Les hypothèses telles que la récurrence au sens de Harris sont pénibles à démontrer mais vérifées dans les cas classiques de la fiabilité.

Dans le même cadre, et en utilisant le même principe, nous avons montré qu'une autre formule bien connue des fiabilistes :

$$R(t) \approx e^{-t/MUT}$$

n'était autre qu'une version simplifiée de l'approximation dite de Vesely, pour laquelle on est capable de prouver dans certains cas qu'elle est pessimiste ([2]).

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On some energy methods applied to free boundary problems and their discrete approximations.

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Abstract

The main goal of this lecture is to present some new results on the theory of free boundaries giving rise by the solutions of some nonlinear PDEs. Most of the results will be derived by using *energy methods* (see, e.g. the recent monograph by S.N. Antontsev, J.I. Díaz and S.I. Shmarev *Energy Methods for Free Boundary Problems. Applications to nonlinear PDEs and fluid mechanics*, Progress in Nonlinear Differential Equations and Their Applications 48, Birkhäuser, Boston, 2002). Several discrete approximations will be considered.

This class of methods are of special interest in the situations where traditional strategies based on the maximum principle fail. A typical example of such a situation is either a higher-order equation or a system of PDEs. Moreover, even when the comparison principle holds, it may be extremely difficult to construct suitable sub or super-solutions if, for instance, the equations contain transport terms, unbounded coefficients, unbounded right-hand side terms, etc.

In a first part of the lecture, a global energy method will be used to prove the finite extinction time of the solutions of a general class of quasilinear parabolic equations. The finite extinction time of the solution of several numerical discretizations will be also discussed.

The application of local energy methods will be illustrated, in the second part of the lecture, by considering the finite speed of propagation and the waiting time properties for the Boussinesq system modeling the heat conduction in a fluid with a thermal conductivity depending on the temperature. Finally, a different proof of the waiting time phenomenon will be given for the special case of scalar nonlinear parabolic problems by means of the, so called, non-diffusion of the support property associated to a family of stationary problems obtained in the time semidiscretization.

Energy methods for free boundary problems: new results and some remarks on numerical algorithms

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Different *energy methods* have been developed since the beginning of the eighties for the study of the free boundaries giving rise by the solutions of nonlinear partial differential equations (see, e.g., the recent monograph [6] by S.N. Antontsev, J.I. Díaz and S.I. Shmarev). In these notes we present some new applications of such methods and make some remarks on the persistence of the free boundaries for the associated discretized problems.

1 Introduction

Energy methods are of special interest in the study of free boundary problems when the traditional methods based on the comparison principles fail. A typical example of such a situation is either a higher-order equation or a system of pde's. Moreover, even when the comparison principle holds, it may be extremely difficult to construct suitable sub or super-solutions if, for instance, the equation under study contains transport terms, variable coefficients, unbounded right-hand side term, etc. Different energy methods (of local or global spatial nature) have been developed since the beginning of the eighties. They were introduced by S.N. Antontsev [4], improved in [23] and extended by many authors (for a systematic presentation including some not published elsewhere results see the monograph by S.N. Antontsev, J.I. Díaz and S.I. Shmarev [6]). One of the main examples of free boundary problem, which in fact motivated the development of the theory, for which energy methods lead to a rich spectrum of qualitative properties is the nonlinear system for the unknowns (s,p) arising in the study of two-phase filtration of immiscible incompressible fluids in a porous medium

$$\begin{cases} m(x)\frac{\partial s}{\partial t} = \operatorname{div}\left(K_0(x)a(x,s)\nabla s + K_1(x,s)\nabla p + \mathbf{f}_0\right) + q, \\ 0 = \operatorname{div}\left(K(x,s)\nabla p + \mathbf{f}(x,s)\right), \end{cases}$$

(see, for instance, [26] for the modelling and Chapter 4 of [6] for energy methods).

In contrast with my talk (of a more pedagogical nature), the main goal of these notes is to present some new applications of energy methods to a variety of problems such as *the obstacle problem*, a doubly nonlinear parabolic problem arising in Glaciology or the problem to stopping a viscous fluid in a channel by an external field. Besides, motivated by the title of this meeting, some remarks are also presented on the persistence of the free boundaries (local or global in the space variable) for the associated discretized problems. In particular we study the meaning of the so called waiting time and extinction in finite time phenomena in the family of elliptic problems obtained by the implicit (and, in the case of the second property, also semi-implicit and explicit) time discretization processes. Some of the above results give answer to some of the open problems possed in the monograph [6]).

2 The formation of the free boundary for the obstacle problem

One of the limitations of the local energy methods, such as described in [6], was their exclusive application to singlevalued pde's. Nevertheless, it is well known that there is a very large class of multivalued problems, such as the Variational Inequalities, which leads to free boundary formulations. In this Section we present some of the results of [20] concerning the formation of a free boundary for local solutions of the obstacle problem

$$\frac{\partial\psi(u)}{\partial t} - \operatorname{div}\mathbf{A}(x,t,u,Du) + B(x,t,u,Du) + C(x,t,u) + \beta(u) \ni f(x,t), \tag{1}$$

where $\beta(u)$ is the maximal monotone graph given by $\beta(u) = \{0\}$ if $u \geq 0$ and $\beta(u) = \phi$ (the empty set) if $u < 0$. The general structural assumptions we shall made are the following

$$|\mathbf{A}(x,t,r,\mathbf{q})| \leq C_1|\mathbf{q}|^{p-1}, C_2\,|\mathbf{q}|^p \leq \mathbf{A}(x,t,r,\mathbf{q})\cdot\mathbf{q}, \tag{2a}$$

$$|B(x,t,r,\mathbf{q})| \leq C_3\,|r|^\alpha\,|\mathbf{q}|^\beta,\ 0 \leq C(x,t,r)\,r, \tag{2b}$$

$$C_6\,|r|^{\gamma+1} \leq G(r) \leq C_5\,|r|^{\gamma+1},\ \text{where } G(r) = \psi(r)\,r - \int_0^r \psi(\tau)\,d\tau.$$

Here $C_1 - C_6$, p, α, β, σ, γ, k are positive constants which will be specified later on. We shall deal with weak solutions satisfying the initial condition $u(x,0) = u_0(x)$ for $x \in \Omega$.

Definition. A function $u(x,t)$, with $\psi(u) \in C\left([0,T] : L^1_{loc}(\Omega)\right)$, is called weak solution of problem (1) if $u \in L^\infty(0,T; L^{\gamma+1}(\Omega')) \cap L^p(0,T; W^{1,p}(\Omega'))$, $\overline{\Omega'} \subset \Omega$, $\mathbf{A}(\cdot,\cdot,u,Du)$, $B(\cdot,\cdot,u,Du)$, $C(\cdot,\cdot,u) \in L^1(Q)$; $\liminf_{t\to 0} G(u(\cdot,t)) = G(u_0)$ in $L^1(\Omega)$; $u(x,t) \geq 0$ and $c(x,t) \in \beta(u(t,x))$ a.e. $(t,x) \in (0,T) \times \Omega$ for some $c \in L^1((0,T)\times\Omega)$, and for every test function $\varphi \in L^\infty\left(0,T; W_0^{1,p}(\Omega)\right) \cap W^{1,2}(0,T; L^\infty(\Omega))$,

$$\int_Q \{\psi(u)\varphi_t - \mathbf{A}\cdot D\varphi - B\,\varphi - C\,\varphi - c\varphi\}\,dxdt - \int_\Omega \psi(u)\varphi\,dx\Big|_{t=0}^{t=T} = -\int_Q f\,\varphi\,dx\,dt. \tag{3}$$

Let us mention that the study of the qualitative behavior of the coincidence set for the parabolic obstacle problem was initiated in 1976 by [16], L. Tartar (personal communication, 1976) and [24] by using the maximum principle. In contrast to considerations on the *finite speed of propagation* (see Section 3) we shall use here some energy functions defined on domains of a special form: given $x_0 \in \Omega$ and the nonnegative parameters ϑ and v, we define the *energy set* $P(t) = \{(x,s)\ |x - x_0| < \rho(s) \equiv \vartheta(s-t)^v, s \in (t,T)\}$. The shape of $P(t)$, *the local energy set*, is determined by the choice of the parameters ϑ and v. Here we shall take $\vartheta > 0$, $0 < v < 1$ and so $P(t)$ becomes a paraboloid (other choices are relevant for the study of different properties: see [6]). We define the *local energy functions*

$$E(P) := \int_{P(t)} |Du(x,\tau)|^p\,dxd\tau, \quad C(P) := \int_{P(t)} |u(x,\tau)|\,dxd\tau$$

$$b(T) := ess\ \sup_{s\in(t,T)} \int_{|x-x_0|<\vartheta(s-t)^v} |u(x,s)|^{\gamma+1}\,dx.$$

Although our results have a local nature (for instance, they are independent of the boundary conditions), we shall need some information on *the global energy function*

$$D(u(\cdot,\cdot)) := ess\ \sup_{s\in(0,T)} \int_\Omega |u(x,s)|^{\gamma+1}\,dx + \int_Q \left(|Du|^p + |u|\right)dxdt. \tag{4}$$

For the sake of simplicity in the exposition, we shall assume the additional condition $\frac{p-1}{p} \leq \gamma \leq p-1$. Our main assumption deals with the forcing term: we assume that there exists $\Theta > 0$ and $\rho > 0$ such that

$$f(x,t) < -\Theta \text{ on } B_\rho(x_0) \subset \Omega, \text{ a.e. } t \in (0,T). \tag{5}$$

In the presence of a first order term, $B(\cdot,\cdot,u,Du)$, we shall need the extra conditions

$$\begin{cases} \alpha = \gamma - (1+\gamma)\beta/p, C_3 < \left(\Theta\frac{p}{p-1}\right)^{(p-\beta)/p}\left(C_2\frac{p}{\beta}\right)^{\beta/p} \text{ if } 0 < \beta < p, \\ C_3 < \Theta \text{ if } \beta = 0 \text{ (respectively } \Theta < C_2 \text{ if } \beta = p). \end{cases} \tag{6}$$

The next result shows how the multivalued term causes the formation of the null-set of the solution, even for positive initial data.

Theorem 1 *There exist some positive constants M, t^*, and $v \in (0,1)$ such that any weak solution of problem (1) with $D(u) \leq M$ satisfies that $u(x,t) \equiv 0$ in $P(t^* : 1, v)$.*

The proof of Theorem 1 consists of several parts: Step 1. *The integration-by-parts formula:*

$$i_1 + i_2 + i_3 + i_4 = \int_{P\cap\{t=T\}} G(u(x,t))dx + \int_P \mathbf{A}\cdot Du\, dxd\theta + \int_P B\, udxd\theta + \int_P C\, udxd\theta -$$
$$\int_P uf dxd\theta \le \int_{\partial_l P} n_x \cdot \mathbf{A}\, u\, d\Gamma d\theta + \int_{\partial_l P} n_\tau G(u(x,t))d\Gamma d\theta + \int_{P\cap\{t=0\}} G(u(x,t))dx = j_1 + j_2 + j_3,$$

were $\partial_l P$ denotes the lateral boundary of P i.e. $\partial_l P = \{(x,s) : |x-x_0| = \vartheta(s-t)^v,\ s\in(t,T)\}$, $d\Gamma$ is the differential form on the hypersurface $\partial_l P \cap \{t = const\}$, n_x and n_τ are the components of the unit normal vector to $\partial_l P$. This inequality can be proved by taking the cutting function $\zeta(x,\theta)=\psi_\varepsilon(|x-x_0|,\theta)\,\xi_k(\theta)\frac{1}{h}\int_\theta^{\theta+h} T_m(u(x,s))\,ds$ as test function,where T_m is the truncation at the level m,

$$\xi_k(\theta) := \begin{cases} 1 & if\ \theta\in[t, T-\frac{1}{k}], \\ k(T-\theta) & for\ \theta\in[T-\frac{1}{k}, T], \\ 0 & otherwise,\quad k\in\mathbb{N}, \end{cases} \qquad \psi_\varepsilon(|x-x_0|,\theta) := \begin{cases} 1 & if\ d>\varepsilon, \\ \frac{1}{\varepsilon}d & if\ d<\varepsilon, \\ 0 & otherwise, \end{cases}$$

with $d = dist((x,\theta),\partial_l P(t))$ and $\varepsilon > 0$. So, $supp\zeta(x,\theta) \equiv P(t)$, $\zeta, \frac{\partial\zeta}{\partial t} \in L^\infty((0,T)\times\Omega)$ and $\frac{\partial\zeta}{\partial x_i} \in L^p((0,T)\times\Omega)$. Using the monotonicity of β and passing to the limits we get the inequality.

Step 2. *A differential inequality for some energy function.* We assume choice P such that it does not touch the initial plane $\{t=0\}$ and $P\subset B_\rho(x_0)\times[0,T]$. Then $i_1+i_2+i_3 \le j_1+j_2$. In order to estimate j_1, let us mention that $\mathbf{n} = (\mathbf{n}_x, \mathbf{n}_\tau) = \frac{1}{(\vartheta^2 v^2 + (\theta-t)^{2(1-v)})^{1/2}}((\theta-t)^{1-v}\mathbf{e}_x - v\mathbf{e}_\tau)$ with $\mathbf{e}_x, \mathbf{e}_\tau$ orthogonal unit vectors to the hyperplane $t=0$ and the axis t, respectively. Then, if we denote by (ρ,ω), $\rho\ge 0$ and $\omega\in\partial B_1$, the spherical coordinate system in $\mathbb{R}^N$ and if $\Phi(\rho,\omega,\theta)$ is the spherical representation of a general function $F(x,t)$, we have

$$I(t) := \int_P F(x,\theta)dxd\theta \equiv \int_t^T d\theta \int_0^{\rho(\theta,t)} \rho^{N-1}d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega,$$

where J is the Jacobi matrix and $\rho(\theta,t) = \vartheta(\theta-t)^v$. So,

$$\frac{dI(t)}{dt} = -\int_0^{\rho(\theta,t)} \rho^{N-1}d\rho \int_{\partial B_1} \Phi(\rho,\omega,\theta)|J|d\omega\Big|_{\theta=t} + \int_t^T \rho_t \rho^{N-1}d\theta \int_{\partial B_1} \Phi(\rho,\omega,t)|J|d\omega = \int_{\partial_l P} \rho_t F(x,\theta)d\Gamma d\theta. \tag{7}$$

Then, by Hölder's inequality, we get

$$\left|\int_{\partial_l P} n_x\cdot\mathbf{A}\, u\, d\Gamma d\theta\right| \le M_2\left(-\frac{dE}{dt}\right)^{(p-1)/p}\left(\int_t^T \frac{|n_x|^p}{|\rho_t|^{p-1}}\left(\int_{\partial B_{\rho(\theta,t)}} |u|^p d\Gamma\right)d\theta\right)^{1/p}. \tag{8}$$

To estimate the right-hand side of this inequality we use the following interpolation inequality ([6]) : if $0\le\sigma\le p-1$, then there exists $L_0>0$ such that $\forall\, v\in W^{1,p}(B_\rho)$

$$\|v\|_{p,S_\rho} \le L_0\left(\|\nabla v\|_{p,B_\rho} + \rho^\delta\|v\|_{\sigma+1,B_\rho}\right)^{\tilde\theta}\cdot\left(\|v\|_{r,B_\rho}\right)^{1-\tilde\theta} \tag{9}$$

$r\in[1,1+\gamma]$, $\tilde\theta = \frac{pN-r(N-1)}{(N+1)p-Nr}$, $\delta = -\left(1+\frac{p-1-\sigma}{p(1+\sigma)}N\right)$. Then, by Hölder's inequality

$$\int_{\partial B_\rho} |u|^p d\Gamma \le K\rho^{\delta\tilde\theta p}\left(E_* + C_*\right)^{\tilde\theta + (1-\tilde\theta)p/qr}\, b^{(q-1)(1-\tilde\theta)p/qr}, \tag{10}$$

where $E_*(t,\rho) := \int_{B_\rho}|\nabla u|^p dx$, $C_*(t,\rho) := \int_{B_\rho}|u|dx$ and K is a suitable positive constant. Taking $r\in\left[\frac{p(\gamma+1)}{p+\gamma}, \gamma+1\right]$ we get that $\mu = \tilde\theta + p\frac{1-\tilde\theta}{qr} < 1$. Applying once again Hölder's inequality with the exponent μ, we have from (10)

$$|j_1| \le L\sigma(t)\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p} b^{(q-1)(1-\tilde\theta)/qr}\,(E+C)^{\frac{\tilde\theta}{p}+\frac{1-\tilde\theta}{qr}} \tag{11}$$

for a suitable positive constant L. We assumed $\sigma(t) := \left(\int_t^T\left(\frac{1}{|\rho_t|^{p-1}}\rho^{\delta\tilde\theta p}(\tau)\right)^{\frac{1}{1-\mu}} d\tau\right)^{\frac{1-\mu}{p}} < \infty$, which is fulfilled if we choose $\nu\in(0,1)$ sufficiently small because the condition of convergence of the integral $\sigma(t)$ has the form $(1-\nu)(p-1)+\nu\delta\tilde\theta p > -(1-\tilde\theta)\left(1-\frac{p}{qr}\right)$. So, we have obtained an estimate of the following type

$$|j_1| \le L_1\Lambda(t)D(u)^{(q-1)(1-\tilde\theta)/qr-\lambda}(E+C+b)^{1-\omega+\lambda}\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p}, \tag{12}$$

where L_1 is a universal positive constant, $D(u)$ is *the total energy* of the solution under investigation, $\lambda\in[0,(q-1)(1-\tilde\theta)/qr]$ and $\omega := 1-\frac{\tilde\theta}{p}-\frac{1-\tilde\theta}{qr} \in \left(1-\frac{1}{p},1\right)$. This allows us to choose λ so that $\frac{p(\omega-\lambda)}{p-1}\in(0,1)$. Let us estimate j_2. Using the expression for n_τ, we have $|j_2| \le C_5\int_{\partial_l P}|u|^{1+\gamma}d\Gamma d\theta$. We apply then the interpolation inequality (for the limit case $\sigma=0$)

$$\|v\|_{\gamma+1,\partial B_\rho} \le L_0\left(\|\nabla v\|_{p,B_\rho} + \rho^\delta\|v\|_{\sigma+1,B_\rho}\right)^s\cdot\|v\|_{r,B_\rho}^{1-s}\quad \forall\, v\in W^{1,p}(B_\rho) \tag{13}$$

with a universal positive constant $L_0>0$ and exponents $s = \frac{(\gamma+1)N-r(N-1)}{(N+r)p-Nr}\frac{p}{\gamma+1}$, $r\in[1+\sigma,1+\gamma]$. Again

$$\int_{\partial B_\rho}|u|^{\gamma+1}dx \le L^{1+\gamma}K^{s(\gamma+1)/\tilde\theta p}\left(\int_{B_\rho}|\nabla u|^p dx + \int_{B_\rho}|u|^{\sigma+1}dx\right)^{s(\gamma+1)/p} \times\left[\left(\int_{B_\rho}|u|^{\sigma+1}dx\right)^{1/qr}\left(\int_{B_\rho}|u|^{\gamma+1}dx\right)^{(q-1)/qr}\right]^{(1-s)(\gamma+1)}. \tag{14}$$

Here K is the same as before. Let $\eta = \frac{s(\gamma+1)}{p} + \frac{(1-s)(\gamma+1)}{qr} < 1$, $\pi = \frac{(q-1)(1-s)(\gamma+1)}{qr}$, $\eta+\pi\ge 1$. Then,

$$|j_2| \le L\,(E+C+b(T,\Omega))\,(b(T,\Omega))^\kappa\left(\int_t^T\left(K^{s(\gamma+1)/\tilde\theta p}\right)^\varepsilon d\tau\right)^{1/\varepsilon}, \tag{15}$$

for some $L = L(C_5, L_0)$ and exponents $\kappa := \eta+\pi-1$, $\varepsilon = 1/(1-\eta)$. Then. we have

$$C_5\int_{P\cap\{t=T\}}|u|^{1+\gamma}dx + E + C\Theta \le i_1+i_2+i_3, \tag{16}$$

$$|i_4| \le \varepsilon C_3\frac{p-\beta}{p}C(\rho,t) + \frac{\beta C_3}{pC_2}\varepsilon^{-(p-\beta)/\beta}E(\rho,t), K\left(\int_{P\cap\{t=T\}}|u|^{1+\gamma}dx + E + C\right) \le i_1+i_2+i_3+i_4, \tag{17}$$

for different positive constants K. Now, assuming that $L\,(b(T,\Omega))^\kappa\left(\int_t^T\left(K^{s(\gamma+1)/\tilde\theta p}\right)^\varepsilon d\tau\right)^{1/\varepsilon} < \frac{K}{2}$ we arrive to

$$E + C + b(T,\Omega) \le L_1\Lambda(t)D(u)^{(q-1)(1-\tilde\theta)/qr-\lambda}(E+C+b(T,\Omega))^{1-\omega+\lambda}\left(-\frac{d(E+C)}{dt}\right)^{(p-1)/p} \tag{18}$$

whence the desired differential inequality for the energy $Y(t) := E + C$, $Y^{(\omega-\lambda)p/(p-1)}(t) \leq c(t)\left(-Y(t)\right)'$,where $c(t) = \left(L_1 \left(D(u)\right)^{(q-1)(1-\tilde{\theta})/q r-\lambda}\sigma(t)\right)^{p/(p-1)}$. Notice that $c(t) \to 0$ as $t \to T$. Moreover, the exponent $(\omega - \lambda)\frac{p}{p-1}$ belongs to the interval $(0, 1)$ which leads to the result (see [6]). ■

Remark 1 *The present technique can be applied to the study of other properties such as the finite speed of propagation, the shrinking of the support, the waiting time or the study of locally vanishing solutions of the associate stationary obstacle problem and of other multivalued problems such as the Stefan problem.*

3 The finite speed of propagation and the waiting time properties for a doubly nonlinear parabolic problem arising in Glaciology

The assumptions on $B(x, u, Du)$ can be improved. We need a sharper argument if we consider, for instance, the problem

$$\left\{\begin{array}{rll} b(u)_t - [\mu\phi(u_x) - u_b b(u)]_x + \beta(u) & \ni a(t,x) & \text{in } Q, \\ u(t,x) & = 0 & \text{on } \Sigma, \\ b(u(0,x)) & = b(u_0(x)) & \text{on } \Omega, \end{array}\right. \tag{19}$$

with $Q = (0,T) \times \Omega$, $\Sigma = \partial Q = (0,T) \times \partial\Omega$, $h = b(u) = u^{1/m}, m = 2(n+1)/n, \phi(u_x) = |u_x|^{p-2} u_x$, $p = n+1, \mu = n^n/[2^n(n+1)^n(n+2)]$ and n represents the so called *Glen's exponent*, usually assumed $n \approx 3$ (but here merely assumed $n > 1$). Function $a = a(t,x)$ is a scaled fixed given accumulation-ablation rate function ($a < 0$ means ablation) and u_b is a (given) function representing the basal velocity. The model was proposed in [25] describing the evolution of the thickness $h(t,x)$ for a two-dimensional plane ice sheet ($z = h(t,x)$ is the top surface of the ice sheet). The mathematical analysis, the qualitative properties and the numerical study of such model was carried out in [17].

When the ice sheet is warm-based, the bed is then temperate and sliding is prescribed (i.e. $u_b = u_b(t,x) \not\equiv 0$). The presence of the convection term makes the method of super and subsolutions very hard to be applied. Thus, in order to prove the existence of the free boundary we shall use the *energy method* (we follows some ideas introduced in [21]). Notice that the equation can be written in terms of a non-conservative transport multivalued equation $b(u)_t + u_b b(u)_x - \mu\phi(u_x)_x + (u_b)_x b(u) + \beta(u) \ni a(t,x)$. In this way, the equation involves the material derivative $b(u)_t + u_b b(u)_x$ which can be associate to a *virtual non-Newtonian fluid with a reactive term* $(u_b)_x b(u) + \beta(u)$. We shall prove the existence of the free boundary in terms of the, so called, *finite speed of propagation* near a given point x_0. We shall assume that u_b is a globally Lipschitz continuous function. So, we can define the characteristics of the associate flow by

$$\frac{d}{dt}X(t,x) = u_b(t, X(t,x)), \quad X(0,x) = x. \tag{20}$$

As usual in Continuum Mechanics, given a ball $B_\rho(x_0) = \{x \in I\!R : |x - x_0| \leq \rho\}$ we denote the transformed set by $B_\rho(x_0)_t = \{\, y \in I\!R : y = X(t,x) \quad \text{for some} \quad x \in B_\rho(x_0)\}$.

Theorem 2 *Let u_b be a globally Lipschitz continuous function on Q. For $\epsilon \geq 0$ let $N_\epsilon(a(t,\cdot)) := \{(t,x) \in \{t\} \times \Omega / \, a(t,x) \leq -\epsilon\}$. Assume also that $\epsilon = 0$ if $m(p-1) > 1$ and $\epsilon > 0$ if $m(p-1) \leq 1$. Let $u_0 = 0$ on a ball $B_{\rho_0}(x_0)$ for some x_0 such that $(t, B_{\rho_T}(x_0)) \subset N_\epsilon(a(t,\cdot))$ for any $t \in [0,T]$ and some $L \geq \rho_0$. Then there exists a $T_\epsilon \in (0,T]$ and a function $\rho : [0, T_\epsilon] \to [0, \rho_0]$ such that $u(t,x) = 0$ a.e. $x \in B_{\rho(t)}(x_0)$ for any $t \in [0, T_\epsilon]$.*

For the proof, we introduce the change of variable $b(w(t,x)) = b(u(t,x))e^{\lambda t}$. Then

$$b(w)_t + u_b b(w)_x - \mu e^{\lambda t(1-(p-1)m)}\phi(w_x)_x + [(u_b)_x + \lambda]b(w) + \beta(w) \ni a(t,x)e^{\lambda t}, \quad \text{in} \quad Q.$$

We take $\lambda > 2C$ with $C = ||(u_b)_x||_{L^\infty(Q)}$. By multiplying, formally, by w we get that if $\rho \leq L$ then

$$\int_{B_\rho(x_0)_t} \frac{\partial}{\partial t}\Psi(w)dx + \int_{B_\rho(x_0)_t} u_b \Psi(w)_x dx + \mu e^{\lambda t(1-(p-1)m)} \int_{B_\rho(x_0)_t} |w_x|^p \, dx \leq$$

$$\leq \mu e^{\lambda t(1-(p-1)m)} w(t,\cdot) \left|w_x(t,\cdot)\right|^{p-1} w_x(t,\cdot) \left|_{\partial B_\rho(x_0)_t}\right. - \epsilon \int_{B_\rho(x_0)_t} w dx$$

where $\Psi(w) := wb(w) - \int_0^w b(s)ds$. Now, by using the Reynolds Transport Lemma

$$\int_{B_\rho(x_0)_t} \frac{\partial}{\partial t}\Psi(w) + \int_{B_\rho(x_0)_t} u_b \Psi(w)_x = \frac{d}{dt}\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy.$$

Thus, integrating in $(0,t)$ and using the information on u_0 we get that

$$\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy + C_1 \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds \leq$$

$$\leq C_2 \int_0^t w(s,\cdot)\left|\left|w_x(s,\cdot)\right|^{p-1} w_x(s,\cdot) \left|_{\partial B_\rho(x_0)_t}\right.\right| ds - \epsilon \int_0^t \int_{B_\rho(x_0)_t} w dx ds \tag{21}$$

with $C_1 = \mu \min_{t\in[0,T]} e^{\lambda t(1-(p-1)m)}$, $\quad C_2 = \mu \max_{t\in[0,T]} e^{\lambda t(1-(p-1)m)}$. Assume now, for the moment, that $1 < (p-1)m$ and $\epsilon = 0$. Then we define the energies

$$B(t,\rho) = \sup_{0\leq s\leq t} \int_{B_\rho(x_0)_s} \Psi(w(s,y))dy, \quad E(t,\rho) = \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds. \tag{22}$$

Using Hölder inequality and the interpolation-trace inequality, we get that

$$B + E \leq K\left(\frac{\partial E}{\partial \rho}\right)^\omega \tag{23}$$

for some positive constant K and some $\omega > 1$ and the result follows in a standard way (see, e.g. [6]). In the case $1 \geq (p-1)m$ and $\epsilon > 0$ we pass the term $\epsilon \int_0^t \int_{B_\rho(x_0)_t} w dx ds$ to the left hand side of the inequality (21) and we introduce the additional energy function defined as

$$C(t,\rho) = \int_0^t \int_{B_\rho(x_0)_t} |w| \, dyds$$

(remember that $|w| = w$). Then, we can apply the interpolation-traze theorem of [6] to arrive to the inequality

$$E + C \leq K\left(\frac{\partial(E + C)}{\partial \rho}\right)^\omega \tag{24}$$

for some positive constant K and some $\omega > 1$ and the theorem holds.

Remark 2 *Notice that, in contrast with the case $u_b = 0$, it may occurs that $T_\epsilon < T$ for any $\epsilon \geq 0$. Moreover, any estimate of the function $\rho(t)$ automatically gives an estimate on the location of the free boundary. Finally, we indicate that it is possible to get global consequences of the above result by estimating (globally) the energies introduced in (22) (for some related arguments see, e.g. [6]).*

The waiting time property can be also studied by energy methods once it is reformulated in terms of the characteristics associated to u_b. Notice that if $u_b \equiv 0$ then the characteristics are vertical lines.

Theorem 3 *Let b, ϕ, β, a, u_b, ϵ, $N_\epsilon(a(t,\cdot))$ and x_0 be as in the previous theorem but now with $L > \rho_0$. Let $u_0(x) = 0$ on a ball $B_{\rho_0}(x_0)$ for some x_0 and satisfying that*

$$\int_{B_\rho(x_0)_t} \Psi(u_0(y))dy \leq \theta[(\rho-\rho_0)^+]^{\omega/(\omega-1)}, \quad \text{for any} \quad \rho_0 \leq \rho \leq L \tag{25}$$

for some small enough $\theta > 0$ and some $L > \rho_0$ with $\omega > 1$ the exponent given in (23) or (24). Then, there exists $T_0 \in (0,T]$ such that $u(t,x) = 0$ a.e. $x \in B_{\rho_0}(x_0)_t$ for any $t \in [0,T_0]$.

For the proof, the integration by parts formula (21) must be replaced by

$$\int_{B_\rho(x_0)_t} \Psi(w(t,y))dy + C_1 \int_0^t \int_{B_\rho(x_0)_t} |w_x|^p \, dyds \leq$$
$$\int_0^t w(s,\cdot) \left| |w_x(s,\cdot)|^{p-1} w_x(s,\cdot) \,|_{\partial B_\rho(x_0)_t} \right| ds - \epsilon \int_{B_\rho(x_0)_t} wdx + \int_{B_\rho(x_0)_t} \Psi(u_0(y))dy.$$

In particular, inequality (23) becomes the non homogeneous one

$$B + E \leq K \left(\frac{\partial E}{\partial \rho}\right)^\omega + \theta(\rho-\rho_0)_+^{\omega/(\omega-1)},$$

and the conclusion holds thanks to a technical lemma (see, [6]).

The numerical study of the ice sheet moving boundary problem was carried out also in [17]. An overview about different numerical strategies to solve free boundary problems (fixed domain methods, front-tracking and front-fixing methods, adaptative algorithms and others) can be found in [36]. The approach of [17] was based on fixed domain methods, upwinding time discretization and duality methods for nonlinearities. We first introduce the total derivative notation in conservative form $\frac{Dh}{Dt} = \frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(u_b h)$. As usual in Glaciology, we consider now

$$\text{Problem (19) with } n = 3\text{: so } m = 2(n+1)/n = 8/3 \text{ and } p = n+1 = 4. \tag{26}$$

Problem (26) is discretized in time using the scheme of characteristics. For this, let T and M be fixed and let Δt be the time step so that $T = M\Delta t$. Our upwinded time scheme is based on the approximation of the total derivative (see [37] for linear convection-diffusion equations). In our case, for $m = 0, 1, \ldots, M$, we consider the approximation:

$$\frac{D}{Dt}(u^{3/8})((m+1)\Delta t, x) \approx \frac{(u^{m+1})^{3/8}(x) - J^m(x)\,(u^m)^{3/8}(\chi^m(x))}{\Delta t} \tag{27}$$

where $u^{m+1}(x) = u((m+1)\Delta t, x)$ and $J^m(x)$ is obtained by numerical quadrature techniques in the expression $J^m(x) = J(t^{m+1}, x; t^m) = 1 - \int_{t^m}^{t^{m+1}} (u_b(\tau, \chi(x, t^{m+1}; \tau)))_x \, d\tau$, where J is the Jacobian associated to the change of variable mapping $x \to \chi(t, x; \tau)$. The value $\chi^m(x)$ is given by $\chi^m(x) = \chi((m+1)\Delta t, x; m\Delta t)$, χ being the solution of the final value problem $\frac{d\chi(t,x;s)}{ds} = u_b(s, \chi(x,t;s))$, $\chi(t,x;t) = x$. Next step consists of the substitution of the approximation (27) in (26) to obtain the following sequence of nonlinear elliptic problems: *for* $m = 0, 1, 2, \ldots, M$, *find* u^{m+1} *such that:*

$$\begin{cases} \dfrac{(u^{m+1})^{3/8} - J^m((u^m)^{3/8} \circ \chi^m)}{\Delta t} - \mu \dfrac{\partial}{\partial x}\left(|u_x^{m+1}|^2 u_x^{m+1}\right) - a^{m+1} \geq 0, u^{m+1} \geq 0, \text{ in } \Omega \\ \left[\dfrac{(u^{m+1})^{3/8} - J^m((u^m)^{3/8} \circ \chi^m)}{\Delta t} - \mu \dfrac{\partial}{\partial x}\left(|u_x^{m+1}|^2 u_x^{m+1}\right) - a^{m+1}\right] u^{m+1} = 0 \text{ in } \Omega \\ u^{m+1} = 0 \text{ in } \partial\Omega, \ u^0(x) = h_0 = (h_0)^{8/3} \text{ in } \Omega, \end{cases} \tag{28}$$

where $a^{m+1}(x) = a((m+1)\Delta t, x)$ and $\circ$ denotes the composition symbol.

In order to present the spatial discretization and solve the nonlinear problem (28) to obtain u^{m+1}, we pose the following equivalent variational inequality formulation: *find* $u^{m+1} \in K$ *such that*

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega (u^{m+1,})^{3/8}\,(\varphi - u^{m+1})\,dx + \mu \int_\Omega |\,u_x^{m+1}\,|^2\, u_x^{m+1}(\varphi - u^{m+1})_x dx \geq \\ &\frac{1}{\Delta t}\int_\Omega J^m\,((u^m)^{3/8} \circ \chi^m)(\varphi - u^{m+1})\,dx + \int_\Omega a^{m+1}\,(\varphi - u^{m+1})dx, \qquad \forall \varphi \in K, \end{aligned} \tag{29}$$

where $K = \{\varphi \in W_0^{1,4}(\Omega) \,/\, \varphi \geq 0 \text{ a.e. in } \Omega\}$. Next, the duality algorithm proposed in [10] is applied to the variational inequality (29). For this, (29) is expressed in terms of the indicatrix function, I_K, of the convex K in the form: *find* $u^{m+1} \in W_0^{1,4}(\Omega)$ *such that*

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega (u^{m+1})^{3/8}(\varphi - u^{m+1}))dx + \mu \int_\Omega |\,u_x^{m+1}\,|^2\, u_x^{m+1}(\varphi - u^{m+1})_x dx + I_K(\varphi) - I_K(u^{m+1}) \\ &\geq \frac{1}{\Delta t}\int_\Omega J^m\,((u^m)^{3/8} \circ \chi^m)(\varphi - u^{m+1})dx + \int_\Omega a^{m+1}(\varphi - u^{m+1})dx, \qquad \forall \varphi \in W_0^{1,4}(\Omega). \end{aligned} \tag{30}$$

Moreover, the use of subdifferentials leads to the formulation $\xi_1^{m+1} = -(\mathcal{A}(u^{m+1})\text{-}f^m) \in \partial I_K(u^{m+1})$ where $\partial I_K(u)$ denotes the subdifferential of the convex function I_K at the point u (see [15]). Here, the operator $\mathcal{A} : W_0^{1,4}(\Omega) \to W^{-1,4/3}(\Omega)$ and the product $< f^m, \psi >$ are defined by

$$< \mathcal{A}(\varphi), \psi > = \frac{1}{\Delta t}\int_\Omega \varphi^{3/8}\psi dx + \mu \int_\Omega |\,\varphi_x\,|^2\, \varphi_x \psi_x dx,$$

$$< f^m, \psi > = \int_\Omega a^{m+1}\psi dx + \frac{1}{\Delta t}\int_\Omega J^m((u^m)^{3/8} \circ \chi^m)\psi dx.$$

Therefore, we arrive to the problem: *find* $u^{m+1} \in W_0^{1,4}(\Omega)$ *such that*

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega (u^{m+1})^{3/8}\,\psi\,dx + \int_\Omega \xi_1^{m+1}\,\psi\,dx + \mu \int_\Omega \xi_2^{m+1}\psi_x dx - \\ &\frac{1}{\Delta t}\int_\Omega J^m((u^m)^{3/8} \circ \chi^m)\psi dx = \int_\Omega a^{m+1}\,\psi\,dx, \qquad \forall \psi \in W_0^{1,4}(\Omega) \end{aligned} \tag{31}$$

$$\xi_1^{m+1} \in \partial I_K\left[u^{m+1}\right], \xi_2^{m+1} = \Lambda\left(\frac{\partial u^{m+1}}{\partial x}\right), \text{ where } \Lambda(v) = |\,v\,|^2\, v = v^3. \tag{32}$$

The application of Bermúdez-Moreno algorithm [10] to solve the above nonlinear problem introduces the following new unknowns (multipliers) q_1^{m+1} and q_2^{m+1}:

$$q_1^{m+1} \in \partial I_K\left[u^{m+1}\right] - \omega_1 u^{m+1}, \ q_2^{m+1} = \Lambda\left(\frac{\partial u^{m+1}}{\partial x}\right) - \omega_2 \frac{\partial u^{m+1}}{\partial x}, \tag{33}$$

defined in terms of the positive parameters ω_1 and ω_2. So, we arrive to

$$\begin{aligned} &\frac{1}{\Delta t}\int_\Omega (u^{m+1})^{3/8}\psi dx + \int_\Omega (q_1^{m+1} + \omega_1 u^{m+1})\,\psi\,dx + \\ &\mu \int_\Omega \left(q_2^{m+1} + \omega_2 \frac{\partial u^{m+1}}{\partial x}\right) \frac{\partial \psi}{\partial x}\,dx = \int_\Omega a^{m+1}\,\psi\,dx + \frac{1}{\Delta t}\int_\Omega J^m\,((u^m)^{3/8} \circ \chi^m)\psi dx, \end{aligned}$$

$\forall \psi \in W_0^{1,4}(\Omega)$. Now, since ∂I_K and Λ are maximal monotone operators, the above definitions can be characterized by their respective identities [10]:

$$q_1^{m+1} = (\partial I_K)_{\lambda_1}^{\omega_1}\left[u^{m+1} + \lambda_1 q_1^{m+1}\right], \ q_2^{m+1} = \Lambda_{\lambda_2}^{\omega_2}\left[\frac{\partial u^{m+1}}{\partial x} + \lambda_2 q_2^{m+1}\right]. \tag{34}$$

Functions $(\partial I_K)^{\omega_1}_{\lambda_1}$ and $\Lambda^{\omega_2}_{\lambda_2}$ denote the Yosida approximations for the operators $(\partial I_K - \omega_1 I)$ and $(\Lambda - \omega_2 I)$ with positive parameters λ_1 and λ_2, respectively (see [15]). For details on the fully discretized problem and some concrete numerical results with comparison tests we send the reader to [17].

Remark 3. *The waiting time property (see Theorem 3) can be proved by means of local comparison arguments in some special cases. This is the case of local solutions of the problem*

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta(|u|^{m-1} u) & in \quad \Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x) & on \quad \Omega, \end{cases} \tag{35}$$

where $\Omega \subset R^N$ is an open set, $m > 1$ and, for simplicity, $u_0(x)$ is a non-negative bounded function. Notice that no boundary condition is prescribed. It is well-known (see, e.g., [30]) that if the initial datum $u_0(x)$ satisfies the growing condition

$$u_0(x) \le C_0 \|x - x_0\|^{\frac{2}{m-1}} \quad in \quad \|x - x_0\| < \delta, \tag{36}$$

for $x_0 \in \partial(supp\ u_0)$ and for some nonnegative constants C_0, δ, then there exists a waiting time $t^* > 0$ such that $u(x_0, t) = 0$, for $0 \le t \le t^*$. This local property can be justified also for weak solutions which are not continuous and it is the key point to obtain global statements in terms of the boundary of the support of u_0. In a short note ([2], see also [1]) we studied the question of when the waiting time property is preserved if we discretize in time equation (35) by the iterative scheme

$$\frac{u_n - u_{n-1}}{\tau} = \Delta u_n^m \quad in \quad \Omega \quad and \quad n \in \mathbb{N}, \tag{37}$$

where $u_n(x)$ represents an approximation of the solution $u(x,t)$ of (35) at $t_n = n\tau$. It turns out that the analogous property to the existence of a waiting time is the nondiffusion of the support (see [19]) of the solutions of (37) in the sense that: there exists $\tau_0, t^* > 0$ such that if $0 < \tau < \tau_0$ then $u_n(x_0) = 0$ for $n \in [0, \frac{t^*}{\tau})$. A fundamental ingredient of the proof is a technical lemma assuring that if $m > 1$, $0 < s \le (\frac{1}{m})^{\frac{1}{m-1}}$ and $\phi(s) = s - s^m$, then $\lim_{n\to\infty} \phi^n(s) n^{\frac{1}{m-1}} = \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}$, where $\phi^n(s) = \overbrace{\phi \circ \phi \circ \ldots\ldots \circ \phi}^{n}(s)$.

4 Stopping a viscous fluid by an external feedback field

As mentioned in the above section, the support of the solution of many stationary problems remains compact once that the source term has compact support. This is also the case in the study of the stopping a planar stationary flow of a incompressible viscous fluid, of velocity $\mathbf{u}(\mathbf{x}) = (u(x,y), v(x,y))$, $\mathbf{x} = (x,y) \in \Omega$, in a semi-infinite strip $\Omega = (0,+\infty) \times (0,L)$, $L > 0$, with an non-zero velocity at the strip entrance

$$\mathcal{P}(\Omega, u_*, \mathbf{f}) \begin{cases} -\nu\Delta\mathbf{u} = \mathbf{f} - \nabla p, \mathrm{div}\mathbf{u} = 0 & \text{in } \Omega, \\ \mathbf{u}(0,y) = \mathbf{u}_*(y), y \in (0,L), & \mathbf{u}(x,0) = \mathbf{u}(x,L) = \mathbf{0},\ x \in (0,+\infty), \\ |\mathbf{u}(x,y)| \to 0, \text{ as } x \to +\infty \text{ and} & y \in (0,L). \end{cases}$$

The main question now is: *can we find an external localized force field* $\mathbf{f}$ *stopping the fluid at a finite distance: i.e. such that* $\mathbf{u}(x,y) = \mathbf{0}$ for $x \ge x_{\mathbf{u}}$ and $y \in (0,L)$ *for some* $x_{\mathbf{u}} > 0$? In the following, we shall denote this property as the *localization effect*. Here, the localization of the external field must be understood in the sense that we search a field $\mathbf{f}$ such that $\mathbf{f}(x,y) = \mathbf{0}$ for $x \ge x_{\mathbf{f}}$ and $y \in (0,L)$, for some $x_{\mathbf{f}} > 0$.We recall that due to the incompressibility condition, the first component of $\mathbf{u}_* = (u_*(y), v_*(y))$ must satisfy $\int_0^L u_*(s)ds = 0$. We also assume the compatibility conditions $\mathbf{u}_*(0,\ 0) = \mathbf{u}_*(0,\ L) = \mathbf{0}$. Although it is well known (see, e.g. [27]) that for the classical Stokes problem (i.e. with $\mathbf{f} = \mathbf{f}(\mathbf{x})$ prescribed) the localization effect fails, it is possible to show ([5]) show the *localization effect* when we assume the external body forces be given in a feedback form, $\mathbf{f} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{f}(\mathbf{x},\mathbf{u}) = (f_1(\mathbf{x},\mathbf{u}), f_2(\mathbf{x},\mathbf{u}))$, and such that, for every $\mathbf{u} \in \mathbb{R}^2$ and for a.e. $\mathbf{x} \in \Omega$,

$$-\mathbf{f}(\mathbf{x},\mathbf{u}) \cdot \mathbf{u} \ge \delta\chi_{(0,x_{\mathbf{f}})}(\mathbf{x})|u(\mathbf{x})|^{1+\sigma} + \mathbf{g}(\mathbf{x}) \cdot \mathbf{u} \tag{38}$$

for some $\delta > 0$, $0 < \sigma < 1$ and $\mathbf{g} \in \mathbf{L}^2(\Omega^{x_{\mathbf{g}}})$, $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ a.e. in $(x_{\mathbf{g}}, +\infty) \times (0,L)$ for some $x_{\mathbf{f}}$, $x_{\mathbf{g}}$ with $0 \le x_{\mathbf{g}} \le x_{\mathbf{f}} \le +\infty$, and $x_{\mathbf{f}}$ large enough. Here, we used the notation $\Omega^R = (0,R) \times (0,L)$, for a given $R > 0$. Function $\chi_{(0,x_{\mathbf{f}})}$ denotes the characteristic function of the interval $(0, x_{\mathbf{f}})$.

The localized effect was proven in [5], by means of the application of an energy method to the associated current function ψ $(\mathbf{u} = (\psi_y, -\psi_x))$. Function ψ satisfies the higher order nonlinear equation

$$\begin{cases} \nu\Delta^2\psi + \frac{\partial f_1}{\partial y}(\mathbf{x}, \psi_y, -\psi_x) - \frac{\partial f_2}{\partial x}(\mathbf{x}, \psi_y, -\psi_x) = 0 & \text{in } \Omega, \\ \psi(x,0) = \psi(x,L) = \frac{\partial \psi}{\partial n}(x,0) = \frac{\partial \psi}{\partial n}(x,L) = 0 & \text{for } x \in [0,+\infty), \\ \psi(0,y) = \int_0^y u_*(s)ds,\ \frac{\partial \psi}{\partial n}(0,y) = v_*(y), \ , & \text{for } y \in (0,L), \\ \psi(x,y),\ |\nabla\psi(x,y)| \to 0, & \text{as } x \to +\infty, \text{for } y \in (0,L). \end{cases} \tag{39}$$

We adapt, in [5], the half-planes technique introduced by F. Bernis in [11] for the study of other higher order equations. We point out that in contrast with the problems considered in the mentioned work, (39) do not contains any zero order term. Our approach is inspired on some previous unidirectional results for anisotropic equations proved in [6] (see Chap.1§4.2) by using a different energy method. In a first (and independent step), the existence of a weak solution having a finite global energy $E := \int_\Omega \left(|\nabla\mathbf{u}|^2 + \chi_{(0,x_{\mathbf{f}})}|u|^{1+\sigma}\right) d\mathbf{x}$ is established by means of the Schauder fixed point theorem. Moreover, we have

Theorem 4. *Assume f satisfies (38). Then: i) if $x_{\mathbf{f}} = +\infty$, u is any weak solution of $P(\Omega, u_*, f)$ with finite energy E, $a_2 = \frac{7+\sigma}{1-\sigma}\left(\frac{2240c\nu^2}{\min^2(\nu,\delta)}\right)^{\frac{2}{3+\sigma}} L^{\frac{1+\sigma}{3+\sigma}} (E)^{\frac{1-\sigma}{2(3+\sigma)}}$, where $c = c(\sigma)$ is a positive constant which depends only on σ, then $u(x,y) = 0$ for $x > a_2$, ii) if $x_{\mathbf{f}} < +\infty$ then there exists at least one weak solution u of $P(\Omega, u_*, f)$, with a finite energy E, such that if $a_2 \le x_{\mathbf{f}}$ then $u(x,y) = 0$ for $x > a_2$, iii) if, in addition, we assume f nonincreasing then conclusion ii) holds for the unique solution of $P(\Omega, u_*, f)$.*

The idea of the proof is the following: assume $x_{\mathbf{f}} = +\infty$ and let ψ be a weak solution of P_ψ with E finite. Then for every $a > x_g$ and every positive integer $m \ge 2$

$$\nu \int_\Omega \Delta\psi\Delta(\psi(x-a)_+^m)\ d\mathbf{x} + \delta \int_\Omega |\psi_y|^{1+\sigma}(x-a)_+^m d\mathbf{x} \le 0. \tag{40}$$

Applying the Leibnitz formula to the first term of the above inequality, it arises an energy type term $E_m(a) = \int_\Omega \left(|D^2\psi|^2 + |\psi_y|^{1+\sigma}\right)(x-a)_+^m d\mathbf{x}$. We observe that $E_0(0) = E$. The mentioned technique has, as main goal, to get a differential inequality for $E_m(a)$ leading to the vanishing of $E_m(a)$ (and then of ψ) for a large enough. Notice that a simple differentiation leads to the relations $\frac{dE_m(a)}{da} = -mE_{m-1}(a)$ and $\frac{d^2E_m(a)}{da^2} = m(m-1)E_{m-2}(a)$. The crucial part of the technique consist in to using the nonlinear structure of the equation in order to get the differential inequality $E_s(a) \le C\left(E_{s-p}(a)\right)^\mu$, for all $a \ge x_{\mathbf{g}} > 0$, where $0 < p \le m$, C is a positive constant and $\mu > 1$. Then, the support of $E_0(a)$ is a bounded interval $[0,\ a_*]$, with $a_* \le a_2$, where $a_2 = (w - m + 1)C^{\frac{1}{(\mu-1)(w-m)}} E_0(0)^{\frac{1}{w-m}}$ and $w = \frac{\mu p}{\mu - 1}$. The proof of the above inequality uses a weighted version of the Gagliardo-Nirenberg inequality and the Hardy type inequality (we send the reader to [5] for details).

5 Global extinction time

We consider the quasilinear problem associated to the nonlinear heat equation with absorption

$$(\mathcal{P}) \begin{cases} \frac{\partial}{\partial t}\left(u\,|u|^{\gamma-1}\right) - \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u|^{\sigma-1}u = f + \mathrm{div}\,\mathbf{g} & \text{in } Q := \Omega \times (0,+\infty), \\ u = 0 & \text{on } \sum_T = \Gamma \times (0,+\infty). \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded of boundary Γ, $\gamma > 0$, $\sigma > 0$, $1 \le p < \infty$, $\lambda \ge 0$ (if $p = 1$, ∇u represents the *total variation*). We shall assume that there exists $T_0 \ge 0$ such that $f(t,.) = 0$, $\mathbf{g}(t,.) = \mathbf{0}$ in Ω, if $t > T_0$. As we shall see, under suitable conditions, the "solution" (notion to be made precise) satisfy an integral energy inequality leading to the extinction in a finite time of the function

Theorem 4. *Let $u \in L^1_{loc}(T_0, +\infty : W_0^{1,p}(\Omega))$ for some $p > 1$ (or $u \in L^1_{loc}(T_0, +\infty : BV_0(\Omega))$, if $p = 1$) such that $\exists\, \gamma, k, c > 0, \lambda \ge 0, \sigma > k - 1$ for which $|u|^{\gamma+k}, |u|^{\sigma+k}, |Du|^p |u|^{k-1} L^1_{loc}(T_0, + : L^1(\Omega))$*

and

$$y(t) + c\int_s^t \int_\Omega |Du|^p \, |u|^{k-1} + \lambda \int_{t_j}^t \int_\Omega |u|^{\sigma+k} \le y(s) \ \text{a.e. } s,t \in (T_0,+\infty), \qquad (41)$$

where $y(t) = \int_\Omega |u|^{\gamma+k}\, dx$. *Assume that*

$$1 \le p < \gamma + 1 \quad \text{and} \quad \lambda = 0 \qquad (42)$$

or

$$1 \le p, \quad \sigma < \gamma \quad \text{and} \quad 0 < \lambda, \qquad (43)$$

and let

$$k = \begin{cases} 1 & \text{if } N \le p \text{ or } (\gamma+1) \le \dfrac{Np}{N-p}, \\ \dfrac{N-p}{p}\left(1+\gamma-\dfrac{p(N-1)}{N-p}\right) > 1 & \text{if } 1 < p < N \text{ and } \gamma + 1 > \dfrac{Np}{(N-p)}. \end{cases}$$

Then $u \in C^{0,\alpha}_{loc}(T_0,+\infty : L^{\gamma+k}(\Omega))$ *for some* $\alpha \in (0,1)$ *and there exists a* $T_e \in (T_0,+\infty)$ *such that* $u(t,.) \equiv 0$ *in* Ω $\forall t \ge T_e$.

The proof uses an integral version of the energy inequality found in [6] (Proposition 1.1. or Theorem 2.1, Chapter 2) for $p > 1$ and [3] for $p = 1$. More precisely $y(t) + C\int_s^t y(\tau)^\mu dt \le y(s)$, for a.e. $s,t \in (T_0,+\infty)$ for some $\mu \in (0,1)$. Some relevant choices of the parameters γ, p, σ which provide the fulfillment of the above conditions are: $p = 2$, $\gamma = 1$ and $\sigma < 1$; $\sigma = 1$, $p = 2$ and $\gamma > 1$; $\sigma = 1$, $\gamma = 1$and $p < 2$.

Several notions of solutions are possible (for simplicity, we assume now $p > 1$). The "variational theory" search for solutions in the "energy space" $u \in L^{p'}(0,T;W_0^{1,p}(\Omega))$, and use that (if $p \ge \frac{2N}{N+2}$) $W_0^{1,p}(\Omega) \subset L^2(\Omega) \subset W^{-1,p'}(\Omega)$. At least for $k = 1, u \in L^p(0,T;W_0^{1,p}(\Omega)), \forall T > 0$ implies that $|Du|^p\,|u|^{k-1} \in L^1_{loc}(0,+\infty : L^1(\Omega))$. A first problem arises with the zero order term $|u|^{\sigma-1}u$ since $u \in L^{p'}(0,T;W_0^{1,p}(\Omega)) \nRightarrow |u|^{\sigma+k} \in L^1_{loc}(0,+\infty : L^1(\Omega))$. Then, if the equation takes place in $D'(\Omega)$ the natural regularity for u_t is $|u_t|^{\gamma-1} u_t \in L^{p'}_{loc}(0,+\infty : W^{-1,p'}(\Omega)) + L^1_{loc}(0,+\infty : L^1(\Omega))$. In that case the test functions must be taken in $L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(0,T : L^\infty(\Omega))$. The existence of solutions in the above framework is due to many authors (Dubinsky, J.L. Lions, Raviart, Bamberger, Grange-Mignot, Benilan, Alt-Luckhaus among others: see references in [6]) assumed $|u_0|^{(\gamma-1)}u_0 \in L^2(\Omega)$ and $f, \mathbf{g} \in L^{p'}_{loc}(0,+\infty : L^{p'}(\Omega))$ and the, so called, *weak solution* satisfies that $u \in C([0,+\infty) : L^2(\Omega))$. The regularity $|u|^{\gamma+k}, |u|^{\sigma+k}, |Du|^p\,|u|^{k-1} \in L^1_{loc}(T_0,+\infty : L^1(\Omega))$ can be obtained by asking some extra regularity to the data (see, e.g. [13]). A nontrivial fact is the justification of the time integration by parts formula

$$\left\langle (|u|^{\gamma-1}u)_t, |u|^{k-1}u \right\rangle = \frac{\gamma}{\gamma+k}\int_0^T \left[\frac{d}{dt}\int_\Omega |u(t,.)|^{\gamma+k}\, dx\right] dt.$$

It could be easily justified for the case of *strong solutions* (i.e. $\frac{\partial}{\partial t}\left(u\,|u|^{\gamma-1}\right) \in L^1(Q)$) but it is know that this class of solutions are quite exceptional. More in general (but for $\gamma = k = 1, \lambda = 0$) this was proved in a pioneering paper by J.L. Lions [32]. For $\gamma \ne 1$ and $k = 1$ the result was proved (under different assumptions) by Bamberger, Grange-Mignot, Alt-Luckhaus, Bernis, Otto, Carrillo and Carrillo-Wittbold; the case $k \ne 1$ is due to Benilan (see references in [18]) .

In some cases the *extinction energy* $y(t) = \int_\Omega |u|^{\gamma+k}\, dx$ may be not well defined for solutions $u(t) \in W_0^{1,p}(\Omega) \subset L^2(\Omega)$. For instance, this is the case if $\gamma = 1, \lambda = 0$ and $1 \le p < \frac{2N}{N+2}$. Due to this difficulty, following to [12], it is useful to justify the energy inequality (for $k = 1$) by working in the space $W = W_0^{1,p}(\Omega) \cap L^{\gamma+1}(\Omega)$ if $\gamma < p - 1$ or $1 < \sigma \le p$ or Ω bounded, otherwise W is defined as the closure of $C_0^\infty(\Omega)$ in the Banach space $\{u \in L^{\gamma+1}(\Omega), Du \in L^p(\Omega)\}, \|u\| = \|u\|_{\gamma+1} + \|Du\|_p$. The existence of *an energy solution* (i.e. $u \in C([0,+\infty) : L^{\gamma+1}(\Omega)) \cap L^p(0,T : W) \cap L^{\sigma+1}((0,T) \times \Omega))$ for any finite T, satisfying the equation in D' and with $u(0,.) = u_0(.)$) was proved by assuming that $u_0 \in L^{\gamma+1}(\Omega)$, and $f + \operatorname{div}\mathbf{g} \in L^{p'}(0,T : W') + L^{(\sigma+1)'}((0,T) \times \Omega)$.

But the notion of solution can be found out of the energy space W. Among the several types of solutions in this framework we could mention, specially, the so called *mild solutions* motivated by the

numerical analysis and the abstract Semigroup Theory: given $\epsilon > 0$ and a time discretization $t_0 = 0 < t_1 < \cdots < t_n \le T, t_i - t_{i-1} < \epsilon, T - t_n < \epsilon$, and given $f_i \in L^\infty(\Omega)$, $w_0 \in L^\infty(\Omega)$ we consider the implicit time-discretization,

$$(DP)\left\{ \frac{b(w_i) - b(w_{i-1})}{t_i - t_{i-1}} - \operatorname{div}\left(|\nabla w_i|^{p-2}\nabla w_i\right) + \lambda|w_i|^{\sigma-1} w_i = f_i \text{ in } D'(\Omega), \right.$$

where $b(u) = |u|^{\gamma-1}u$. Notice that $w_i \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Now, let $u_0 \in L^\gamma(\Omega), f \in L^1(0,T : L^1(\Omega)), \mathbf{g} = \mathbf{0}$.

Definition. A *mild solution of (P)* is a function u such that $b(u) \in C([0,+\infty) : L^1(\Omega)), u(0,.) = u_0(.)$, and, for any $\epsilon > 0$ there exists $(t_0, t_1, \cdots, t_n,\ f_0, f_1, \cdots, f_n,\ w_0, w_1, \cdots, w_n)$ satisfying (DP) with $\|b(u_0) - b(w_0)\|_1 \le \epsilon$, $\sum_i \int_{t_{i-1}}^{t_i} \|f(t) - f_i\|_1\, dt \le \epsilon$ and $\|b(u(t)) - b(w_0)\|_1 \le \epsilon$ for any $t \in (t_{i-1}, t_i], i = 1, ..., n$.

The existence of a mild solution was due to [8]. Moreover, it was proved there that if, in addition, $u_0 \in L^{\gamma+1}(\Omega), f \in L^{p'}(0,T : W') + L^{(\sigma+1)'}((0,T) \times \Omega)$ then the mild solution is also an energy solution.

Now we can study the finite extinction time for the step function $w_\epsilon(t) := w_i$ if $t \in (t_{i-1}, t_i], i = 1, ..., n$.

Definition. *We say that* $w_\epsilon(t)$ *extincts in a finite time if there exists* $T_{\epsilon,e} = t_j$, *for some* $j \le n$ *such that* $\|w_\epsilon(t)\|_\infty > 0$ *for* $t \in [0, T_{\epsilon,e})$ *and* $\|w_\epsilon(t)\|_\infty = 0$ *for* $t \in [T_{\epsilon,e}, T]$.

Since $w_\epsilon(t)$ satisfies the integral energy inequality (41) we get to the following result (due to [7] for $p = 2, \lambda = 0$ and [28] for $p > 1$ and $\lambda = 0$)

Corollary 1. *Assume that there exists* $T_0 = t_m$, $m \le n$ *such that* $f_\epsilon(t,.) = 0$,*in* Ω,*if* $t > T_0$ ($f_\epsilon(t,.)$ *defined in a similar way to* $w_\epsilon(t)$). *Then, under the assumption of Theorem 4 on* γ, p, k, *and* σ, *function* $w_\epsilon(t)$ *extincts in a finite time* $T_{\epsilon,e}$. *Moreover, if* u *is a mild solution and assume that there exists* $T_0 \ge 0$ *such that* $f(t,.) = 0, \mathbf{g}(t,.) = \mathbf{0}$ *in* Ω, *if* $t > T_0$, *and that* $u(T_0) \in L^{\gamma+k}(\Omega)$) *then* $u(t)$ *extincts in a finite time* T_e(*only dependent on* $\|u(T_0)\|_{\gamma+k}$).

Notice that due to the regularizing effects (see [8]), it is possible to have a finite energy at time T_0 ($u(T_0) \in L^{\gamma+k}(\Omega)$)) even if $u_0 \in L^\gamma(\Omega)$.

Remark 4. *An unpleasant fact of mild solutions is the lake of an easy characterization in terms of test functions and the lake of information on their spatial regularity. A different notion of solutions corresponds to the so called renormalized solutions (see [13]). Since the general integral energy holds, the finite extinction time phenomenon can be obtained also for such solutions assumed, again,* $u(T_0) \in L^{\gamma+k}(\Omega)$).

Remark 5. *The assumption* $u(T_0) \in L^{\gamma+k}(\Omega)$ *is, in some sense, necessary. A counterexample can be done in other case: take* $\gamma = 1, \lambda = 0$, $p = 1$, *assume that* $0 \in \Omega$ *and* $u(0,.) = \delta_0$ *(the Dirac delta at the origin). Then, it is possible to show (V. Caselles: personal communication: 2002) that there is not any regularizing effect and* $u(t,.) = C(t)\delta_0$ *with* $C(t) > 0$.

The extinction time also exists for other time-discretizations (now of semi-implicit type). We write (assuming now $w \ge 0$)

$$(w^\gamma)_t = \frac{\gamma}{\gamma+1-p}(w^{p-1})(w^{\gamma+1-p})_t \approx \frac{\gamma}{\gamma+1-p}(w_i^{p-1})\frac{(w_i^{\gamma+1-p} - w_{i-1}^{\gamma+1-p})}{t_i - t_{i-1}}.$$

Given $\epsilon > 0$, a time discretization $t_0 = 0 < t_1 < \cdots < t_n \le T, t_i - t_{i-1} < \epsilon, T - t_n < \epsilon$, and given $f_i \in L^\infty(\Omega)$, $w_0 \in L^\infty(\Omega)$ we consider the semi-implicit time-discretization,

$$\frac{\gamma}{\gamma+1-p}(w_i^{p-1})\frac{(w_i^{\gamma+1-p} - w_{i-1}^{\gamma+1-p})}{t_i - t_{i-1}} - \operatorname{div}\left(|\nabla w_i|^{p-2}\nabla w_i\right) + \lambda|w_i|^{\sigma-1} w_i = f_i \text{ in } D'(\Omega).$$

When f_i and w_0 are nonnegative, the existence and uniqueness of a nonnegative w_i is consequence of the results of [22]. The convergence of the scheme was given in [31] for $p = 2, \lambda = 0$ and in [34] for $p \ne 2$ and $\lambda = 0$. We have

Corollary 2. *Assume that there exists* $T_0 = t_m$, $m \le n$ *such that* $f_\epsilon(t,.) = 0$, *in* Ω,*if* $t > T_0$ ($f_\epsilon(t,.)$ *defined as done for* $w_\epsilon(t)$). *Then, under the assumption of the Theorem 4 on* γ, p, k, *and* σ *function* $w_\epsilon(t)$ *extincts in a finite time* $T_{\epsilon,e}$.

Remark 6. *The existence of a finite extinction time can be also proved for another type of semi-implicit time-discretization (see [9])*

$$\frac{b(w_i) - b(w_{i-1})}{t_i - t_{i-1}} - \operatorname{div}\left(|\nabla w_{i-1}|^{p-2}\nabla w_i\right) + \lambda|w_i|^{\sigma-1} w_i = f_i \text{in } D'(\Omega).$$

Now we consider other type of schemes in which the discretization is not in time but in space. For simplicity, we assume $\Omega = (0,1)$. Following [38] (see also [33]) we introduce the notations

$$\varphi(v) = |v|^{\alpha-2}v,\ \psi(v) = |v|^{(\alpha-2)/2}v,\ \theta(v) = \lambda v|v|^{q-1},\ A(u) = -\left(|u_x|^{p-2}u_x\right)_x$$

So, the equation can be written in terms of $v_t + A(\varphi(v)) + \theta(v) = f$. Given $M \in \mathbb{N}$ we call $h =$ $1/(M+1)$ and introduce the sequence spaces $V_h = \{\mathbf{v}_h = (v_i),\ i = 0,...,M+1\}$, $v_0 = v_{M+1} = 0$. We introduce the following product and norms $(\mathbf{v}_h, \mathbf{u}_h) = h\sum_{i=1}^{M} v_i u_i$, $|\cdot|_{2,h}$ corresponding norm, $|\mathbf{v}_h|_{\alpha,h} =$ $(h\sum_{i=1}^{M}|v_i|^{\alpha})^{1/\alpha}$, $\|\mathbf{v}_h\|_h = (h\sum_{i=1}^{M}\left|\frac{v_{i+1}-v_i}{h}\right|^p)^{1/p}$. The discretizations of the nonlinear operators will be given, for $i = 0,...,M+1$, by

$$\left\{\begin{array}{l} \varphi(\mathbf{v}_h) \in V_h,\ \varphi(\mathbf{v}_h)_i = |v_i|^{\alpha-2}v_i = \varphi(v_i), \theta(\mathbf{v}_h) \in V_h,\ \theta(\mathbf{v}_h)_i = \lambda v_i|v_i|^{q-1} = \theta(v_i), \psi(\mathbf{v}_h) \\ = |v_i|^{(\alpha-2)/2}v_i = \psi(v_i),\ A_h(\mathbf{v}_h)_i = \frac{1}{h^p}[|v_{i+1}-v_i|^{p-2}(v_{i+1}-v_i) - |v_i - v_{i-1}|^{p-2}(v_i - v_{i-1})], \end{array}\right.$$

The discretized problem is $\frac{d\mathbf{v}_h(t)}{dt} + A_h(\varphi(\mathbf{v}_h(t))) + \theta(\mathbf{v}_h) = \mathbf{f}_h(t)$, $\mathbf{v}_h(0) = \mathbf{v}_{0h}$, where $\mathbf{f}_h(t)$ and $\mathbf{v}_{0h}$ are approximations of $f(t)$ and v_0. The convergence of the scheme was due to [38]. Assume again that there exists $T_0 \geq 0$, such that $\mathbf{f}_h(t) = \mathbf{0}$, in Ω, if $t > T_0$.

Definition. *We say that $\mathbf{v}_h(t)$ extincts in a finite time if there exists $T_{h,e} \geq T_0$, such that $\|\mathbf{v}_h(t)\|_1 >$ 0 for $t \in [0, T_{\epsilon,e})$ and $\|\mathbf{v}_h(t)\|_\infty = 0$ for $t \in [T_{h,e}, T]$.*

Since it was shown in [38] that $\mathbf{v}_h(t)$ satisfies the integral energy inequality (41) we get

Corollary 3. *Under the assumption of the Theorem 4 on γ, p, k, and σ function $\mathbf{v}_h(t)$ extincts in a finite time $T_{h,e}$.*

Remark 7. *In [14] it is considered the special case $\gamma = 1$ and $p = 2$. The finite extinction time is obtained by means of a comparison argument (with solutions of the ode $\frac{d\mathbf{W}_h(t)}{dt} + \theta(W_h) = 0$ and for bounded initial data). It is not difficult to show that the estimates on $T_{h,e}$ obtained by the energy method (which takes into consideration the spatial operator $A_h(\varphi(v_h(t))))$ are sharper than the ones of [14] when $\|\mathbf{v}_{0h}\|_\infty$ is large enough. We also point out that the integral energy inequality (41) is satisfied for the solution obtained via Galerkin for the approximation by finite elements and so the extinction time phenomenon also holds for such a discrete solution. Different full discretizations (in time and space) where introduced by several authors (see, e.g., [38], [35], [29], [14]). Although some of these algorithms are explicit (but convergent under suitable stability assumptions), the finite extinction time holds once that the general Theorem 4 can be applied.*

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Equations satisfaites par des limites de solutions approchées

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Résumé

Dans de nombreuses situations "industrielles", des schèmas numériques efficaces ont été developpés par des équipes d'ingénieurs, souvent après de nombreux essais infructueux. L'efficacité du schéma apparaîssant dans le fait que la solution trouvée semble "raisonnable" (elle satisfait, par exemple, des contraintes naturelles, comme des contraintes de bornes, et semble concordante avec des observations expérimentales). Dans de nombreux cas, il n'est toutefois pas clair de voir de quel problème est solution la limite des solutions approchées lorsque les pas de discrétisation tendent vers 0, donc, finalement, de comprendre quelles équations (et quelle conditions aux limites) ont été discrétisées par le schéma numérique. Il peut s'agir, par exemple, de comprendre une condition aux limites satisfaite par la limite de solutions approchées ou le sens avec lequel une équation est satisfaite (condition d'entropie, par exemple) ou encore de trouver l'équation elle même ...

Dans cet exposé, quelques exemples de telles situations seront présentés :

- Ecoulement diphasique avec "forcing" dans un milieu poreux,
- Ecoulement diphasique dans une conduite,
- Simulation de colonnes à distiller,
- Ecoulement gravitaire dans un milieu poreux hétérogène,
- Ecoulement diphasique multidimensionnel en milieux poreux,
- Simulation de la sédimentation et de l'érosion dans les basins sédimentaires.

Pour les 3 premiers exemples, il s'agit de comprendre la condition aux limites satisfaites par les limites de solutions approchées données par des schémas "naturels". Pour le quatrième, il s'agit de comprendre le sens de l'équation (condition d'entropie) et, pour le cinquième et le sixième, de trouver l'équation elle même.