea already adopted in D. Callerie, CRAS (1996) and which will play an important role also in the next Section.

The previous lemma allows to define the following norm

$$\|v\|_{V_A} = \|Av\|_{V'} ,$$

and we denote

$$V_A \text{ the completion of } V \text{ with } \|\cdot\|_{V_A} .$$

We have

Lemma 2 *The range of* A, $\mathcal{R}(A)$, is dense in $V\prime$

Then we have

Corollary 1. *The operator* A *extends as an isomorphism from* V_A *onto* V' *(which we denote again as* A*). In particular, for any* $F \in V'$ *there exists a unique solution* $u \in V_A$ *of*

$$Au = F.$$

We return now to the consideration of the evolution problem

$$(EP)\begin{cases} \frac{du}{dt}(t) + Au = f_\infty & t > 0, \\ u(0) = 0 \end{cases}$$

where

$$f_\infty \in V'$$

with $V = H^m(\mathbb{T})$, i.e. we know that

$$f_\infty = \sum_{-\infty}^{+\infty} b_k e^{ikx} \text{ with } \sum_{k=-\infty}^{+\infty} \frac{|b_k|^2}{(1+k^{2m})} < \infty.$$

If we assume that $u(t,x) = \sum_{-\infty}^{+\infty} u_k(t) w_k$ satisfies (EP) then we get that

$$(EP_k)\begin{cases} \frac{du_k}{dt}(t) + e^{-|k|}u_k(t) = b_k & t > 0, \\ u_k(0) = 0 \end{cases}$$

and so,

$$u_k(t) = b_k e^{|k|}(1 - e^{-e^{-|k|}t}).$$

It is not difficult to show that the function

$$u(t,x)=\sum_{-\infty}^{+\infty} b_k e^{|k|}(1-e^{-e^{-|k|}t})w_k(x)$$

satisfies that $u \notin L^{\infty}(0,+\infty : H^m(\mathbb{T}))$ since

$$\|u(t,.)\|_{H^{-m}}^2=\sum_{-\infty}^{+\infty}\frac{|b_k|^2\, e^{2|k|}(1-e^{-e^{-|k|}t})^2}{(1+k^{2m})}$$

is unbounded as $t \to +\infty$.

We introduce the space V_A as the completion of V with $\|v\|_{V_A}=\|Av\|_{V'}$ for $A=S^*S$. Then, the norm of the space V_A is given by

$$\|v\|_{V_A}^2=\sum_{k=-\infty}^{+\infty} e^{-2|k|}\,|v_k|^2$$

and so we have that

$$\|u(t,.)\|_{V_A}^2=\sum_{-\infty}^{+\infty}|b_k|^2\,(1-e^{-e^{-|k|}t})^2 \le \|f_\infty\|_{H^0}^2 .$$

Moreover

$$\left|\frac{\partial u(t,x)}{\partial t}\right| = \sum_{-\infty}^{+\infty} b_k e^{-e^{-|k|}t} \Rightarrow \left\|\frac{\partial u(t,.)}{\partial t}\right\|_{H^0}^2 \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Then (at least), $u(t,.) \rightharpoonup u_\infty$, weakly in V_A, as $t \rightarrow +\infty$, for some $u_\infty \in V_A$. As a matter of fact, if we write the (unique) solution u_∞ of the stationary limit problem

$$(SP)\ \ Au_\infty = f_\infty$$

as

$$u_\infty = \sum_{-\infty}^{+\infty} u_{\infty,k} w_k$$

we get that

$$Au_\infty = f_\infty \text{ iff } u_{\infty,k} = b_k e^{|k|}$$

and we see that

$$u_k(t) = b_k e^{|k|}(1 - e^{-e^{-|k|}t}) \rightarrow u_{\infty,k}$$

as $t \rightarrow +\infty$. Since

$$\|u(t,.)\|_{V_A}^2 \rightarrow \|u_\infty\|_{V_A}^2$$

we get that $u(t,.) \rightarrow u_\infty$, strongly in V_A, as $t \rightarrow +\infty$.

Remark 1. we show that if (for simplicity) $f(t) = f_\infty \in H^{-m}(\mathbb{T}) - C^\infty(\mathbb{T})$, the associate solution of (SP), u_∞, is not a distribution ($u_\infty \notin \mathcal{D}'(\mathbb{T})$).

4. On some quasi-static problems in shell theory

Motivated by the arguments mentioned in Section 2, a "trace general formulation" can be obtained by starting with, $p(x, D)$ and $q(x, D)$, be two elliptic pseudo differential operators of order $m \in \mathbb{R}^+$ with real symbols $p(x, \xi)$ and $q(x, \xi)$ continuous and coercive on $V = H^m(\Gamma)$, i.e. satisfying

$$\begin{aligned} c\,\|v\|_m \leq \|p(x,D)v\|_0 \leq C\,\|v\|_m \quad \forall v \in V \\ c\,\|v\|_m \leq \|q(x,D)v\|_0 \leq C\,\|v\|_m \quad \forall v \in V \end{aligned} \tag{3.5}$$

with $c, C > 0$. Let S be a (linear) smoothing operator, i.e. $S(H^s(\Gamma)) \subset H^r(\Gamma))$ for any $s, r \in \mathbb{R}^+$ where now Γ is a given one-dimensional compact manifold without boundary.We assume that S is injective

$$v \in V,\ Sv = 0 \implies v = 0.$$

We then define on V the Hermitian forms

$$a(u,v) = \int_\Gamma Su\,\overline{Sv}dx$$

$$b(u,v) = \int_\Gamma p(x,D)u\,\overline{p(x,D)v}dx$$

$$c(u,v) = \int_\Gamma q(x,D)u\,\overline{q(x,D)v}dx$$

We start with an abstract formulation: given $\varepsilon \geq 0$, $u_0 \in V$ and $f \in H^1(0,T:V')$ for any $T > 0$, find $u^\varepsilon \in L^2(0,T:V)$ with $\frac{du}{dt} \in L^2(0,T:V')$ such that for any $v \in V$

$$(QSP)\begin{cases} \frac{d}{dt}c(u(t),v) + a(u(t),v) + \varepsilon^2 b(u(t),v) = \langle f(t),v\rangle, \\ u(0) = u_0 \end{cases}$$

where the bracket denote the duality between V' and V (it may depend on the small parameter $\varepsilon \geq 0$). In order to show that the problem is well posed we shall need to make some additional assumptions (see Showalter (1996) for other alternatives). It is useful to reformulate (QSP) in terms of operators

$$(QSP)\begin{cases} \frac{dCu}{dt}(t) + Au(t) + \varepsilon^2 Bu(t) = f(t), \quad \text{in } V' \\ Cu(0) = Cu_0 \end{cases}$$

where the operators A, B, C of $\mathcal{L}(V,V')$ are given by

$$
\begin{aligned}
\langle Au, v\rangle &= a(u,v) \quad \forall u, v \in V,\\
\langle Bu, v\rangle &= b(u,v) \quad \forall u, v \in V,\\
\langle Cu, v\rangle &= c(u,v) \quad \forall u, v \in V.
\end{aligned}
$$

Notice that they can be written in the form

$$
A = S^* S,\ B = p^* p \text{ and } C = q^* q
$$

where we understand that S and its adjoint S^* are considered as

$$
S \in \mathcal{L}(H^m, H^0),\ S^* \in \mathcal{L}(H^0, H^{-m}).
$$

As in the precedent Section, the operator $A \in L(V, V')$ is injective, The operator A extends as an isomorphism from V_A onto V' (which we denote again as A). In particular, for any $F \in V'$ there exists a unique solution $u \in V_A$ of $Au = F$.

From the assumptions on $q(x, D)$ we deduce, from Lax-Milgram's Lemma, that operators C admits an inverse $C^{-1} \in \mathcal{L}(V', V)$ and so, by introducing $w := Cu$ we can reformulate problem (QSP) as find $w \in C([0, +\infty) : V')$ solution of the Cauchy problem

$$
(\widetilde{QSP}) \begin{cases} \frac{dw}{dt}(t) + AC^{-1}w(t) + \varepsilon^2 BC^{-1}w(t) = f(t), & \text{in } V' \\ w(0) = w_0, \end{cases}
$$

with $w_0 := Cu_0$.

Lemma 3 *Assume that*

$$q(x,D) \text{ commutes with } S \text{ and } p(x,D), \tag{3.6}$$

$$Su=\overline{Su},\ p(x,D)u=\overline{p(x,D)u} \ \text{ and } q(x,D)u=\overline{q(x,D)u} \ \text{ for any } V. \tag{3.7}$$

Then, for any $\epsilon \geq 0$ the operator $\mathcal{A}\in L(V',V')$ defined as $\mathcal{A}w = AC^{-1}w + \varepsilon^2 BC^{-1}w$, for any $w\in V'$, is a maximal monotone operator on V'. Moreover, $\mathcal{A}$ is a selfadjoint and $\mathcal{A}=\partial\varphi$, the subdifferential of the convex lower continuous function

$$\varphi(w)=\frac{1}{2}\left\|\mathcal{A}^{1/2}w\right\|^2 \text{ for any } w\in V'.$$

Proof. We first notice that $\mathcal{A}$ is (single valued) defined in the whole Hilbert space V'. The monotonicity of $\mathcal{A}$ comes from the fact that, if we denote by $((\cdot,\cdot))$ to the scalar product in V' and if $w := Cu = q^*qu$

$$((\mathcal{A}w,w)) = \left\langle S^*Su+\varepsilon^2p^*pu, q^*qu\right\rangle = (qSu,qSu)_{H^0}+\varepsilon^2\,(qpu,qpu)_{H^0}\geq 0.$$

Moreover $\mathcal{A}$ is a maximal operator since, for any $F\in V'$ and $\lambda>0$ there exists a (unique) $w\in V'$ solution of

$$\mathcal{A}w+\lambda w=F.$$

Indeed, this is equivalent to solve the equation

$$Au + \varepsilon^2 Bu + \lambda Cu = F$$

which has a solution (even if $\epsilon = 0$) via Lax-Milgram' Lemma. Finally, $\mathcal{A}$ is a *selfadjoint* operator since operators A and C verify that

$$\langle Au, v\rangle = a(u,v) = \int_\Gamma Su\,\overline{Sv}dx = \langle Av, u\rangle$$

$$\langle Cu, v\rangle = c(u,v) = \int_\Gamma q(x,D)u\,\overline{q(x,D)v}dx = \langle Cv, u\rangle$$

And so,

$$\big((AC^{-1}w, v)\big) = \big((w, (C^{-1})Av)\big) \text{ for any } v, w \in V'.$$

Analogously $\big((BC^{-1}w, v)\big) = \big((w, (C^{-1})Bv)\big)$ for any $v, w \in V'$. Then, by Proposition 2.5 of Brezis (1972) $\mathcal{A} = \partial\varphi$.

Remark. In the concrete examples, the commutativity assumption (3.6) holds when the pseudo-differential operators $p(x,D)$ and $q(x,D)$ are x-independent. Other assumptions (different to (3.6)) implying the monotonicity of $\mathcal{A}$ are possible.

Concerning the stabilization we have:

Theorem *Assume the conditions (3.6) and (3.7) of the above Lemma. For* $u_0 \in V$ *and,* $\epsilon \geq 0$ *let* $u \in C([0, +\infty) : V)$ *be the (unique) solution of*

$$(QSP) \begin{cases} \frac{dCu}{dt}(t) + Au(t) + \varepsilon^2 Bu(t) = f(t), \quad \text{in } V', \\ Cu(0) = Cu_0. \end{cases}$$

Then:

i) if $\epsilon > 0$ *and* $f(t) - f_\infty \in L^1(0, +\infty : V')$ *for some* $f_\infty \in V'$ *then* $lim_{t\to\infty} u(t) = u_\infty$ *in* V *with* u_∞ *the unique solution of*

$Au_\infty + \varepsilon^2 Bu_\infty = f_\infty$.

ii) if $\epsilon = 0$ *and* $f(t) \equiv f_\infty$ *for some* $f_\infty \in V'$ *then* $\left\| \frac{dCu}{dt}(t) \right\|_{V'} = O(\frac{1}{t})$ *and* $lim_{t\to\infty} u(t) = u_\infty$ *in* V_A *with* $u_\infty \in V_A$ *the unique solution of* $Au_\infty = f_\infty$.

Proof. Part i) is a consequence of the application of Theoreme 3.11 of Brezis (1972) since, from the coercivity assumptions (3.5) on $p(x, D)$ and $q(x, D)$, the set $\{w \in V' : \left\| \mathcal{A}^{1/2} w \right\|^2 + \|w\|^2 \leq C\}$ is (strongly) compact in V'.

To show Part ii) we use that, as $f(t) \equiv f_\infty$, and problem (QSP) can be written as

$$(\widetilde{QSP}) \begin{cases} \frac{dw}{dt}(t) + \partial\varphi(w(t)) = f(t), & \text{in } V' \\ w(0) = w_0, \end{cases}$$

then we get that $\left\| \frac{dCu}{dt}(t) \right\|_{V'} = O(\frac{1}{t})$ (see Theoreme 3.10 of Brezis). In contrast to the case $\epsilon > 0$, the compactness of the set $\{w \in V' : \|\varphi(w(t))\|^2 + \|w\|^2 \leq C\}$ fails and we can not apply the abstract result implying the convergence in V.

Nevertheless some direct arguments lead to the conclusion. Indeed, let $u_\infty \in V_A$ be the unique solution of $Au_\infty = f_\infty$. Then, $lim_{n\to\infty} Au(t_n) = Au_\infty$ in V', which, by construction, implies that $lim_{n\to\infty} u(t_n) = u_\infty$ in V_A. Moreover, from the uniqueness of the solution of $Au_\infty = f_\infty$ we get that the limit takes place for any $t \to +\infty$.

Thanks for your attention