Beyond the unique continuation:

“flat solutions" for reactive slow diffusions and the infinite well and Hardy potential for Schrödinger equation

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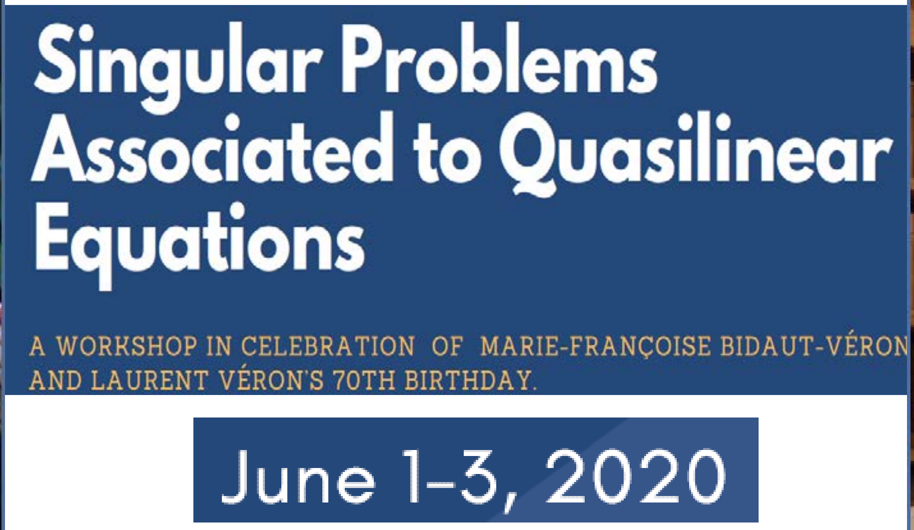


Souvenirs

Bruxelles, September, 11-30, 1975, International meeting on nonlinear monotone operators,

Bruges

Brezis, Temam,… Attouch, Damlamian, Haraux,….
Marie Françoise and Laurent

Participation in many other international meetings: Besançon (1977),…

J. I. Díaz and L. Véron. Existence, uniqueness and qualitative properties of the solutions of some first order quasilinear equations. Indiana University Mathematics Journal, Vol 32, No3, 319-361, 1983

Santander (Spain) 1982

In my home, Madrid, December 1981

First France-Spain Colloquium on Nonlinear Partial Differential Equations, Madrid, December 14 -18, 1981.

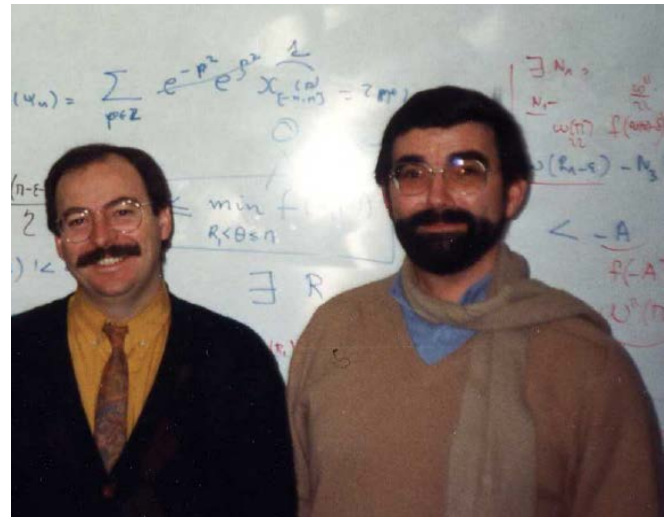
J. I. Díaz and L. Véron. Compacité du support des solutions d'équations quasilinéaires elliptiques ou paraboliques. C.R.Acad. Sc. París, t.297, Série I, 149-152, 1983

J. I. Díaz and L. Véron. Local vanishing properties of solutions of elliptic and parabolic quasilinear equations. Trans. of Am. Math. Soc., Vol 290, Nº 2, 787-814, 1985.

Nonlinear Evolution Equations, Visegrad (Hungary) (organized by D.G.Aronson, R.Kersner, V.P.Maslov and O.A.Oleinik), May, 1988.

Many other meetings: Oxford (1982), Paris (1985), …

One month at Tours: Professeur de première classe, Université de Tours, December, 1990.



Unpublished manuscript
Asymptotic behaviour for problems with a continuum of stationary solutions.

Frédéric (13), Fabian (10) and Ophélie (8)

The Second World Congress of Nonlinear Analysis, Athens, July, 10-17, 1996

Journées “Analyse non linéaire”, 60º anniversaire Philippe Bénilan, October 20-26, 2000: Les Moussières, Jura, France

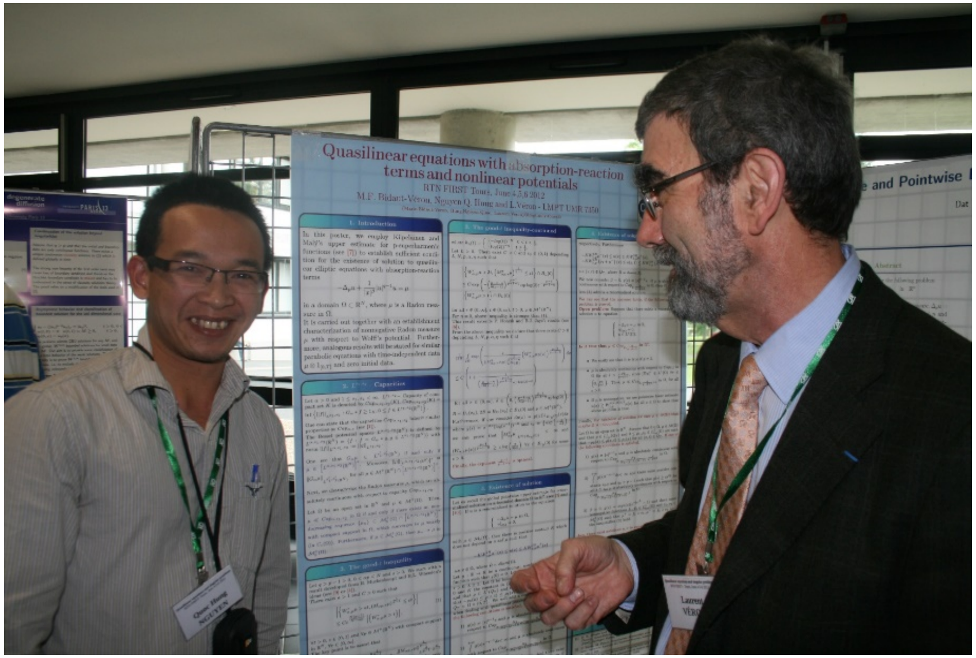
Jim Serrin honorary doctorate from the University François-Rabelais at Tours in 2005

Conference in honor of Marie-Françoise and Laurent Véron
Haifa, March 1-5, 2010

Unable to travel there by a transitory illness

Initial Training Network FIRST (2009-2013)

Workshop on Quasilinear PDEs, Tours, June 04-06, 2012



Laurent and Hung

Many other pictures,...

……… for their 75 birthday

0. *Plan* of the lecture

My main subject today:

***"flat solutions"* of linear and non-linear PDEs**

[u **flat** (on a part of the boundary) if u and its normal derivative vanish on this part of the boundary]

Plan:

1. Introduction: a change of mentality concerning the Unique Continuation Principle.
2. Stable flat solutions of slow diffusion with absorption and forcing terms.
3. Flat solutions for the Schrödinger equation with a nonnegative very singular potential.

1. Introduction: a change of mentality concerning the Unique Continuation Principle.

Analytical Unique Continuation Principle,...

1894. Émile Borel (These: Poincaré), Non analytical Unique Continuation Principle,..., Denjoy (1921), Hadamard (1912), Bernstein (1926), ...

1908 **Holmgren**,The Unique Continuation Principle for *parabolic equations*,..., **Carleman (1933)**, Tychonoff (1935),..., Aronszajin (1957), Nirenberg (1957), Mizohata (1958), Calderon (1958), Friedman (1958), Protter (1959), Hörmander (1963), ,. . . and many others.

The Unique Continuation Principle in our days

D. Tataru (2004), In *Geometric Methods in Inverse Problems and PDE Control.* IMA-Springer.

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open set. Mostly we will consider linear operators

$$Pu = a^{jk}\partial_{jk}u + b^j\partial_j u + cu,$$

where the coefficients satisfy

$$a^{jk} \in W^{1,\infty}(\Omega), \quad b^j \in L^\infty(\Omega),$$

A priori, no sign is prescribed on $c(x)$:
[Hopf strong maximum principle, spectral problems,...]

and (a^{jk}) is a symmetric matrix satisfying the uniform ellipticity condition for some constant $\lambda > 0$,

$$a^{jk}(x)\xi_j\xi_k \geq \lambda|\xi|^2 \quad \text{for all } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

The unique continuation principle (UCP) comes in several different forms:

THEOREM 1.1. *(Weak UCP) If* $u \in H^2(\Omega)$ *satisfies*

$$Pu = 0 \qquad \text{in } \Omega$$

and

$$u = 0 \text{ in some ball } B \text{ contained in } \Omega,$$

then $u = 0$ *in* Ω.

THEOREM 1.2. *(Strong UCP) If* $u \in H^2(\Omega)$ *satisfies*

$$Pu = 0 \qquad \text{in } \Omega$$

and if u *vanishes to infinite order at* $x_0 \in \Omega$ *in the sense that*

$$\lim_{r \to 0} \frac{1}{r^N} \int_{B(x_0,r)} |u|^2 \, dx = 0 \quad \text{for all } N \geq 0,$$

then $u = 0$ *in* Ω.

THEOREM 1.4. *(UCP for local Cauchy data) Let* $\Omega \subset \mathbb{R}^n$ *have smooth boundary, and let* Γ *be a nonempty open subset of* $\partial\Omega$. *If* $u \in H^2(\Omega)$ *satisfies*

$$Pu = 0 \quad in\ \Omega$$

and

$$u|_\Gamma = \partial_\nu u|_\Gamma = 0,$$

then $u = 0$ *in* Ω.

REMARKS.

1. Note that since P is linear, weak UCP implies that any two solutions u_1 and u_2 that coincide in a small ball must be equal in the whole domain. This explains the name "unique continuation principle".
2. Once clearly has

$$\text{strong UCP} \implies \text{weak UCP} \implies \text{UCP for Cauchy data.}$$

3. The above theorems remain valid if $u \in H^1(\Omega)$.
4. In the above theorems, the condition $Pu = 0$ in Ω can be replaced by the differential inequality

$$|a^{jk}\partial_{jk}u| \leq C(|\nabla u| + |u|) \quad \text{a.e. in } \Omega.$$

5. The assumption that the coefficients (a^{jk}) are Lipschitz continuous is optimal for $n \geq 3$, in the sense that for any $\alpha < 1$ there exist C^α coefficients (a^{jk}) such that UCP does not hold for the corresponding operator. (If $n = 2$ the UCP holds for $a^{jk} \in L^\infty$.) However, the assumptions for the first and zeroth order terms can be improved, and in fact UCP holds if $b^j \in L^n(\Omega)$ and $c \in L^{n/2}(\Omega)$ (at least when $n \geq 3$).

We shall see (in Section 3) assumptions on $c(x) \equiv V(x)$, the (stationary) Schrödinger equation

$$\begin{aligned} -\Delta u + V(x)u = \lambda u \qquad & \text{in } \Omega, \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned}$$

under which the weak and strong versions of the UCP fail.

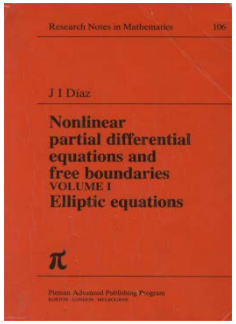
6. The UCP for uniformly elliptic nonlinear equations often reduces to the linear case.

The UCP also holds for suitable quasilinear non-uniformly elliptic (and parabolic) equations: see, e.g.

Bobkov, Vladimir; Takáč, Peter; On maximum and comparison principles for parabolic problems with the p-Laplacian. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* RACSAM 113 (2019), no. 2, 1141–1158

The class of nonlinear elliptic problems for which there is a lack of the UCP is very large:

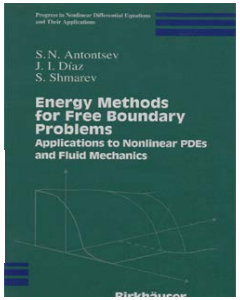
- Degenerate equations (p-Laplacian operators, porous media operators, Monge-Ampere with zero order perturbations,...):



i) method of super and subsolutions: solutions with compact support, dead core problems,...

ii) energy local methods: higher order equations, systems of equations,...

(alternative, in some sense, to the integral Carleman inequalities to prove the UCP in the linear case).



- Nonlinear equations with *suitable singular terms* (as, e.g. in the **Space Charge Problem**: lack of UCP for local Cauchy data),

- Non-Lipschiz perturbations: in Section 2 we shall show that there is a lack of the weak UCP for the problem

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u = \lambda |u|^{\beta-1}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \qquad 0 < \alpha < \beta < 1 \end{cases} \qquad P(\alpha, \beta, \lambda)$$

Change of mentality. Instead to looking for isolated counterexamples to the UCP, try to show that some opposite property to the UCP holds for a wide class of spatial domains (flat solutions, compact support solutions, dead cores,...).

2 . Stable flat solutions of slow diffusion with absorption and forcing terms

$$PP(m,\alpha,\beta,\lambda,v_0)\quad \begin{cases} \left(|v|^{\frac{1}{m}-1}v\right)_t - \Delta v + |v|^{\alpha-1}v = \lambda|v|^{\beta-1}v & \text{in } (0,+\infty)\times\Omega \\ v=0 & \text{on } (0,+\infty)\times\partial\Omega \\ v(0,x)=v_0(x) & \text{on } \Omega, \end{cases}$$

$$m>0 \qquad \boxed{0<\alpha<\beta<1}$$

$m \neq 1 \quad w := |v|^{\frac{1}{m}-1}v$

$$\overline{PP}(m,a,b,\lambda,w_0)\quad \begin{cases} w_t - \Delta|w|^{m-1}w + |w|^{a-1}w = \lambda|w|^{b-1}w & \text{in } (0,+\infty)\times\Omega \\ w=0 & \text{on } (0,+\infty)\times\partial\Omega \\ w(0,x)=w_0(x) & \text{on } \Omega, \end{cases}$$

$$a=\alpha m,\ b=\beta m \text{ and } w_0 := |v_0|^{\frac{1}{m}-1}v_0.$$

The pionering work Fujita (1966) for $m=1$ [without absorption $|w|^{a-1}w$]

i) $b\in(0,1)$ all the solutions are globally defined in time,

ii) $b\in(1,1+2/N)$ all the solutions blow up in a finite time,

iii) $b\in(1+2/N,+\infty)$ there exist both global solutions and blowing-up solutions according to the initial datum w_0.

The quasilinear case $m > 0$ was considered by many other authors, **but most of them without the absorption term** $|w|^{a-1}w$. In that case the Fujita exponent separating the three regimes is $m + 2/N$.

A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Nauka, Moscow, 1987; English translation: Walter de Gruyter, Berlin/New York, 1995.

V.A. Galaktionov and J.L. Vázquez. *A Stability Technique for Evolution Partial Differential Equations. A Dynamical Systems Approach*. Progress in Non-Linear Differential Equations and Their Applications 56, Birkhäuser Verlag, 2003.

Our main interest here is on non-negative stationary solutions

$$\begin{cases} -\Delta u + |u|^{\alpha-1}u = \lambda|u|^{\beta-1}u \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases} \qquad P(\alpha,\beta,\lambda)$$

J. I. D., J. Hernández, and Y. Ilyasov, **Chinese Ann. Math**. (2017).

J. I. D., J. Hernández and Y. Ilyasov, **Advances in Nonlinear Analysis** (2020)

Crucial assumption: $0 < \alpha < \beta < 1$

P. Drabek and P. Takáč, Convergence to travelling waves in Fisher's population genetics model with a non-Lipschitzian reaction term. *J. Math. Biol.* 75 (2017), no. 4, 929–972

$$-\Delta u = u^{\alpha} - \lambda u^{\beta}$$

Th. Cazenave, M. Escobedo, M.A. Pozio, Some stability properties for minimal solutions of $-\Delta u = \lambda g(u)$, *Port. Math. (N.S.)* 59 (2002), no. 4, 373–391.

$$-\Delta u - \lambda u^{\alpha} = \lambda u^{\beta} \qquad 0 < \alpha < 1 < \beta$$

Here Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 3$, with a smooth boundary $\partial\Omega$, which is strictly star-shaped with respect to a point $x_0 \in \mathbb{R}^N$ (which will be identified as the origin of coordinates if no confusion may arise), λ is a real parameter, $0 < \alpha < \beta < 1$. By a weak solution of $P(\alpha, \beta, \lambda)$ we mean a critical point $u \in H_0^1 := H_0^1(\Omega)$ of the energy functional

$$E_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1} dx - \frac{\lambda}{\beta+1}\int_\Omega |u|^{\beta+1} dx,$$

We are interested in *ground states*

a weak solution u_λ of $P(\alpha, \beta, \lambda)$ which satisfies the inequality

$$E_\lambda(u_\lambda) \leq E_\lambda(w_\lambda)$$

for any non-zero weak solution w_λ of $P(\alpha, \beta, \lambda)$.

Since the diffusion-reaction balance $-\Delta u = f(\lambda, u)$ involves the non-linear reaction term

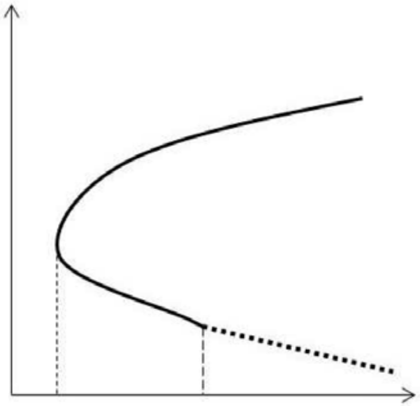
$$f(\lambda, u) := \lambda |u|^{\beta-1} u - |u|^{\alpha-1} u,$$

and it is a non-Lipschitz function at zero (since $\alpha < 1$ and $\beta < 1$) important peculiar behavior of solutions of these problems arises. For instance, that may lead to the violation of the Hopf maximum principle on the boundary and the existence of compactly supported solutions as well as the so called *flat solutions* which correspond to weak solutions $u > 0$ in Ω such that

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \tag{1}$$

where ν denotes the unit outward normal to $\partial\Omega$. When the additional information (1) holds but the weak solution may vanish in a positively measured subset of Ω, i.e. $u \geq 0$ in Ω, we shall call it as a *compact support solution* of $P(\alpha, \beta, \lambda)$ (sometimes also called as a *free boundary solution*, since the boundary of its support is not a priori known). Notice that in that case the support of u is strictly included in $\overline{\Omega}$. If u is a weak solution such that property (1) is not satisfied we shall call it as an *"classical" weak solution* (since, at least for the associated linear problem and for Lipschitz non-linear terms, the strong maximum principle due to Hopf, implies that (1) cannot be verified). However, we cannot exclude a priori the existence of solutions where (1) is only satisfied on part of $\partial\Omega$.

Main motivation: The onedimensional case $\Omega = (-R, R)$



J.I. Díaz and J. Hernández,, Global bifurcation and continua of nonnegative solutions for a quasilinear elliptic problem, *C.R. Acad. Sci. Paris*, **329**, (1999), 587-592.

For sufficiently large λ the existence of a *compactly supported solution* of $P(\alpha,\beta,\lambda)$ follows from

F. Gazzola, J. Serrin and M. Tang, Existence of ground states and free boundary problems for quasilinear elliptic operators, *Advances in Differential Equations* **5** (2000) 1-30.

J. Serrin and H. Zou, Symmetry of ground states of quasilinear elliptic equations. *Archive for Rational Mechanics and Analysis*, **148** 4 (1999) 265-290.

Indeed, by

H. Kaper and M. Kwong, Free boundary problems for Emden-Fowler equation, *Differential and Integral Equations*, **3** (1990) 353-362.

H. Kaper, M. Kwong and Y. Li, Symmetry results for reaction-diffusion equations, *Differential and Integral Equations*, **6** (1993) 1045-1056.

$P(\alpha,\beta,1)$ considered in $\mathbb{R}^N$ has a unique (up to translation in $\mathbb{R}^N$) compactly supported solution u^*, moreover u^* is radially symmetric such that supp(u^*)=$\overline{B}_{R^*}$ for some $R^*>0$. Hence since the support of $u^*_\sigma(x) := u^*(x/\sigma)$, $x \in B_{\sigma R^*}$ is contained in Ω, for sufficiently small σ, the function $w^\sigma_\lambda(x) = \sigma^{-\frac{2}{1-\alpha}} \cdot u^*_\sigma(x)$ weakly satisfies $P(\alpha,\beta,\lambda)$ in Ω with $\lambda = \sigma^{-\frac{2(\beta-\alpha)}{1-\alpha}}$.

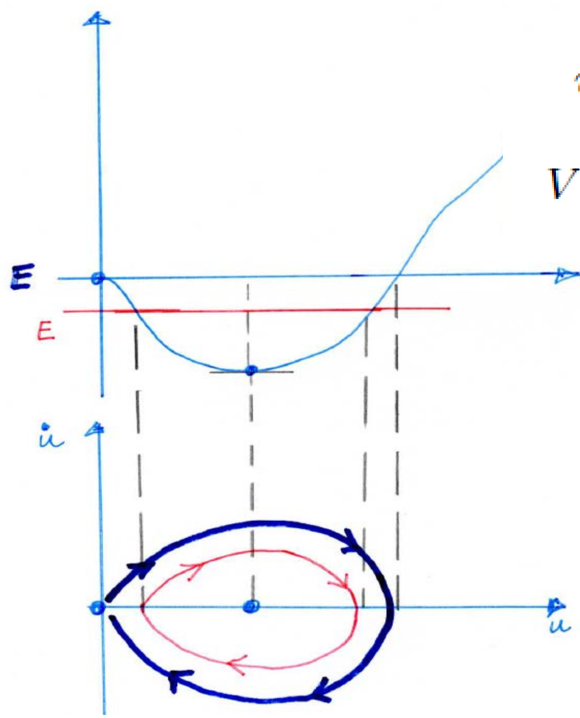
However, in general (for all sufficiently large λ), weak solutions w_λ are not ground states.

The existence of *flat* and *compact support ground states*, for certain λ^* of $P(\alpha,\beta,\lambda)$ was first obtained in

Y. Sh. Ilyasov and Y. Egorov, Hopf maximum principle violation for elliptic equations with non-Lipschitz nonlinearity, *Nonlin. Anal.* **72** (2010) 3346-3355.

We recall that, as a matter of fact, flat solutions of $P(\alpha, \beta, \lambda^*)$ only may arise if Ω is the ball B_{R^*} mentioned before. For the rest of domains, and values of $\lambda \geq \lambda^*$, any weak solution which is not a "*classical*" *weak solution* should be radially symmetric and has compact support.

By using phase plane arguments for conservative forces, it is easy to see that if $\mathbf{N = 1}$ any flat solution is unstable.



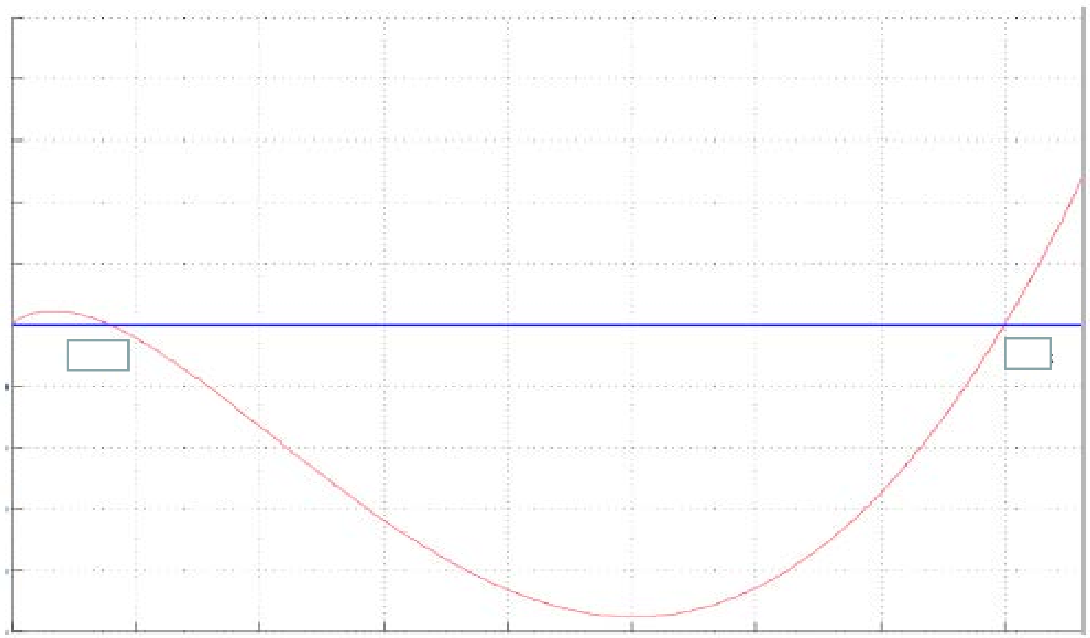
$$u'' = -V'(u)$$

$$V(u) = \frac{\lambda}{(\beta+1)} u^{\beta+1} - \frac{1}{(\alpha+1)} u^{\alpha+1}$$

Let us state our main results. For given $u \in H_0^1(\Omega)$, the *fibrering mappings* are defined by $\phi_u(t) = E_\lambda(tu)$ so that from the variational formulation of $P(\alpha, \beta, \lambda)$ we know that $\phi_u'(t)|_{t=1} = 0$ for solutions, where we use the notation

$$\phi_u'(t) = \frac{\partial}{\partial t} E_\lambda(tu).$$

If we also define $\phi_u''(t) = \frac{\partial^2}{\partial t^2} E_\lambda(tu)$, then, since $\beta < 1$ the equation $\phi_u'(t) = 0$ may have at most two nonzero roots $t_{\min}(u) > 0$ and $t_{\max}(u) > 0$ such that $\phi_u''(t_{\max}(u)) \leq 0$, $\phi_u''(t_{\min}(u)) \geq 0$ and $0 < t_{\max}(u) \leq t_{\min}(u)$. This implies that any weak solution of $P(\alpha, \beta, \lambda)$ (any critical point of $E_\lambda(u)$) corresponds to one of the cases $t_{\min}(u) = 1$ or $t_{\max}(u) = 1$.



As in Ilyasov and Egorov (2010) for the study of flat solutions it is useful to use some *Pohozhaev's identity* for weak solutions of the problem.

Denote

$$P_\lambda(u) := \frac{1}{2^*}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \frac{1}{\alpha+1}\int_\Omega |u|^{\alpha+1}\,\mathrm{d}x - \lambda\frac{1}{\beta+1}\int_\Omega |u|^{\beta+1}\,\mathrm{d}x,$$

where

$$2^* = \frac{2N}{N-2} \quad \text{for } N \geq 3.$$

Lemma 2.1 *Assume that $\partial\Omega$ is a C^2-manifold, $N \geq 3$. Let $u \in C^1(\overline{\Omega})$ be a weak solution of $P(\alpha,\beta,\lambda)$. Then there holds the Pohozaev identity*

$$P_\lambda(u) = -\frac{1}{2N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 (x\cdot\nu(x))\mathrm{d}\sigma(x).$$

Notice that

$$E_\lambda(u) = P_\lambda(u) + \frac{1}{N}\int_\Omega |\nabla u|^2 dx, \quad \forall u \in H_0^1(\Omega).$$

Corollary 2.1 *Let Ω be a bounded star-shaped domain in $\mathbb{R}^N$ with a C^2-manifold boundary $\partial\Omega$. Then any weak solution $u \in C^1(\overline{\Omega})$ of $P(\alpha,\beta,\lambda)$ satisfies $P_\lambda(u) \leq 0$. Moreover, if u is a flat solution or it has a compact support then $P_\lambda(u) = 0$. Furthermore, in the case Ω is strictly star-shaped, the converse is also true: if $P_\lambda(u) = 0$ and $u \in C^1(\overline{\Omega})$ is a weak solution of $P(\alpha,\beta,\lambda)$, then u is flat or it has a compact support.*

We use the additional notations

$$T(u) = \int_\Omega |\nabla u|^2\,dx,\ A(u) = \int_\Omega |u|^{\alpha+1}\,dx,\ B(u) = \int_\Omega |u|^{\beta+1}\,dx.$$

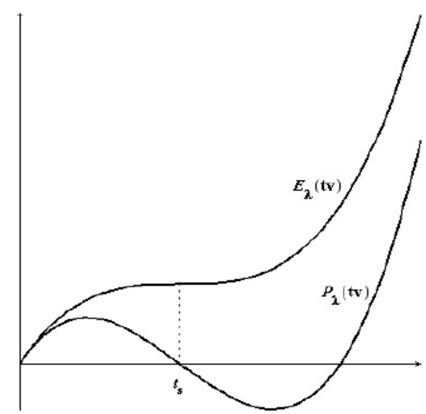
Then

$$E_\lambda(u) = \frac{1}{2}T(u) + \frac{1}{\alpha+1}A(u) - \lambda\frac{1}{\beta+1}B(u).$$

$$P_\lambda(u) := \frac{N-2}{2N}T(u) + \frac{1}{\alpha+1}A(u) - \lambda\frac{1}{\beta+1}B(u), \quad u \in H_0^1(\Omega).$$

By some quite technical arguments (spectral analysis with respect to the fiber procedure, nonlinear generalized Rayleigh quotient, constrained minimization problem, Lagrange multipliers for unilateral problems,...) we show that (even if $0 < \alpha < \beta < 1$) it is useful to look for solutions such that $E''_\lambda(u) = 0$.

$$\begin{cases} E'_\lambda(u) := T(u) + A(u) - \lambda B(u) = 0 \\ P_\lambda(u) := \dfrac{N-2}{2N}T(u) + \dfrac{1}{\alpha+1}A(u) - \lambda\dfrac{1}{\beta+1}B(u) = 0 \\ E''_\lambda(u) := T(u) + \alpha A(u) - \lambda\beta B(u) = 0. \end{cases}$$



This system is solvable with respect to the variables $T(u), A(u), B(u)$ if the corresponding determinant

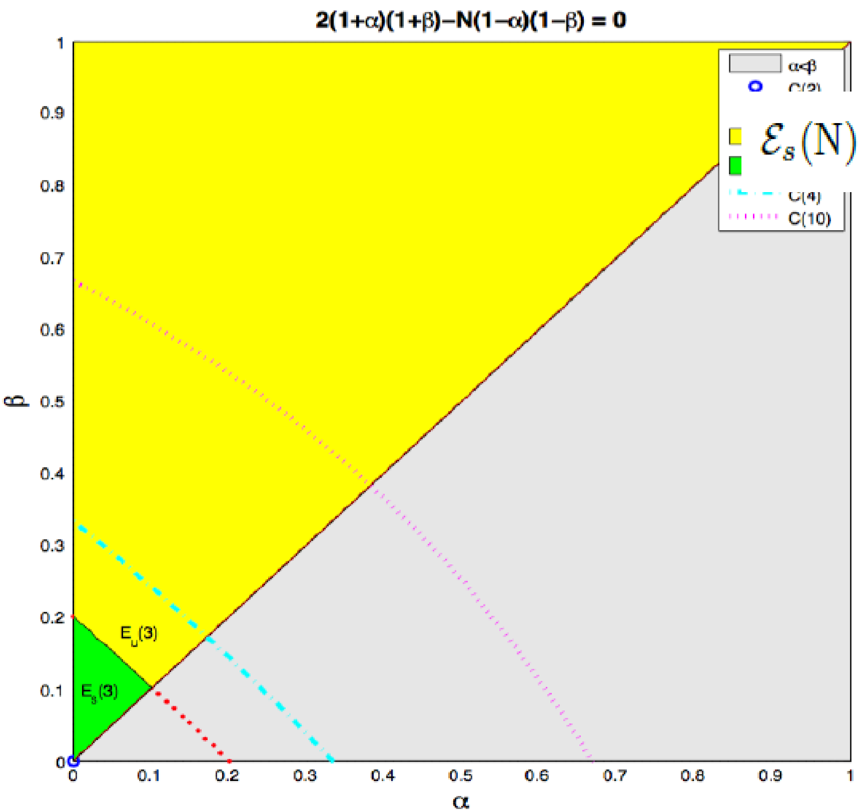
$$D = \frac{(\beta - \alpha)(2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta))}{2\mathrm{N}(1+\alpha)(1+\beta)}.$$

is non-zero.

On the other hand $D = 0$ if and only if $(\alpha, \beta) \in \mathcal{C}(\mathrm{N})$. $\mathcal{E} := \{(\alpha, \beta) : \ 0 < \ \alpha < \beta \leq 1\}$

$$\mathcal{C}(\mathrm{N}) := \{(\alpha, \beta) \in \mathcal{E} : \ 2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta) = 0\}.$$

This curve exists if and only if $\mathrm{N} \geq 3$ and it separates two sets of exponents in $\mathcal{E}$

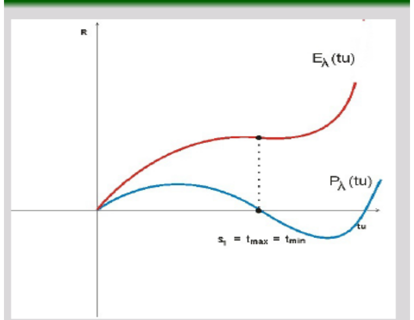


$$\mathcal{E}_s(\mathrm{N}) := \{(\alpha, \beta) \in \mathcal{E} : \ 2(1+\alpha)(1+\beta) - \mathrm{N}(1-\alpha)(1-\beta) < 0\},$$

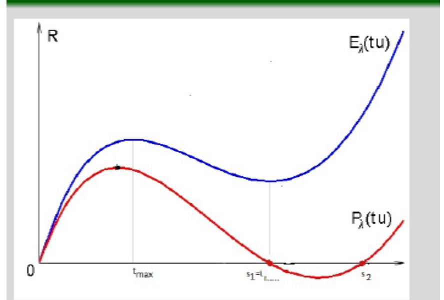
Sets $\mathcal{E}_s(\mathrm{N})$ and $\mathcal{E}_u(\mathrm{N})$ for $\mathrm{N} = 3, \ 4$ and 10

Thus we have 4 structurally different cases: the more interesting cases

$n = \frac{2(1+\alpha)(1+\beta)}{(1-\alpha)(1-\beta)}$



$n > \frac{2(1+\alpha)(1+\beta)}{(1-\alpha)(1-\beta)}$

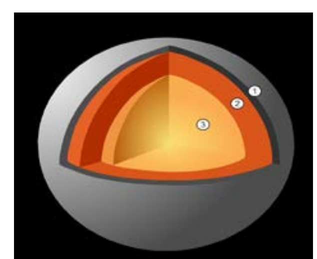


$\Longrightarrow$ stability of flat ground states

Theorem 1.1 *Let $N \geq 3$ and let Ω be a bounded strictly star-shaped domain in $\mathbb{R}^N$ with C^2-manifold boundary $\partial\Omega$. Assume that $(\alpha, \beta) \in \mathcal{E}_s(N)$. Then there exists $\lambda^* > 0$ such that for any $\lambda \geq \lambda^*$ problem $P(\alpha, \beta, \lambda)$ possess a ground state u_λ. Moreover $E''_\lambda(u_\lambda) > 0$, $u_\lambda \in C^{1,\gamma}(\overline{\Omega})$ for some $\gamma \in (0, 1)$ and $u_\lambda \geq 0$ in Ω. For any $\lambda < \lambda^*$, problem $P(\alpha, \beta, \lambda)$ has no weak solution.*

Our second main result deals with the existence (or not) of flat or compactly supported ground states.

Theorem 1.2 *Under the same assumptions of the above theorem, there is a non-negative ground state u_{λ^*} which is flat or has compact support. Moreover, u_{λ^*} is radially symmetric about some point of Ω, and supp(u_{λ^*})=$\overline{B}_{R(\Omega)}$ is an inscribed ball in Ω. For all $\lambda > \lambda^*$, any ground state u_λ of $P(\alpha, \beta, \lambda)$ is a ““classical” weak solution.*



Summarizing for the case of the slow diffusion, we have stable flat ground states under suitable circumstances for $N \geq 3$:

$$\overline{PP}(m,a,b,\lambda,w_0) \quad \begin{cases} w_t - \Delta|w|^{m-1}w + |w|^{a-1}w = \lambda|w|^{b-1}w & \text{in } (0,+\infty)\times\Omega \\ w = 0 & \text{on } (0,+\infty)\times\partial\Omega \\ w(0,x) = w_0(x) & \text{on } \Omega, \end{cases}$$

Theorem 7.1 *Assume* $0 < a < b < m$ *such that*

$$2(m+a)(m+b) - N(m-a)(m-b) < 0, \tag{34}$$

then, if $\alpha = a/m$ *and* $\beta = b/m$*, the stationary ground state* $u_{\lambda^*} \in H_0^1(\Omega)$ *of problem* $P(\alpha,\beta,\lambda^*)$ *is a* H_0^1*-stable solution of* $\overline{PP}(m,a,b,\lambda^*,w_0)$*, i.e., given any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that*

$$||u_{\lambda^*} - |w|^{m-1}w(t;w_0)||_1 < \varepsilon \text{ for any } w_0 \text{ s.t. } ||u_{\lambda^*} - |w_0|^{m-1}w_0||_1 < \delta, \ \forall t > 0, \tag{35}$$

where $w(t;w_0)$ *is the weak solution of* $\overline{PP}(m,a,b,\lambda^*,w_0)$.

There are two different kinds of arguments in the proof. On one hand, we first prove that under the assumptions the ground state u_{λ^*} is $H_0^1(\Omega)$-isolated. The second ingredient is to prove that the energy $E_\lambda(u)$ (here $\lambda > 0$ is arbitrary) is a Lyapounov function in the sense that

$$\frac{\partial}{\partial t}E_\lambda(v(t)) \leq 0 \quad \text{in} \quad (0,T)$$

for any $T > 0$ and for any $v(t)$ weak solution of $PP(m,\alpha,\beta,\lambda,v_0)$. The case $m \neq 1$ requires to use some regularity results, as in:

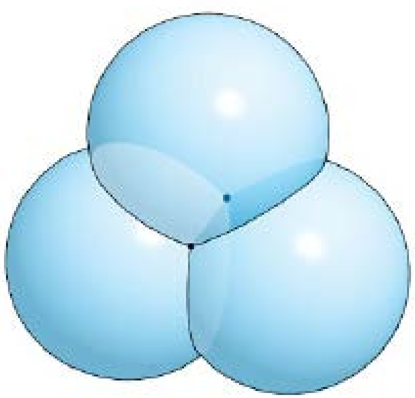
Ph. Bénilan, Sur un problème d'évolution non monotone dans $L^2(\Omega)$, *Publications Mathématiques de la Faculté des Sciences de Besançon*. Fascicule n° 2 (1975-76).

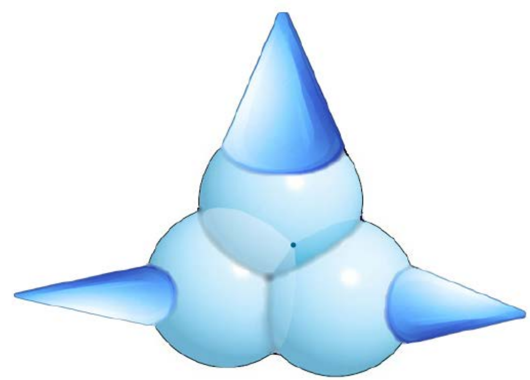
M. Efendiev and S. Zelik, Finite- and infinite-dimensional attractors for porous media equations. *Proceedings of the London Mathematical Society* **96** (2008) 51–77.

The result concerning the exact number of stable non-negative flat (or compact support) solutions is, at the best of our knowledge, new, and the same can be said of the introduction of the classes of strictly star-shaped domains.

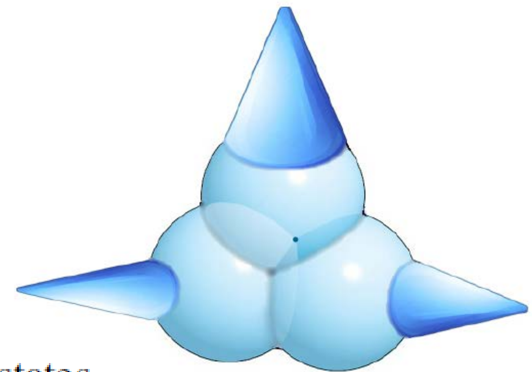
In order to present our exact multiplicity results we introduce the *geometrical reflection* across a given hyperplane H by the usual isometry $R_H : \mathbb{R}^N \to \mathbb{R}^N$. Remember that any point of H is a fixed point of R_H. Now we shall introduce some classes of starshaped sets Ω for which we can obtain the exact multiplicity of flat stable ground solutions of problem $P(\alpha, \beta, \lambda^*)$. We say that Ω is of *Strictly Starshaped Class* m, if it is a strictly starshaped domain and contains exactly m inscribed balls of the same radius $R(\Omega)$ such that each of them can be obtained from any other by $k \in \{1, ..., m\}$ reflections of Ω across some hyperplanes $H_i, i = 1, ..., k$.

Schwarz reflection principle

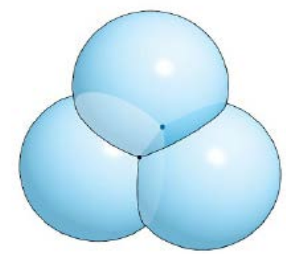




Theorem 1.3 *Assume $N \geq 3$, $(\alpha, \beta) \in \mathcal{E}_s(N)$. Let Ω be a domain of Strictly Starshaped Class $m > 1$ with a C^2-manifold boundary $\partial\Omega$. Then there exist exactly m stable nonnegative flat or compact supported ground states $u^1_{\lambda^*}$, $u^2_{\lambda^*}$,..., $u^m_{\lambda^*}$ of problem $P(\alpha, \beta, \lambda^*)$ and m sets of "classical" ground states $(u^1_{\lambda_n})^\infty_{n=1}$, $(u^2_{\lambda_n})^\infty_{n=1}$,..., $(u^m_{\lambda_n})^\infty_{n=1}$ of $P(\alpha, \beta, \lambda_n)$, , with $\lim_{n\to\infty} \lambda_n = \lambda^*$, $\lambda_n > \lambda^*$, $n = 1, 2, ...$ and such that $u^i_{\lambda_n} \to u^i_{\lambda^*}$, strongly in H^1_0 as $n \to \infty$, for any $i = 1, ..., m$.*



Domain generating exactly three ground states



Union of the supports of the three radially symmetric ground states corresponding to the domain

Adv. Nonlinear Anal. 2020; 9: 1046–1065 DE GRUYTER

J.I. Díaz*, J. Hernández, and Y.Sh. Ilyasov

On the exact multiplicity of stable ground states of non-Lipschitz semilinear elliptic equations for some classes of starshaped sets

3. On the confinement for Schroedinger equation

We recall that in Quantuum Mechanics,

$\psi : [0,\infty) \times \mathbb{R}^N \to \mathbb{C}$ the matter wave function (L. de Broglie 1924: wave-particle duality)

$\hbar > 0$ renormalized Plank constant, m mass of the elementary particle,

$V(x) \in \mathbb{R}$ the external potential

Crucial fact: $|\psi(x,t)|^2$ represents the probability density (Max Born 1926) to find the particle at point x and time t :

Question: how to "confine" (or "localize") the particle (and how to measure its linear momentum **p**) ??

Famous **negative** answers: 1927 W.Heisenberg "uncertainty principle" [1932 Nobel Prize]

Unique continuation results:

* Reed-Simon, Methods of modern Mathematical Physics (1975), Vol. 4, Theorem XIII.57

$V(x)$ is bonded on any compact interval of $\mathbb{R} - \mathcal{S}$ with $|\mathcal{S}| = 0$

* Escauriaza L, Kenig C. E., Ponce G., Vega, L. Hardy's uncertainty principle, convexity and Schrödinger evolutions J. Eur. Math. Soc. (JEMS) 10 (2008), no. 4, 883–907 ($V(x) \sim$ bonded on compact intervals of $\mathbb{R}$)

H. Brezis and M. Marcus (1997), …, L.Orsina and A. C. Ponce (2018).

A pioneering **partially positive** answer: *the infinite well potential*

Simplifications for the linear Schrödinger equation (attributed by him, in 1935, to George Gamow (1904-1968) and repeated in any text book in Quantum Mechanics):

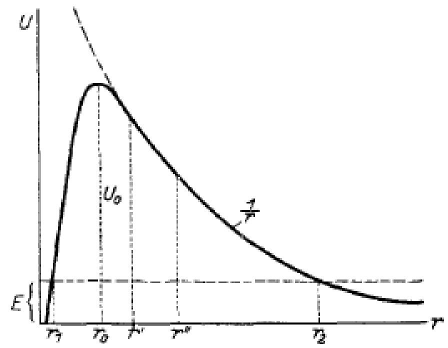


Fig. 1.

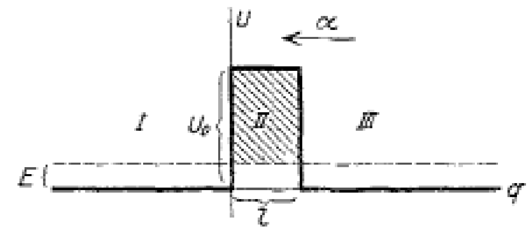


Fig. 2.

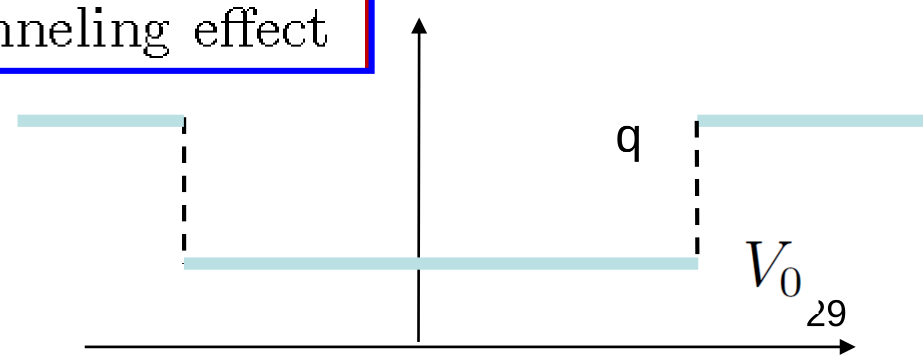
$$\psi(x,t) = e^{-\mathbf{i}Et}u(x)$$

$$\mathrm{i}\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi, \text{ in } (0,\infty)\times\mathbb{R}^N,$$

For simplicity $m = 1$, $\hbar = 1$ and $E = \lambda$

$$-\Delta u + V(x)u = \lambda u \quad \text{in } \mathbb{R}^N,$$

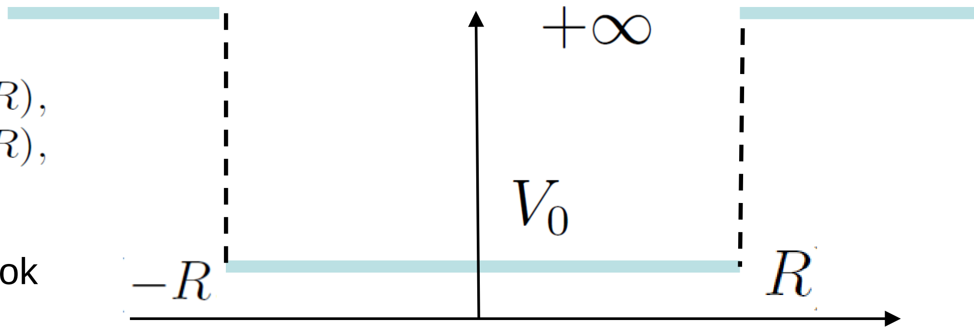
"the square well potential": tunneling effect



$$V_q(x:R,V_0) = \begin{cases} V_0 & \text{if } x\in(-R,R), \\ q & \text{if } x\notin(-R,R). \end{cases}$$

The infinite well potential: ambiguity in the standard presentation.

$$V_\infty(x:R,V_0)=\begin{cases} V_0 & \text{if } x\in(-R,R),\\ +\infty & \text{if } x\notin(-R,R),\end{cases}$$



Mandatory as one of the first examples in any book of Quantuum Mechanics.

Survey:
Belloni-Robinett, The infinite well and Dirac delta function potentials as pedagogical, mathematical and physical models in quantum mechanics, **Physics Reports** (2014)

Curiously, in this important survey the first work dealing with the infinite square well is not attributed to Gamow but **to N.F. Mott**
[book of **1930**] [1977 Nobel Prize].

Teaching Mechanics, on 2012 (and papers Bégout-Díaz on the Nonlinear Schrödinger Eq.), **I** realized some ambiguities which were the starting point of an important part of my research in the last 6 years.

In many textbooks this case is presented as a limit case of the associate *finite well potential*. In fact, there is an abuse of the notation in the above terminology.

We can introduce as definition of solution u of the *infinite well potential problem* to any function $u = \lim_{q\to\infty} u_q$ with u_q solution associated to the potential $V_q(x : R, V_0)$.

It is usually claimed that $u = \lim_{q\to\infty} u_q$ satisfies (at least in a weak sense) equation for the *infinite well potential* but, as we shall explain this is not correct since some other terms appear in the limit equation (which, in fact must be understood in distributional sense).

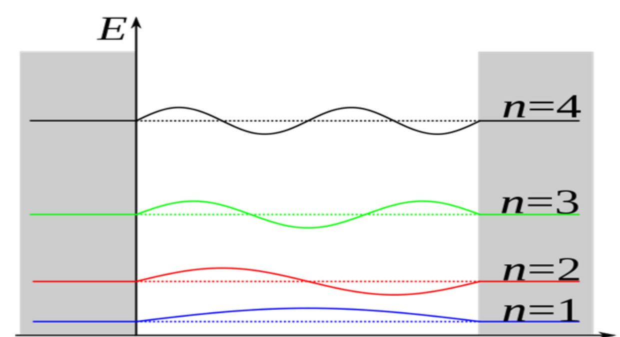
LEMMA 2.1 Given $q > 0$ and $V_q(x : R, V_0)$ defined by (1.2) problem (1.3), with $N = 1$, has a numerable sequence of eigenvalues $\lambda_n(q)$ and eigenfunctions $u_{q,n}(x)$ (renormalized such that $\|u_{q,n}\|_{L^2(\mathbb{R})} = 1$). Moreover, as $q \to +\infty$,

$$\lambda_n(q) \to \left(\frac{\pi}{2R}\right)^2 n^2, \quad \text{with } n \in \mathbb{N},$$

and $u_{q,n} \to u_n$ weakly in $H^1(\mathbb{R})$, with u_n given by (1.5) and u_n extended by zero on $\mathbb{R} - (-R, R)$. Finally, $(u_n)_{xx}$ generate two family of Dirac deltas (depending on $n \in \mathbb{N}$): one at $x = R$ and the other at $x = -R$.

$$E_n := \frac{\hbar^2}{2m}\lambda_n$$

$$\begin{cases} u_n(x) = C \sin \dfrac{n\pi}{2R}(x+R), \\ \lambda_n - V_0 = \left(\dfrac{\pi}{2R}\right)^2 n^2, \quad n = 1, 2, \ldots \end{cases}$$



The ambiguity in this mathematical treatment arises because the derivatives of such u_n are discontinuous at the points $x = \pm R$, and thus such u_n are not solutions of the equation in the whole domain $\mathbb{R}$ in the sense of distributions

$$-\frac{\hbar^2}{2m}\frac{d^2 u_n}{dx^2} + V(x)u_n = E_n u_n, \quad \text{in } \mathbb{R},$$

but they satisfy a different equation

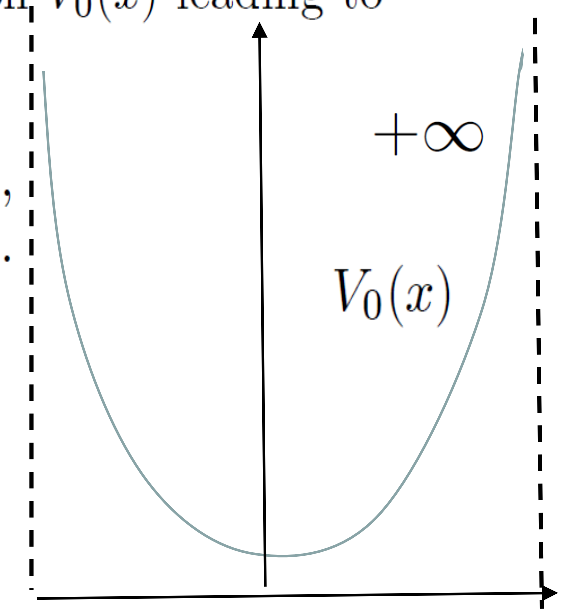
$$-\frac{\hbar^2}{2m}\frac{d^2 u_n}{dx^2} + V(x)u_n = E_n u_n + k_n(R)\delta_{\{R\}} - k_n(-R)\delta_{\{-R\}}, \quad \text{in } \mathbb{R}, \tag{1}$$

since the second derivative develops two Dirac deltas (see Lemma above). Here

$$k_n(-R) = \frac{\hbar^2}{2m}\frac{\sqrt{2}}{R^{3/2}} n\pi \quad \text{and} \quad k_n(R) = \frac{\hbar^2}{2m}\frac{\sqrt{2}}{R^{3/2}} n\pi(-1)^n.$$

As a matter of fact, after the work by Gamow, several authors considered many generalizations of the *infinite well potential* corresponding to the case in which the constant value V_0 is replaced by a general function $V_0(x)$ leading to the potential

$$V_\infty(x : R, V_0(\cdot)) = \begin{cases} V_0(x) & \text{if } x \in (-R, R), \\ +\infty & \text{if } x \notin (-R, R). \end{cases}$$



Some singular concrete potentials in the Quantum Mechanics literature:

* **Pösch-Teller potential**

$$V(x) = \frac{1}{2}V_0\{\frac{k(k-1)}{\sin^2 \alpha x} + \frac{\mu(\mu-1)}{\cos^2 \alpha x}\}, \ x \in [0, \frac{\pi}{2\alpha}]$$

Pöschl, G.; Teller, E. (1933), Zeitschrift für Physik, 83, 143–151.

***Supersymmetric potentials (SUSY)**

$$V(x) = \frac{k(k-1)}{\sin^2 x}, \ x \in [0, \pi]$$

(F. Cooper *et al.* (1995) Phys. Rep. 251 , 267-385)

Main program (started on 2013): consider a class of nonnegative potentials $V(x)$, being suitably singular on $\partial\Omega$, prove the existence of flat solutions $u(x)$ for some *energy* values λ, and then, extend them by zero on the rest of $\mathbb{R}^N$.

The key assumption

$$\frac{\underline{C}}{\delta(x)^\alpha} \leq V(x) \quad \left[\leq \frac{\overline{C}}{\delta(x)^\alpha} \text{ for simplicity!!}\right] \text{ a.e. } x \in \Omega,$$

for some $\overline{C} > \underline{C} \geq 0$ and $\alpha > 0$. Here, $\delta(x) = d(x, \partial\Omega)$.

Contrast with the case of negative singular potentials

Roughly speaking: **Theorem.** *The solutions are flat if and only if* $\alpha \geq 2$.

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via flat solutions: the one-dimensional case. **Interfaces and Free Boundaries**, **17** (2015).

J. I. Díaz, On the ambiguous treatment of the Schrödinger equation for the infinite potential well and an alternative via singular potentials: the multi-dimensional case, **SeMA-Journal** (2017).

J.I. Díaz, Correction **SeMA-Journal** (2018).

J. I. Díaz, D. Gómez – Castro, J.M. Rakotoson and R. Temam,**Discrete and Continuous Dynamical Systems**(2018).

J. I. Díaz, D. Gómez-Castro, and J.-M. Rakotoson. **Differential Equations and Applications**, (2018), .

J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez (fractional Schrödinger eq.) **Nonlinear Analysis**, (2018).

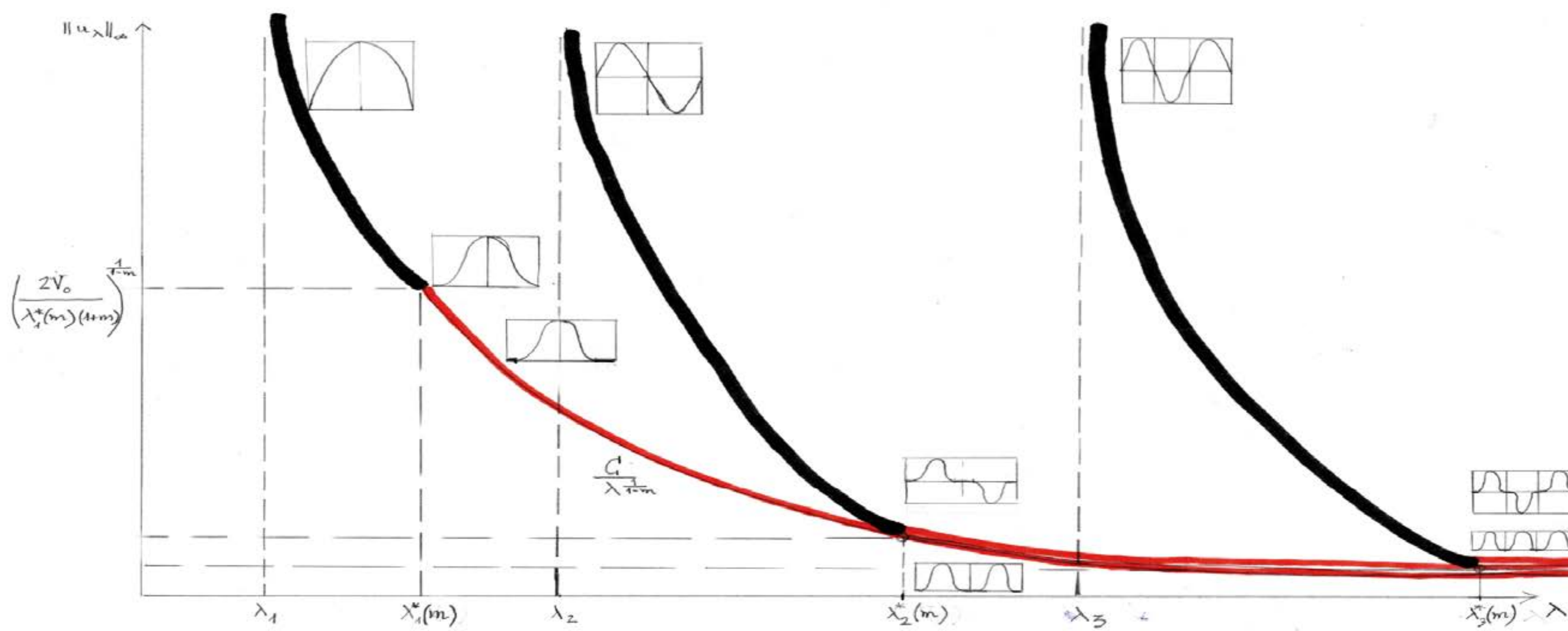
J. I. Díaz, D. Gómez-Castro, and J. L. Vázquez (Infinite order on the boundary)**. In preparation** 2020.

Case N=1 and Hardy potentials (first method of proof)

A **nonlinear** problem as a *basic tool* to build super and subsolutions !!

$$P(R,m,V_0,\lambda) \equiv \begin{cases} -\dfrac{d^2v}{dx^2} + V_0 v^m = \lambda v, \quad v \geq 0 \text{ in } (-R,R), \\ v(\pm R) = 0, \end{cases}$$

for a given $V_0 > 0$ and $m \in (0,1)$ J. I. D. and Hernández, **Portugaliae Math.**(2015)



Proposition 3 *For any* $\lambda \geq \left(\frac{\pi}{2R}\right)^2$ *there exists a unique nonnegative solution* v_m *of* $P(R, m, V_0, \lambda)$. *Moreover, there exists a* $\lambda^*(m) > \left(\frac{\pi}{2R}\right)^2$ *such that: a) If* $\lambda \geq \lambda^*(m)$ *then*

$$v_m(x) \leq \overline{K} d(x, \partial\Omega)^{2/(1-m)} \quad \text{for any } x \in \overline{\Omega} = [-R, R], \tag{5}$$

for some constant $\overline{K}$. *In particular* $\frac{dv_m}{dx}(\pm R) = 0$. *b) If* $\lambda \leq \lambda^*(m)$ *then*

$$\underline{K} d(x, \partial\Omega)^{2/(1-m)} \leq v_m(x) \quad \text{for any } x \in \overline{\Omega} = [-R, R], \tag{6}$$

for some constant $\underline{K}$. *In particular* $v_m > 0$ *in* Ω. *c) If* $\lambda = \lambda^*(m)$ *inequalities (5) and (6) hold for some* $\overline{K} > \underline{K} > 0$.

Main idea of the application to the **linear** Schrödinger equation:

$$\frac{\overline{V}_0}{\left|v_{\lambda^*(m^\#)}(x)\right|^{1-m^\#}} \leqslant \frac{\overline{V}_0}{(\underline{K}^\#)^{1-m^\#}} \frac{1}{d(x, \partial\Omega)^2} \leqslant V(x),$$

then

$$\lambda^*(m^\#) v_{\lambda^*(m^\#)} = -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \overline{V}_0 v_{\lambda^*(m^\#)}^{m^\#} = -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \frac{\overline{V}_0}{\left|v_{\lambda^*(m^\#)}(x)\right|^{1-m^\#}} v_{\lambda^*(m^\#)}$$

$$\leqslant -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + \frac{\overline{V}_0}{(\underline{K}^\#)^{1-m^\#}} \frac{v_{\lambda^*(m^\#)}}{d(x, \partial\Omega)^2} \leqslant -\frac{d^2 v_{\lambda^*(m^\#)}}{dx^2} + V(x) v_{\lambda^*(m^\#)},$$

which proves that $v_{\lambda^*(m^\#)}(x : \overline{V}_0)$ is a supersolution (notice that for the moment $\overline{V}_0$ is arbitrary).

Corollary 4 *Estimates (5) and (6) also apply to the nodal solutions of the semilinear problem* $P(R, m, V_0, \lambda)$ *corresponding to suitable values* $\lambda_n^*(m)$ *of the parameter* λ *in branches bifurcating at the infinity from the simple eigenvalues* λ_n *for any* $n \in \mathbb{N}$.

As a particular consequence of the above Corollaries and it is possible to offer a correct alternative to the "localizing" process suggested by Gamow in his 1928 paper.

Corollary 5 *For any* $R > 0$, $n \in \mathbb{N}$ *and* $m \in (0, 1)$ *there exists a countable set of values of the parameter* $\lambda = \lambda_n^*(m)$ *(in branches bifurcating at the infinity from the simple eigenvalues* λ_n*, of the linear Schroedinger equation, and there exists a countable set of infinite well type potentials* $V_{n,m}(x) = V_\infty(x : R, V_{0,n,m}(.))$ *such that the associated Schrödinger equation*

$$-\Delta u + V_{n,m}(x)u = \lambda_n^*(m)u \quad in\ \mathbb{R},$$

admits a solution $u_{n,m} \in C^{\frac{2}{1-m}}(\mathbb{R})$*, changing sign* n*-times, such that* $u_{n,m}(x) = 0$ *for any* $x \notin (-R, R)$ *(and in particular* $u'_{n,m}(\pm R) = 0$*). Moreover* $V_\infty\big(x : R, V_{0,n,m}(x)\big)u_{n,m}(x) = 0$ *for any* $x \notin (-R, R)$ *(i.e. no Dirac delta is generated on the boundaries* $x = \pm R$*).*

Remark. Different view point (semilinear problem with a singular elliptic operator)
C. Bandle and M. A. Pozio, Sublinear elliptic problems with a Hardy potential. *Nonlinear Anal.* 119 (2015), 149–166.

Case N>1 and Hardy potentials (second method of proof)

Hardy inequality + compactness $H_0^1(\Omega) \subset L^2(\Omega)$

Proposition 2.1 *Assume (10), then there exists a sequence of eigenvalues $\lambda_n \to +\infty$, $\lambda_1 > \lambda_{1,\Omega}$ (the first eigenvalue for the Dirichlet problem for the $-\Delta$ operator on Ω), λ_1 is isolated and $u_1 > 0$ on Ω.*

Flat solutions through the local comparison with a *direct local barrier* function

$$U(x) = C_U\,|x - x_0|^{\beta} \qquad -\Delta U + VU = (-\beta(\beta + N - 2) + \underline{C})C_U|x - x_0|^{\beta-2}$$

Theorem 2.1* *Assume (2) and let u_n be an eigenfunction of $DP(V, \lambda_n, \Omega)$ associated to the eigenvalue λ_n.*

(a) *There exists $\varepsilon \in [0, 2)$, $\varepsilon = \varepsilon(\underline{C}, N, n)$ and $\overline{K}_n = \overline{K}_n(\underline{C}, N, n, \Omega) > 0$ such that*

$$|u_n(x)| \leq \overline{K}_n d(x, \partial\Omega)^{2-\epsilon} \quad \text{a.e. } x \in \Omega. \tag{3}$$

(b) *If*

$$\underline{C} > N - 1, \tag{4}$$

then (3) holds for some $\varepsilon \in [0, 1)$. In particular, u_n is a flat solution.

(c) *If*

$$\underline{C} > 2N, \tag{5}$$

then (3) holds for $\varepsilon = 0$.

The flatness exponent grows with the constant $\underline{C}$!!

J. I. Díaz, SeMA (2017) 74:255–278, (2018) 75:563–568.

Easely adaptable (after a redefinition of the notion of very weak solution) to the case $\alpha > 2$:

J. I. Díaz, D. Gómez – Castro, J.M. Rakotoson (2017).

As a particular consequence of Theorem 2.1 it is possible to offer a correct alternative to the "localizing" process suggested by Gamow in his paper [40].

Corollary 2.1 *Let Ω be an open regular bounded set of $\mathbb{R}^N$, $N \geq 1$. For any $q \in [0, +\infty)$ consider the potential*

$$V_{q,\Omega}(x) = \begin{cases} V(x) & \text{if } x \in \Omega, \\ q & \text{if } x \in \mathbb{R}^N - \Omega. \end{cases}$$

Assume (10). *Then there exists a countable set of eigenvalues λ_n and eigenfunctions $\widetilde{u}_{n,q}$ of the Schrö dinger equation*

$$-\Delta u + V_{q,\Omega}(x)u = \lambda_n u \ \ \text{in } \mathbb{R}^N, \tag{18}$$

such that

$$\widetilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^N - \Omega, \end{cases}$$

where λ_n and $u_n(x)$ are the eigenvalues and eigenfunctions of the Dirichlet problem $DP(V, \lambda, \Omega)$. Moreover the same conclusion holds for $q = +\infty$ if we define the corresponding solution as $\widetilde{u}_{n,\infty}(x) = \lim_{q \nearrow +\infty} \widetilde{u}_{n,q}(x)$.

When V is super-singular, $V(x) \geq c_V d(x, \partial\Omega)^{-(2+\varepsilon)}$ then u is flat near $\partial\Omega$ to the *infinite order*, i.e.

$$|u(x)| \leq C d(x, \partial\Omega)^{\beta}, \qquad \forall \beta > 0.$$

In fact, we can prove an exponential decay:

Theorem (D-Gómez-Vázquez (2020)). *Assume* $V(x) \geq c_V d(x,\partial\Omega)^{-(2+\varepsilon)}$, *then*

$$|u(x)| \leq C \exp\left(-d(x,\partial\Omega)^{-\varepsilon/4}\right)$$

in a neighbourhood of $\partial\Omega$.

Construction of a barrier function. $U(x) = \exp(g(x))$ where $g(x) = -\delta(x)^{-\varepsilon/4}$

$\Delta U + VU = \exp(-g(x))(|\nabla g|^2 + \Delta g(x) + V)$ Delicate (and technical) estimates...

The above results on flat solutions prove a lack of the weak and strong versions of the UCP for singular Schrödinger equations.

Corollaries for the associated **complex evolution problem** (for initial data *with compact support*) in the mentioned papers

$$\begin{cases} \mathbf{i}\frac{\partial\psi}{\partial t} = -\Delta\psi + V(x)\psi & \text{in } (0,\infty)\times\mathbb{R}^N, \\ \psi(0,x) = \psi_0(x) & \text{on } \mathbb{R}^N. \end{cases}$$

Partial confinement (support$\psi(t,.) \subset \overline{\Omega}$ for any $t > 0$) for (nonnegative) potentials $V(x)$, being sufficiently singular near $\partial\Omega$, for initial wave functions $\psi_0 \in H^1(\mathbb{R} : \mathbb{C})$ with support $\psi_0 \subset \overline{\Omega}$.

No ambiguity, in contrast to the (Gamow-Mott) infinite well potential case !!

Thanks for your attention