

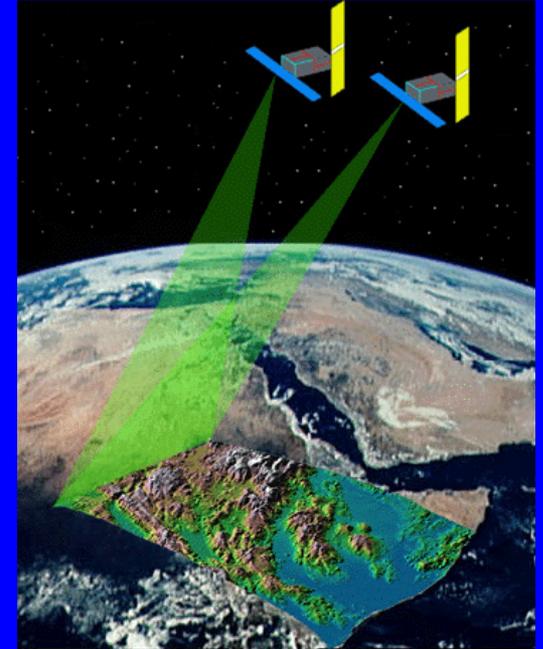
Un viaje por los sistemas dinámicos del planeta Tierra

J.I. Díaz

Valencia, 26 de abril de 2005



1. Introducción. El Planeta Tierra



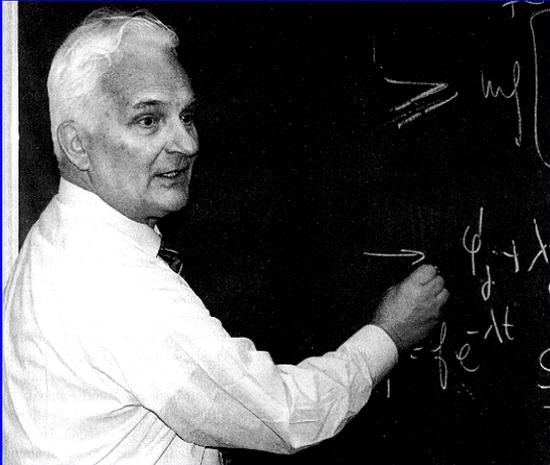
Siglo XXI

4.500 millones de años

Viaje de la mano de un matemático

- **Guía poco frecuente**
- **Geólogo o Físico con otra visión**
- **Bagaje del matemático**
- **Kepler, Descartes, Newton, Fourier, Gaus, Lord Kelvin, ...**
- **Conferencia de divulgación (Programa RAC/Ministerio de Educación)**
- **Distintos niveles de la audiencia (mínimo/máximo)**

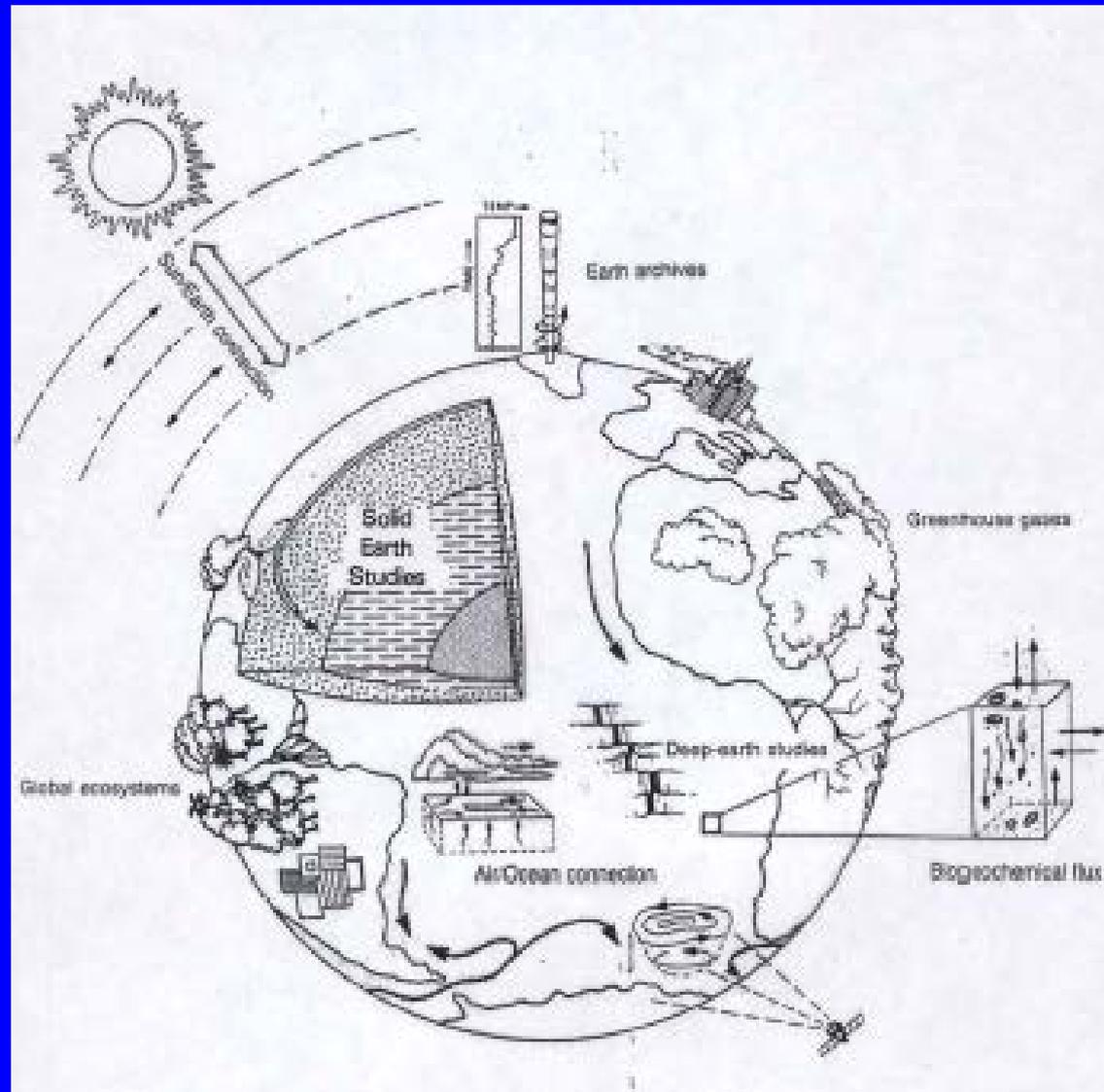
2. Guía para el viaje: La “Trilogía Universal” de la Matemática Aplicada



Sistema real

J.L. Lions
(1928-2001)

Planeta Tierra: sistema complejo



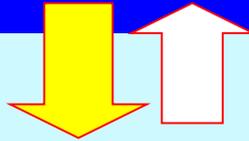
Natural disaster 1517-18, Black chalk, pen and ink on paper, 162 x 203 mm
Royal Library, Windsor



La “Trilogía Universal” de la Matemática Aplicada

Sistema real

Predicción



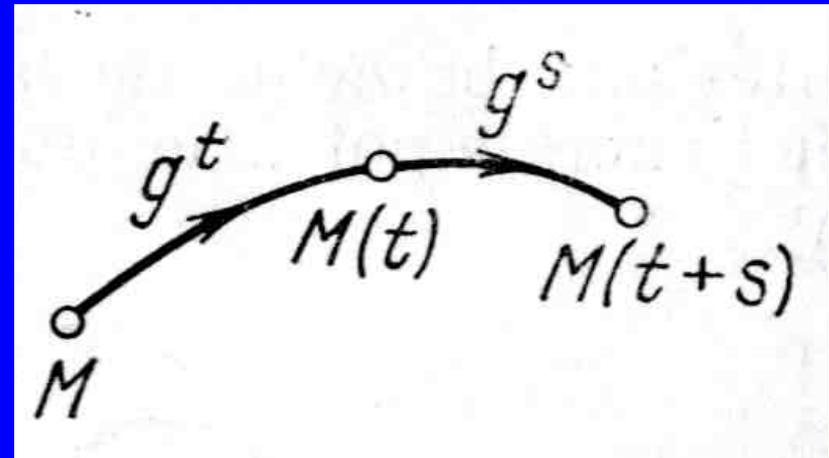
Modelización
matemática

Modelos matemáticos

- Deterministas
- Estocásticos
 - Mixtos
- Régimen transitorio (sistemas dinámicos)
- Régimen estacionario (estados de equilibrio)

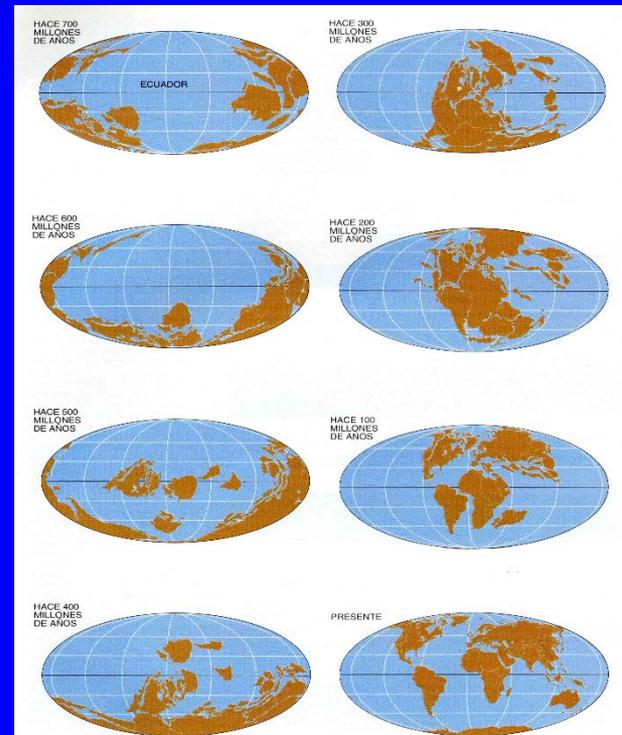
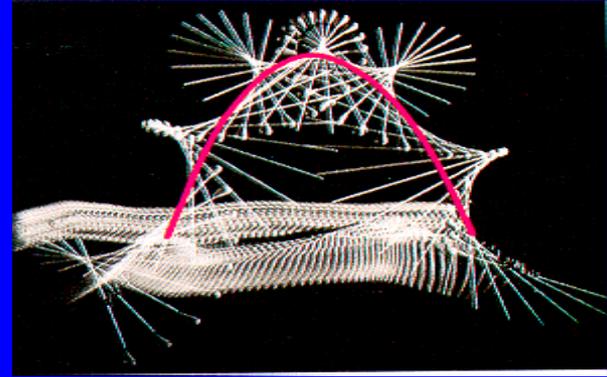
sistemas dinámicos

- Sistema
- Dinámica



- Espacio de los estados

Contínuos/Discretos

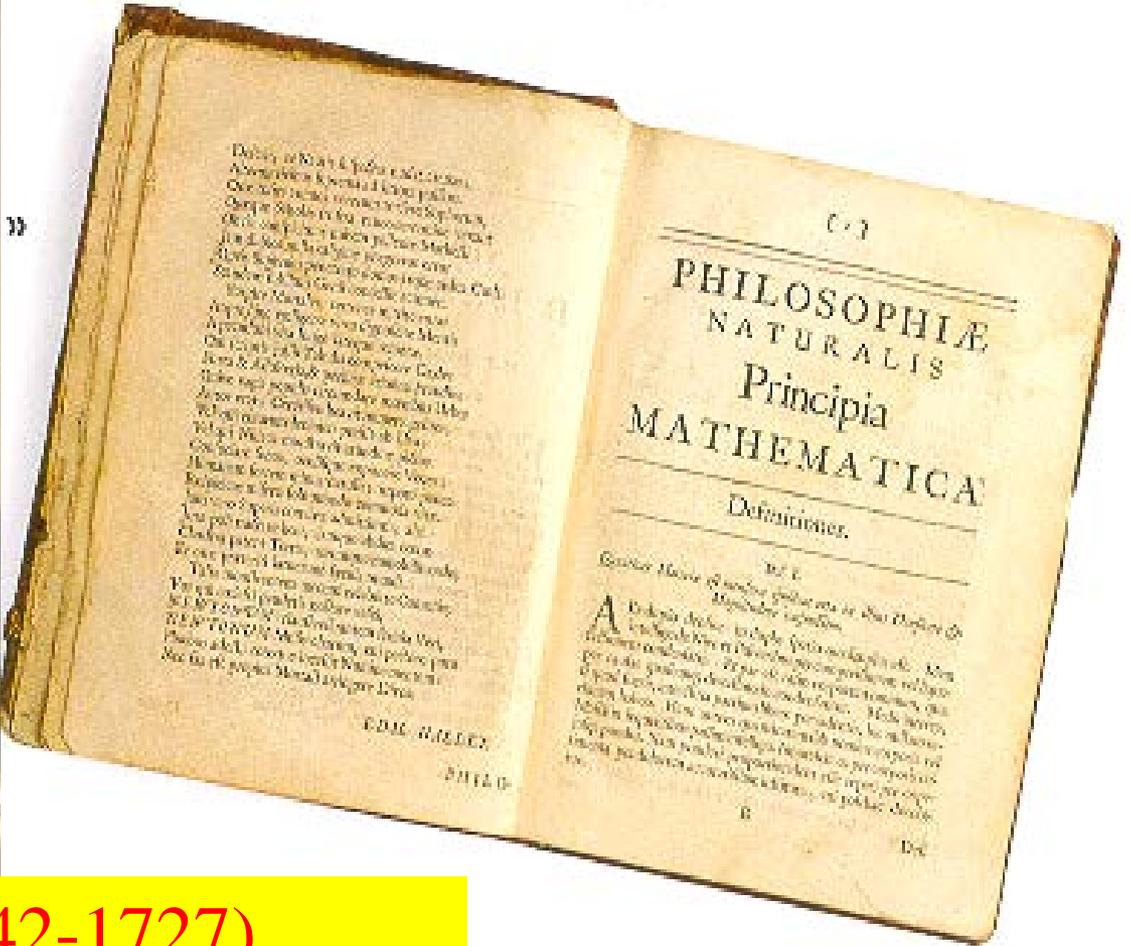
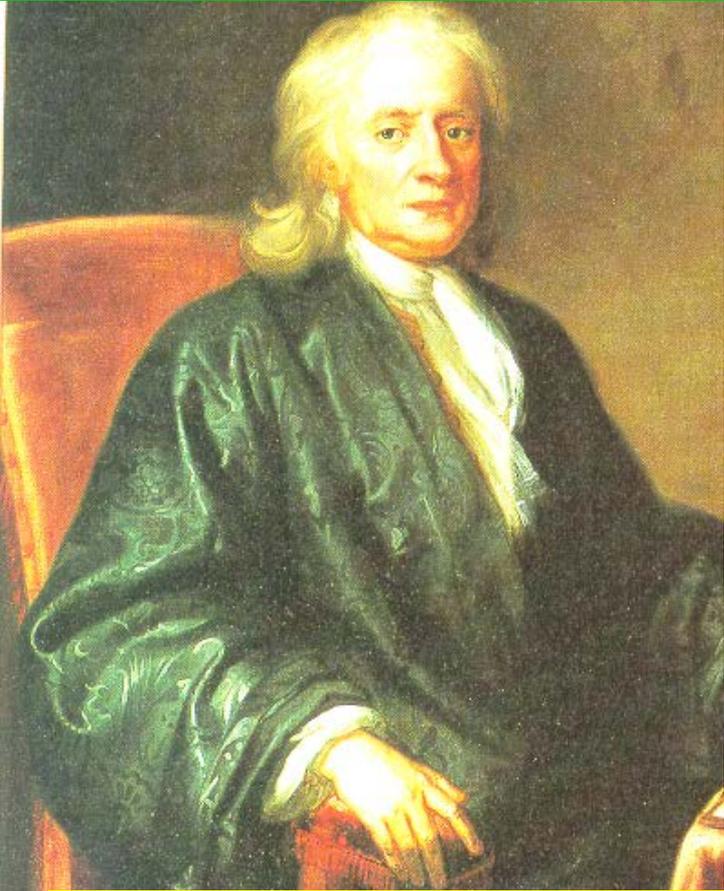


Takoma Narrows (Washington, 1940)

24/08/2023

J.I. Díaz

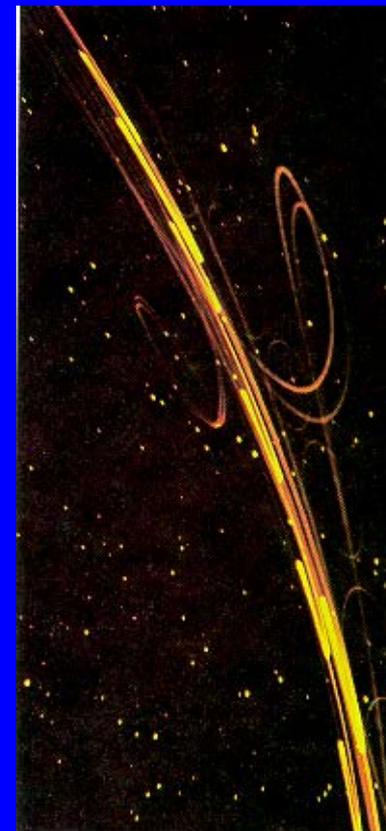
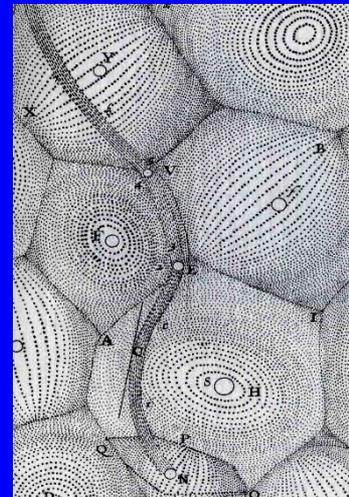
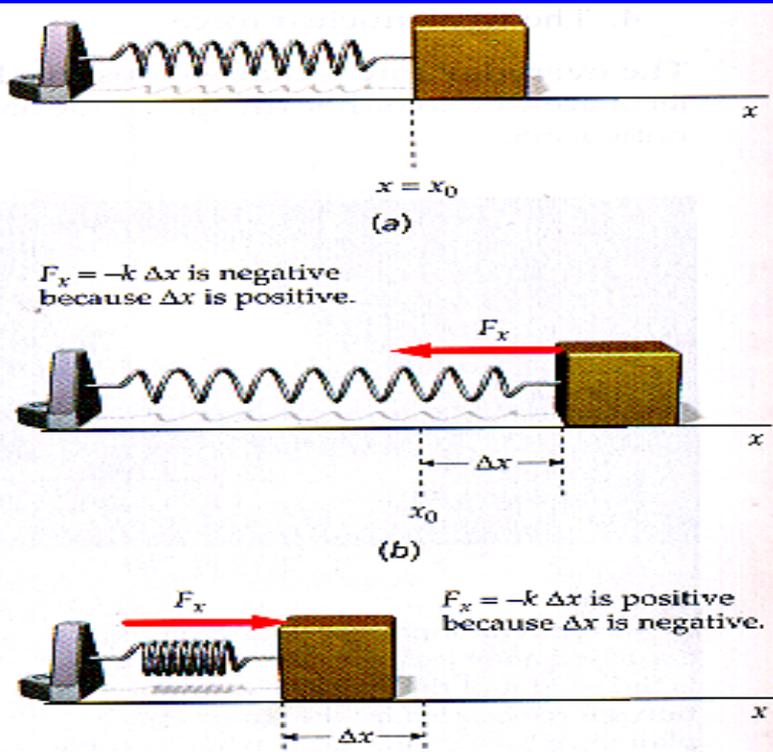
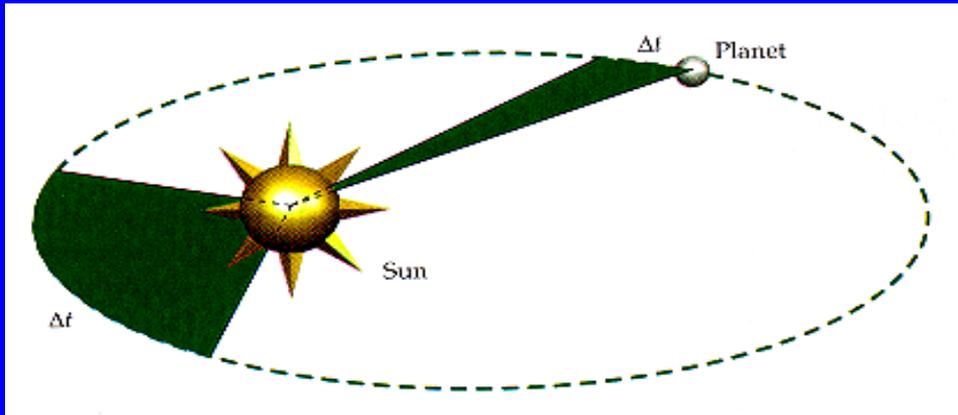




Sir Isaac Newton (1642-1727)

1687

Mecánica Clásica



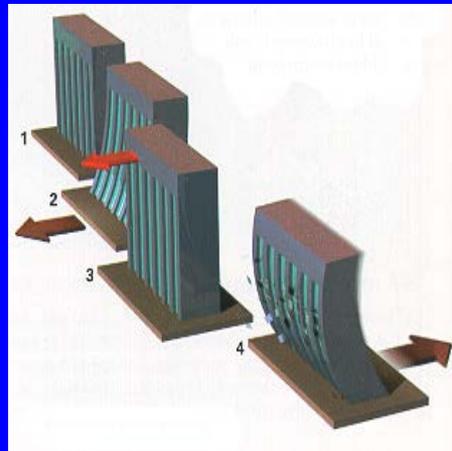
The apparent motion of the planets.

Mecánica de Medios Continuos

Fluidos



Elasticidad



Termodinámica

Ecuaciones: Sistemas dinámicos

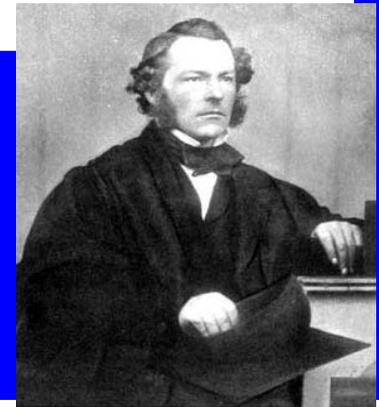
Modelos de Circulación General

Claude Navier (1785-1836), Sir George Gabriel Stokes (1819-1903)

$$\frac{\partial v_i}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} - g\delta_{i3} + f_{ri}, \quad i = 1, 2, 3,$$

$$\frac{\partial \rho}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial \rho}{\partial x_j} = -\rho \operatorname{div}(\vec{v}).$$

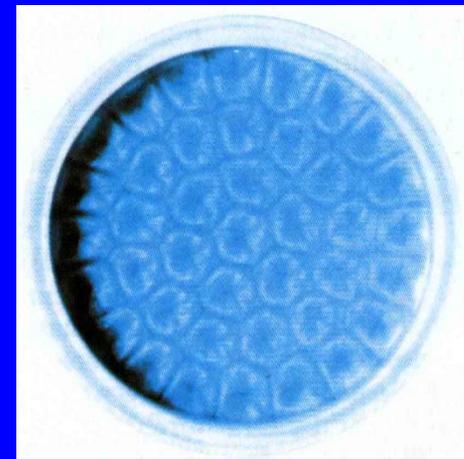
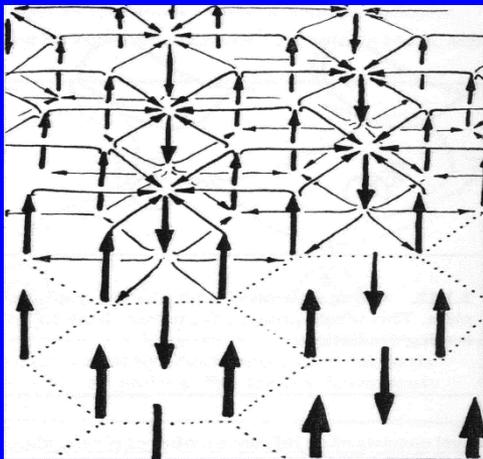
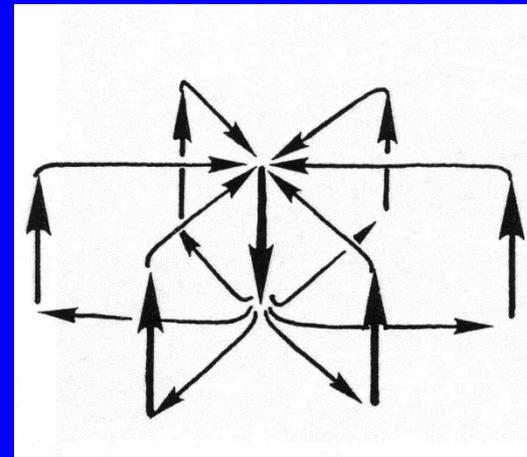
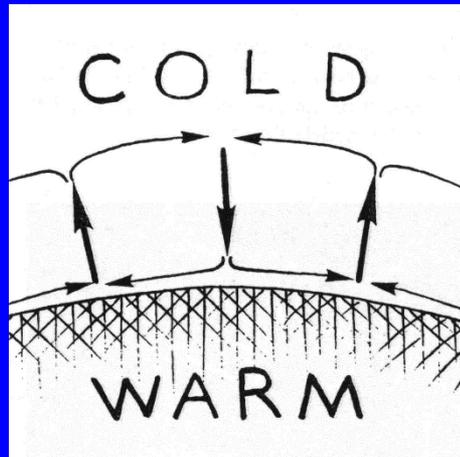
$$p = \rho R T.$$



$$C_p \left(\frac{\partial T}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial T}{\partial x_j} \right) = \frac{RT}{p} \left(\frac{\partial p}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial p}{\partial x_j} \right) + Q,$$

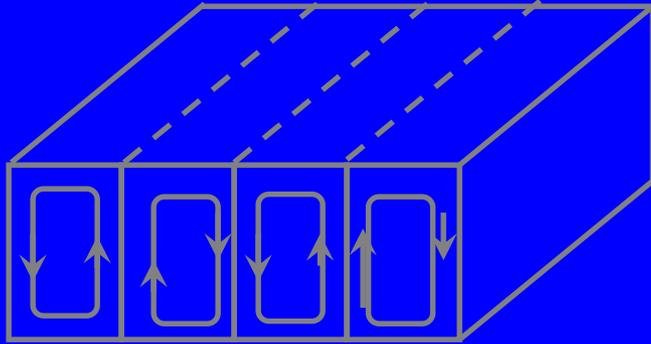
$$\frac{\partial q}{\partial t} + \sum_{j=1}^3 v_j \frac{\partial q}{\partial x_j} = Q',$$

Células de Rayleigh-Benard



[26]

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla p + \frac{1}{R_e} \Delta u + (T - T_0) g \quad \text{N-S}$$

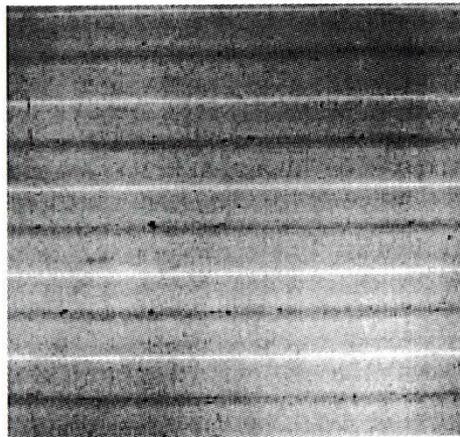


$$\frac{dy}{dt} = -xz + rx - y$$

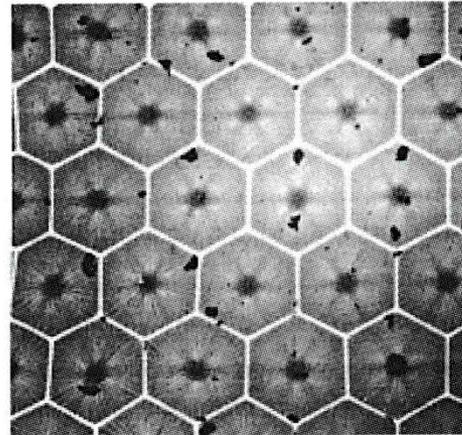
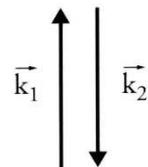
$$\frac{dx}{dt} = -\beta(x - y)$$

$$\frac{d^2 z}{dt} = xy - bz$$

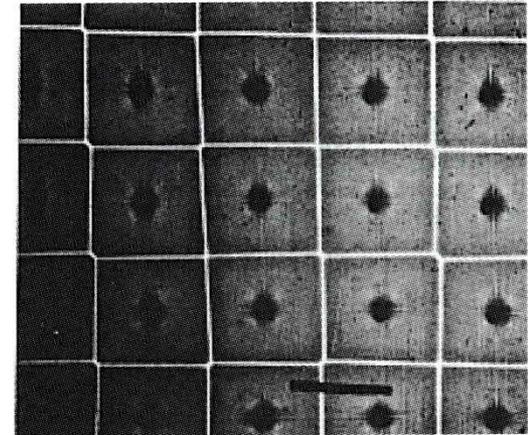
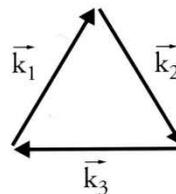
LORENTZ



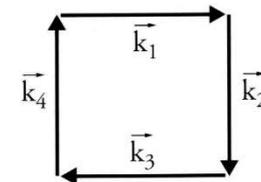
(a)



(b)

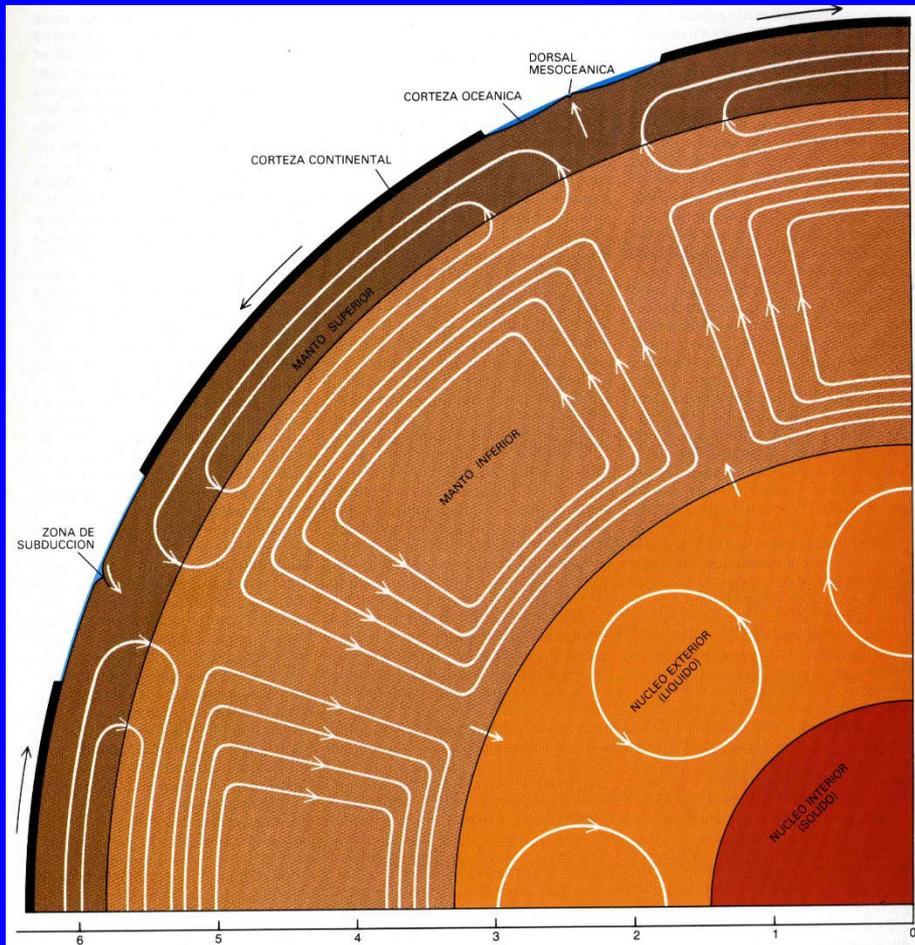


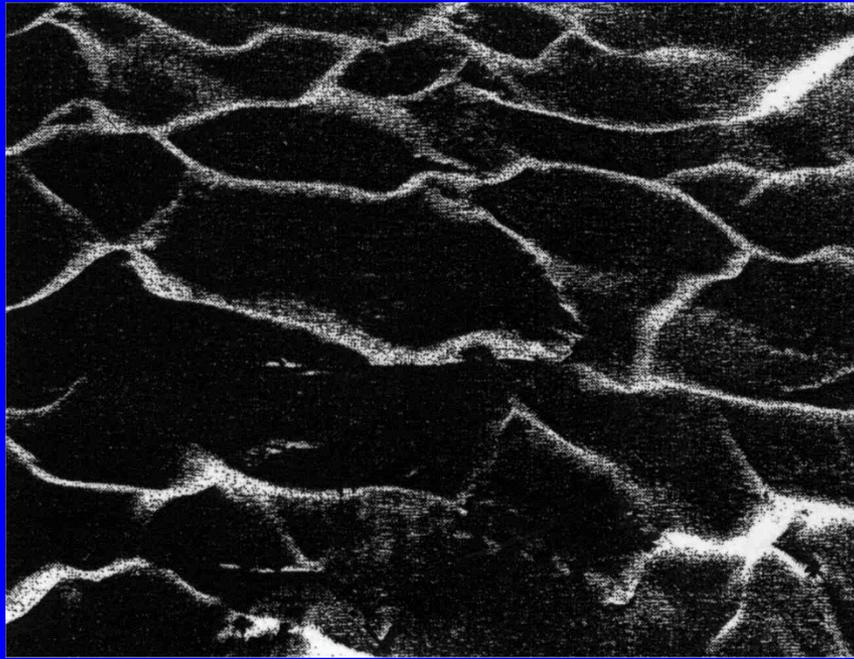
(c)



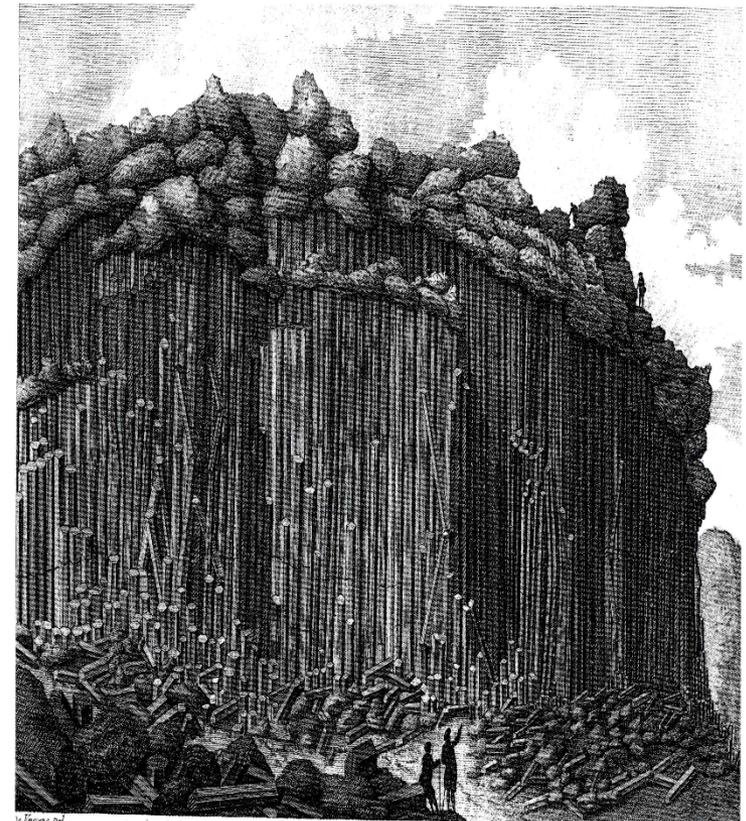
2 | Cellules convectives de Rayleigh-Bénard près du seuil d'instabilité dans un sirop dont la viscosité diminue fortement avec la température (visualisation dans le plan horizontal par ombroscopie). (a) Rouleaux obtenus lorsque la variation de viscosité est faible, de sorte que les deux parois jouent un rôle symétrique. La formation des rouleaux s'explique théoriquement par l'interaction de deux perturba-

tions de vecteur d'onde opposé $\vec{k}_1 + \vec{k}_2 = \vec{0}$. (b) Hexagones obtenus pour une viscosité variant fortement avec la température (dans un rapport 50), dont la formation s'explique par l'interaction de trois perturbations de vecteurs d'onde tels que $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 = \vec{0}$. (c) Carrés obtenus dans un cas intermédiaire, et expliqués par l'interaction de 4 vecteurs d'onde tels que $\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4 = \vec{0}$. D'après D. B. White, 1988.





Dune fields with a regular pattern, Lençóis Maranhenses National Park, on Brazil's north coast. Image courtesy of Earth Sciences and Image Analysis Laboratory, NASA Johnson Space Center, ISS007-E-15177. (See **Dune Formation; Geomorphology and tectonics; Pattern formation.**)



de Feytaud del.

PAVÉ DES GÉANS DE CHENAVARI.

de Feytaud del.

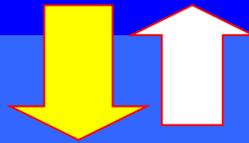
3 | Les cellules hexagonales des orgues de basalte ont de tout temps frappé l'imagination. Leur régularité est quelque peu idéalisée sur cette gravure tirée d'un ouvrage de Faujas de Saint-Fond (1778), un des premiers géologues ayant étudié les volcans d'Auvergne. Les orgues de basalte résultent des

craquelures formées (à l'état solide) par refroidissement d'une coulée de lave. Ces craquelures tendent à s'organiser en un réseau hexagonal, à la manière des cellules de convection. Collection P. Thomas.

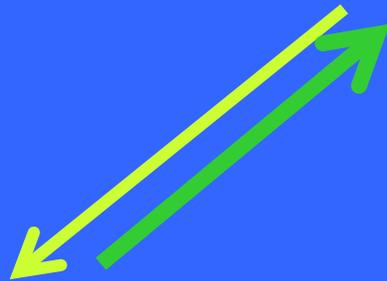
La “Trilogía Universal” de la Matemática Aplicada

Sistema real

Predicción



**Modelos
matemáticos**



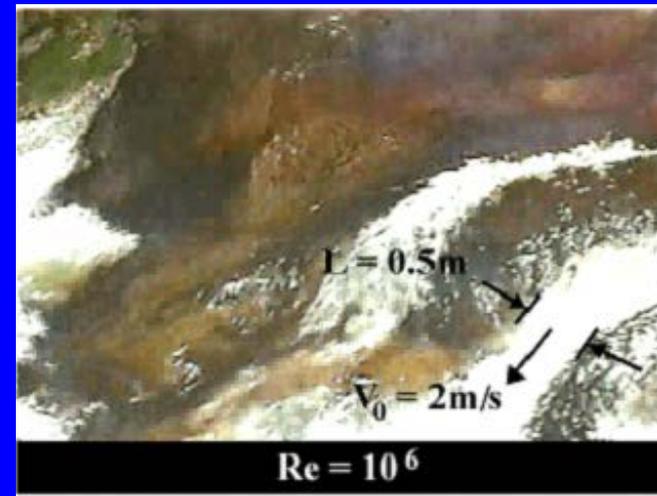
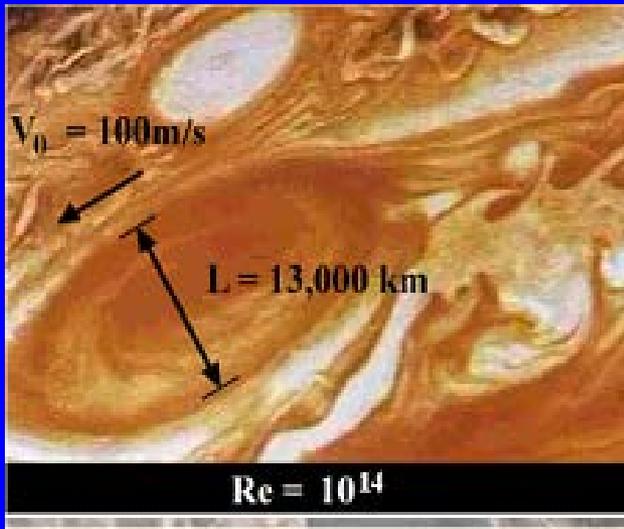
**Análisis
matemático**

Cuestión de escalas

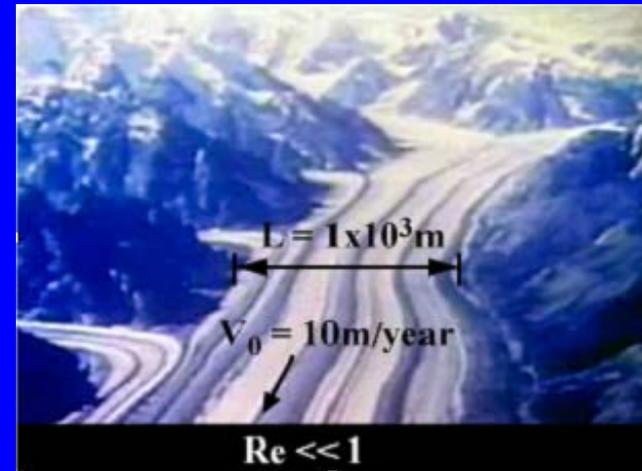
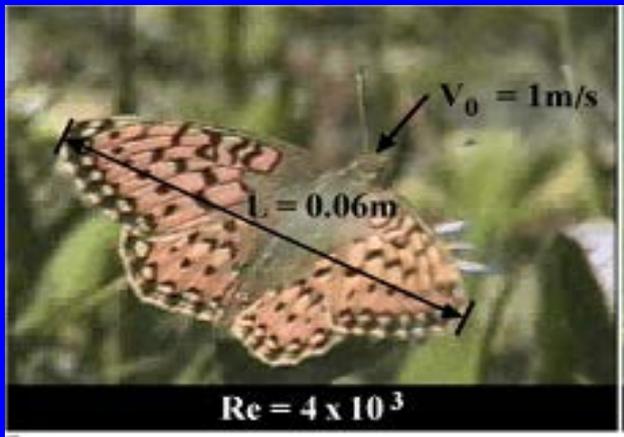
Richard Feymann:

Peso/tensión superficial

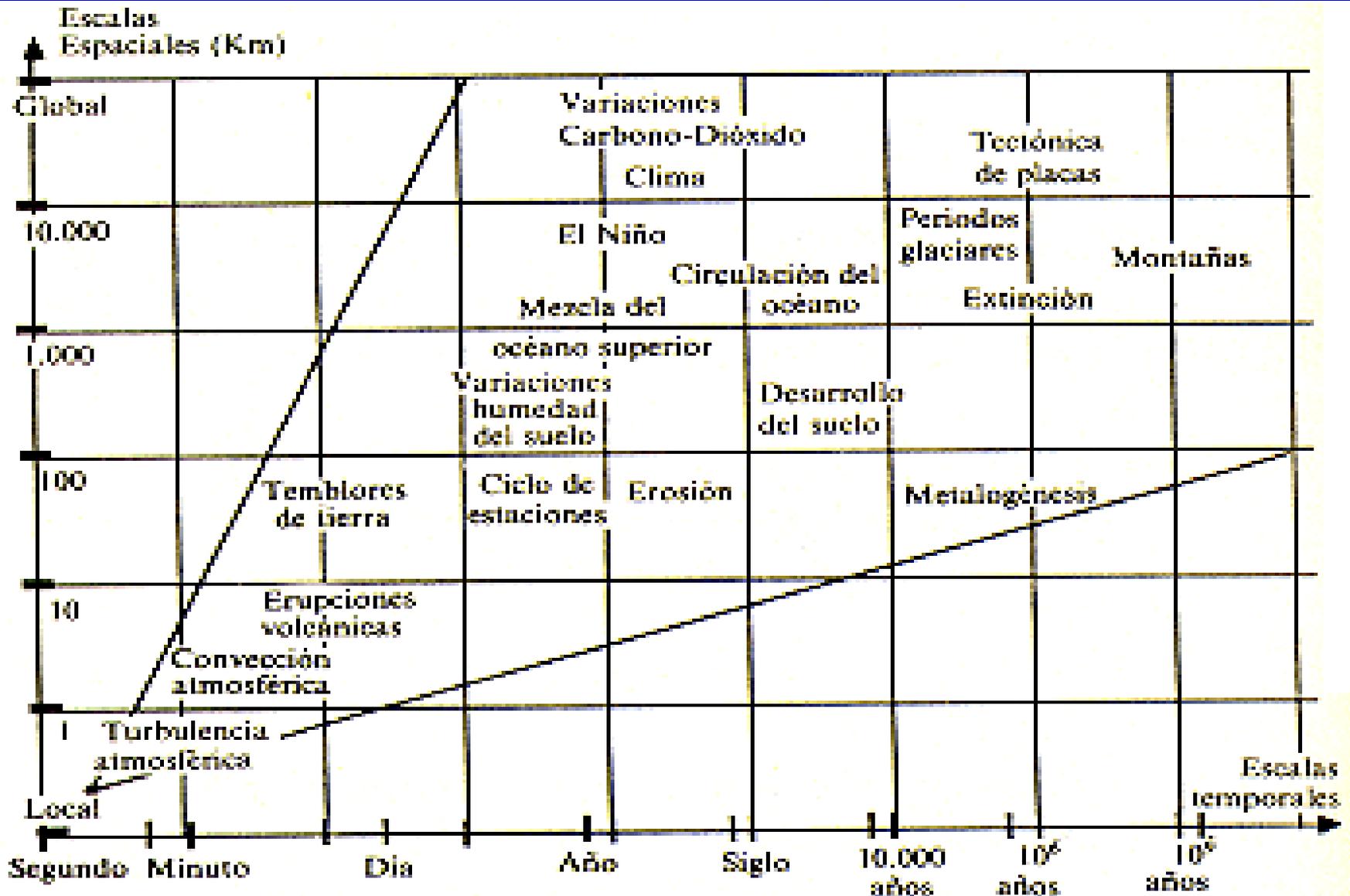




$$Re = \frac{VL}{\nu}$$



Escalas características



Análisis dimensional

Órdenes de magnitud de fluidos en Ciencias de la Tierra

Symbole	Nom	Unité	Atmosphère	Océan	Man-teau	Noyau
d	épaisseur de fluide	m	10^3 à 10^4	10^3	10^6	10^6
α	coefficient de dilatation	K^{-1}	10^{-5}	10^{-4}	10^{-5}	10^{-5}
κ	diffusivité thermique	$m^2 s^{-1}$	10^{-5}	10^{-7}	10^{-6}	10^{-5}
ν	viscosité cinématique	$m^2 s^{-1}$	10^{-5}	10^{-6}	10^{-17}	10^{-6}
ΔT_{pot}	différence de température potentielle	K	1	10	10^3	1
V	vitesse du fluide	$m s^{-1}$	10	10^{-2}	10^{-8}	10^{-4}
ρ	flux convectif	W/m^2	100	100	10^{-1}	10^{-2}

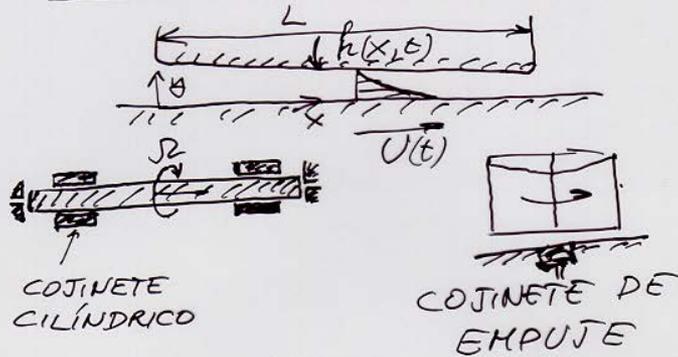
Números adimensionales y su magnitud

Symbole	Nom	Expres-sion	Atmos-phère	Océan	Man-teau	Noyau
Ra	Rayleigh	$\frac{\alpha g \Delta T_{pot} d^3}{\kappa \nu}$	$0-10^{20}$	$0-10^{20}$	10^6	0 à 10^{25}
Pr	Prandtl	ν/κ	1	10	10^{23}	10^{-1}
Re	Reynolds	Vd/ν	10^9 à 10^{10}	10^7	10^{-19}	10^8
E	Ekman	$\nu/\Omega d^2$	10^{-9} à 10^{-7}	10^{-8}	10^9	10^{-14}

El arte de la modelización

TEORIA DE LA LUBRICACIÓN FLUIDODINÁMICA (REYNOLDS)

LUBRICACIÓN HIDRODINÁMICA



ZAPATA BIDIMENSIONAL

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$U_c/L = V_c/h_0$

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} \right)$$

$\rho U_c t_0$ $\rho U_c^2/L$ $\rho x P/L$ $\mu U_c/h_0^2$ $\mu U_c/L^2$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial x^2} \right)$$

$\rho V_c t_0$ $\rho U_c V_c/L$ $\rho y P/h_0$ $\mu V_c/h_0^2$ $\mu V_c/L^2$

$$y=0: u=U(t), v=0$$

$$y=h(x,t): u=0, v=\frac{\partial h}{\partial t}$$

$U(t) \xrightarrow{U_c} : h(x,t) \xrightarrow{h_0, t_0}$

$$h_0/L \ll 1, h_0^2/\lambda t_0 \ll 1; h_0^2/\nu U_c \ll 1$$

HIPÓTESIS DE LA TEORÍA DE LA LUBRICACIÓN

$$h_0/L \ll 1 \quad (\text{CAPAS DELGADAS})$$

$$h_0^2/\lambda t_0 \ll 1 \quad (\text{ACELERACIÓN LOCAL DESPREC.})$$

$$\frac{U h_0}{\nu} \frac{h_0}{L} \ll 1 \quad (\text{ACELERACIÓN CONVECTIVA DESPRECIABLE})$$

ECUACIONES (DE STOKES) SIMPLIFICADAS

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}$$

$$0 = \frac{\partial p}{\partial y} \rightarrow p = p(x, t)?$$

$$y=0: u=U(t), v=0$$

$$y=h(x,t): u=0, v=\frac{\partial h}{\partial t}$$

$$u = \underbrace{U \left(1 - \frac{y}{h}\right)}_{\text{Couette}} + \underbrace{\frac{1}{2\mu} \frac{\partial p}{\partial x} y(y-h)}_{\text{Poiseuille}}$$

$$v = -\int_0^h \frac{\partial u}{\partial x} dy \rightarrow \frac{\partial h}{\partial t} = -\int_0^h \frac{\partial u}{\partial x} dy$$

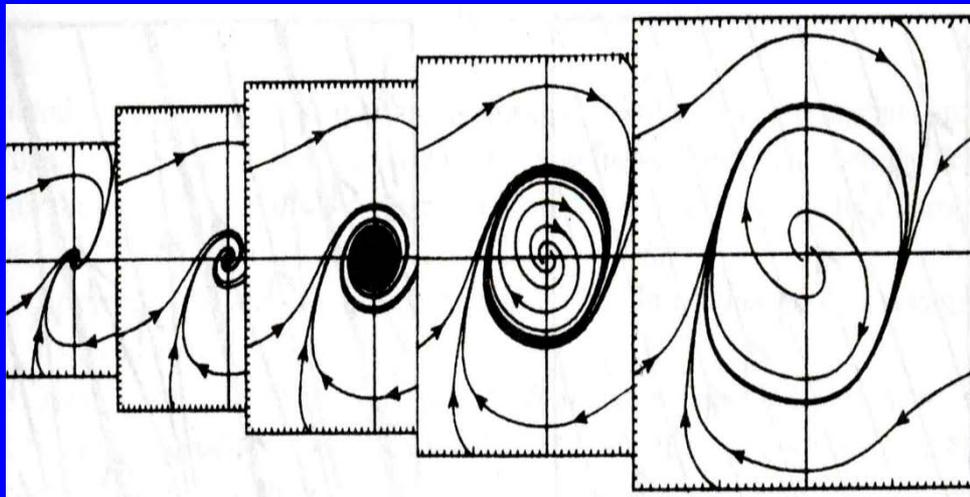
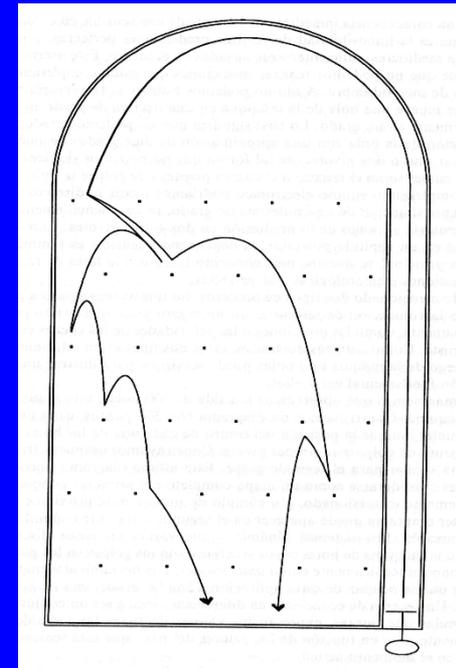
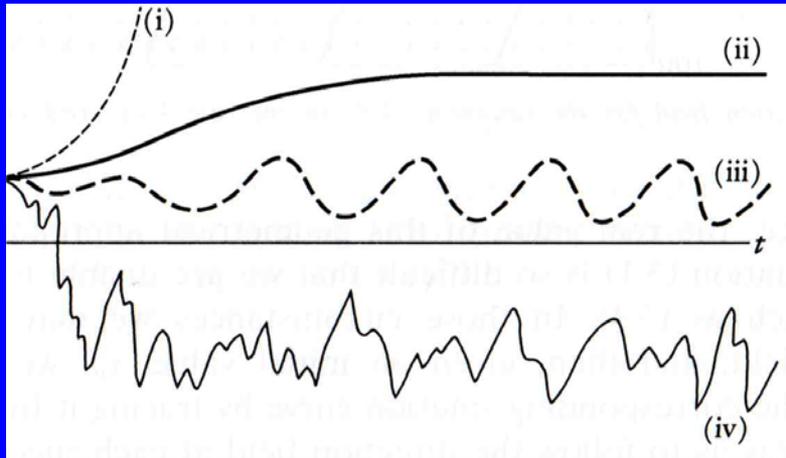
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \int_0^h u dy = 0 \quad q_x = \int_0^h u dy$$

$$q_x = \frac{U h}{2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial x}$$

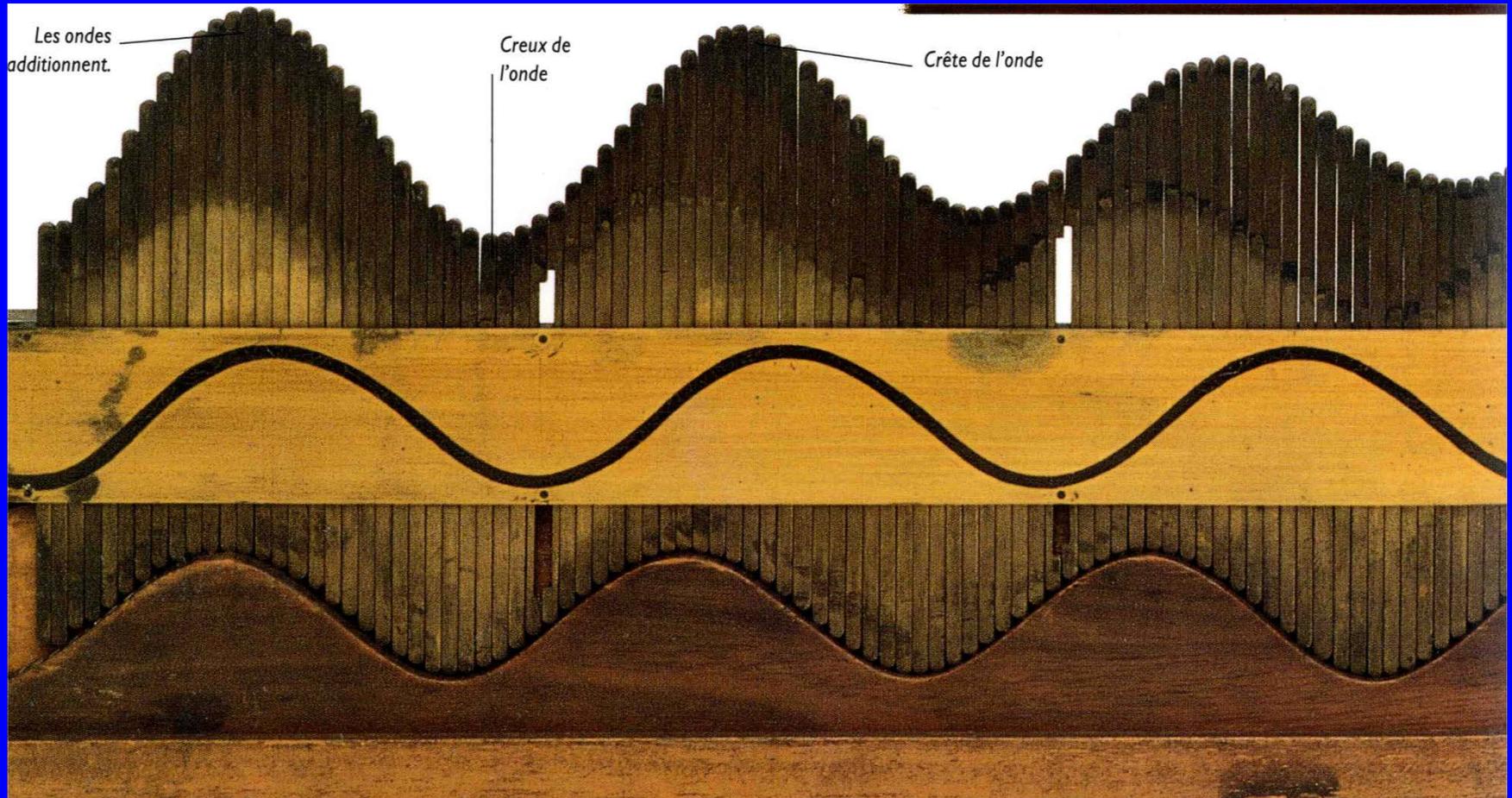
$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x} \left(\frac{U h}{2} - \frac{h^3}{12\mu} \frac{\partial p}{\partial x} \right) = 0$$

(ECUACIÓN DE REYNOLDS DE LA LUBRICACIÓN)

Análisis matemático

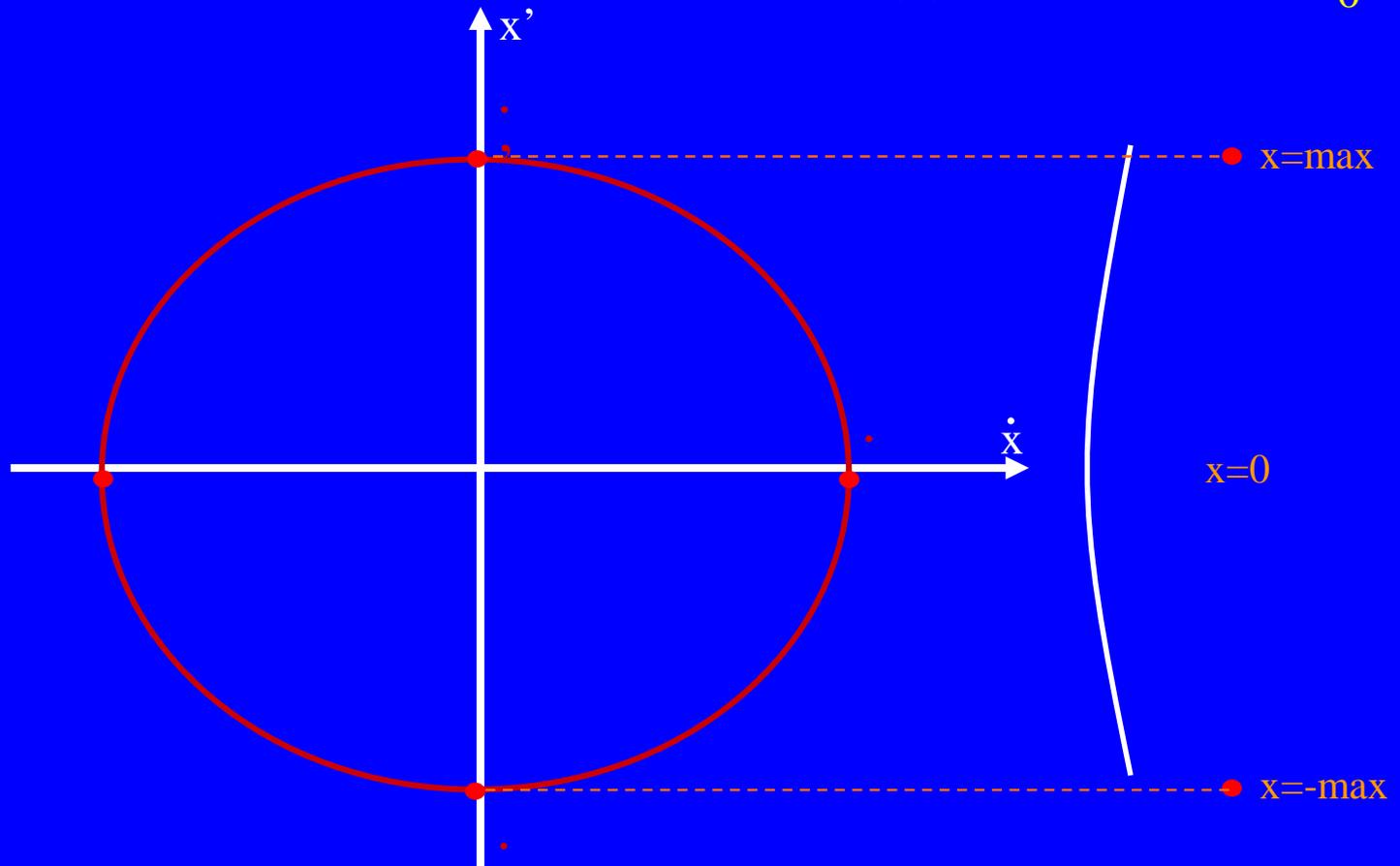


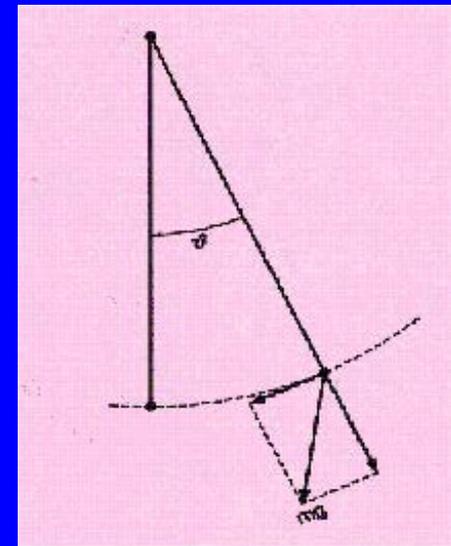
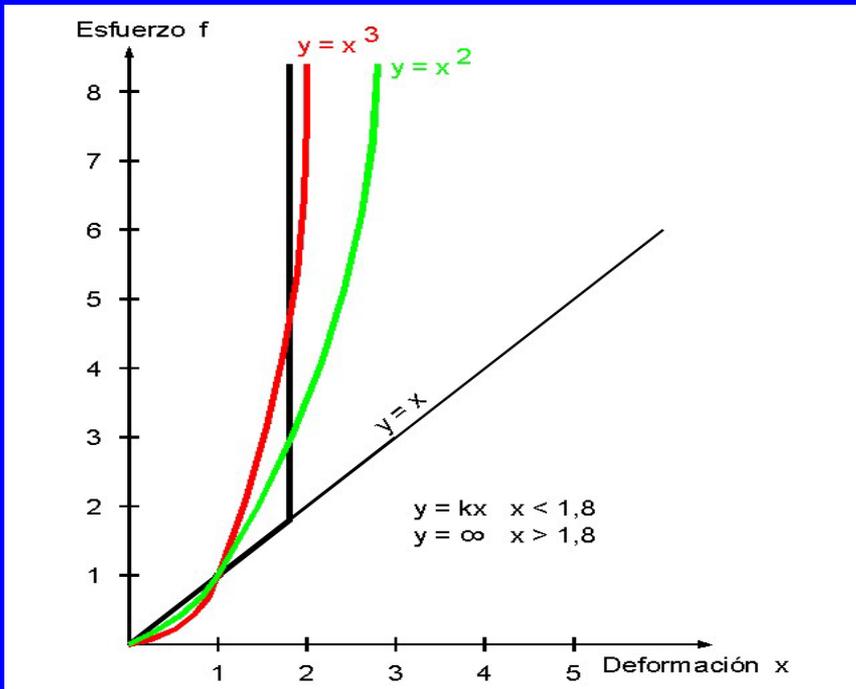
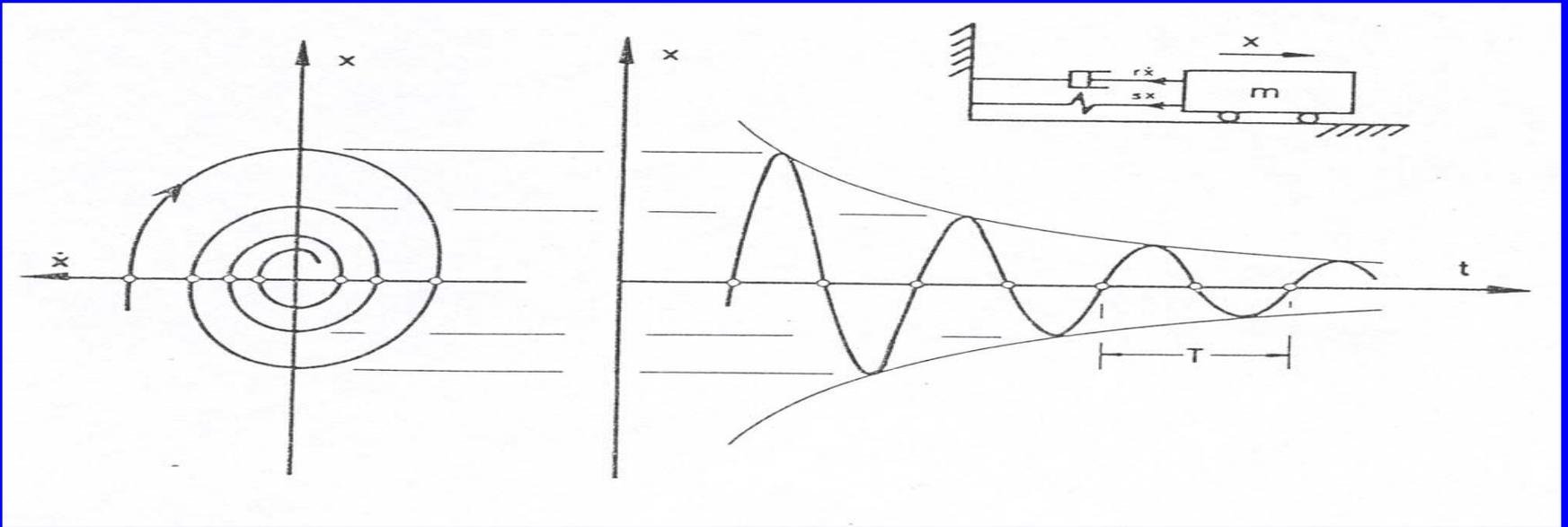
Modelos Lineales / no lineales



$$\frac{d^2 x}{dt^2} + 2\omega_0 \lambda \frac{dx}{dt} + \omega_0^2 x = 0$$

$$\lambda = 0 \quad x(t) = \text{sen } \omega_0 t \quad \lambda \neq 0 \quad x(t) = e^{-kt} \text{sen } \omega_0 t$$





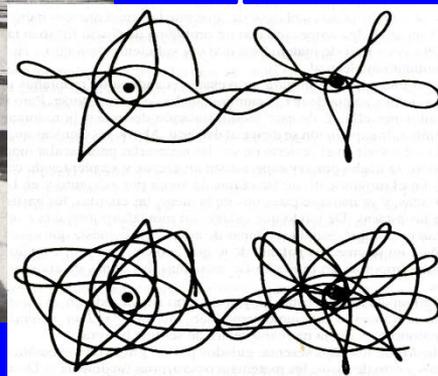
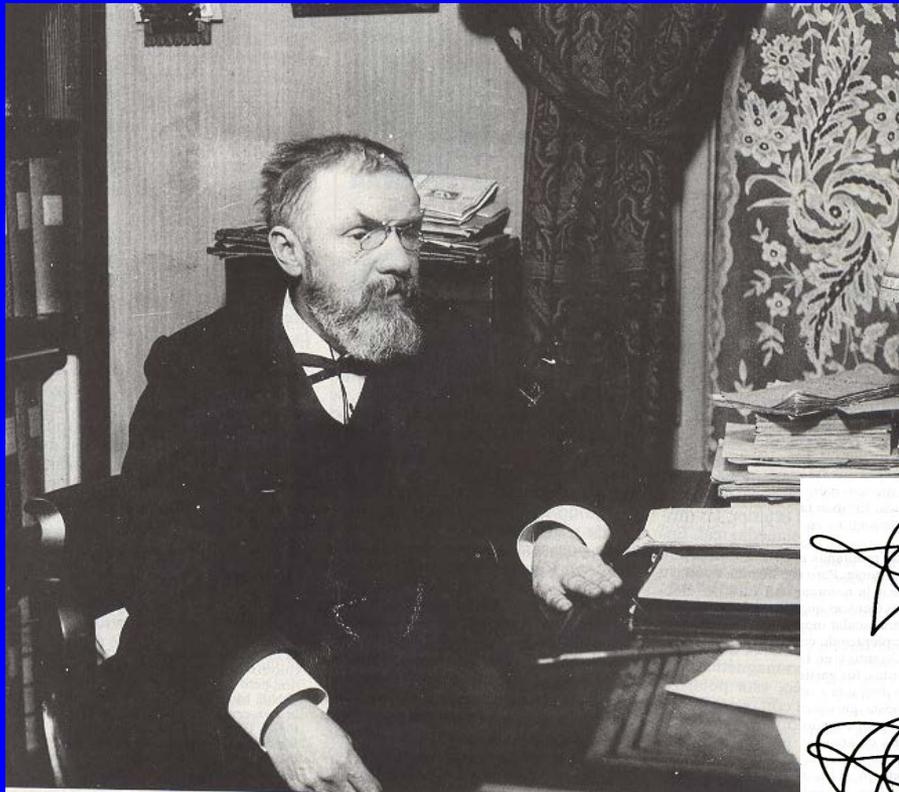
Sistemas dinámicos caóticos

Problema de los
3 cuerpos, ...

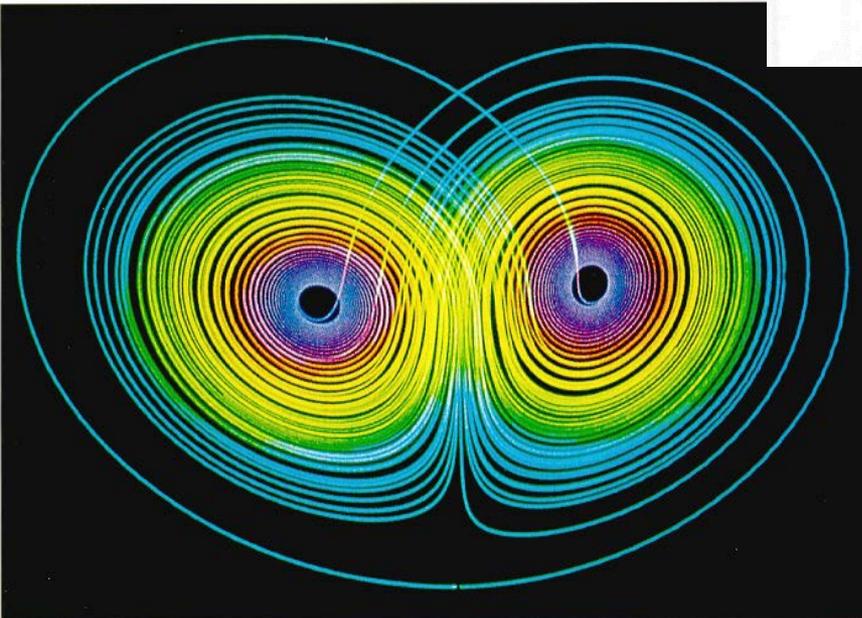
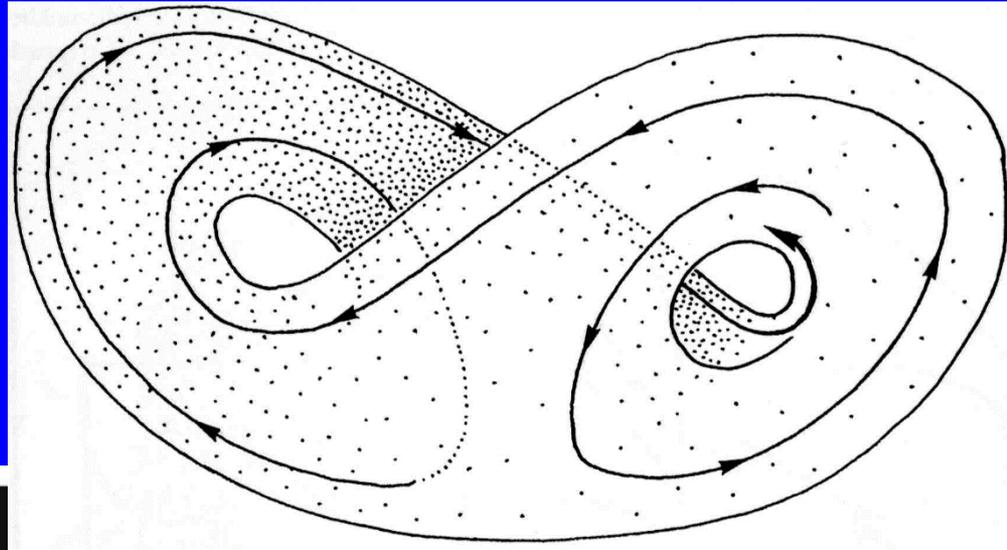
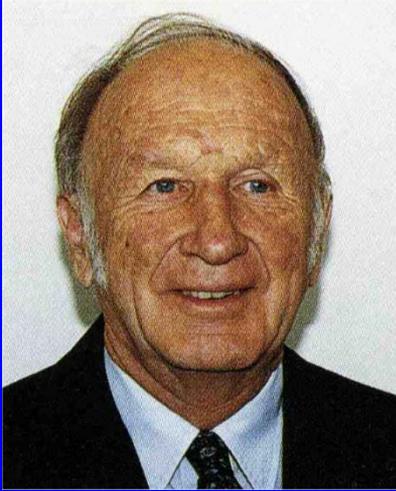
Henri Poincaré (1854-1941)

En 1908, H. Poincaré (1854-1912),

“puede ocurrir que pequeñas diferencias en las condiciones iniciales produzcan otras muy grandes en el fenómeno final. Un pequeño error en las primeras producirá otro enorme en las últimas. La predicción resulta imposible”



1917-



El atractor de Lorenz. Ilustración de Chaos and Fractals. New Frontiers of Science. Edit. Springer-Verlag.

$$\frac{dx}{dt} + \sigma(x - y) = 0$$

$$\frac{dy}{dt} + y - rx + xz = 0$$

$$\frac{dz}{dt} + bz - xy = 0$$

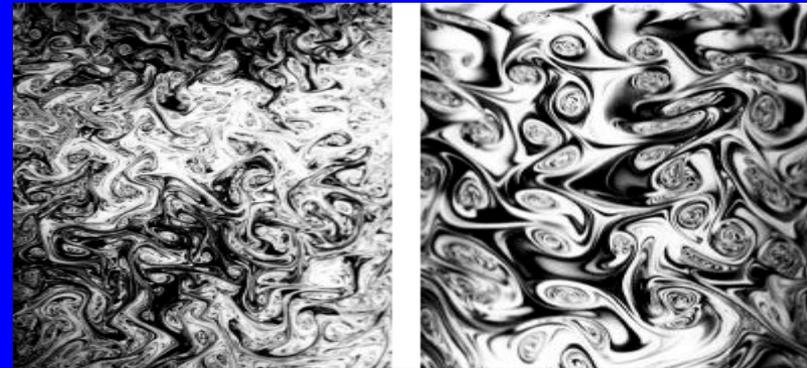
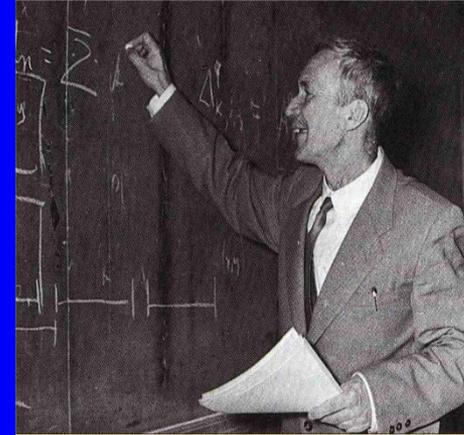
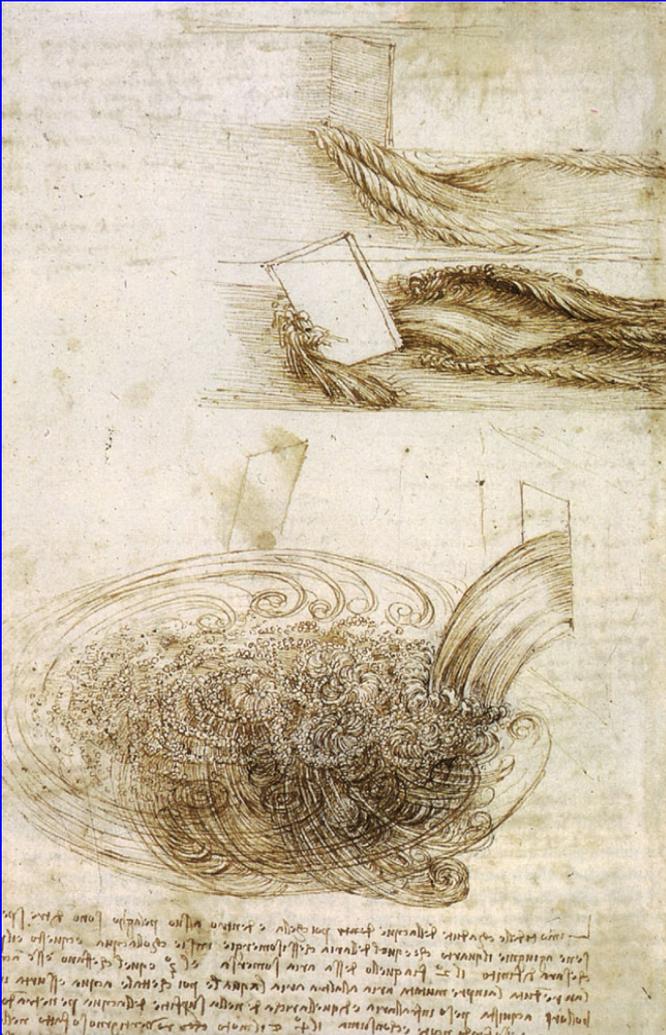
Atractor Extraño

\dot{X}

X

$$\ddot{X} + 0.05 \dot{X} + X^3 = 0.75 \cos t$$

Turbulencia (Kolmogorov)



Características del Caos

Proceso determinista

Sistemas dinámicos no lineales

Trayectoria errática

Atractor

Hipersensible a las condiciones iniciales

Objeto fractal

S. IX – XX

HENRI POINCARÉ (1854-1912)

ESCUELA RUSA

S. XX

KOLMOGOROV (1903 – 1987)

LORENZ (USA) 1961

1961 – 75

EXPLOSIÓN DE LA TEORÍA DEL CAOS

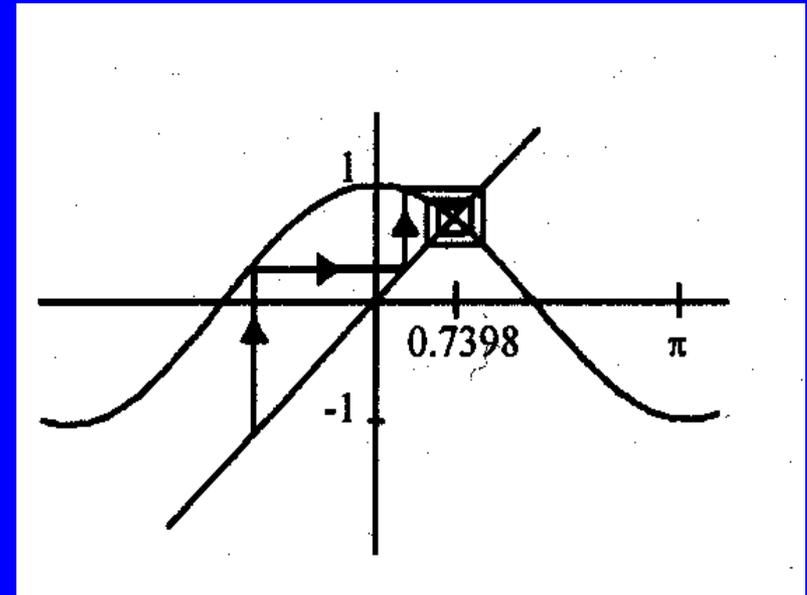
1975 - 2000

POPULARIZACIÓN

El ejemplo más simple de sistema dinámico discreto definido en un conjunto X es el obtenido iterando una aplicación de X en sí mismo:

Si $f: X \longrightarrow X$ y se define

$$\begin{aligned}\Phi(n, x) &= f(f(f(\dots(f(x))\dots))) \\ &= f^n(x)\end{aligned}$$



obtenemos un sistema dinámico sobre X .

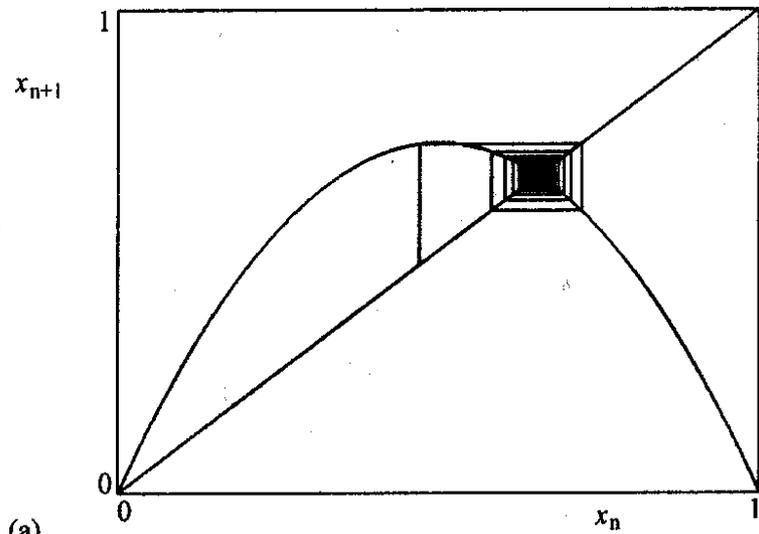
Sistemas caóticos

Un sistema dinámico discreto es caótico si:

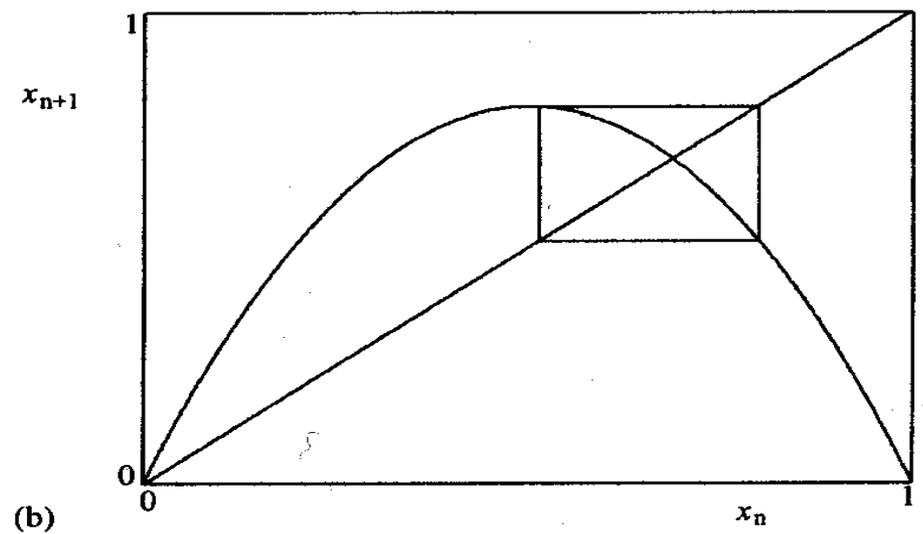
- Es sensible a las condiciones iniciales
- Es transitivo
- Las órbitas periódicas son densas en el espacio

$$x_{n+1} = ax_n(1 - x_n)$$

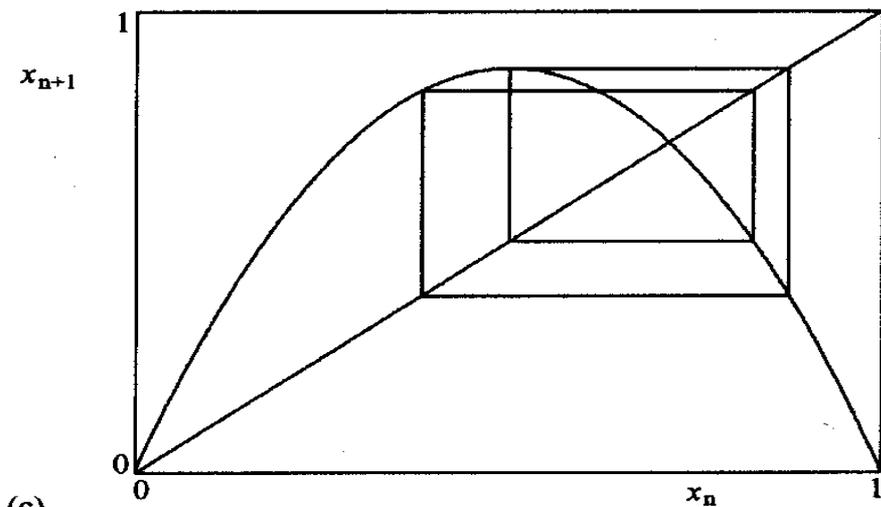
Ecuación logística



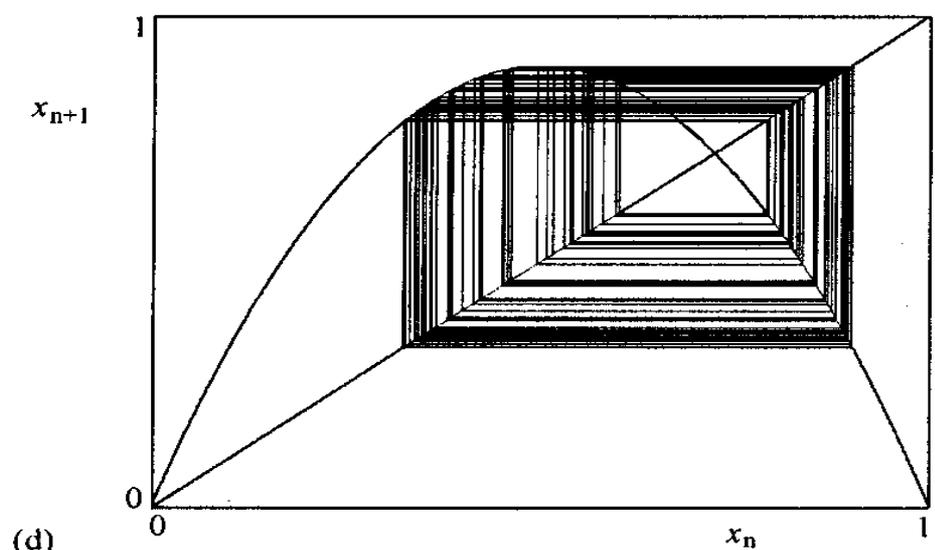
(a)



(b)



(c)



(d)

