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STRANGE NON-LOCAL OPERATORS HOMOGENIZING THE POISSON EQUATION WITH DYNAMICAL UNILATERAL BOUNDARY CONDITIONS: ASYMMETRIC PARTICLES OF CRITICAL SIZE

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Abstract. We study the homogenization of a nonlinear problem given by the Poisson equation, in a domain with arbitrarily shaped perforations (or particles) and with a dynamic unilateral boundary condition (of Signorini type), with a large coefficient, on the boundary of these perforations (or particles). This problem arises in the study of chemical reactions of zero order. The consideration of a possible asymmetry in the perforations (or particles) is fundamental for considering some applications in nanotechnology, where symmetry conditions are too restrictive. It is important also to consider perforations (or particles) constituted by small different parts and then with several connected components. We are specially concerned with the so-called critical case in which the relation between the coefficient in the boundary condition, the period of the basic structure, and the size of the holes (or particles) leads to the appearance of an unexpected new term in the effective homogenized equation. Because of the dynamic nature of the boundary condition this "strange term" becomes now a non-local in time and non-linear operator. We prove a convergence theorem and find several properties of the "strange operator" showing that there is a kind of regularization through the homogenization process.

1. Introduction

This article studies the asymptotic behavior, as $\varepsilon \to 0$, of the solution u_ε to a problem given by the Poisson equation in a domain Ω_ε which we can understand as either to be given as an initial bounded regular domain Ω of $\mathbb{R}^n$, $n \geq 3$ which is perforated by many periodical cavities of an arbitrary shape, or that Ω_ε is the part of Ω which is exterior to a periodical distribution of many particles G_ε of arbitrary shape. On the boundary S_ε of these perforations, or of the particles G_ε, we impose a dynamic unilateral boundary condition (of Signorini type) and we obtain the

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formulation

$$
\begin{gathered}
-\Delta_x u_\varepsilon = f(x,t), \quad (x,t) \in Q_\varepsilon^T, \\
u_\varepsilon \geq 0, \quad \varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon \geq 0, \quad (x,t) \in S_\varepsilon^T, \\
u_\varepsilon(\varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon) = 0, \quad (x,t) \in S_\varepsilon^T, \\
u_\varepsilon(x,t) = 0, \quad (x,t) \in \Gamma^T, \\
u_\varepsilon(x,0) = 0, \quad x \in S_\varepsilon.
\end{gathered} \tag{1.1}
$$

Here, we assume to be given a time $T > 0$ and which leads to define the sets $Q_\varepsilon^T = \Omega_\varepsilon \times (0,T)$, $S_\varepsilon^T = S_\varepsilon \times (0,T)$, $\Gamma^T = \partial\Omega \times (0,T)$. Function $f(x,t)$ is a datum of the problem and we are assuming an identically zero initial datum ($u_\varepsilon(x,0) = 0$ for $x \in S_\varepsilon$) just for simplicity in the formulation (see Remark 5.3 below for a more general case). Our main interest concerns the so-called "critical case", corresponding to the situation in which G_ε is the set of translations of a cell perforation (or particle) $a_\varepsilon G_0$ where the homothety is of the order

$$
a_\varepsilon = C_0\varepsilon^\gamma, \quad \gamma = \frac{n}{n-2}, \quad C_0 > 0. \tag{1.2}
$$

(notice that the big parameter $\varepsilon^{-\gamma}$ appears in the boundary condition in order to get some relevant problems: for a general exposition see [13]). Although the detailed presentation on the geometric aspects of the domains and notations will be presented in the next Section, we point out that the consideration of a possible asymmetry in the perforations (or particles) is fundamental in order to consider some applications in nanotechnology (see, e.g., [28, 6]) were symmetry conditions are too restrictive. For instance, in the case of a reactive flow through particulate filters of fixed bed at nanosccale it is inevitable to produce particles that have symmetry defects and therefore the mathematical treatment must be justified assuming that the particles are asymmetric. As a matter of facts, it is also important to consider perforations (or particles) constituted by small different parts and then with several connected components. Several positive properties that only take place at the nanometer scale correspond faithfully to the case of homogenization at the critical scale (see, for instance, [13, Section 4.9.4]).

One of the many applications in which such formulation arises is in Chemical Engineering. In that framework u_ε represents the concentration of some chemical substance which is distributed in a permanent flow of some Newtonian fluid. This explains the use of a linear elliptic equation in the exterior of the granular chemical particles: here the Poisson equation is a simplification of the stationary Navier-Stokes equation (see, e.g. [8, 22, 26] and its references) and with a dynamic chemical reaction on the boundary S_ε of the particles G_ε (since the concentration on S_ε decreases with the time). An usual kinetics reaction is given by the so-called reactions of order $p \in [0, +\infty)$

$$
\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}(\partial_t u_\varepsilon + \lambda\sigma(u_\varepsilon)) = 0 \quad \text{on } S_\varepsilon^T,
$$

with $\sigma(u_\varepsilon) = u_\varepsilon^p$. Since u_ε represents a concentration then we can assume that $u_\varepsilon \geq 0$. The case of reactions of order $p \in [1, +\infty)$ corresponds to the case in which we assume $\sigma(u)$ Lipschitz continuous and increasing in u (in this setting it is natural to assume that u_ε is a bounded function). The case of reactions of order $p \in (0, +1)$ leads to the more general assumption of $\sigma(u)$ Hölder continuous and the important case of the so-called zero-order reactions corresponds to the discontinuous function $\sigma(u) = 1$ if $u > 0$ and $\sigma(0) = 0$. It is easy to see (see

Remark 5.4) that the solutions of the zero-order reactions satisfy, with a slight modification, the unilateral formulation (1.1)).

For the purely stationary problem the "critical case" is characterized by the appearance of a new non-local term in the effective equation as $\varepsilon \to 0$. We send the reader to the monograph [13] where many references on the pioneering papers are given. In the literature, this term is usually called as "strange" (see [7]) because nothing similar happens when the exponent γ in (1.2) is different. For the case of "big particles" of arbitrary shape the effective diffusion operator depends crucially of the shape of the particle and the "effectiveness" of the chemical reaction can be optimized by a suitable choice of the shape of the particles (see, e.g., [16] and its references).

We also point out that dynamic Signorini boundary conditions also appear while studying different physical and chemical processes, see, e.g. [21, 2, 4, 5, 3]. The homogenization of problems with unilateral (dynamical or stationary) boundary conditions attracted the attention of many researchers: here, we refer to [18, 29, 1, 9, 20, 15, 12, 27, 11, 10, 31].

Here we obtain the homogenized problem containing a non-local "strange" because the size a_ε is critical, and we prove the convergence of the original problem's solution to the solution of the elliptic homogenized one in which the time plays the role of a parameter.

Theorem 1.1. *Let $n \geq 3$, $a_\varepsilon = C_0\varepsilon^\gamma$, $\gamma = \frac{n}{n-2}$. Assume $f \in H^1(0,T;L^2(\Omega))$. Let u_ε be the strong solution of the problem* (1.1)*, and let $P_\varepsilon u_\varepsilon$ be a suitable extension of u_ε to the whole domain Ω. Then $P_\varepsilon u_\varepsilon \rightharpoonup u_0$ weakly in $L^2(0,T;H_0^1(\Omega))$ and $u_0 \in L^2(0,T;H_0^1(\Omega))$ is characterized as the unique weak solution to the problem*

$$\begin{gathered} -\Delta_x u_0 + C_0^{n-2} H[u_0] = f(x,t), \quad x \in \Omega,\ t \in (0,T), \\ u_0 = 0, \quad x \in \partial\Omega,\ t \in (0,T), \end{gathered} \tag{1.3}$$

where the "strange operator" $H[u_0]$ is defined by the expression (4.1) *below.*

The detailed expression of the non-local in time (and non-linear) strange operator $H[\cdot]$ is rather technical as to be detailed here but we will devote a subsection to get a series of properties which explains that there is a kind of regularization through the homogenization process. Indeed, the non-linear operator $H[\cdot]$ is bounded, monotone and Lipschitz continuous (see Theorem 4.1), in contrast with the unilateral nature of the original problem. In addition, it allows to get solutions u_0 changing sign on Ω when the datum $f(t,x)$ is negative in some part of Q_ε^T (something which is impossible for the original solutions u_ε on the boundary S_ε^T of so many particles, when $\varepsilon \to 0$): see Remark 5.2.

The study of dynamical boundary conditions under the assumption of particles of critical size was already initiated with the series of papers [14, 15, 18, 20, 31]. What distinguishes this article from the previous ones is that here we address the homogenization of the problem with the dynamic Signorini boundary condition in the critical case for *arbitrarily shaped* particles or perforations. In the case of symmetric particles the "strange operator" $H[u_0]$ is given in terms of the solution of a quite simple unilateral problem (see Remark 5.1 and [18]). The main difficulty of the asymmetric case, as it arises in many applications of nanotechnology, is that the particles G_ε have arbitrary shape and then the corrections functions which are used in the proof of the "method of oscillating test functions" must depend of the exact nonlinear term arising in the boundary condition. That was carried out in the case of $\sigma(u)$ Hölder continuous in the paper [17] but, in contrast to the case of the purely stationary problem, the extension to the case in which σ is a general maximal monotone operator (as it is the case of the Signorini boundary conditions) made in [11] does not work when the particles are not symmetric and the boundary conditions are dynamic. So, this paper covers such an important lack.

The organization of this paper is the following: Section 2 is devoted to present the geometric aspects of the domains, some useful notations and the derivation of suitable a priori estimates on the solutions implying the weak convergence in some functional space. Two important auxiliary problems are introduced in Section 3: they play a fundamental role for the definition of the "strange operator" $H[\cdot]$ which is carried out in Section 4. Finally, the proof of Theorem 1.1, the comparison with the results for symmetric particles and other remarks are presented in Section 5.

2. Statement of the problem and a priori estimates of solutions

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 3$, with Lipschitz boundary $\partial\Omega$. In the cube $Y = (-1/2, 1/2)^n$, we consider a subdomain G_0, $\overline{G_0} \subset Y$, which, for simplicity, is star-shaped with respect to a ball $T^0_\rho \subset G_0$ of radius ρ with the center at the origin. Our treatment remains valid if G_0 consists of a finite number of disjoint connected components satisfying the same geometric property. Let $\delta B = \{x : \delta^{-1}x \in B\}$, $\delta > 0$. For $\varepsilon > 0$, we define

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}.$$

Denote by $\mathbb{Z}^n$ the set of all vectors $j = (j_1, \dots, j_n)$ with integer coordinates j_i, $i = 1, \dots, n$. We consider a set

$$G_\varepsilon = \cup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \cup_{j\in\Upsilon_\varepsilon} G^j_\varepsilon,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n | \overline{G^j_\varepsilon} \subset Y^j_\varepsilon = \varepsilon Y + \varepsilon j, G^j_\varepsilon \cap \overline{\widetilde{\Omega}_\varepsilon} \neq \emptyset\}$. Our assumption on the size of the inclusions is

$$a_\varepsilon = C_0\varepsilon^\gamma, \quad \text{with } \gamma = \frac{n}{n-2} \text{ and } C_0 > 0. \tag{2.1}$$

It is easy to see that $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, with d a positive constant. Note that $\overline{G^j_\varepsilon} \subset T^j_{Ca_\varepsilon} \subset T^j_{\varepsilon/4} \subset Y^j_\varepsilon$, where T^j_r is a ball in $\mathbb{R}^n$ of radius r with the center at $P^j_\varepsilon = \varepsilon j$ (the center of the cell Y^j_ε), C is a positive constant independent of ε. We introduce sets

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon}, \quad S_\varepsilon = \partial G_\varepsilon, \quad \partial\Omega_\varepsilon = S_\varepsilon \cup \partial\Omega,$$
$$Q^T_\varepsilon = \Omega_\varepsilon \times (0, T), \quad S^T_\varepsilon = S_\varepsilon \times (0, T), \quad \Gamma^T = \partial\Omega \times (0, T).$$

We define

$$K_\varepsilon = \{v \in H^1(\Omega_\varepsilon, \partial\Omega) : v \geq 0 \text{ a.e. } x \in S_\varepsilon\}.$$

As usual, we denote by $H^1(\Omega_\varepsilon, \partial\Omega)$ the completion with respect to the norm in $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable functions in $\overline{\Omega}_\varepsilon$, vanishing in a neighborhood of $\partial\Omega$. We also introduce the convex closed set

$$\mathcal{K}_\varepsilon = \{v \in L^2(0, T; H^1(\Omega_\varepsilon, \partial\Omega)) : v(\cdot, t) \in K_\varepsilon \text{ for a.e. } t \in [0, T]\}.$$

Given $f \in H^1(0,T;L^2(\Omega))$ we say that $u_\varepsilon \in \mathcal{K}_\varepsilon$ is a strong solution to (1.1) if $\partial_t u_\varepsilon \in L^2(0,T;L^2(S_\varepsilon))$, $u_\varepsilon(x,0)=0$ and we have

$$\begin{aligned} &\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon(\phi-u_\varepsilon)\,ds\,dt+\int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla(\phi-u_\varepsilon)\,dx\,dt\\ &\geq\int_0^T\int_{\Omega_\varepsilon}f(\phi-u_\varepsilon)\,dx\,dt, \end{aligned} \tag{2.2}$$

for all $\phi \in \mathcal{K}_\varepsilon$. Notice that (2.2) is a variational formulation of the unilateral problem with the Signorini dynamic boundary conditions.

Theorem 2.1. *For any $\varepsilon > 0$ problem (1.1) has a unique strong solution u_ε. Moreover u_ε satisfies the following estimates*

$$\begin{gathered} \|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))}+\varepsilon^{-\gamma/2}\|u_\varepsilon\|_{C([0,T];L^2(S_\varepsilon))}\leq K\|f\|_{L^2(Q^T)},\\ \varepsilon^{-\gamma/2}\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))}+\|\nabla u_\varepsilon\|_{C([0,T],L^2(\Omega_\varepsilon))}\leq K\|f\|_{H^1(0,T;L^2(\Omega))}, \end{gathered} \tag{2.3}$$

where $K > 0$ is a constant independent of ε and f.

Proof. We use the penalty method (see, e.g., [24]). Given a parameter $\delta > 0$, the penalized problem associated to the original problem (1.1) has the form

$$\begin{gathered} -\Delta_x u_\varepsilon^\delta=f(x,t),\quad (x,t)\in Q_\varepsilon^T,\\ \varepsilon^{-\gamma}\partial_t u_\varepsilon^\delta+\partial_\nu u_\varepsilon^\delta+\varepsilon^{-\gamma}\delta^{-1}(u_\varepsilon^\delta)^-=0,\quad (x,t)\in S_\varepsilon^T,\\ u_\varepsilon^\delta(x,t)=0,\quad (x,t)\in\Gamma^T,\\ u_\varepsilon^\delta(x,0)=0,\quad x\in S_\varepsilon, \end{gathered} \tag{2.4}$$

where $u^+ = \sup(0,u(x,t))$, $u^- = u - u^+$. Note that function $\sigma(u) = u^-$ is a monotone Lipschitz continuous function that satisfies

$$|u^- - v^-| \leq |u-v|, \quad \forall u,v \in \mathbb{R}.$$

We say that a function $u_\varepsilon^\delta \in C([0,T];L^2(S_\varepsilon))$ is a strong solution to the problem (2.4) if $u_\varepsilon^\delta \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, $\partial_t u_\varepsilon^\delta \in L^2(0,T;L^2(S_\varepsilon))$, and it satisfies the integral identity

$$\begin{aligned} &\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon^\delta v\,ds\,dt+\int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon^\delta\nabla v\,dx\,dt\\ &+\varepsilon^{-\gamma}\delta^{-1}\int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^- v\,ds\,dt\\ &=\int_{Q_\varepsilon^T}fv\,dx\,dt, \end{aligned} \tag{2.5}$$

for arbitrary functions $v \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, and the initial condition $u_\varepsilon^\delta(x,0) = 0$ holds for a.e. $x \in S_\varepsilon$.

By applying the results from [20], we conclude that for any $\delta > 0$ the problem (2.4) has a unique strong solution and the following estimates hold

$$\begin{gathered} \|u_\varepsilon^\delta\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))}+\varepsilon^{-\gamma/2}\max_{t\in[0,T]}\|u_\varepsilon^\delta\|_{L^2(S_\varepsilon)}\leq K\|f\|_{L^2(Q^T)},\\ \varepsilon^{-\gamma/2}\|(u_\varepsilon^\delta)^-\|_{L^2(0,T;L^2(S_\varepsilon))}\leq K\sqrt{\delta}\|f\|_{L^2(Q^T)},\\ \varepsilon^{-\gamma/2}\|\partial_t u_\varepsilon^\delta\|_{L^2(0,T;L^2(S_\varepsilon))}+\max_{t\in[0,T]}\|\nabla u_\varepsilon^\delta\|_{L^2(\Omega_\varepsilon)}\leq K\|f\|_{H^1(0,T;L^2(\Omega))}, \end{gathered} \tag{2.6}$$

From (2.6), we derive that there exists a subsequence such that

$$\begin{gathered} u_\varepsilon^\delta \rightharpoonup u_\varepsilon \quad \text{weakly in } L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega)), \\ \partial_t u_\varepsilon^\delta \rightharpoonup \partial_t u_\varepsilon \quad \text{weakly in } L^2(0,T;L^2(S_\varepsilon)), \\ u_\varepsilon^\delta \rightharpoonup u_\varepsilon \quad \text{weakly in } L^2(0,T;L^2(\Omega_\varepsilon)), \\ (u_\varepsilon^\delta)^- \to 0 \quad \text{in } L^2(0,T;L^2(S_\varepsilon)), \end{gathered}$$

as $\delta \to 0$. Then by the compactness result of [5, Theorem 2.1] we conclude that

$$u_\varepsilon^\delta \to u_\varepsilon \quad \text{in } C([0,T];L^2(S_\varepsilon)),$$

as $\delta \to 0$. Next, we show that u_ε is a solution to the variational inequality (2.2). Indeed, we take $v = \phi - u_\varepsilon^\delta$, where $\phi \in \mathcal{K}_\varepsilon$, as a test function in (2.5) and obtain

$$\begin{aligned} &\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon^\delta(\phi - u_\varepsilon^\delta)\,ds\,dt + \int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon^\delta\nabla(\phi - u_\varepsilon^\delta)\,dx\,dt \\ &+ \varepsilon^{-\gamma}\delta^{-1}\int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^-(\phi - u_\varepsilon^\delta)\,ds\,dt \\ &= \int_0^T\int_{\Omega_\varepsilon} f(\phi - u_\varepsilon^\delta)\,dx\,dt. \end{aligned}$$

Applying the inequality

$$\|\nabla u_\varepsilon\|_{L^2(Q_\varepsilon^T)} \le \lim_{\delta\to 0}\|\nabla u_\varepsilon^\delta\|_{L^2(Q_\varepsilon^T)},$$

we obtain

$$\lim_{\delta\to 0}\int_0^T\int_{\Omega_\varepsilon}\nabla u_\varepsilon^\delta\nabla(\phi - u_\varepsilon^\delta)\,dx\,dt \le \int_0^T\int_{\Omega}\nabla u_\varepsilon\nabla(\phi - u_\varepsilon)\,dx\,dt.$$

Then, using

$$\|u_\varepsilon(x,T)\|^2_{L^2(S_\varepsilon)} \le \lim_{\delta\to 0}\|u_\varepsilon^\delta(x,T)\|^2_{L^2(S_\varepsilon)},$$

we obtain

$$\varepsilon^{-\gamma}\lim_{\delta\to 0}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon^\delta(\phi - u_\varepsilon^\delta)\,ds\,dt \le \varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon(\phi - u_\varepsilon)\,ds\,dt.$$

Taking into account that $\phi \in \mathcal{K}_\varepsilon$, we conclude that

$$\int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^-(\phi - u_\varepsilon^\delta)\,ds\,dt = \int_0^T\int_{S_\varepsilon}(u_\varepsilon^\delta)^-\phi ds dt - \int_0^T\int_{S_\varepsilon}|(u_\varepsilon^\delta)^-|^2\,ds\,dt \le 0.$$

Combining the derived inequalities, we obtain that u_ε satisfies (2.2). Finally, the estimates (2.6) imply (2.3). This concludes the proof. □

We recall that by [25], there exists a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon,\partial\Omega) \to H_0^1(\Omega)$, such that

$$\|\nabla(P_\varepsilon u)\|_{L^2(\Omega)} \le K\|\nabla u\|_{L^2(\Omega_\varepsilon)},\ \|P_\varepsilon u\|_{H_0^1(\Omega)} \le K\|u\|_{H^1(\Omega_\varepsilon)},$$

where constant $K > 0$ is independent of ε. Then, the estimate from Theorem 2.1 implies

$$\|P_\varepsilon u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \le K.$$

Therefore, for some subsequence (still denoted with the subindex ε), we have as $\varepsilon \to 0$

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } L^2(0,T;H_0^1(\Omega)), \tag{2.7}$$

for some $u_0 \in L^2(0,T;H_0^1(\Omega))$.

3. Adaptation of global test functions

3.1. First auxiliary problem: a different unilateral problem in the macroscopic variables. Let $\phi(x,t) = \psi(x)\eta(t)$, where $\psi \in C^\infty(\overline{\Omega})$, $\eta \in C^1([0,T])$. For every $j \in \Upsilon_\varepsilon$, we consider the auxiliary elliptic problem with the dynamic unilateral boundary condition given by

$$
\begin{gathered}
\Delta_x w^j_{\varepsilon,\phi}(x,t) = 0, \quad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ t \in (0,T),\\
w^j_{\varepsilon,\phi} \le \phi(P^j_\varepsilon, t), \quad x \in \partial G^j_\varepsilon,\ t \in (0,T),\\
\partial_\nu w^j_{\varepsilon,\phi} \le \varepsilon^{-\gamma}\partial_t(\phi(P^j_\varepsilon,t) - w^j_{\varepsilon,\phi}), \quad x \in \partial G^j_\varepsilon,\ t \in (0,T),\\
(w^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon,t))(\partial_\nu w^j_{\varepsilon,\phi} - \varepsilon^{-\gamma}\partial_t(\phi(P^j_\varepsilon,t) - w^j_{\varepsilon,\phi}) = 0,\\
x \in \partial G^j_\varepsilon,\ t \in (0,T),\\
w^j_{\varepsilon,\phi}(x,t) = 0, \quad x \in \partial T^j_{\varepsilon/4},\ t \in (0,T),\\
w^j_{\varepsilon,\phi}(x,0) = \phi(P^j_\varepsilon, 0), \quad x \in \partial G^j_\varepsilon.
\end{gathered}
\tag{3.1}
$$

Notice that the above unilateral boundary condition has a different nature to the one in the original problem (1.1). To define the notion of solution, we introduce the convex closed sets:

$$
\begin{gathered}
K_{j,\varepsilon} = \{g \in H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4}) : g \le \phi(P^j_\varepsilon, t) \text{ a.e. on } \partial G^j_\varepsilon\},\\
\mathcal{K}_{j,\varepsilon} = \{g \in L^2(0,T; H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4})) : g \in K_{j,\varepsilon} \text{ for a.e. } t \in [0,T]\}.
\end{gathered}
$$

We say that a function $w^j_{\varepsilon,\phi} \in \mathcal{K}_{j,\varepsilon}$ is a strong solution to the problem (3.1), if $\partial_t w^j_{\varepsilon,\phi} \in L^2(0,T;L^2(\partial G^j_\varepsilon))$, $w^j_{\varepsilon,\phi}(x,0) = \phi(P^j_\varepsilon, 0)$ for a.e. $x \in \partial G^j_\varepsilon$ and if it satisfies the variational inequality

$$
\begin{aligned}
&\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^j_{\varepsilon,\phi} \nabla (v - w^j_{\varepsilon,\phi})\,dx\,dt\\
&\ge \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t(\phi(P^j_\varepsilon,t) - w^j_{\varepsilon,\phi})(v - w^j_{\varepsilon,\phi})\,ds\,dt,
\end{aligned}
\tag{3.2}
$$

for an arbitrary function $v \in \mathcal{K}_{j,\varepsilon}$.

Notice that the inequality (3.2) is equivalent to the two following relations: the first one is

$$
\begin{aligned}
&\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^j_{\varepsilon,\phi} \nabla (v - \phi_{j,\varepsilon})\,dx\,dt\\
&\ge \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t(\phi(P^j_\varepsilon,t) - w^j_{\varepsilon,\phi})(v - \phi(P^j_\varepsilon,t))\,ds\,dt,
\end{aligned}
\tag{3.3}
$$

where $v \in \mathcal{K}_{j,\varepsilon}$, $\phi_{j,\varepsilon}(x,t) = \phi(P^j_\varepsilon, t)\psi_{j,\varepsilon}(x)$, $\psi_{j,\varepsilon} \in C_0^\infty(T^j_{\varepsilon/4})$, $\psi_{j,\varepsilon}(x) \equiv 1$, $x \in T^j_{Ca_\varepsilon}$, $C > 1$, $\psi_{j,\varepsilon} \equiv 0$, $x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{2Ca_\varepsilon}}$, $|\nabla \psi_{j,\varepsilon}| \leq \frac{K}{a_\varepsilon}$. The second relation is

$$\begin{aligned} &\int_0^T \int_{T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}} \nabla w^j_{\varepsilon,\phi} \nabla (w^j_{\varepsilon,\phi} - \phi_{j,\varepsilon})\, dx\, dt \\ &= \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t (\phi(P^j_\varepsilon, t) - w^j_{\varepsilon,\phi})(w^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))\, ds\, dt. \end{aligned} \tag{3.4}$$

Indeed, to show this equivalency, we start by pointing out that by subtracting from (3.3) the equality (3.4), we obtain (3.2). Thus, to get the reverse implication, we set $v = \phi_{j,\varepsilon} \in \mathcal{K}_{j,\varepsilon}$ in (3.2) and obtain

$$\begin{aligned} &\int_0^T \int_{T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}} \nabla w^j_{\varepsilon,\phi} \nabla (\phi_{j,\varepsilon} - w^j_{\varepsilon,\phi})\, dx\, dt \\ &\geq \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t (\phi(P^j_\varepsilon, t) - w^j_{\varepsilon,\phi})(\phi(P^j_\varepsilon, t) - w^j_{\varepsilon,\phi})\, ds\, dt. \end{aligned} \tag{3.5}$$

Then, if we take $v = 2w^j_{\varepsilon,\phi} - \phi_{j,\varepsilon}$, since on ∂G^j_ε, we have $v = 2w^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t) \leq 2\phi(P^j_\varepsilon, t) - \phi(P^j_\varepsilon, t) = \phi(P^j_\varepsilon, t)$, then $v \in \mathcal{K}_{j,\varepsilon}$ is an admissible test function. Setting v as a test function in (3.2), we obtain

$$\begin{aligned} &\int_0^T \int_{T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}} \nabla w^j_{\varepsilon,\phi} \nabla (w^j_{\varepsilon,\phi} - \phi_{j,\varepsilon})\, dx\, dt \geq \\ &\geq \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t (\phi(P^j_\varepsilon, t) - w^j_{\varepsilon,\phi})(w^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))\, ds\, dt. \end{aligned} \tag{3.6}$$

From (3.5) and (3.6), we derive equality (3.4). Then, taking in (3.2) $v = h + w^j_{\varepsilon,\phi} - \phi_{j,\varepsilon}$, where $h \in \mathcal{K}_{j,\varepsilon}$, we obtain (3.3). Thus, we have proved the equivalence of the two formulations.

Using the penalty method we will prove the following result.

Theorem 3.1. *Let $\phi(x,t) = \psi(x)\eta(t)$, where $\psi \in C^\infty(\overline{\Omega})$ and $\eta \in C^1([0,T])$. Then, the problem (3.1) has a unique solution that satisfies the estimates*

$$\begin{gathered} \|\nabla w^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}))} + \varepsilon^{-\gamma} \max_{[0,T]} \|w^j_{\varepsilon,\phi}\|^2_{L^2(\partial G^j_\varepsilon)} \leq K\varepsilon^n \|\phi\|^2_{L^2(0,T;C(\overline{\Omega}))}, \\ \|w^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon}))} \leq K\varepsilon^{n+2} \|\phi\|^2_{L^2(0,T;C(\overline{\Omega}))}, \end{gathered} \tag{3.7}$$

and the estimate

$$\varepsilon^{-\gamma} \|\partial_t w^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(\partial G^j_\varepsilon))} \leq K\varepsilon^n \|\phi\|^2_{L^2(0,T;C(\overline{\Omega}))}. \tag{3.8}$$

Proof. We consider the penalty problem associated with the auxiliary problem (3.1)

$$\begin{gathered} \Delta w^{j,\delta}_{\varepsilon,\phi} = 0, \quad x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ t \in (0,T), \\ \varepsilon^{-\gamma} \partial_t w^{j,\delta}_{\varepsilon,\phi} + \partial_\nu w^{j,\delta}_{\varepsilon,\phi} - \varepsilon^{-\gamma} \partial_t \phi(P^j_\varepsilon, t) \\ - \delta^{-1} \varepsilon^{-\gamma} (\phi(P^j_\varepsilon, t) - w^{j,\delta}_{\varepsilon,\phi})^- = 0, \quad x \in \partial G^j_\varepsilon,\ t \in (0,T), \\ w^{j,\delta}_{\varepsilon,\phi}(x,t) = 0, \quad x \in \partial T^j_{\varepsilon/4},\ t \in (0,T), \\ w^{j,\delta}_{\varepsilon,\phi}(x,0) = \phi(P^j_\varepsilon, 0), \quad x \in \partial G^j_\varepsilon. \end{gathered} \tag{3.9}$$

We say that a function $w^{j,\delta}_{\varepsilon,\phi} \in L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4}))\cap C([0,T];L^2(\partial G^j_\varepsilon))$ is a strong solution of the penalized problem if $\partial_t w^{j,\delta}_{\varepsilon,\phi} \in L^2(0,T;L^2(\partial G^j_\varepsilon))$, $w^{j,\delta}_{\varepsilon,\phi}(x,0) = \phi(P^j_\varepsilon,0)$ on ∂G^j_ε, and it satisfies the integral identity

$$\begin{aligned}
&\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^{j,\delta}_{\varepsilon,\phi}\nabla v\,dx\,dt + \varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon} \partial_t w^{j,\delta}_{\varepsilon,\phi} v\,ds\,dt\\
&- \varepsilon^{-\gamma}\delta^{-1}\int_0^T\int_{\partial G^j_\varepsilon}(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^- v\,ds\,dt \qquad (3.10)\\
&= \varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t\phi(P^j_\varepsilon,t)v\,ds\,dt,
\end{aligned}$$

for all arbitrary functions $v \in L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4}))$.

This problem was investigated in [17, Theorem 5.1]. It was proved that there exists a unique solution and it satisfies the estimates

$$\begin{aligned}
&\varepsilon^{-\gamma}\|w^{j,\delta}_{\varepsilon,\phi}\|^2_{C([0,T];L^2(\partial G^j_\varepsilon))} + \|w^{j,\delta}_{\varepsilon,\phi}\|^2_{L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4}))}\\
&+ \delta^{-1}\varepsilon^{-\gamma}\|(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^-\|^2_{L^2(0,T;L^2(\partial G^j_\varepsilon))}\\
&\le K\varepsilon^n \max_{\overline{\Omega}}\|\phi\|^2_{L^2(0,T)},\\
&\quad\varepsilon^{-\gamma}\|\partial_t w^{j,\delta}_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(\partial G^j_\varepsilon))} + \max_{[0,T]}\|\nabla w^{j,\delta}_{\varepsilon,\phi}\|^2_{L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})}\\
&\quad+ \delta^{-1}\varepsilon^{-\gamma}\max_{[0,T]}\|(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^-\|^2_{L^2(\partial G^j_\varepsilon)}\\
&\quad\le K\varepsilon^n(\max_{\overline{Q^T}}|\phi|^2 + \max_{\overline{\Omega}}\|\partial_t\phi\|^2_{L^2(0,T)}).
\end{aligned}$$

From these estimates, we have that there exists a subsequence (still denoted as the original one) such that, as $\delta \to 0$, we have

$$\begin{gathered}
w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup w^j_{\varepsilon,\phi} \quad \text{weakly in } L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4})),\\
\partial_t w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup \partial_t w^j_{\varepsilon,\phi} \quad \text{weakly in } L^2(0,T;L^2(\partial G^j_\varepsilon)),\\
(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^- \to 0 \quad \text{in } L^2(0,T;L^2(\partial G^j_\varepsilon)),\\
w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup w^j_{\varepsilon,\phi} \quad \text{weakly in } L^2(0,T;L^2(\partial G^j_\varepsilon)),\\
w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup w^j_{\varepsilon,\phi} \quad \text{weakly in } L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})).
\end{gathered}$$

Then, by the compactness result of [5, Theorem 2.1], we conclude that

$$w^{j,\delta}_{\varepsilon,\phi} \to w^j_{\varepsilon,\phi} \quad \text{in } C([0,T];L^2(\partial G^j_\varepsilon)),$$

as $\delta \to 0$. From the above convergences we conclude that the limit function $w^j_{\varepsilon,\phi}$ satisfies the analogous estimates as $w^{j,\delta}_{\varepsilon,\phi}$.

Now we show that $w^j_{\varepsilon,\phi}$ is the strong solution to the problem (3.1). Indeed, using the convergence of $(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^-$ to zero, it is easy to see that $w^j_{\varepsilon,\phi} \in \mathcal{K}_{j,\varepsilon}$. Next, we take $v - w^{j,\delta}_{\varepsilon,\phi}$, where $v \in \mathcal{K}_{j,\varepsilon}$ as a test function in the integral identity for $w^{j,\delta}_{\varepsilon,\phi}$, we obtain

$$\begin{aligned}&\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^{j,\delta}_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt\\&+\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^{j,\delta}_{\varepsilon,\phi}\nabla(v-w^{j,\delta}_{\varepsilon,\phi})\,dx\,dt\\&=\delta^{-1}\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})^-(v-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt\ge 0.\end{aligned}$$

From the above convergences we derive that

$$\begin{aligned}&\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})dsdt\\&=\frac{\varepsilon^{-\gamma}}{2}\|\phi(P^j_\varepsilon,T)-w^j_{\varepsilon,\phi}(x,T)\|^2_{L^2(\partial G^j_\varepsilon)}\\&\le\frac{\varepsilon^{-\gamma}}{2}\lim_{\delta\to0}\|\phi(P^j_\varepsilon,T)-w^{j,\delta}_{\varepsilon,\phi}(x,T)\|^2_{L^2(\partial G^j_\varepsilon)}\\&=\lim_{\delta\to0}\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt.\end{aligned}$$

From this and the above convergences we conclude that

$$\begin{aligned}&\lim_{\delta\to0}\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^{j,\delta}_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt\\&=\lim_{\delta\to0}\Big(\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^{j,\delta}_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt\\&\quad+\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^{j,\delta}_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-\phi(P^j_\varepsilon,t))\,ds\,dt\Big)\\&\le\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})\,ds\,dt\\&\quad+\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-\phi(P^j_\varepsilon,t))\,ds\,dt\\&=\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^j_{\varepsilon,\phi})\,ds\,dt.\end{aligned}$$

Thus, we have proved that

$$\begin{aligned}&\lim_{\delta\to0}\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^{j,\delta}_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^{j,\delta}_{\varepsilon,\phi})\,ds\,dt\\&\le\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^j_{\varepsilon,\phi})\,ds\,dt.\end{aligned}\tag{3.11}$$

The above convergences also imply that

$$\|\nabla w^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}\le\lim_{\delta\to0}\|\nabla w^{j,\delta}_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}.$$

From this inequality we deduce that

$$\lim_{\delta\to 0}\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^{j,\delta}_{\varepsilon,\phi}\nabla(v-w^{j,\delta}_{\varepsilon,\phi})\,dx\,dt\le\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^{j}_{\varepsilon,\phi}\nabla(v-w^{j}_{\varepsilon,\phi})\,dx\,dt.$$

Combining the above results we derive that $w^j_{\varepsilon,\phi}$ satisfies

$$\begin{aligned}&\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))(v-w^j_{\varepsilon,\phi})\,ds\,dt\\&+\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^j_{\varepsilon,\phi}\nabla(v-w^j_{\varepsilon,\phi})\,dx\,dt\ge 0,\end{aligned}$$

for any arbitrary function $v\in\mathcal{K}_{j,\varepsilon}$. Hence, $w^j_{\varepsilon,\phi}$ is a strong solution of the problem (3.1). The uniqueness of the strong solution is consequence of the monotonicity of the associate operator (as in [17]). ☐

The next theorem gives a pointwise estimate for $w^j_{\varepsilon,\phi}$.

Theorem 3.2. *The solution to the problem* (3.1) *satisfies the estimate*

$$\sup_{(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon})\times(0,T)}|w^j_{\varepsilon,\phi}|\le 2\max_{\overline{Q^T}}|\phi(x,t)|. \tag{3.12}$$

Proof. We denote $h^{j,\delta}_{\varepsilon,\phi}(x,t)=\phi(P^j_\varepsilon,t)-w^{j,\delta}_{\varepsilon,\phi}$. It is easy to see that $h^{j,\delta}_{\varepsilon,\phi}$ satisfies the problem

$$\begin{gathered}\Delta h^{j\delta}_{\varepsilon,\phi}=0,\quad x\in T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\ t\in(0,T),\\ \varepsilon^{-\gamma}\partial_t h^{j,\delta}_{\varepsilon,\phi}+\partial_\nu h^{j,\delta}_{\varepsilon,\phi}+\delta^{-1}\varepsilon^{-\gamma}(h^{j,\delta}_{\varepsilon,\phi})^-=0,\quad x\in\partial G^j_\varepsilon,\ t\in(0,T),\\ h^{j,\delta}_{\varepsilon,\phi}(x,t)=\phi(P^j_\varepsilon,t),\quad x\in\partial T^j_{\varepsilon/4},\ t\in(0,T),\\ h^{j,\delta}_{\varepsilon,\phi}(x,0)=0,\quad x\in\partial G^j_\varepsilon.\end{gathered}\tag{3.13}$$

We define $K=\max_{\overline{Q^T}}|\phi(x,t)|$. Then take $v=(K-h^{j,\delta}_{\varepsilon,\phi})^-\in L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4}))$ as a test function in the integral identity for problem (3.13),

$$\begin{aligned}&-\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}|\nabla(K-h^{j,\delta}_{\varepsilon,\phi})^-|^2\,dx\,dt-\frac{\varepsilon^{-\gamma}}{2}\|(K-h^{j,\delta}_{\varepsilon,\phi})^-(T)\|^2_{L^2(\partial G^j_\varepsilon)}\\&+\delta^{-1}\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}(h^{j,\delta}_{\varepsilon,\phi})^-(K-h^{j,\delta}_{\varepsilon,\phi})^-\,ds\,dt=0.\end{aligned}\tag{3.14}$$

Suppose now that $h^{j,\delta}_{\varepsilon,\phi}\le 0$, then $K-h^{j,\delta}_{\varepsilon,\phi}\ge 0$. So, $(h^{j,\delta}_{\varepsilon,\phi})^-(K-h^{j,\delta}_{\varepsilon,\phi})^-=0$ on ∂G^j_ε. Thus, from (3.14), we deduce that $(K-h^{j,\delta}_{\varepsilon,\phi})^-\equiv 0$ for $x\in T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}$, $t\in(0,T)$. Hence

$$w^{j,\delta}_{\varepsilon,\phi}\ge -K+\phi\ge -2K.$$

Similarly, taking $v = (K + h^{j,\delta}_{\varepsilon,\phi})^- \in L^2(0,T;H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \partial T^j_{\varepsilon/4}))$ as a test function in the integral identity for problem (3.13), we obtain

$$\begin{aligned} &\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} |\nabla(K + h^{j,\delta}_{\varepsilon,\phi})^-|^2 \,dx\,dt + \frac{\varepsilon^{-\gamma}}{2}\|(K + h^{j,\delta}_{\varepsilon,\phi})^-(T)\|^2_{L^2(\partial G^j_\varepsilon)} \\ &+ \delta^{-1}\varepsilon^{-\gamma}\int_0^T \int_{\partial G^j_\varepsilon} (h^{j,\delta}_{\varepsilon,\phi})^-(K + h^{j,\delta}_{\varepsilon,\phi})^- \,ds\,dt = 0. \end{aligned} \tag{3.15}$$

All of the terms in (3.15) are non-negative, thus, we conclude that $K + h^{j,\delta}_{\varepsilon,\phi} \geq 0$ and so $w^{j,\delta}_{\varepsilon,\phi} \leq \phi(P^j_\varepsilon, t) + K \leq 2K$.

Using standard arguments of the penalty method, one can show that $w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup w^j_{\varepsilon,\phi}$ weakly in $L^2(0,T;H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}))$ as $\delta \to 0$. Now, consider a function $g \in L^2(0,T;H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}))$ such that $g \geq 0$ and multiply by it in both sides of the inequality $w^{j,\delta}_{\varepsilon,\phi} - 2K \leq 0$. Then, we integrate the obtained expression over $(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}) \times (0,T)$ and obtain

$$\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} (w^{j,\delta}_{\varepsilon,\phi} - 2K)g \,dx\,dt \leq 0.$$

Using the weak convergence, we pass to the limit as $\delta \to 0$ in the above inequality and derive

$$\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} (w^j_{\varepsilon,\phi} - 2K)g \,dx\,dt \leq 0,$$

where g is an arbitrary non-negative function as before. Taking $g = (w^j_{\varepsilon,\phi} - 2K)^+$, we obtain

$$\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} |(w^j_{\varepsilon,\phi} - 2K)^+|^2 \,dx\,dt \leq 0.$$

Thus, we conclude that $w^j_{\varepsilon,\phi} - 2K \leq 0$ for a.e. $x \in \Omega$, $t \in (0,T)$. Analogously, we can get the opposite estimate and then we arrive to (3.12). □

Remark 3.3. From the penalty method we have the weak convergence of $w^{j,\delta}_{\varepsilon,\phi}$ to $w^j_{\varepsilon,\phi}$ in $L^2(0,T;H^1(T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}))$. Nevertheless, we can show that the above convergence is in fact strong. Indeed, the penalty method also implies that $\partial_t w^{j,\delta}_{\varepsilon,\phi} \rightharpoonup \partial_t w^j_{\varepsilon,\phi}$ weakly in $L^2(0,T;L^2(\partial G^j_\varepsilon))$. As mentioned before we have $w^{j,\delta}_{\varepsilon,\phi} \to w^j_{\varepsilon,\phi}$ in $C([0,T];L^2(\partial G^j_\varepsilon))$ as $\delta \to 0$. From the integral equality for problem (3.9), we obtain

$$\begin{aligned} &\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^{j,\delta}_{\varepsilon,\phi} \nabla(\psi - w^{j,\delta}_{\varepsilon,\phi}) \,dx\,dt \\ &+ \varepsilon^{-\gamma}\int_0^T \int_{\partial G^j_\varepsilon} \partial_t(w^{j,\delta}_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))(\psi - w^{j,\delta}_{\varepsilon,\phi}) \,ds\,dt \\ &= \delta^{-1}\varepsilon^{-\gamma}\int_0^T \int_{\partial G^j_\varepsilon} (\phi(P^j_\varepsilon, t) - w^{j,\delta}_{\varepsilon,\phi})^-(\psi - w^{j,\delta}_{\varepsilon,\phi}) \,ds\,dt \end{aligned}$$

$$= \delta^{-1}\varepsilon^{-\gamma}\int_0^T \int_{\partial G_\varepsilon^j}((\phi(P_\varepsilon^j,t) - w_{\varepsilon,\phi}^{j,\delta})^- - (\phi(P_\varepsilon^j,t)-\psi)^-)(\psi - w_{\varepsilon,\phi}^{j,\delta})\,ds\,dt$$

for an arbitrary function $\psi \in \mathcal{K}_{j,\varepsilon}$ (i.e. $(\phi(P_\varepsilon^j,t)-\psi)^- \equiv 0$ on ∂G_ε^j). The function $\lambda \to -(C-\lambda)^-$ is monotone non-decreasing, hence, the right-hand side of the derived expression is non-negative. Thus, we have the same inequality as (3.2) for the $w_{\varepsilon,\varphi}^{j,\delta}$. From here, we obtain

$$\begin{aligned}
&\|\nabla w_{\varepsilon,\phi}^{j,\delta}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}\\
&\le \int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w_{\varepsilon,\phi}^{j,\delta}\nabla\psi\,dx\,dt + \varepsilon^{-\gamma}\int_0^T\int_{\partial G_\varepsilon^j}\partial_t(w_{\varepsilon,\phi}^{j,\delta} - \phi(P_\varepsilon^j,t))(\psi - w_{\varepsilon,\phi}^{j,\delta})\,ds\,dt
\end{aligned}$$

For the second integral on the right-hand side of the inequality, we use (3.11), and passing to the limit as $\delta \to 0$, we obtain

$$\begin{aligned}
&\lim_{\delta\to 0}\|\nabla w_{\varepsilon,\phi}^{j,\delta}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}\\
&\le \int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w_{\varepsilon,\phi}^{j}\nabla\psi\,dx\,dt + \varepsilon^{-\gamma}\int_0^T\int_{\partial G_\varepsilon^j}\partial_t(w_{\varepsilon,\phi}^{j} - \phi(P_\varepsilon^j,t))(\psi - w_{\varepsilon,\phi}^{j})\,ds\,dt
\end{aligned}$$

Taking $\psi = w^j_{\varepsilon,\phi}$, we obtain

$$\lim_{\delta\to 0}\|\nabla w_{\varepsilon,\phi}^{j,\delta}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))} \le \|\nabla w_{\varepsilon,\phi}^{j}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}$$

Using this estimate and the properties of the weak convergence, we derive

$$\lim_{\delta\to 0}\|\nabla w_{\varepsilon,\phi}^{j,\delta}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))} = \|\nabla w_{\varepsilon,\phi}^{j}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}$$

Thus, $w_{\varepsilon,\phi}^{j,\delta}$ converges to $w^j_{\varepsilon,\phi}$ strongly in $L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))$ as $\delta\to 0$.

3.2. **Second auxiliary problem: a different unilateral exterior problem on the cell variables.** To define the notion of a strong solution to the exterior Signorini problem, we denote by $\mathfrak{M}$ the set of functions $w \in C^\infty(\mathbb{R}^n\setminus\overline{G_0})$ such that $w(y)=0$ for $y\in\mathbb{R}^n\setminus\overline{T_R^o}$ for some ball T_R^0 such that $\overline{G_0}\subset T_R^0$. We denote by $\mathcal{M}$ the closure of $\mathfrak{M}$ with respect to the norm $\|w\|_{\mathcal{M}} = \|\nabla w\|_{L^2(\mathbb{R}^n\setminus\overline{G_0})}$.

Given $\phi \in H^1(0,T;H^1(\Omega))$ let us consider the exterior problem with dynamic Signorini boundary condition

$$\begin{gathered}
\Delta_y w_\phi(x,y,t) = 0, \quad y\in\mathbb{R}^n\setminus\overline{G_0},\ t\in(0,T),\\
w_\phi \le \phi(x,t), \quad y\in\partial G_0,\ t\in(0,T),\\
\partial_\nu w_\phi \le C_0(\partial_t\phi - \partial_t w_\phi), \quad y\in\partial G_0,\ t\in(0,T),\\
(w_\phi-\phi)(\partial_\nu w_\phi - C_0(\partial_t\phi-\partial_t w_\phi)) = 0, \quad y\in\partial G_0,\ t\in(0,T),\\
w_\phi(x,y,0) = \phi(x,0), \quad y\in\partial G_0,\\
w_\phi(x,y,t)\to 0, \quad |y|\to\infty,
\end{gathered} \tag{3.16}$$

where now $x\in\Omega$ is a parameter.

For $x\in\Omega$ and $t\in[0,T]$, we define the closed convex sets

$$\begin{gathered}
K_\phi(x) = \{v\in\mathcal{M} : v\le\phi \text{ for a.e. } y\in\partial G_0\},\\
\mathcal{K}_\phi(x) = \{v\in L^2(0,T;\mathcal{M}) : v\in K_\phi(x) \text{ for a.e. } t\in[0,T]\}.
\end{gathered}$$

By a strong solution to the problem (3.16), we mean a function $w_\phi(x,\cdot,\cdot) \in \mathcal{K}_\phi(x)$ such that $w_\phi \in C([0,T];L^2(\partial G_0))$, $\partial_t w_\phi \in L^2(0,T;L^2(\partial G_0))$ and $w_\phi(x,y,0) = \phi(x,0)$ for a.e. $x \in \Omega$, and w_ϕ satisfies the variational inequality

$$\int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla w_\phi \nabla(v - w_\phi)\,dy\,dt \geq C_0 \int_0^T \int_{\partial G_0} \partial_t(\phi - w_\phi)(v - w_\phi)\,ds\,dt, \quad (3.17)$$

for any arbitrary test function $v \in \mathcal{K}_\phi(x)$, for a.e. $x \in \Omega$.

We define now the function

$$\hat{w}_\phi(x,y,t) = \phi(x,t)\kappa(y) - w_\phi(x,y,t), \quad (3.18)$$

where $\kappa(y)$ is the solution to the G_0 − capacity problem in the cell variables

$$\begin{gathered} \Delta_y \kappa = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0}, \\ \kappa(y) = 1, \quad y \in \partial G_0, \\ \kappa(y) \to 0, \quad |y| \to \infty. \end{gathered} \quad (3.19)$$

Clearly, the function $\hat{w}_\phi(x,y,t)$ is a solution to the unilateral problem in the cell variables

$$\begin{gathered} \Delta_y \hat{w}_\phi(x,y,t) = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0},\ t \in (0,T), \\ \hat{w}_\phi \geq 0,\ \partial_\nu \hat{w}_\phi + C_0 \partial_t \hat{w}_\phi \geq \phi(x,t)\partial_\nu \kappa(y), \quad y \in \partial G_0,\ t \in (0,T), \\ \hat{w}_\phi(\partial_\nu \hat{w}_\phi + C_0 \partial_t \hat{w}_\phi - \phi(x,t)\partial_\nu\kappa(y)) = 0, \quad y \in \partial G_0,\ t \in (0,T), \\ \hat{w}_\phi(x,y,0) = 0, \quad y \in \partial G_0, \\ \hat{w}_\phi \to 0, \quad |y| \to +\infty. \end{gathered} \quad (3.20)$$

This reformulation of the auxiliary function w_ϕ reduces the conditions imposed on the function ϕ as the problem (3.20) doesn't contain $\partial_t \phi$. Thus, here and in the theorems below, we consider $\phi \in L^2(0,T;L^2(\Omega))$ unless otherwise stated explicitly.

Theorem 3.4. *Problem* (3.20) *has a unique strong solution and, for a.e. $x \in \Omega$, the following estimates hold*

$$\begin{gathered} \|\hat{w}_\phi(x,\cdot,\cdot)\|_{C([0,T];L^2(\partial G_0))} + \|\hat{w}_\phi(x,\cdot,\cdot)\|_{C([0,T];\mathcal{M})} \leq K\|\phi(x,\cdot)\|_{L^2(0,T)}, \\ \|\partial_t \hat{w}_\phi(x,\cdot,\cdot)\|_{L^2(0,T;L^2(\partial G_0))} \leq K\|\phi(x,\cdot)\|_{L^2(0,T)}. \end{gathered} \quad (3.21)$$

Proof. Given $\delta > 0$, let us consider, again, the auxiliary penalty formulation in the cell variables

$$\begin{gathered} \Delta_y \hat{w}^\delta_\phi(x,y,t) = 0, \quad y \in \mathbb{R}^n \setminus \overline{G_0},\ t \in (0,T), \\ \partial_\nu \hat{w}^\delta_\phi + C_0 \partial_t \hat{w}^\delta_\phi + \delta^{-1}(\hat{w}^\delta_\phi)^- = \phi \partial_\nu \kappa, \quad y \in \partial G_0,\ t \in (0,T), \\ \hat{w}^\delta_\phi(x,y,0) = 0, \quad y \in \partial G_0, \\ \hat{w}^\delta_\phi \to 0, \quad |y| \to \infty, \end{gathered} \quad (3.22)$$

where $x \in \Omega$ is taken as a parameter. It is well known (see [20, 24]) that the problem (3.22) has a unique strong solution, and that for a.e. $x \in \Omega$, the following estimates hold

$$\begin{gathered} \|\hat{w}^\delta_\phi(x,\cdot,\cdot)\|_{C([0,T];L^2(\partial G_0))} + \|\hat{w}^\delta_\phi(x,\cdot,\cdot)\|_{C([0,T];\mathcal{M})} \leq K\|\phi(x,\cdot)\|_{L^2(0,T)}, \\ \|\partial_t \hat{w}^\delta_\phi(x,\cdot,\cdot)\|_{L^2(0,T;L^2(\partial G_0))} \leq K\|\phi(x,\cdot)\|_{L^2(0,T)}. \end{gathered} \quad (3.23)$$

Also, it is easy to obtain that

$$\|(\hat{w}^\delta_\phi)^-(x,\cdot,\cdot)\|_{L^2(0,T;L^2(\partial G_0))} \leq K\sqrt{\delta}\|\phi(x,\cdot)\|_{L^2(0,T)}. \quad (3.24)$$

From estimates (3.23), (3.24), we conclude that, for some subsequence, as $\delta \to 0$ we have

$$\begin{gathered} \hat{w}_\phi^\delta \rightharpoonup \hat{w}_\phi \quad \text{weakly in } L^2(0,T;\mathcal{M}), \\ \partial_t \hat{w}_\phi^\delta \rightharpoonup \partial_t \hat{w}_\phi \quad \text{weakly in } L^2(0,T;L^2(\partial G_0)), \\ \hat{w}_\phi^\delta \rightharpoonup \hat{w}_\phi \quad \text{weakly in } L^2(0,T;L^2(\partial G_0)). \end{gathered} \tag{3.25}$$

Then, by the compactness result of [5, Theorem 2.1], we conclude that

$$\hat{w}_\phi^\delta \to \hat{w}_\phi \quad \text{in } C([0,T];L^2(\partial G_0)),$$

as $\delta \to 0$. The integral identity for problem (3.22) takes the form

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla \hat{w}_\phi^\delta \nabla(\psi - \hat{w}_\phi^\delta)\,dy\,dt + C_0 \int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi^\delta (\psi - \hat{w}_\phi^\delta) ds_y dt \\ &+ \delta^{-1} \int_0^T \int_{\partial G_0} (\hat{w}_\phi^\delta)^- (\psi - \hat{w}_\phi^\delta) ds_y dt \\ &= \int_0^T \phi(x,t) \int_{\partial G_0} \partial_\nu \kappa(y)(\psi - \hat{w}_\phi^\delta) ds_y dt, \end{aligned} \tag{3.26}$$

where $\psi \in L^2(0,T;\mathcal{M})$, $\psi \geq 0$ on ∂G_0 for a.e. $t \in [0,T]$, $x \in \Omega$ is a parameter. We rewrite it in the form

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla \hat{w}_\phi^\delta \nabla(\psi - \hat{w}_\phi^\delta)\,dy\,dt + C_0 \int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi^\delta (\psi - \hat{w}_\phi^\delta) ds_y dt \\ &- \int_0^T \phi(x,t) \int_{\partial G_0} \partial_\nu \kappa(y)(\psi - \hat{w}_\phi^\delta) ds_y dt \\ &= \delta^{-1} \int_0^T \int_{\partial G_0} (\psi^- - (\hat{w}_\phi^\delta)^-)(\psi - \hat{w}_\phi^\delta) ds_y dt \geq 0, \end{aligned} \tag{3.27}$$

where we have used that $\psi \geq 0$ on ∂G_0 a.e. $t \in [0,T]$ and that the real function $\lambda \to \lambda^-$ is a Lipschitz and monotone function. Now, we are going to pass to the limit as $\delta \to 0$ in the inequality (3.27). Taking into account that

$$\|\hat{w}_\phi(x,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})} \leq \varliminf_{\delta\to 0} \|\hat{w}_\phi^\delta(x,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})},$$

we conclude that

$$\varlimsup_{\delta\to 0} \int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla \hat{w}_\phi^\delta \nabla(\psi - \hat{w}_\phi^\delta)\,dy\,dt \leq \int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla \hat{w}_\phi \nabla(\psi - \hat{w}_\phi)\,dy\,dt. \tag{3.28}$$

From the convergences (3.25), we have

$$\begin{aligned} &C_0 \varlimsup_{\delta\to 0} \int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi^\delta (\psi - \hat{w}_\phi^\delta) ds_y dt \\ &= C_0 \varlimsup_{\delta\to 0} \Big(\int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi^\delta \psi ds_y dt - \frac{1}{2} \|\hat{w}_\phi^\delta(\cdot,T)\|^2_{L^2(\partial G_0)} \Big) \\ &\leq C_0 \Big(\int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi \psi ds_y dt - \frac{1}{2} \|\hat{w}_\psi(\cdot,T)\|^2_{L^2(\partial G_0)} \Big) \\ &= C_0 \int_0^T \int_{\partial G_0} \partial_t \hat{w}_\phi (\psi - \hat{w}_\phi) ds_y dt. \end{aligned} \tag{3.29}$$

From (3.29)-(3.31) we conclude that

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla \hat{w}_\phi \nabla(\psi-\hat{w}_\phi)\,dy\,dt + C_0\int_0^T\int_{\partial G_0}\partial_t\hat{w}_\phi(\psi-\hat{w}_\phi)\,ds\,dt \\ &\geq \int_0^T \phi(x,t)\int_{\partial G_0}\partial_\nu\kappa(y)(\psi-\hat{w}_\phi)ds_y dt, \end{aligned} \tag{3.30}$$

where $\psi\in L^2(0,T;\mathcal{M})$ and $\psi\geq 0$ for a.e. $y\in\partial G_0$, $t\in[0,T]$. The inequality (3.30) is a variational formulation of the problem (3.20), hence, $\tilde{w}_\phi$ is a solution.

Then, estimates (3.21) are a consequence of (3.23), (3.24) and the weak convergences (3.25). □

Remark 3.5. Notice that we can transform the inequality (3.27) into

$$\begin{aligned} &\|\hat{w}^\delta_\phi(x,\cdot,\cdot)\|^2_{L^2(0,T;\mathcal{M})} + \frac{C_0}{2}\|\hat{w}^\delta_\phi(x,\cdot,T)\|^2_{L^2(\partial G_0)} \\ &\leq \int_0^T\int_{\mathbb{R}^n\setminus\overline{G_0}}\nabla w^\delta_\phi\nabla\psi dydt + C_0\int_0^T\int_{\partial G_0}\partial_t\hat{w}^\delta_\phi\psi ds_y dt \\ &\quad -\int_0^T\phi(x,t)\int_{\partial G_0}\partial_\nu\kappa(y)(\psi-\hat{w}^\delta_\phi)ds_y dt. \end{aligned}$$

Using Hardy's inequality, we have $\hat{w}^\delta_\phi(x,\cdot,T)$ is uniformly bounded in δ in $H^1(T^0_{R_0}\setminus\overline{G_0})$, where $R_0>0$ is large enough to have $\overline{G_0}\subset T^0_{R_0}$. From the embedding theorem, we have that there exists a subsequence (for which we preserve the notation of the original) such that $\hat{w}^\delta_\phi(x,\cdot,T)$ converges to $\hat{w}_\phi(x,\cdot,T)$ in $L^2(\partial G_0)$ as $\delta\to 0$. Passing to the limit as $\delta\to 0$, we obtain

$$\begin{aligned} &\lim_{\delta\to 0}(\|\hat{w}^\delta_\phi(x,\cdot,\cdot)\|^2_{L^2(0,T;\mathcal{M})} + \frac{C_0}{2}\|\hat{w}^\delta_\phi(x,\cdot,T)\|^2_{L^2(\partial G_0)}) \\ &\leq \int_0^T\int_{\mathbb{R}^n\setminus\overline{G_0}}\nabla \hat{w}_\phi\nabla\psi dydt + C_0\int_0^T\int_{\partial G_0}\partial_t\hat{w}_\phi\psi\, ds_y\, dt \\ &\quad -\int_0^T\phi(x,t)\int_{\partial G_0}\partial_\nu\kappa(y)(\psi-\hat{w}_\phi)\,ds_y\, dt. \end{aligned}$$

Taking $\psi=\hat{w}_\phi$ as a test function in this inequality, we derive

$$\lim_{\delta\to 0}\|\hat{w}^\delta_\phi(x,\cdot,\cdot)\|^2_{L^2(0,T;\mathcal{M})} \leq \|\hat{w}_\phi(x,\cdot,\cdot)\|^2_{L^2(0,T;\mathcal{M})}.$$

Hence, we actually have the strong convergence $\hat{w}^\delta_\phi\to\hat{w}_\phi$ in $L^2(0,T;\mathcal{M})$, as $\delta\to 0$.

Theorem 3.6. *Let ϕ_1, $\phi_2\in L^2(0,T;L^2(\Omega))$. Then, for a.e. $x\in\Omega$, we have*

$$\begin{aligned} &\|(\hat{w}_{\phi_1}-\hat{w}_{\phi_2})(x,\cdot,\cdot)\|_{C([0,T];L^2(\partial G_0))} + \|(\hat{w}_{\phi_1}-\hat{w}_{\phi_2})(x,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})} \\ &\leq K\|(\phi_1-\phi_2)(x,\cdot)\|_{L^2(0,T)}. \end{aligned} \tag{3.31}$$

In addition, we have the following estimate for the time derivative of the functions

$$\|(\partial_t\hat{w}_{\phi_1}-\partial_t\hat{w}_{\phi_2})(x,\cdot,\cdot)\|_{L^2(0,T;L^2(\partial G_0))} \leq K\|(\phi_1-\phi_2)(x,\cdot)\|_{L^2(0,T)}, \tag{3.32}$$

for a.e. $x\in\Omega$.

Proof. Let $\hat{v} = \hat{w}_{\phi_1} - \hat{w}_{\phi_2}$. We use again the penalized problem (3.22). If we define $\hat{v}^\delta = \hat{w}^\delta_{\phi_1} - \hat{w}^\delta_{\phi_2}$ then, according to (3.25), we have

$$\begin{gathered} \hat{v}^\delta \rightharpoonup \hat{v} \quad \text{weakly in } L^2(0,T;\mathcal{M}), \\ \partial_t \hat{v}^\delta \rightharpoonup \partial_t \hat{v} \quad \text{weakly in } L^2(0,T;L^2(\partial G_0)), \\ \hat{v}^\delta \rightharpoonup \hat{v} \quad \text{weakly in } L^2(0,T;L^2(\partial G_0)). \end{gathered} \tag{3.33}$$

For the solution of the function $\hat{v}^\delta$, we have some estimates similar to the estimates obtained in [20],

$$\begin{gathered} \|\hat{v}^\delta(x,\cdot,\cdot)\|_{C([0,T];L^2(\partial G_0))} + \|\hat{v}^\delta(x,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})} \\ \le K\|(\phi_1-\phi_2)(x,\cdot)\|_{L^2(0,T)}, \\ \|\partial_t \hat{v}^\delta(x,\cdot,\cdot)\|_{L^2(0,T;L^2(\partial G_0))} \le K\|(\phi_1-\phi_2)(x,\cdot)\|_{L^2(0,T)}, \end{gathered} \tag{3.34}$$

where the constant K does not depend on δ neither on ϕ_i, $i=1,2$. Then, using the convergence (3.33), we conclude that the estimates (3.31), (3.32) follow from (3.34). □

Theorem 3.7. *Consider a test function of the form $\phi(x,t) = \psi(x)\eta(t)$, with $\psi(x) \in C^\infty(\overline{\Omega})$ and $\eta \in C^1([0,T])$. Then, for a.e. $t \in [0,T]$ and $x \in \Omega$, we have*

$$|\hat{w}_\phi(x,y,t)| \le \frac{K \max_{\overline{Q}^T} |\phi(x,t)|}{|y|^{n-2}} \quad \text{for } y \in \mathbb{R}^n \setminus \overline{G_0}, \tag{3.35}$$

$$|\nabla_y w_\phi(x,y,t)| \le \frac{K \max_{\overline{Q}^T} |\phi(x,t)|}{|y|^{n-1}}, \quad \text{for } y \in \mathbb{R}^n \setminus \overline{G_0}, \tag{3.36}$$

where the positive constant K is independent of ϕ, $x \in \Omega$ and $t \in [0,T]$.

Proof. Once again, we consider the solution $\hat{w}^\delta_\phi$ of the associate penalized problem (3.22). Then we have the point-wise estimates (see [20])

$$|\hat{w}^\delta_\phi(x,y,t)| \le \frac{K \max_{\overline{Q}^T} |\phi(x,t)|}{|y|^{n-2}}, \quad \forall y \in \mathbb{R}^n \setminus \overline{G_0}, \tag{3.37}$$

$$|\nabla_y \hat{w}^\delta_\phi(x,y,t)| \le \frac{K \max_{\overline{Q}^T} |\phi(x,t)|}{|y|^{n-1}}, \quad \forall y \in \mathbb{R}^n \setminus \overline{G_0}, \tag{3.38}$$

where K is independent of ϕ, δ, $x \in \Omega$ and $t \in [0,T]$.

By Remark 3.5 we know the strong convergence of $\hat{w}^\delta_\phi$ to $\hat{w}_\phi$ in $L^2(0,T;\mathcal{M})$. Thus, we can extract a subsequence converging almost everywhere. Passing to the limit in (3.37) for this subsequence we obtain the first statement of this theorem. Using the same arguments, we can take a subsequence converging almost everywhere for the gradient of $\hat{w}^\delta_\phi$ and this proves the second part of the statement. □

Theorem 3.8. *Let ϕ as in Theorem 3.7 and let $\hat{w}_\phi$ be the solution to the problem* (3.20)*. Then, for a.e. $x_1, x_2 \in \Omega$*

$$\|\hat{w}_\phi(x_1,\cdot,\cdot) - \hat{w}_\phi(x_2,.,.)\|_{L^2(0,T;\mathcal{M})} \le K\|\phi(x_1,\cdot) - \phi(x_2,\cdot)\|_{L^2(0,T)}. \tag{3.39}$$

Proof. We use the following estimate for the solution of the penalized problem (3.22) proved in [20]

$$\|\hat{w}^\delta_\phi(x_1,\cdot,\cdot) - \hat{w}^\delta_\phi(x_2,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})} \le K\|\phi(x_1,\cdot) - \phi(x_2,\cdot)\|_{L^2(0,T)}.$$

From here and the weak convergence (3.33), we derive

$$\begin{aligned}&\|\hat{w}_\phi(x_1,\cdot,\cdot)-\hat{w}_\phi(x_2,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})}\le\\&\le\lim_{\delta\to 0}\|\hat{w}^\delta_\phi(x_1,\cdot,\cdot)-\hat{w}^\delta_\phi(x_2,\cdot,\cdot)\|_{L^2(0,T;\mathcal{M})}\le K\|\phi(x_1,\cdot)-\phi(x_2,\cdot)\|_{L^2(0,T)}.\end{aligned}$$

□

3.3. **Asymptotic similarity between the two types of auxiliary functions after a suitable substitution.** We use a natural substitution to define the function $\widetilde{w}^j_{\varepsilon,\phi}(x,t)=w_\phi(P^j_\varepsilon,\frac{x-P^j_\varepsilon}{a_\varepsilon},t)$. Then we are in conditions to get some asymptotic estimates on the difference of the two types of auxiliary functions. We define $v^j_{\varepsilon,\phi}(x,t)=\widetilde{w}^j_{\varepsilon,\phi}(x,t)-w^j_{\varepsilon,\phi}(x,t)$. Then, we have the following result.

Theorem 3.9. *Let $\phi(x,t)=\psi(x)\eta(t)$, with $\psi\in C^\infty(\overline{\Omega})$ and $\eta\in C^1([0,T])$. Then we have the following estimate for the function $v^j_{\varepsilon,\phi}$ expressing the difference of the two auxiliary functions*

$$\sup_{(T^j_{\varepsilon/4}\setminus\overline{G}^j_\varepsilon)\times(0,T)}|v^j_{\varepsilon,\phi}|\le\sup_{\partial T^j_{\varepsilon/4}\times(0,T)}|\widetilde{w}^j_{\varepsilon,\phi}|. \tag{3.40}$$

Proof. Given $\delta>0$, we consider the solutions to the penalized problems (3.9), and (3.22) and we define

$$v^{j,\delta}_{\varepsilon,\phi}(x,t)=\widetilde{w}^{j,\delta}_{\varepsilon,\phi}(x,t)-w^{j,\delta}_{\varepsilon,\phi}(x,t),$$

where

$$\widetilde{w}^{j,\delta}_{\varepsilon,\phi}(x,t)=\widetilde{w}^\delta_\phi(P^j_\varepsilon,\frac{x-P^j_\varepsilon}{a_\varepsilon},t),\quad\widetilde{w}^\delta_\phi(x,y,t)=k(y)\phi(x,t)-\hat{w}^\delta_\phi(x,y,t),$$

where $\hat{w}^\delta_\phi(x,y,t)$ was defined similarly to (3.18). From Remark 3.5, we conclude that $\widetilde{w}^{j,\delta}_{\varepsilon,\phi}$ converges to $\tilde{w}^j_{\varepsilon,\phi}$ in $L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))$ as $\delta\to 0$. From Remark 3.3, we have $w^{j,\delta}_{\varepsilon,\phi}\to w^j_{\varepsilon,\phi}$ in $L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))$ as $\delta\to 0$. Therefore, $v^{j,\delta}_{\varepsilon,\phi}\to v^j_{\varepsilon,\phi}$ in $L^2(0,T;H^1(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))$. Hence, we can obtain a sub-sequence (that we denote as the original one) that converges almost everywhere. Moreover, concerning the function $v^{j,\delta}_{\varepsilon,\phi}$, we have the estimate

$$\sup_{(T^j_{\varepsilon 4}\setminus\overline{G}^j_\varepsilon)\times(0,T)}|v^{j,\delta}_{\varepsilon,\phi}|\le\sup_{\partial T^j_{\varepsilon/4}\times(0,T)}|\widetilde{w}^{j,\delta}_{\varepsilon,\phi}|,$$

(see. [20]). From here, we derive the estimate (3.40). □

Theorem 3.10. *Let $\phi=\psi(x)\eta(t)$, $\psi\in C^\infty(\overline{\Omega})$, $\eta\in C^1([0,T])$. The following global estimates hold*

$$\begin{gathered}\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\max_{t\in[0,T]}\|v^j_{\varepsilon,\phi}\|^2_{L^2(\partial G^j_0)}+\sum_{j\in\Upsilon_\varepsilon}\|\nabla v^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}\le K\varepsilon^2\max_{\overline{Q^T}}\phi^2(x,t),\\\sum_{j\in\Upsilon_\varepsilon}\|v^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))}\le K\varepsilon^4\max_{\overline{Q^T}}\phi^2(x,t).\end{gathered} \tag{3.41}$$

Proof. Recall that $\phi_{j,\varepsilon}(x,t)=\phi(P^j_\varepsilon,t)\psi_{j,\varepsilon}(x)$, where $\psi_{j,\varepsilon}\in C^\infty_0(T^j_{\varepsilon/4})$, with $\psi_{j,\varepsilon}(x)\equiv 1$, if $x\in T^j_{Ca_\varepsilon}$, $C>1$ and $\psi_{j,\varepsilon}\equiv 0$, if $x\in T^j_{\varepsilon/4}\setminus\overline{T^j_{2Ca_\varepsilon}}$, and that $|\nabla\psi_{j,\varepsilon}|\le\frac{K}{a_\varepsilon}$.

Taking into account the definition of the $\widetilde{w}^j_{\varepsilon,\phi}$ and using that $w^j_{\varepsilon,\phi}-\phi_{j,\varepsilon}\in H^1(T^j_{\varepsilon/4}\setminus \overline{G^j_\varepsilon},\partial T^j_{\varepsilon/4})$, we obtain

$$
\begin{aligned}
&\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla\widetilde{w}^j_{\varepsilon,\phi}\nabla(w^j_{\varepsilon,\phi}-\phi_{j,\varepsilon})\,dx\,dt\\
&=-\int_0^T\int_{\partial G^j_\varepsilon}\partial_\nu\widetilde{w}^j_{\varepsilon,\phi}(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})\,ds\,dt\\
&\geq -\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(\phi(P^j_\varepsilon,t)-\widetilde{w}^j_{\varepsilon,\phi})(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})\,ds\,dt\\
&=\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(\phi(P^j_\varepsilon,t)-\widetilde{w}^j_{\varepsilon,\phi})(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))\,ds\,dt.
\end{aligned}
$$

Additionally, we have (see (3.4))

$$
\begin{aligned}
&\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla w^j_{\varepsilon,\phi}\nabla(w^j_{\varepsilon,\phi}-\phi_{j,\varepsilon})\,dx\,dt\\
&=\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))\,ds\,dt.
\end{aligned}
$$

Subtracting one expression from the other, we obtain

$$
\begin{aligned}
&\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla v^j_{\varepsilon,\phi}\nabla(w^j_{\varepsilon,\phi}-\phi_{j,\varepsilon})\,dx\,dt\\
&\geq\varepsilon^{-\gamma}\int_{\partial G^j_\varepsilon}\partial_t(w^j_{\varepsilon,\phi}-\widetilde{w}^j_{\varepsilon,\phi})(w^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))\,ds\,dt\\
&=\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t v^j_{\varepsilon,\phi}(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})\,ds\,dt.
\end{aligned}
$$

From here, we derive

$$
\begin{aligned}
&-\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}|\nabla v^j_{\varepsilon,\phi}|^2\,dx\,dt+\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla v^j_{\varepsilon,\phi}\nabla(\widetilde{w}^j_{\varepsilon,\phi}-\phi_{j,\varepsilon})\,dx\,dt\\
&\geq\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t v^j_{\varepsilon,\phi}(\phi(P^j_\varepsilon,t)-w^j_{\varepsilon,\phi})\,ds\,dt.
\end{aligned}
$$

Next, we transform this inequality into

$$
\begin{aligned}
&\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}|\nabla v^j_{\varepsilon,\phi}|^2\,dx\,dt+\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t v^j_{\varepsilon,\phi}v^j_{\varepsilon,\phi}\,ds\,dt\\
&\leq\int_0^T\int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}}\nabla v^j_{\varepsilon,\phi}\nabla(\widetilde{w}^j_{\varepsilon,\phi}-\phi_{j,\varepsilon})\,dx\,dt\\
&\quad+\varepsilon^{-\gamma}\int_0^T\int_{\partial G^j_\varepsilon}\partial_t v^j_{\varepsilon,\phi}(\widetilde{w}^j_{\varepsilon,\phi}-\phi(P^j_\varepsilon,t))\,ds\,dt=\mathcal{I}_\varepsilon.
\end{aligned}
\tag{3.42}
$$

Now, we need to estimate the right-hand side of the inequality, i.e. $\mathcal{I}_\varepsilon$. Using Green's formula, we obtain

$$\begin{aligned}
\mathcal{I}_\varepsilon &= \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt + \int_0^T \int_{\partial G^j_\varepsilon} \partial_\nu v^j_{\varepsilon,\phi} (\widetilde{w}^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))\, ds\, dt \\
&\quad + \varepsilon^{-\gamma} \int_0^T \int_{\partial G^j_\varepsilon} \partial_t v^j_{\varepsilon,\phi} (\widetilde{w}^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))\, ds\, dt \\
&= \int_0^T \int_{\partial G^j_\varepsilon} (\partial_\nu v^j_{\varepsilon,\phi} + \varepsilon^{-\gamma} \partial_t v^j_{\varepsilon,\phi})(\widetilde{w}^j_{\varepsilon,\phi} - \phi(P^j_\varepsilon, t))\, ds\, dt \\
&\quad + \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt \le \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt.
\end{aligned} \tag{3.43}$$

Here, we used that if $\widetilde{w}^j_{\varepsilon,\phi} < \phi(P^j_\varepsilon, t)$, then

$$\varepsilon^{-\gamma} \partial_t \widetilde{w}^j_{\varepsilon,\phi} + \partial_\nu \widetilde{w}^j_{\varepsilon,\phi} - \varepsilon^{-\gamma} \partial_t \phi(P^j_\varepsilon, t) = 0. \tag{3.44}$$

Additionally, on $\partial G^j_\varepsilon \times (0,T)$, we have

$$\varepsilon^{-\gamma} \partial_t w^j_{\varepsilon,\phi} + \partial_\nu w^j_{\varepsilon,\phi} - \varepsilon^{-\gamma} \partial_t \phi(P^j_\varepsilon, t) \le 0. \tag{3.45}$$

Subtracting from the equality (3.44) the inequality (3.45), for $(x,t) \in \partial G^j_\varepsilon \times (0,T)$ we obtain the inequality

$$\varepsilon^{-\gamma} \partial_t v^j_{\varepsilon,\phi} + \partial_\nu v^j_{\varepsilon,\phi} \ge 0. \tag{3.46}$$

From (3.42) and (3.43), we deduce

$$\begin{aligned}
&\int_0^T \int_{T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}} |\nabla v^j_{\varepsilon,\phi}|^2\, dx\, dt + \frac{\varepsilon^{-\gamma}}{2} \|v^j_{\varepsilon,\phi}(x,T)\|^2_{L^2(\partial G^j_\varepsilon)} \\
&\le \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt \\
&= \int_0^T \int_{T^j_{\varepsilon/4} \setminus \overline{T^j_{\varepsilon/8}}} \nabla v^j_{\varepsilon,\phi} \nabla \widetilde{w}^j_{\varepsilon,\phi}\, dx\, dt - \int_0^T \int_{\partial T^j_{\varepsilon/8}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt.
\end{aligned} \tag{3.47}$$

Applying Theorems 3.4 and 3.9, we obtain

$$|v^j_{\varepsilon,\phi}(x,t)| \le K\varepsilon^2 \max_{\overline{Q^T}} |\phi(x,t)|, \tag{3.48}$$

where K is a constant independent of ε and ϕ. From here, for any $x_0 \in \partial T^j_{\varepsilon/8}, t \in (0,T)$, we deduce the estimate

$$\begin{aligned}
\partial_{x_i} v^j_{\varepsilon,\phi}(x_0,t)| &= |T^{x_0}_{\varepsilon/16}|^{-1} \Big| \int_{T^{x_0}_{\varepsilon/16}} \partial_{x_i} v^j_{\varepsilon,\phi}(x,t) dx \Big| \\
&= |T^{x_0}_{\varepsilon/16}|^{-1} \Big| \int_{\partial T^{x_0}_{\varepsilon/16}} v^j_{\varepsilon,\phi} \nu_i ds \Big| \le K\varepsilon \max_{\overline{Q^T}} |\phi(x,t)|.
\end{aligned}$$

Consequently,

$$\Big| \int_0^T \int_{\partial T^j_{\varepsilon/8}} \partial_\nu v^j_{\varepsilon,\phi} \widetilde{w}^j_{\varepsilon,\phi}\, ds\, dt \Big| \le K\varepsilon^{n+2} \max_{\overline{Q^T}} |\phi(x,t)|^2.$$

From the estimate

$$|\nabla_y w_\phi(x,y,t)| \le \frac{K \max_{\overline{Q^T}} |\phi(x,t)|}{|y|^{n-1}},$$

we obtain

$$|\nabla_x \widetilde{w}^j_{\varepsilon,\phi}(x,t)| \le K \max_{\overline{Q^T}} |\phi(x,t)|\varepsilon,$$

if $x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{\varepsilon/8}}, t \in [0,T]$. Thus, we have

$$\begin{aligned}
&\Big|\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{T^j_{\varepsilon/8}}} \nabla v^j_{\varepsilon,\phi} \nabla w^j_{\varepsilon,\phi}\, dx\, dt\Big| \\
&\le \frac{1}{2}\int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} |\nabla v^j_{\varepsilon,\phi}|^2\, dx\, dt + K\varepsilon^{n+2} \max_{\overline{Q^T}} |\phi(x,t)|^2.
\end{aligned}$$

From this and (3.47), we conclude that

$$\varepsilon^{-\gamma} \max_{[0,T]} \|v^j_{\varepsilon,\phi}\|^2_{L^2(\partial G^j_\varepsilon)} + \|\nabla v^j_{\varepsilon,\phi}\|^2_{L^2(0,T;L^2(T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}))} \le K\varepsilon^{n+2} \max_{\overline{Q^T}} |\phi(x,t)|^2.$$

Using Friedrichs inequality, we obtain the second estimate in (3.41). □

4. Definition and properties of the strange non-local operator $H[\cdot]$

We are now in a position to define the important nonlinear nonlocal operator $H : L^2(0,T;L^2(\Omega)) \to L^2(0,T;L^2(\Omega))$ which arises in the main Theorem 1.1. We start by defining the operator on a class of smoother functions $\phi \in H^1(0,T;H^1(\Omega))$, but, by density, we can extend it to the general space $L^2(0,T;L^2(\Omega))$. We define

$$H[\phi](x,t) = \int_{\partial G_0} \partial_\nu w_\phi(x,y,t) ds_y, \tag{4.1}$$

where w_ϕ is the solution to the problem (3.16).

It is easy to rewrite $H[\phi]$ in an equivalent form

$$H[\phi](x,t) = \phi(x,t)\lambda_{G_0} - \int_{\partial G_0} \partial_\nu \hat{w}_\phi(x,y,t) ds_y, \tag{4.2}$$

where $\hat{w}_\phi$ is the solution to problem (3.20) and where it appears the important notion of the capacity of the model set G_0

$$\lambda_{G_0} = \int_{\partial G_0} \partial_\nu \kappa(y) ds_y = \mathrm{Cap}(G_0).$$

Theorem 4.1. *Assume $\phi \in L^2(0,T;L^2(\Omega))$. Then*

$$\|H[\phi]\|_{L^2(0,T;L^2(\Omega))} \le K\|\phi\|_{L^2(0,T;L^2(\Omega))}. \tag{4.3}$$

Also we have Lipschitz continuity with respect to ϕ: for $\phi_1, \phi_2 \in L^2(0,T;L^2(\Omega))$, we have

$$\|H[\phi_1] - H[\phi_2]\|_{L^2(0,T;L^2(\Omega))} \le K\|\phi_1 - \phi_2\|_{L^2(0,T;L^2(\Omega))}. \tag{4.4}$$

In addition, we have the following monotone time-dependence on ϕ: given $\phi_1, \phi_2 \in L^2(0,T;L^2(\Omega))$, we have

$$\int_0^T \int_\Omega \Big(H[\phi_1] - H[\phi_2]\Big)(\phi_1 - \phi_2)\, dx\, dt \ge 0. \tag{4.5}$$

Proof. Given $\delta > 0$ we consider the penalized version of the operator $H[\phi]$ as the one given by

$$H_\delta[\phi](x,t) = \phi(x,t)\lambda_{G_0} - \int_{\partial G_0} \partial_\nu \hat{w}^\delta_\phi ds_y, \tag{4.6}$$

where $\hat{w}^\delta_\phi$ is the solution to the penalized problem (3.22). The properties of the operator $H_\delta[\phi]$ were studied in the previous paper [20] and it was shown there that it satisfies properties (4.3)-(4.5). Thus, the estimates (4.3), (4.4) are a direct consequence of (3.21), (3.31), respectively, and the monotonicity property can be derived also by using the integral identity satisfied by $\hat{w}^\delta_{\phi_1} - \hat{w}^\delta_{\phi_2}$. Hence, to prove the theorem, we only need to show the convergence of $H_\delta[\phi]$ to $H[\phi]$, in $L^2(0,T;L^2(\Omega))$. To do that, we first point out that, by using the definition of κ, we can rewrite (4.6) in the form

$$H_\delta[\phi](x,t) = \phi(x,t)\lambda_{G_0} - \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla_y \hat{w}^\delta_\phi \nabla\kappa dy. \tag{4.7}$$

Then, for an arbitrary function $\psi \in L^2(0,T;L^2(\Omega))$, we have

$$\int_0^T \int_\Omega H_\delta[\phi]\psi(t)dxdt = \lambda_{G_0}\int_0^T \int_\Omega \phi\psi\, dx\, dt - \int_0^T \int_\Omega \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla_y \hat{w}^\delta_\phi \nabla_y(\kappa\psi)\, dy\, dx\, dt.$$

Using that $\hat{w}^\delta_\phi \rightharpoonup \hat{w}_\phi$ weakly in $L^2(0,T;\mathcal{M})$, as $\delta \to 0$, we conclude

$$\lim_{\delta\to 0}\int_0^T \int_\Omega H_\delta[\phi]\psi\, dx\, dt = \int_0^T \int_\Omega H[\phi]\psi\, dx\, dt, \tag{4.8}$$

for all arbitrary functions $\psi \in L^2(0,T;L^2(\Omega))$. Hence, $H_\delta[\phi] \rightharpoonup H[\phi]$ weakly in $L^2(0,T;L^2(\Omega))$ and thus

$$\|H[\phi]\|_{L^2(0,T;L^2(\Omega))} \le \lim_{\delta\to 0}\|H_\delta[\phi]\|_{L^2(0,T;L^2(\Omega))} \le K\|\phi\|_{L^2(0,T;L^2(\Omega))}. \tag{4.9}$$

$$\begin{aligned}\|H[\phi_1] - H[\phi_2]\|_{L^2(0,T;L^2(\Omega))} &\le \lim_{\delta\to 0}\|H_\delta[\phi_1] - H_\delta[\phi_2]\|_{L^2(0,T;L^2(\Omega))}\\ &\le K\|\phi_1 - \phi_2\|_{L^2(0,T;L^2(\Omega))},\end{aligned} \tag{4.10}$$

and we obtain

$$\begin{aligned}0 &\le \lim_{\delta\to 0}\int_0^T \int_\Omega \big(H_\delta[\phi_1] - H_\delta[\phi_2]\big)(\phi_1 - \phi_2)\, dx\, dt\\ &= \int_0^T \int_\Omega \big(H[\phi_1] - H[\phi_2]\big)(\phi_1 - \phi_2)\, dx\, dt.\end{aligned} \tag{4.11}$$

□

Theorem 4.2. *For a.e.* $x_1,\, x_2 \in \Omega$*, we have the estimate*

$$\|H[\phi](x_1,\cdot) - H[\phi](x_2,\cdot)\|_{L^2(0,T)} \le K\big\{\|\phi(x_1,\cdot) - \phi(x_2,\cdot)\|_{L^2(0,T)}\big\}. \tag{4.12}$$

Proof. We consider again $H_\delta[\phi]$. Using Theorem 3.8, we have

$$\begin{aligned}&H_\delta[\phi](x_1,t) - H_\delta[\phi](x_2,t)\\ &= (\phi(x_1,t) - \phi(x_2,t))\lambda_{G_0} - \int_{\mathbb{R}^n\setminus\overline{G_0}} \nabla(\hat{w}^\delta_\phi(x_1,y,t) - \hat{w}^\delta_\phi(x_2,y,t))\nabla\kappa(y)\, dy\end{aligned}$$

Taking the square of this identity, integrating it in t, from 0 to T, and applying Theorem 3.8, we arrive exactly to the estimate (4.12). □

Remark 4.3. To bring a connection with some previous works (specially with [18]), we now consider G_0 to be given by the unit ball, $G_0 = \{|x| < 1\}$. Notice that this is a small abuse of notation since $\overline{G_0} \not\subseteq Y$, but this is not any problem since for any small $\varepsilon > 0$ we have $\overline{G^j_\varepsilon} \subset T^j_{\varepsilon/4} \subset Y^j_\varepsilon$. Hence, we are under the same conditions as above. Given $\phi \in L^2(0,T)$. We define the function H_ϕ as the unique solution to the unilateral problem

$$\begin{gathered}
H'_\phi + \mathcal{B}_n H_\phi \geq \mathcal{B}_n \phi, \quad H_\phi \geq 0, \quad t \in (0,T), \\
H_\phi(H'_\phi + \mathcal{B}_n H_\phi - \mathcal{B}_n \phi) = 0, \quad t \in (0,T), \\
H_\phi(0) = 0,
\end{gathered}$$

where $\mathcal{B}_n = (n-2)C_0^{-1}$. We reformulate this problem as a variational inequality. We search for a function $H_\phi \in H^1(0,T)$ such that $H_\phi \geq 0$ on $(0,T)$ and satisfying the integral inequality

$$\int_0^T (H'_\phi + \mathcal{B}_n H_\phi - \mathcal{B}_n \phi)(v - H_\phi) dt \geq 0,$$

for arbitrary function $v \in L^2(0,T)$, $v \geq 0$ a.e. $t \in (0,T)$. It is well-known (see, e.g., [24]) that this problem has a unique solution. Moreover, it satisfies

$$\begin{gathered}
\|H_\phi\|_{L^2(0,T)} \leq \|\phi\|_{L^2(0,T)}\,, \\
\|H_{\phi_1} - H_{\phi_2}\|_{L^2(0,T)} \leq \|\phi_1 - \phi_2\|_{L^2(0,T)}\,, \\
\int_0^T (H_{\phi_1} - H_{\phi_2})(\phi_1 - \phi_2) dt \geq 0.
\end{gathered}$$

Now we compare these conclusions with the ones given in Theorem 4.1. Then, if we consider $\phi \in L^2(0,T; L^2(\Omega))$, we can understand $x \in \Omega$ as a parameter in the problem for H_ϕ and thus $H_\phi(x,t)$ is the unique solution to the problem

$$\begin{gathered}
\partial_t H_\phi + \mathcal{B}_n H_\phi \geq \mathcal{B}_n \phi(x,t), \quad H_\phi \geq 0, \\
(\partial_t H_\phi + \mathcal{B}_n (H_\phi - \phi(x,t)) H_\phi = 0, \quad H_\phi(x,0) = 0,
\end{gathered} \tag{4.13}$$

Now, we can use spherical symmetry properties to search for the solution of (3.19) in the form $\kappa(r)$, where r is the radial coordinate. We get that $\kappa(r) = r^{2-n}$. Hence, $\partial_\nu \kappa = \frac{d}{dr}\kappa(r) = const$ on ∂G_0. Further, we can search for the function $\hat{w}_\phi$, solving the problem (3.20), in the form $\hat{w}_\phi = r^{2-n} H_\phi(x,t)$. A direct computation shows that $\hat{w}_\phi$ satisfies (3.20). The last conclusion is a consequence of that H_ϕ satisfies (4.13). Actually, we have $H[\phi] = (n-2)|\partial G_0|(\phi(x,t) - H_\phi)$. We also notice that, in this case, $\lambda_{G_0} = (n-2)|\partial G_0|$.

5. Proof of the main result

Before proving Theorem 1.1 it is useful to make some remarks.

Remark 5.1. As in Remark 4.3, if we consider the case in which G_0 is a unit ball, $G_0 = \{|x| < 1\}$, then the homogenized problem corresponding to problem (1.1) is

$$\begin{gathered}
-\Delta_x u_0 + \mathcal{A}_n (u_0 - H_{u_0}) = f, \quad (x,t) \in Q^T, \\
\partial_t H_{u_0} + \mathcal{B}_n H_{u_0} \geq \mathcal{B}_n u_0, \quad (x,t) \in Q^T, \\
H_{u_0} \geq 0,\ H_{u_0}(\partial_t H_{u_0} + \mathcal{B}_n (H_{u_0} - u_0)) = 0, \quad (x,t) \in Q^T, \\
H_{u_0}(x,0) = 0, \quad x \in \Omega,
\end{gathered}$$

$$u_0 = 0, \quad (x,t) \in \partial\Omega \times (0,T),$$

where the constants are given by $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$, $\omega_n = |\partial G_0|$ and $\mathcal{B}_n = (n-2)C_0^{-1}$ respectively (see [18]). By using the relation between H_ϕ and $H[\phi]$ of Remark 4.3, we obtain exactly problem (1.3).

Remark 5.2. As in the case in which the particles are radially symmetric, it is possible to prove that for suitable source negative function $f(t,x)$, the solution of the homogenized problem (1.1) may become negative in some parts of the domain Ω, even if for each ϵ, the approximate solution u_ε is non-negative on the many points of the boundary of the particles contained in Ω_ε. See [18], for a detailed proof for the radial symmetric case. This unexpected property also holds in the asymmetric case but, as it is natural, with some additional difficulties. For instance, the boundedness of the solution u_0 (when the datum $f(t,x)$ is assumed to be bounded, or in some $L^p(Q^T)$ with p large enough, requires to get, previously, some L^∞-estimates on $\partial_t w^j_{\varepsilon,\phi}$. This can be justified by working with Lipschitz solutions of the auxiliary unilateral problems as, for instance, in [23]. The rest of details are very similar to the symmetric case.

Proof of Theorem 1.1. Let $\phi(x,t) = \psi(x)\eta(t)$, where $\psi \in C_0^\infty(\overline{\Omega})$, $\eta \in C^1([0,T])$. To adapt any test function of the limit equation to the heterogeneous original problem we introduce our last auxiliary function

$$W_{\varepsilon,\phi}(x,t) = \begin{cases} w^j_{\varepsilon,\phi}(x,t) - (\phi(P^j_\varepsilon,t) - \phi(x,t))\kappa^j_\varepsilon(x), \\ \quad \text{if } x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ t \in (0,T),\ j \in \Upsilon_\varepsilon, \\ 0, \quad \text{if } x \in \mathbb{R}^n \setminus \overline{\cup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}},\ t \in (0,T), \end{cases} \tag{5.1}$$

where $\kappa^j_\varepsilon(x)$ is the unique solution to the problem

$$\Delta\kappa^j_\varepsilon = 0,\ x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \quad \kappa^j_\varepsilon = 1,\ x \in \partial G^j_\varepsilon, \quad \kappa^j_\varepsilon = 0,\ x \in \partial T^j_{\varepsilon/4}. \tag{5.2}$$

Notice that the function $v = \phi(x,t) - W_{\varepsilon,\phi}(x,t)$ is a good test function since $v \in \mathcal{K}_\varepsilon$. Indeed, if $x \in \partial G^j_\varepsilon$, then

$$v(x,t) = \phi(x,t) - w^j_{\varepsilon,\phi}(x,t) + \phi(P^j_\varepsilon,t) - \phi(x,t) = \phi(P^j_\varepsilon,t) - w^j_{\varepsilon,\phi}(x,t) \geq 0. \tag{5.3}$$

Therefore, we can take the function v as a test function in the integral inequality of the original problem

$$\begin{aligned} &\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon} \partial_t v(v-u_\varepsilon)\,ds\,dt + \int_0^T\int_{\Omega_\varepsilon} \nabla v\nabla(v-u_\varepsilon)\,dx\,dt \geq \\ &\geq \int_0^T\int_{\Omega_\varepsilon} f(v-u_\varepsilon)\,dx\,dt - \frac{\varepsilon^{-\gamma}}{2}\|v(x,0)\|^2_{L^2(S_\varepsilon)}. \end{aligned} \tag{5.4}$$

Thus, we obtain

$$
\begin{aligned}
&\varepsilon^{-\gamma}\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G_\varepsilon^j}(\partial_t\phi(P_\varepsilon^j,t)-\partial_t w_{\varepsilon,\phi}^j(x,t))(\phi(P_\varepsilon^j,t)-w_{\varepsilon,\phi}^j(x,t)-u_\varepsilon)\,ds\,dt\\
&+\int_0^T\int_{\Omega_\varepsilon}\nabla\phi\nabla(\phi(x,t)-W_{\varepsilon,\phi}(x,t)-u_\varepsilon)\,dx\,dt\\
&-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{T_{\varepsilon/4}^j\setminus\overline{G_\varepsilon^j}}\nabla w_{\varepsilon,\phi}^j\nabla(\phi(x,t)-w_{\varepsilon,\phi}^j+\kappa_\varepsilon^j(\phi(P_\varepsilon^j,t)-\phi(x,t))-u_\varepsilon)\,dx\,dt\\
&+\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{T_{\varepsilon/4}^j\setminus\overline{G_\varepsilon^j}}\nabla(\kappa_\varepsilon^j(x)(\phi(P_\varepsilon^j,t)-\phi(x,t)))\nabla(\phi(x,t)-w_{\varepsilon,\phi}^j\\
&\quad+\kappa_\varepsilon^j(\phi(P_\varepsilon^j,t)-\phi(x,t))-u_\varepsilon)\,dx\,dt\\
&\geq\int_0^T\int_{\Omega_\varepsilon}f(\phi(x,t)-W_{\varepsilon,\phi}-u_\varepsilon)\,dx\,dt.
\end{aligned}
\tag{5.5}
$$

Here, we have used that $v(x,0)=0$ for $x\in S_\varepsilon$. Moreover, we have

$$
\begin{aligned}
&-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{T_{\varepsilon/4}^j\setminus\overline{G_\varepsilon^j}}\nabla w_{\varepsilon,\phi}^j\nabla(\phi(x,t)-w_{\varepsilon,\phi}^j+\kappa_\varepsilon^j(x)(\phi(P_\varepsilon^j,t)-\phi(x,t))-u_\varepsilon)\,dx\,dt\\
&=-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G_\varepsilon^j}\partial_\nu w_{\varepsilon,\phi}^j(\phi(P_\varepsilon^j,t)-w_{\varepsilon,\phi}^j-u_\varepsilon)\,ds\,dt\\
&\quad-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T_{\varepsilon/4}^j}\partial_\nu w_{\varepsilon,\phi}^j(\phi(x,t)-u_\varepsilon)\,ds\,dt.
\end{aligned}
\tag{5.6}
$$

Combining all of the integrals over ∂G_ε^j, $j\in\Upsilon_\varepsilon$, we obtain

$$
J_\varepsilon\equiv\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G_\varepsilon^j}\Big(\varepsilon^{-\gamma}(\partial_t\phi(P_\varepsilon^j,t)-\partial_t w_{\varepsilon,\phi}^j)-\partial_\nu w_{\varepsilon,\phi}^j\Big)((\phi(P_\varepsilon^j,t)-w_{\varepsilon,\phi}^j)-u_\varepsilon)\,ds\,dt.
\tag{5.7}
$$

Using that $w_{\varepsilon,\phi}^j$ is a solution to the problem (3.1), we conclude that

$$
J_\varepsilon=-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G_\varepsilon^j}\Big(\varepsilon^{-\gamma}\partial_t(\phi(P_\varepsilon^j,t)-w_{\varepsilon,\phi}^j)-\partial_\nu w_{\varepsilon,\phi}^j)u_\varepsilon\,ds\,dt\leq 0.
\tag{5.8}
$$

Taking into account (5.6)-(5.8), we derive from (5.5) that

$$
\begin{aligned}
&\int_0^T\int_{\Omega_\varepsilon}\nabla\phi\nabla(\phi-W_{\varepsilon,\phi}-u_\varepsilon)\,dx\,dt-\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial T_{\varepsilon/4}^j}\partial_\nu w_{\varepsilon,\phi}^j(\phi-u_\varepsilon)\,ds\,dt\\
&+\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{T_{\varepsilon/4}^j\setminus\overline{G_\varepsilon^j}}\nabla(\kappa_\varepsilon^j(x)(\phi(P_\varepsilon^j,t)-\phi(x,t)))\nabla(\phi(x,t)-w_{\varepsilon,\phi}^j\\
&+\kappa_\varepsilon^j(\phi(P_\varepsilon^j,t)-\phi(x,t))-u_\varepsilon)\,dx\,dt\\
&\geq\int_0^T\int_{\Omega_\varepsilon}f(\phi-W_{\varepsilon,\phi}-u_\varepsilon)\,dx\,dt.
\end{aligned}
\tag{5.9}
$$

If we define the function

$$\kappa_\varepsilon(x) = \begin{cases} \kappa_\varepsilon^j(x), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in G^j_\varepsilon,\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\cup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}, \end{cases} \tag{5.10}$$

we have that $\kappa_\varepsilon \rightharpoonup 0$ weakly in $H^1_0(\Omega)$ and that $\kappa_\varepsilon \to 0$ strongly in $L^2(\Omega)$. Using this and the estimate $|\phi(P^j_\varepsilon, t) - \phi(x,t)| \le K\varepsilon$ for $x \in T^j_{\varepsilon/4}$, we derive that the sum of the integrals over $T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}$ in (5.9) converges to zero as $\varepsilon \to 0$.

Using Theorem 3.10 and results from [14] (see also the "from the surface to the volume integrals" [13, Theorem 4.5]), we obtain the limit of the integrals over the "big" balls

$$\begin{aligned} &\lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_{\varepsilon,\phi}(\phi - u_\varepsilon)\, ds\, dt \\ &= \lim_{\varepsilon\to 0} \sum_{j\in\Upsilon_\varepsilon} \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w_\phi(P^j_\varepsilon, \frac{x - P^j_\varepsilon}{a_\varepsilon}, t)(\phi - u_\varepsilon)\, ds\, dt \\ &= -C_0^{n-2} \int_0^T \int_\Omega H[\phi](x,t)(\phi - u_0)\, dx\, dt. \end{aligned} \tag{5.11}$$

Thus, from (5.5)- (5.11), we conclude that u_0 satisfies

$$\begin{aligned} &\int_{Q^T} \nabla\phi\nabla(\phi - u_0)\, dx\, dt + C_0^{n-2} \int_{Q^T} H[\phi](x,t)(\phi - u_0)\, dx\, dt \\ &\ge \int_{Q^T} f(\phi - u_0)\, dx\, dt \end{aligned} \tag{5.12}$$

for any smooth test function $\phi(x,t) = \psi(x)\eta(t)$, $\psi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$.

Taking into account that the linear span of functions $\{\psi(x)\eta(t) : \psi \in C_0^\infty(\Omega),\ \eta \in C^1([0,T])\}$ is dense in the space $L^2(0,T;H^1_0(\Omega))$, we derive that inequality (5.12) is valid for an arbitrary function $\phi \in L^2(0,T;H^1_0(\Omega))$. Then we take $\phi = u_0 \pm \lambda\theta$, where $\lambda \ge 0$ and $\theta \in L^2(0,T;H^1_0(\Omega))$, as a test function in the inequality (5.12) and pass to the limit as $\lambda \to 0$. Notice that $H[u_0 \pm \lambda\theta] \to H[u_0]$ in $L^2(Q^T)$ as $\lambda \to 0$. Combining the two limit inequalities, we conclude that u_0 satisfies

$$\int_{Q^T} \nabla u_0 \nabla\theta\, dx\, dt + C_0^{n-2} \int_{Q^T} H[u_0]\theta\, dx\, dt = \int_{Q^T} f\theta\, dx\, dt$$

for any $\theta \in L^2(0,T;H^1_0(\Omega))$. Hence, u_0 satisfies the problem (1.3). The uniqueness of solutions of problem (1.3) is a trivial consequence of the monotonicity properties of the operator $H[u_0]$ (see the fourth step of the proof of [17, Theorem 3.2]). □

Remark 5.3. The case of non-zero initial datum in the original problem, $u_\varepsilon(x,0) = u^0(x)$ for $x \in S_\varepsilon$ can be also treated with the arguments of this paper. In fact, that was detailed in [18] for the case of symmetric particles. The important modification is that now the "strange operator" also depends on u^0 (since u^0 appears in the definition of $H[u_0]$). We leave the details to the interested reader.

Remark 5.4. Remark that the solutions of the zero-order reactions satisfy, with a slight modification, the unilateral formulation (1.1). Indeed, the boundary conditions are now

$$\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}(\partial_t u_\varepsilon + \lambda\sigma(u_\varepsilon)) \ni 0 \text{ on } S_\varepsilon^T,$$

with $\sigma(s)$ the maximal monotone graph of $\mathbb{R}^2$ given by

$$\sigma(s) = \begin{cases} 0 & \text{if } s < 0, \\ [0,1] & \text{if } s = 0, \\ 1 & \text{if } s > 0. \end{cases}$$

We recall that in this case, automatically $u_\varepsilon \geq 0$ on S_ε^T since u_ε represents a concentration. Then, we obtain that, if $\sigma_S(s)$ is the maximal monotone graph corresponding to the Signorini type boundary conditions, then

$$\partial_\nu u_\varepsilon + \varepsilon^{-\gamma}(\partial_t u_\varepsilon + \lambda(\sigma_S(u_\varepsilon) + 1) \ni 0 \text{ on } S_\varepsilon^T,$$

and thus we arrive that if u_ε satisfies the zero-order reaction on S_ε^T then u_ε satisfies also the non-homogeneous unilateral boundary conditions similar to the ones considered in this paper

$$\begin{gathered} u_\varepsilon \geq 0, \, \varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon \geq -\varepsilon^{-\gamma}\lambda, \quad (x,t) \in S_\varepsilon^T, \\ u_\varepsilon(\varepsilon^{-\gamma}\partial_t u_\varepsilon + \partial_\nu u_\varepsilon - \varepsilon^{-\gamma}\lambda) = 0, \quad (x,t) \in S_\varepsilon^T. \end{gathered}$$

It is easy to see that the arguments of this paper can be adapted to the case in which there is a non-homogeneous boundary data $g(x) = -\varepsilon^{-\gamma}\lambda$ (see, for instance, the exposition made in [13, Section 4.7.1] for the symmetric case and stationary boundary conditions). We leave the details to the interested reader.

Remark 5.5. In the case $n = 2$ the critical case must be written in a different way (see, the general exposition made in [13, Section 4.7.2]). In fact, it can be shown (see [19]) that in that case it is possible to prove an universal homogenized nonlocal problem when the particles G_ε have different geometrical shapes but having the same perimeter on their boundary.

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