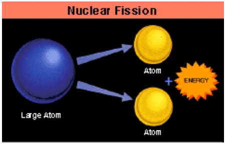
On some nonlinear and nonlocal elliptic and parabolic problems arising in Stellerator nuclear fusion devices

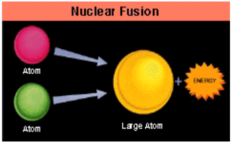
J.I. Diaz

December 12, 2012

What is the Nuclear fusion?

- The nuclear fusion:





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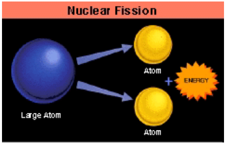
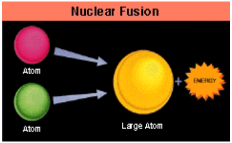
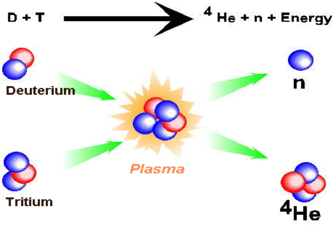
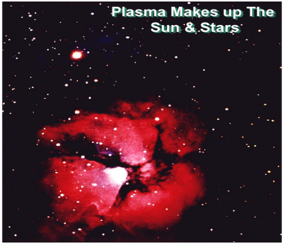




- The plasma: A mixture of particles of positive, negative and neuter electrical charge can be consider as an ideal fluid for determining the macroscopic properties.

Particles of low mass: Deuterium, Tritium, He,...

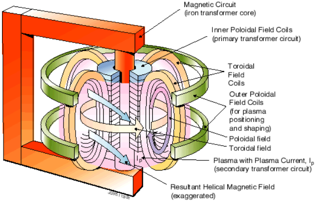




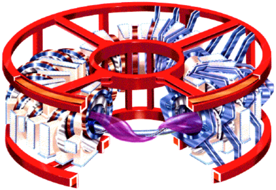
- Magnetic confinement: Need $> 100 * 10^6 C^o$ to obtain an equilibrium state.
- Confinement: magnetic or inertial (not presented in this lecture).Many general expositions on the physical and engineering phenomenology (see e.g. [3]).

Axisymmetric geometry:
Tokamak devices

Non axisymmetric geometry:
Stellarator devices

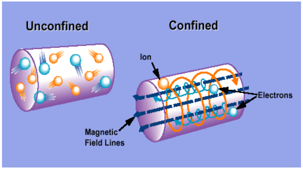


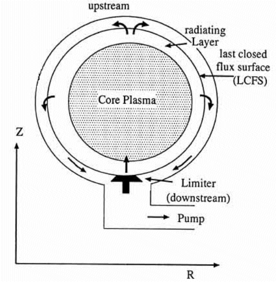
Sketch of a Tokamak

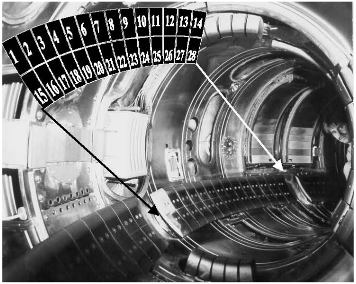


Sketch of TJ-II in the Ciemat-Madrid

- Difficulties: to determine the conditions on the magnetic field and on the current density in order **to keep the plasma far from the camera walls.**







A way to prevent mechanically this is to introduce a *limiter:* a solid object which determines the boundary of the plasma (limiter plays the role of a *thin obstacle* for the plasma).

The 3D stationary model

The plasma as a ideal fluid and use the ideal MHD model.

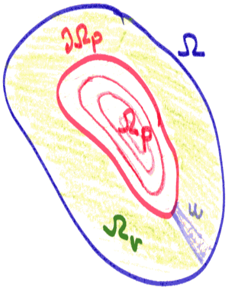
- Assume that the plasma is a perfect conductor (Ohm's Law).

$$\nabla \cdot \mathbf{B} = 0, \quad (\textit{Conservation of } \mathbf{B}),$$
$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (\textit{Ampère's Law}),$$
$$\nabla P = \mathbf{J} \times \mathbf{B} \text{ in } \Omega_p, \quad (\textit{conservation of momentum})$$

The electromagnetic variables are:	The fluid variables are:
• the magnetic field **B** and • the current density **J**	• the pressure P. • magnetic permeability μ_0.

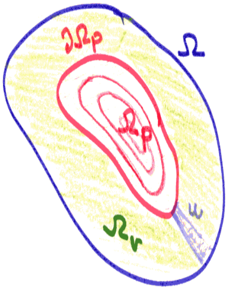
are satisfied in plasma region.

- Sketch:



- Sketch:
- $\mathbb{R}^3 \supset \Omega = \Omega_p \cup \Omega_v$

$$\begin{cases} \Omega_p & := \text{plasma region } (\textit{unknown}) \\ \partial\Omega_p & := \textbf{the free boundary} \\ \Omega_v = \{x : \mathbf{J}(x) = 0\} & := \text{vacuum region} \\ \omega & := \text{the limiter} \end{cases}$$



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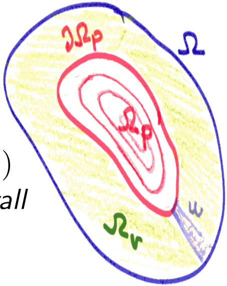

- **Boundary Conditions**:

$$\mathbf{n}^3 \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega_p = \{x : P(x) = 0\}$$
$$(\Longleftarrow \nabla P || \mathbf{n}^3 \text{ and } \nabla P(x) \perp \mathbf{B})$$
$$\mathbf{n}^3 \cdot \mathbf{B} = 0 \quad \text{on } \partial\Omega. \quad \textit{perfectly conducting wall}$$

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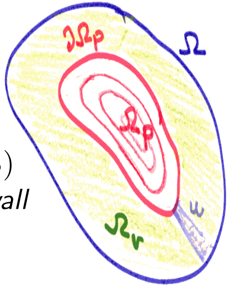

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- **One Integral Condition**:

"*the current carrying*" into the plasma.

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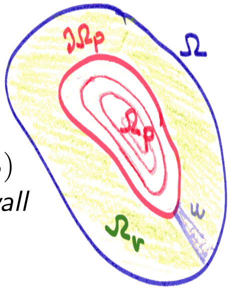

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- **One Integral Condition**:

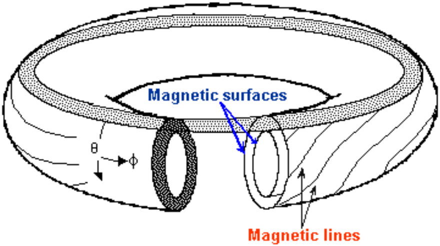
  "*the current carrying*" into the plasma.

- **The problem is to find**

$$\boxed{P : \Omega \subset \mathbb{R}^3 \to \mathbb{R}, \ \ \mathbf{B}, \mathbf{J} : \Omega \subset \mathbb{R}^3 \to \mathbb{R}^3?}$$

- From $\nabla P = \mathbf{J} \times \mathbf{B}$ in Ω_p, it follows that

$$\boxed{\mathbf{B} \cdot \nabla P = 0 \quad \text{and} \quad \mathbf{J} \cdot \nabla P = 0.}$$



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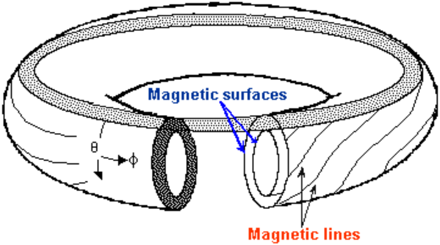
  - Then **the pressure is constant on each magnetic surface.**



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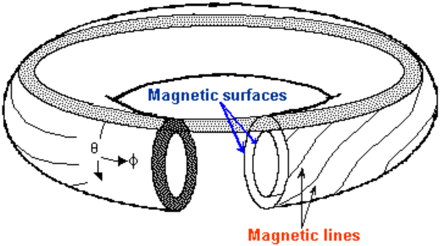
  - Then **the pressure is constant on each magnetic surface.**
  - If a surface lies in a bounded volume then it must be a toroid, i.e. a topological torus. (Due to Alexandroff and Hopf).



The 2D stationary models

Axisymmetric geometry (Tokamak)

As the magnetic field lines are in toroidal nested surfaces, it is useful to *introduce a new coordinates system:*

- Axisymmetric geometry (*Tokamak* devices):

Cylindrical coordinates system (r, φ, z): Let be ψ the magnetic surface, then

$$\underbrace{\begin{array}{c} \mathbf{B}\cdot\nabla\psi = 0 \\ (\text{MHD}) \left\{ \begin{aligned} \nabla\cdot\mathbf{B} &= 0 \\ \nabla\times\mathbf{B} &= \mu_0\mathbf{J} \\ \nabla P &= \mathbf{J}\times\mathbf{B} \ \ \textbf{in } \Omega_p \text{ (plasma region)} \end{aligned} \right. \\ \mathbf{B} = (B_r, B_\varphi, B_z) \ \text{(covariant coordinates)} \end{array}}$$

$$\Downarrow$$

$$-\left[\frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\frac{\partial\psi}{\partial z}\right)\right]=\frac{1}{2\mu_0 r^2}\frac{\partial F^2(\psi)}{\partial\psi}+r\frac{\partial p(\psi)}{\partial\psi}$$

Grad-Shafranov equation

Operator: $-\mathcal{L}\psi := -\mu_0 r \left[\frac{\partial}{\partial r}\left(\frac{1}{\mu_0 r}\frac{\partial \cdot}{\partial r}\right) + \frac{\partial}{\partial z}\left(\frac{1}{\mu_0}\frac{\partial \cdot}{\partial z}\right)\right] \psi,$

Grad-Shafranov equation ( )

$$\boxed{-\mathcal{L}\psi = \frac{1}{2}\left(F^2(\psi)\right)' + \mu_0 r^2 p'(\psi)} \tag{1}$$

- $\psi := \psi(r, z)$ (is a potential flux and *unknown* function),

(3)

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- $P := p(\psi)$ the pressure. In the plasma region $p(p) \geq 0$ and in the vacuum region $p(\psi) \leq 0$. (p is a *prescribed* function: $p(\psi) \sim \frac{\lambda}{2}\psi_+^2$).

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- **Boundary Conditions**: $\psi = \gamma$ on $\partial\Omega$, and γ is a negative constant.
- **One Integral Conditions**: The known *total current carrying* $I_p$into the plasma: ( )

$$\boxed{\int_\Omega \left\{ \frac{1}{2\mu_0 r^2}\left(F^2(\psi)\right)' + p'(\psi)\right\} rdrdz = I_p} \qquad (2)$$

(3)

- Some free boundary problems in Tokamak:machine:
  To find a function $u : \Omega \to \mathbb{R}$ ($\Omega \subset \mathbb{R}^2$, bounded regular open set), u "*regular enough*" such that

$$\begin{cases} -\Delta u + \lambda G(x,u) & = 0 & \text{in } \Omega, \\ u & = \gamma \text{ (unknown constant } > 0) & \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial u}{\partial n} & = I_p > \text{ given.} \end{cases}$$

  - Friedman–Liu95: $-\Delta u + \lambda \max(0, -u) \quad = 0$ ←existence of solution and regularity of free boundary $\partial\Omega_p$ (analytic).
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  - Temam77, Temam78, Mossino–Temam81:

$$G(x,u) = \begin{cases} 0 & \text{if } u \geq 0, \\ g\left(x, u, S(u(x)), \dfrac{du}{dS}(S(u(x))), \dfrac{d^2u}{dS^2}(S(u(x)))\right) & \text{if } u < 0. \end{cases}$$

where $S(u(x))$ =meas$\{y \in \Omega : u(y) < u(x)\}$.
Existence of solution (rearrangement of a function and variational methods).

Non axisymmetric geometry (Stellatator)

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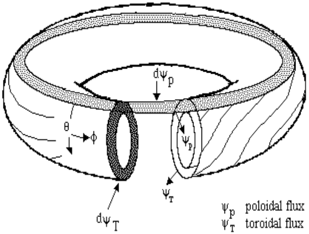
1. *Boozer vacuum coordinates system* (ρ, θ, ϕ)[Boozer82]. The the magnetic field lines becomes "*straights*" in the (θ, ϕ)-plane:



- $\rho = \rho(x, y, z) > 0$ and $\rho = 0$ on the magnetic axis
  ρ is constant on each nested toroid.
- $\theta = \theta(x, y, z)$ is the poloidal angle, is constant on any toroidal circuit but changes by 2π over a poloidal circuit
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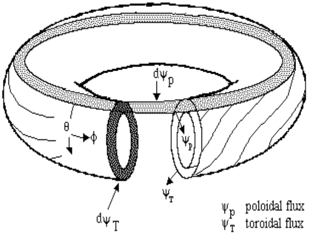

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2. Pass form a 3D to 2D problem**:** *averaging methods* were used [GreeneJohnson84], [HenderCarreras84].

- **In the plasma region,** the following **Grad–Shafranov equation is satisfies**:( )

$$\boxed{-\mathcal{L}\psi = a(\rho,\theta)F(\psi) + \frac{1}{2}\left(F^2(\psi)\right)' + b(\rho,\theta)p'(\psi) \quad \text{in } \Omega_p} \qquad (3)$$

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- The *limiter* ω is modeled by the *multivalued maximal monotone graph* β:

the problem is to find ψ and F, such that

$$(\mathcal{P})\begin{cases} -\mathcal{L}\psi+\beta\left(\psi\chi_\omega\right) \ni a(\rho,\theta)F(\psi)+\frac{1}{2}\left(F^2(\psi)\right)'+b(\rho,\theta)p'(\psi) \text{ in } \Omega \\ \qquad +\text{Boundary Condition} \quad + \quad \text{One Integral Condition} \end{cases}$$

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- **One Integral Condition,** "*The current carrying*" into the plasma: ( ) for any $s \in [essinf\ \psi,\ esssup\ \psi]$

$$\boxed{\int_{\{\psi > s\}} \left[\tfrac{1}{2}\left(F^2(\psi)\right)' + bp'(\psi)\right] \rho d\rho d\theta = j\left(s_+, \|\psi_+\|_{L^\infty(\Omega)}\right).} \qquad (4)$$

"We will replacethe $\mathcal{L}$ operator by the *Laplacian* one, Δ."

The inverse thin obstacle problem

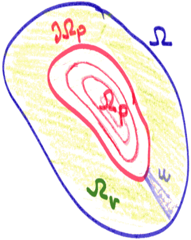
We assume that:

$\Omega \subset \mathbb{R}^2$, a open bounded regular set,
ω (*the limiter*)$\subset \Omega$, connected subset, $\emptyset \neq \bar{\omega} \cap \partial\Omega$ connected subset,
β (*bounded multivalued maximal monotone graph*):
$\beta(r) = 0$ if $r > 0$, $\beta(0) = [0, +\infty)$.

$\gamma < 0$, $F_v > 0$, $a, b \in L^\infty(\Omega)$, $b > 0$ a.e. in Ω,

$$p \begin{cases} p \in \mathcal{C}^1(\mathbb{R}), p(0) = 0 \\ 0 \leq p'(t) \leq \lambda t_+, \text{ Hölder continuous functions} \\ (\lambda > 0) \end{cases}$$

$$j \begin{cases} j \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^+), \; j(s,s) = 0, \; (s > 0) \\ j'_t \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^+), \; \eta := \|j'_t\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R}^+)} < \infty \end{cases}$$



To find: (u, F) u: $\Omega \to \mathbb{R}$, $F : \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ such that $F(s) = F_v$ for any $s \leq 0$ and satisfying

$$(\mathcal{P}) \begin{cases} -\Delta u + \beta(u\chi_\omega(x)) \ni a(x) F(u(x)) + \frac{1}{2}\left(F(u(x))^2\right)' \\ \qquad + b(x) p'(u(x)) \quad \text{in } \Omega, \\ u - \gamma \in H_0^1(\Omega), \\ \displaystyle\int_{\{x:u(x)>s\}} \frac{1}{2}\left(F(u(x))^2\right)' + b(x) p'(u(x))\, dx = j(s_+, \|u_+\|_{L^\infty(\Omega)}) \\ \qquad \text{for any } s \in \left[\operatorname{ess}\inf_\Omega u, \operatorname{ess}\sup_\Omega u\right], \end{cases} \tag{5}$$

References: *Without limiter*:

1. The case $j \equiv 0$: Díaz91, Padial92, Díaz-Rakotoson93,94, Díaz-Galiano-Padial96.

$(\,)$ (6)

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2. The case $j \not\equiv 0$: Díaz-Padial-Rakotoson98.

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$$(\mathcal{P})\begin{cases} -\Delta u+\beta(u\chi_{\omega}(x)) \ni a(x)F(u(x))+\frac{1}{2}\left(F(u(x))^{2}\right)' \\ \qquad +b(x)p'(u(x)) \quad \text{in } \Omega, \\ u-\gamma\in H_0^1(\Omega), \\ \displaystyle\int_{\{x:u(x)>s\}} \frac{1}{2}\left(F(u(x))^{2}\right)'+b(x)p'(u(x))\,dx = j(s_+,\|u_+\|_{L^\infty(\Omega)}) \\ \qquad \text{for any } s\in\left[ess\inf_{\Omega} u, ess\sup_{\Omega} u\right], \end{cases} \tag{5}$$

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1. The case $j\equiv 0$: Díaz91, Padial92, Díaz-Rakotoson93,94, Díaz-Galiano-Padial96.
2. The case $j\not\equiv 0$: Díaz-Padial-Rakotoson98.
3. Evolution case, with $j\equiv 0$, Díaz-Lerena-Padial02, Díaz-Lerena-Padial-Rakotoson04.

Existence of solutions.

Theorem

Suppose that $\gamma \leq 0$. *Then there exist* $\Lambda_1, \Lambda_2 > 0$ *such that if*

$$\lambda \|b\|_{L^\infty(\Omega)} + \eta < \Lambda_1 \quad \text{and} \quad \Lambda_2 < \inf_{\Omega} |a| F_v,$$

there exist a couple (u, F) *with*

$$u \in V(\Omega) := \left\{ v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \leq 0 \right\},$$

$$F \in W^{1,\infty}(] \inf_{\Omega} u, \sup_{\Omega} u[), \qquad F(t) = F_v, \ \forall t \leq 0$$

solution of $(\mathcal{P})$. *Moreover,* u *satisfies that*

$$meas\{x \in \Omega : \nabla u(x) = 0\} = 0$$

and F *is entirely determined by* u.

Steps of the proof:

a) **Eliminating** the unknown F by a term involving u:
The *non local problem* $(\mathcal{P}^*)$

b) $(\mathcal{P}) \Longleftrightarrow (\mathcal{P}^*) +$ the assumption $(\mathcal{H})$.

c) Looking for a week solution of $(\mathcal{P}^*)$ verifying $(\mathcal{H})$.

One dimensional rearrangement

Definition

Let $u : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ be a measurable function and let $\Omega_* :=]0, |\Omega|[$. The **Decreasing Rearrangement** of u is the following decreasing real function $u_* : \Omega_* \to \mathbb{R}$:

$$m_u(\sigma) := \text{meas}\{x \in \Omega : u(x) > \sigma\} = |u > \sigma| \text{ (distribution function of } u\text{)}$$

$$u_*(s) := \inf\{t \in \mathbb{R} : m_u(\sigma) \leq s\} \text{ (decreasing rearrangement of } u\text{)}$$

$$u_*(0) := \operatorname*{esssup}_{\Omega} u := \|u_+\|_{L^\infty(\Omega)} = u_{+*}(0),$$

$$u_*(|\Omega|) := \operatorname*{essinf}_{\Omega} u, \quad \hat{m} := \operatorname*{essinf}_{\Omega} u, \; M := \operatorname*{esssup}_{\Omega} u \; .$$

$$m_u(\sigma) := \text{meas}\{x \in \Omega : u(x) > \sigma\} = |u > \sigma| \ (\textit{distribution function of } u)$$

$$u_*(s) := \inf\{t \in \mathbb{R} : m_u(\sigma) \leq s\} \ (\textit{decreasing rearrangement of } u)$$

Example

Let be $u : \Omega = (-2, 5) \to R$, such that

$$u(x) = \begin{cases} (x+2)(x+1), & x < -1 \\ 0.3 & -1 \leq x \leq 1 \\ \frac{1}{2} + e^{-x} & 1 < x \end{cases}, \quad u_* : \Omega_* = (0, |\Omega|) \to [\underset{\Omega}{\text{ess inf}}\, u, \underset{\Omega}{\text{ess sup}}\, u]$$

The function u — The rearrangement u_*

Graphics: *P.Galán*

Relative rearrangement

Definition ([MossinoTemam81])

Let $b \in L^1(\Omega)$ and a measurable function u in Ω, we set $w : \bar{\Omega}_* =]0, |\Omega|[\to \mathbb{R}$

$$w(s) = \int_{\{x:u>u_*(s)\}} b(x)\,dx + \int_0^{s-|u>u_*(s)|} \left(b|_{\{u=u_*(s)\}}\right)_* (t)\,dt \ , \quad \text{for } s \in \Omega_*.$$

The **Relative Rearrangement** of b with respect to u is

$$\boxed{b_{*u}(s) := \frac{dw(s)}{ds} = \lim_{\sigma \to 0} \frac{(u+\sigma b)_*(s) - u_*(s)}{\sigma} \qquad \text{in } \Omega_* \ .}$$

Remark: If u has not flat region then $s - |u > u_*(s)| = 0$ and
$$b_{*u}(s) := \frac{d}{ds}\int_{\{x:u>u_*(s)\}} b(x)\,dx.$$

a) The non local problem (P*)
Eliminating the unknown F by a term involving u

Theorem

Let (u, F) *verify the integral condition, with* $u \in W^{2,p}(\Omega)$, $p \geq 1$, $F \in W^{1,\infty}(]\hat{m}, M[)$, $F(t) = F_v$ *if* $t \leq 0$ *and such that*

$$\boxed{meas\{x \in \Omega : \nabla u(x) = 0\} = 0}.$$

Then, for all $t \in [\hat{m}, M]$

1. $$\boxed{\frac{1}{2}\left(F^2(t)\right)' := -p'(t)b_{*u}(|u > t|) + j_t'(t_+, u_{+*}(0))u_{+*}'(|u > t|)} \quad (5)$$

a) The non local problem (P*)
Eliminating the unknown F by a term involving u

Theorem

Let (u, F) verify the integral condition, with $u \in W^{2,p}(\Omega)$, $p \geq 1$, $F \in W^{1,\infty}(]\hat{m}, M[)$, $F(t) = F_v$ if $t \leq 0$ and such that

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Then, for all $t \in [\hat{m}, M]$

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2. $$F(t) := \mathcal{F}(t) = \left[F_v^2 - 2\int_0^{t_+} p'(\sigma) b_{*u}(|u > \sigma|) d\sigma + 2\int_0^{t_+} j_t'(\sigma_+, u_{+*}(0)) u_{+*}'(|u > \sigma|) d\sigma\right]_+^{\frac{1}{2}}$$

Theorem

3. $\boxed{F(u(x)) = \mathcal{F}_u(x) \quad a.e.\ x \in \bar{\Omega}}$, *with*

$$\boxed{\begin{aligned}\mathcal{F}_u(x) := \quad & \Big[F_v^2 - 2\int_{|u>0|}^{|u>u_+(x)|} [p(u_*(s))]' \, b_{*u}(s) ds \\ & + 2\int_{|u>0|}^{|u>u_+(x)|} j_t'(u_{+*}(s), u_{+*}(0)) \, (u_{+*}'(s))^2 ds \Big]_+^{\frac{1}{2}}\end{aligned}}$$

Theorem

3. $\boxed{F(u(x)) = \mathcal{F}_u(x) \quad a.e.\ x \in \bar{\Omega}}$, *with*

$$\mathcal{F}_u(x) := \left[F_v^2 - 2\int_{|u>0|}^{|u>u_+(x)|} [p(u_*(s))]' \, b_{*u}(s) ds + 2\int_{|u>0|}^{|u>u_+(x)|} j_t'\left(u_{+*}(s), u_{+*}(0)\right)\left(u_{+*}'(s)\right)^2 ds\right]_+^{\frac{1}{2}}$$

4. $$\frac{1}{2}\left(F(u(x))^2\right)' = -p'(u(x))b_{*u}(|u > u(x)|) + j_t'(u_+(x), u_{+*}(0))u_{+*}'(|u > u(x)|).$$

Now, we consider the following non local problem: to find $u \colon \Omega \to \mathbb{R}$, such that

$$(\mathcal{P}^*) \begin{cases} -\Delta u + \beta(u\chi_\omega) \ni & a\mathcal{F}_u(x) + H(u(x), b_{*u}) + J(u(x)) \text{ in } \Omega \\ u - \gamma \in & H_0^1(\Omega) \end{cases} \tag{6}$$

with

$$H(u(x), b_{*u}) := p'(u(x))[b(x) - b_{*u}(|u > u(x)|)]$$
$$J(u(x)) := j_t'(u_+(x), u_{+*}(0))u_{+*}'(|u > u(x)|)$$

(Recall the problem $(\mathcal{P})$ ( ))

b) (P)<=>(P*) + the assumption (H)

Theorem (Equivalent Problems)

- *Let* (u, F) *be one weak solution of* $(\mathcal{P})$ *such that*

$$meas\{x \in \Omega : \nabla u(x) = 0\} = 0;$$

then u *is weak solution of* $(\mathcal{P}^*)$.

b) (P)<=>(P*) + the assumption (H)

Theorem (**Equivalent Problems**)

- *Let* (u, F) *be one weak solution of* $(\mathcal{P})$ *such that*

$$meas\{x \in \Omega : \nabla u(x) = 0\} = 0;$$

then *u is weak solution of* $(\mathcal{P}^*)$.

- *Reciprocally, let* $u \in V(\Omega) := \{v \in H^1(\Omega) : \Delta v \in L^\infty(\Omega), v_{|\partial\Omega} \leq 0\}$ *solution of* $(\mathcal{P}^*)$ *such that*

$$(\mathcal{H}) \begin{cases} meas\{x \in \Omega : \nabla u(x) = 0\} = 0 \\ \min\{\mathcal{F}(t) : t \in [\hat{m}, M]\} > 0 \,, \end{cases}$$

holds. **Then** $(u, \mathcal{F})$ *is a solution of problem* $(\mathcal{P})$.

c) Looking for a week solution of (P*) verifying (H)

Steps:

c.1) The approximate problem $(\mathcal{P}_\epsilon^*)$.

c.2) The *Galerkin Method*:

Existence of solution for a finite dimensional problems $\left(\mathcal{P}_{\epsilon,m}^*\right)$, $m \in \mathbb{N}$.

c.3) A priori estimates uniformly in m and past to the limit $m \longrightarrow \infty$: Existence of solution of $(\mathcal{P}_\epsilon^*)$.

c.4) The property $(\mathcal{H})$.

c.5) A priori estimates uniformly in ϵ and past to the limit $\epsilon \longrightarrow 0$: Existence of solution of $(\mathcal{P}_*)$.

We assume $\epsilon > 0$. Let $\beta_\epsilon(\cdot)$ be a Yosida approximation of $\beta(\cdot)$. We used the truncation functions

$$h_\epsilon(t) = \frac{t^2}{1+\epsilon t^2} \qquad \xi_\epsilon(t) = \frac{t}{1+\epsilon|t|} \qquad (\leq 1/\epsilon)$$

$$F_\epsilon(x, v, b_{*v}) := \Bigg[\underbrace{F_v^2 - 2\int_{|v>0|}^{|v>v_+(x)|} [p(v_*(s))]' b_{*v}(s) ds}_{:=F_1(x,v,b_{*v})} + 2 \underbrace{\int_{|v>0|}^{|v>v_+(x)|} j_t'(v_{+*}(s), v_{+*}(0)) h_\epsilon(v_{+*}'(s)) ds}_{:=F_{\epsilon,2}(x,v)} \Bigg]_+^{\frac{1}{2}}$$

$$J_\epsilon(v(x)) := \xi_\epsilon(v_{+*}'(|v > v_+(x)|)) j_t'(v_+(x), v_{+*}(0))$$

We introduce the approximate problem $(\mathcal{P}_\epsilon^*)$:

Find u^ϵ such that $u^\epsilon - \gamma \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$; $\forall\ p \geq 1$ and

$$(\mathcal{P}_\epsilon^*) \begin{cases} -\Delta u^\epsilon + \beta_\epsilon\left(u^\epsilon \chi_\omega\right) = & aF_\epsilon(x, u^\epsilon, b_{*u^\epsilon}) + H(u^\epsilon(x), b_{*u^\epsilon}) \\ & + J_\epsilon(u^\epsilon(x)) \qquad \text{in } \Omega, \\ u^\epsilon - \gamma \in & H_0^1(\Omega). \end{cases}$$

We recall that

$$(\mathcal{P}^*) \begin{cases} -\Delta u + \beta\left(u \chi_\omega\right) \ni & a\mathcal{F}_u(x) + H(u(x), b_{*u}) \\ & + J(u(x)) \qquad \text{in } \Omega, \\ u - \gamma \in & H_0^1(\Omega). \end{cases}$$

We consider:

- Let V_m =span$\{\varphi_1, \ldots, \varphi_m\}$, $(\lambda_k, \varphi_k)_{k\geq 1}$ eigenvalues and eigenfunctions:

$$\boxed{-\Delta\varphi_k = \lambda_k\varphi_k, \;\; \varphi \in H_0^1(\Omega).}$$

- On V_m, we define $[v, w] := \sum_{k=1}^{m} v^k w^k$ where

$$v = \sum_{k=1}^{m} v^k \varphi_k andw = \sum_{k=1}^{m} w^k \varphi_k.$$

- For $\gamma \leq 0$ fixed, we consider $T_m^\epsilon : V_m \to V_m$ defined as

$$\boxed{\begin{aligned}[T_m^\epsilon v, \varphi] = \int_\Omega \nabla v \cdot \nabla\varphi dx + \int_\Omega \beta_\epsilon\left((v+\gamma)\chi_\omega\right)\varphi dx \\ -\int_\Omega aF_\epsilon(x, v+\gamma, b_{*(v+\gamma)})\varphi dx - \int_\Omega H(v+\gamma, b_{*(v+\gamma)})\varphi dx \\ -\int_\Omega J_\epsilon(v+\gamma)\varphi dx \qquad \forall v, \varphi \in V_m .\end{aligned}}$$

We shall prove that:

- T_m^ϵ operator attains zero for some $w_m^\epsilon \in V_m \setminus \{0\}$, i.e.

$$T_m^\epsilon w_m^\epsilon = 0 \text{ in } V_m,$$

We shall prove that:

- T_m^ϵ operator attains zero for some $w_m^\epsilon \in V_m \setminus \{0\}$, i.e.

$$T_m^\epsilon w_m^\epsilon = 0 \text{ in } V_m,$$

- if $T_m^\epsilon w_m^\epsilon = 0 \Longrightarrow w_m^\epsilon$ satisfies the finite dimensional problem

$$(\mathcal{P}_{\epsilon,m}^*) \begin{cases} -\Delta(w_m^\epsilon+\gamma)=P_m[-\beta_\epsilon\left(\left(w_m^\epsilon+\gamma\right)\chi_\omega\right)+aF_\epsilon(x, w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)}) \\ \quad +H(w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)})+J_\epsilon(w_m^\epsilon+\gamma)] \text{ in } \Omega \text{ with } w_m^\epsilon \in V_m \end{cases}$$

where P_m is the orthogonal projection operator from $L^2(\Omega)$ onto V_m.

Thus w_m^ϵ is a weak solution of $(\mathcal{P}_{\epsilon,m}^*) \Leftrightarrow T_m^\epsilon w_m^\epsilon = 0$

Theorem

Assume $\lambda_1 - \lambda \operatorname*{osc}_{\Omega} b > 0$. *Then there exists at least* $w_m^\epsilon \in V_m$ *solution of problem* $(\mathcal{P}^*_{\epsilon,m})$, *i.e. satisfying* $\forall \varphi \in V_m$

$$[T_m^\epsilon w_m^\epsilon, \varphi] = \int_\Omega \nabla w_m^\epsilon \cdot \nabla \varphi dx + \int_\Omega \beta_\epsilon \left(\left(w_m^\epsilon + \gamma \right) \chi_\omega \right) \varphi dx$$
$$- \int_\Omega a F_\epsilon (x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon + \gamma)}) \varphi dx$$
$$- \int_\Omega H(w_m^\epsilon + \gamma, b_{*(w_m^\epsilon + \gamma)}) \varphi dx - \int_\Omega J_\epsilon (w_m^\epsilon + \gamma) \varphi dx = 0 .$$

Proof. Solution of problems $(\mathcal{P}^*_{\epsilon,m})$: Brouwer Fixed Point Theorem (J.L. Lions 1969):

- T^ϵ_m is a continuous map. ($\Leftarrow$ + technical Lemma).

$T^\epsilon_m v$ can be expressed as $T^\epsilon_m v = \sum_{k=1}^m [T^\epsilon_m v, \varphi_k]\varphi_k$ where $\varphi \in V_m$ is an arbitrary function and so it is enough to use the continuity of the different functions appearing in the definition.

- T^ϵ_m is a coercive map when $\lambda \operatorname{osc}_{\Omega} b < \lambda_1$.

The assumption implies the coercivity of T^ϵ_m since

$$[T^\epsilon_m v, v] = \int_\Omega |\nabla v|^2 dx + \int_\Omega \beta_\epsilon\left(\left(v+\gamma\right)\chi_\omega\right) vdx - \int_\Omega aF_\epsilon(x, v+\gamma, b_{*(v+\gamma)})vdx$$
$$-\int_\Omega H(v+\gamma, b_{*(v+\gamma)})vdx - \int_\Omega J_\epsilon(v+\gamma)vdx \qquad \forall\ v \in V_m$$

and $\int_\Omega \beta_\epsilon\left(\left(v+\gamma\right)\chi_\omega\right) vdx$ is minored by zero (the rest, as in Proposition 1 of [DPR98]). ■

Let $\varphi = w_m^\epsilon$, then

$$0 = [T_m^\epsilon w_m^\epsilon, w_m^\epsilon] \geq \int_\Omega |\nabla w_m^\epsilon|^2 dx + \int_\Omega \beta_\epsilon \left(\left(w_m^\epsilon + \gamma \right) \chi_\omega \right) w_m^\epsilon dx$$
$$- C_\epsilon \int_\Omega |w_m^\epsilon| dx + \lambda \operatorname*{osc}_\Omega b \int_\Omega |\nabla w_m^\epsilon|^2 dx$$
$$\geq \left(\lambda_1 - \lambda \operatorname*{osc}_\Omega b - \delta \right) \int_\Omega |w_m^\epsilon|^2 dx - C_{\epsilon\delta},$$

$$\Rightarrow \boxed{\|w_m^\epsilon\|_{L^2(\Omega)} \leq C_\epsilon \text{ (a estimate in } L^2(\Omega)\text{uniformly in } m)}$$

$$\Downarrow$$

$$\boxed{\|\nabla w_m^\epsilon\|^2_{L^2(\Omega)} = \int_\Omega |\nabla w_m^\epsilon|^2 dx \leq C_\epsilon \text{ (a estimate in } H_0^1(\Omega)\text{uniformly in } m).}$$

Finally,

$$\|\Delta w_m^\epsilon\|_{L^2(\Omega)} \leq \|\beta_\epsilon \left(\left(w_m^\epsilon + \gamma \right) \chi_\omega \right)\|_{L^2(\Omega)} + \|aF_\epsilon(x, w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^2(\Omega)}$$
$$+ \|H(w_m^\epsilon + \gamma, b_{*(w_m^\epsilon+\gamma)})\|_{L^2(\Omega)} + \|J_\epsilon(w_m^\epsilon + \gamma)\|_{L^2(\Omega)}$$
$$\leq M\,|\omega| + \|a\|_{L^\infty(\Omega)} \left[F_v^2 + \frac{2\eta}{\epsilon} |\Omega| \right]^{\frac{1}{2}} |\Omega|^{\frac{1}{2}} + \lambda \operatorname*{osc}_\Omega b \|w_m^\epsilon\|_{L^2(\Omega)}$$

By standard regularity results, $(w_m^\epsilon)_{m\geq 1}$ is uniformly bounded in $W^{2,2}(\Omega)$.

For any $\epsilon > 0$ fixed, there exist a subsequence $\{w_m^\epsilon\}$ and one function $w^\epsilon \in H^2(\Omega)$, such that

$$\begin{aligned} w_m^\epsilon &\rightharpoonup w^\epsilon && \text{weakly in } H^2(\Omega), \text{ and so} \\ w_m^\epsilon &\to w^\epsilon && \text{strongly in } W^{1,p}(\Omega)\ \forall p \geq 1 (N=2) \text{ and in } C(\bar{\Omega}). \end{aligned}$$

By using technical result on *relative rearrangement*,

$$b_{*(w_m^\epsilon+\gamma)} \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \hat{b}^\epsilon \text{ weakly}^* \text{ in } L^\infty(\Omega_*),$$

$$b_{*(w_m^\epsilon+\gamma)}\left(|w_m^\epsilon+\gamma > (w_m^\epsilon+\gamma)(\cdot)|\right) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} \tilde{b}^\epsilon \text{ weakly}^* \text{ in } L^\infty(\Omega_*)$$

for some $\hat{b}^\epsilon, \tilde{b}^\epsilon \in L^\infty(\Omega_*)$, and

$$F_\epsilon(x, w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)}) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} F_\epsilon(x, w^\epsilon+\gamma, \hat{b}^\epsilon) \text{ weakly}^* \text{ in } L^\infty(\Omega),$$

$$H(x, w_m^\epsilon+\gamma, b_{*(w_m^\epsilon+\gamma)}) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} H(x, w^\epsilon+\gamma, \hat{b}^\epsilon) \text{ weakly}^* \text{ in } L^\infty(\Omega),$$

$$J_\epsilon(w_m^\epsilon+\gamma) \underset{m\to\infty}{\to} J_\epsilon(w^\epsilon+\gamma) \text{ strongly in } L^1(\Omega).$$

Now, for any $\epsilon > 0$ we have that

Proposition

If

$$\boxed{meas\{x \in \Omega : \nabla w^{\epsilon}(x) = 0\} = 0}$$

then

$$\hat{b}^{\epsilon} = b_{*(w^{\epsilon}+\gamma)} \text{ in } \Omega_{*}$$

and

$$b^{\epsilon} = b_{*(w^{\epsilon}+\gamma)}(|w^{\epsilon} + \gamma > (w^{\epsilon}+\gamma)(x)|) \text{ in } \Omega.$$

That implies that

$$F_{\epsilon}(x, w_m^{\epsilon} + \gamma, b_{*(w_m^{\epsilon}+\gamma)}) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} F_{\epsilon}(x, w^{\epsilon} + \gamma, b_{*(w_m^{\epsilon}+\gamma)}) \text{ weakly}^{*} \text{ in } L^{\infty}(\Omega) .$$

$$H(x, w_m^{\epsilon} + \gamma, b_{*(w_m^{\epsilon}+\gamma)}) \underset{m\to\infty}{\overset{*}{\rightharpoonup}} H(x, w^{\epsilon} + \gamma, b_{*(w_m^{\epsilon}+\gamma)}) \text{ weakly}^{*} \text{ in } L^{\infty}(\Omega) .$$

Let $T^\epsilon : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by

$$[T^\epsilon v, \varphi] = \int_\Omega \nabla v \cdot \nabla \varphi dx + \int_\Omega \beta_\epsilon((v+\gamma)\chi_\omega)\varphi dx$$
$$- \int_\Omega aF_\epsilon(x, v+\gamma, b_{*(v+\gamma)})\varphi dx - \int_\Omega H(v+\gamma, b_{*(v+\gamma)})\varphi dx$$
$$- \int_\Omega J_\epsilon(v+\gamma)\varphi dx \qquad \text{if } v, \varphi \in H_0^1(\Omega).$$

the last convergence implies that $T^\epsilon w^\epsilon = 0$ and so, $w^\epsilon + \gamma$ will be a solution of

$$(\mathcal{P}_\epsilon^*) \begin{cases} -\Delta w^\epsilon + \beta_\epsilon(u^\epsilon \chi_\omega) = & aF_\epsilon(x, w^\epsilon+\gamma, \hat{b}^\epsilon) + p'(w^\epsilon+\gamma)[b - b^\epsilon] \\ & +J_\epsilon(w^\epsilon+\gamma) \qquad \text{in } \Omega \\ w^\epsilon + \gamma \in & H_0^1(\Omega) \cap H^2(\Omega) \end{cases}$$

for any $\epsilon > 0$.

Notation: $u^\epsilon := w^\epsilon + \gamma$, (recall that $u^\epsilon - \gamma \in W_0^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$)

Lemma

If $\nu := \frac{1}{4\pi}\left[2^{1/2}\eta^{1/2}|\Omega|^{1/2}\|a\|_{L^\infty(\Omega)} + \lambda|\Omega| \operatorname*{osc}_{\Omega} b + \eta\right] < 1$, *then uniformly in* ϵ

$$\|\Delta u^\epsilon\|_{L^\infty(\Omega)} \leq \frac{\|\beta\|_\infty + \|a\|_{L^\infty(\Omega)} F_\nu}{1-\nu}.$$

$$\|u_+^\epsilon\|_{L^\infty(\Omega)} \leq \frac{|\Omega|}{4\pi}\left(\frac{\|\beta\|_\infty + \|a\|_{L^\infty(\Omega)} F_\nu}{1-\nu}\right) := S\ .$$

Proof. Following Lemma 23 of [DPR98], we get the conclusion from the estimate

$$\|\Delta u^\epsilon\|_{L^\infty(\Omega)} \leq \|\beta_\epsilon(u^\epsilon \chi_\omega)\|_{L^\infty(\Omega)} + \|a\|_{L^\infty(\Omega)} F_\nu + \frac{1}{4\pi}\left[2^{\frac{1}{2}}\eta^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}\|a\|_{L^\infty(\Omega)} + \lambda|\Omega| \operatorname*{osc}_{\Omega} b + \eta\right]\|\Delta u^\epsilon\|_{L^\infty(\Omega)}\ . \blacksquare$$

Theorem

(When does $(\mathcal{H})$ hold?). *If*

$$\boxed{\lambda\|b\|_{L^\infty(\Omega)} \text{ and } \eta \text{ are small enough}}$$

and

$$\boxed{\inf_\Omega |a| \text{ and } F_v \text{ large enough}}$$

such that

$$\|\beta\|_\infty + \left[\lambda\|b\|_{L^\infty(\Omega)} + \frac{\eta}{|\Omega|}\right] S < \inf_\Omega |a| \left[F_v^2 - 2\lambda\|b\|_{L^\infty(\Omega)} S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}},$$

then

$$\boxed{meas\{x \in \Omega : \nabla u^\epsilon(x) = 0\} = 0.}$$

In particular, u^ϵ *satisfies problem* $(\mathcal{P}_\epsilon^*)$.

Proof. We argue by contradiction. Suppose that

$$\mathrm{meas}\{x \in \Omega : \nabla u^\epsilon(x) = 0\} \neq 0 \ .$$

Then, from the equation of $(\mathcal{P}_\epsilon^*)$

$$\begin{aligned}\beta_\epsilon\left(u^\epsilon\chi_\omega\right) = a[F_v^2 - 2F_1(u^\epsilon, \hat{b}^\epsilon) + 2F_{\epsilon,2}(u^\epsilon)]_+^{\frac{1}{2}} \\ + H(u^\epsilon, \tilde{b}^\epsilon) + J_\epsilon(u^\epsilon) \text{ a.e. on } \{x \in \Omega : \nabla u^\epsilon(x) = 0\}.\end{aligned}$$

By the last estimates, we get that

$$\begin{aligned}\beta_\epsilon\left(u^\epsilon\chi_\omega\right) + \lambda S \operatorname*{osc}_\Omega b &\geq \beta_\epsilon\left(u^\epsilon\chi_\omega\right) + |H(u^\epsilon, \tilde{b}^\epsilon)| \\ &\geq \inf_\Omega |a| [F_v^2 - 2F_1(u^\epsilon, \hat{b}^\epsilon) + 2F_{\epsilon,2}(u^\epsilon)]_+^{\frac{1}{2}} - J_\epsilon(u^\epsilon),\end{aligned}$$

$$\beta_\epsilon\left(u^\epsilon\chi_\omega\right) + \left[\lambda \operatorname*{osc}_\Omega b + \frac{\eta}{|\Omega|}\right] S \geq \inf_\Omega |a| \left[F_v^2 - 2\lambda \|b\|_{L^\infty(\Omega)} S - \frac{2\eta S^2}{|\Omega|}\right]_+^{\frac{1}{2}} .$$

This contradicts the assumption in lemma, proving result. ■

Theorem

Assume $\gamma \in \mathbb{R}^-$ *and that* $\lambda\|b\|_{L^\infty(\Omega)} + \eta < \Lambda_1$ *and* $\inf_\Omega |a|F_v > \Lambda_2$ *for a suitable positive constant* Λ_1 *and* $\Lambda_2 > 0$*. Then there is u solution of* $(\mathcal{P}_*)$*. Moreover u*$\in V(\Omega)$*.*

Proof. Our aim is to let $\epsilon \to 0$. We use the uniform estimates obtained before.

By the uniform estimate on $\|\Delta u^\epsilon\|_{L^\infty(\Omega)}$ given before, there exists some subsequence of (u^ϵ) (which we will again denote by u^ϵ) and a function $\alpha \in L^\infty(\Omega)$ such that $\Delta u^\epsilon \underset{\epsilon\to 0}{\overset{*}{\rightharpoonup}} \alpha$ weakly* in $L^\infty(\Omega)$. By standard regularity, u^ϵ belongs to a bounded set of $W^{2,p}(\Omega)$, for all $p \in [1, +\infty[$. Then, we have (for some subsequence) that

$$u^\epsilon \underset{\epsilon\to 0}{\rightharpoonup} u \text{ weakly in } W^{2,p}(\Omega) \quad \text{and} \quad u^\epsilon \underset{\epsilon\to 0}{\longrightarrow} u \text{ strongly in } \mathcal{C}^1(\bar{\Omega})$$

In particular, $\alpha = \Delta u$, $\Delta u \in L^\infty(\Omega)$, $u \in V(\Omega)$ and the estimates, and the technical result used for the pas to the limit in problem $(\mathcal{P}^*_{\epsilon,m})$ remain true replacing u^ϵ by u. Moreover $\beta_\epsilon(u^\epsilon \chi_\omega) \rightharpoonup B$ weakly in $L^p(\Omega)$ for any $p \in (1, +\infty)$ and as β is maximal monotone we get that $B(x) \in \beta(u\chi_\omega)$ a.e. $x \in \Omega$. Arguing like before, we prove the convergence of equation of problem $(\mathcal{P}^*_\epsilon)$ term by term to the equation of problem $(\mathcal{P}_*)$. Analogously, we obtain that $\mathrm{meas}\{x \in \Omega : \nabla u(x) = 0\} = 0$. and thus we can identify all the terms which appear after to take the limit. In this way, we get the conclusion that u is a solution of $(\mathcal{P}_*)$. ■

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