ДОКЛАДЫ РОССИЙСКОЙ АКАДЕМИИ НАУК. МАТЕМАТИКА, ИНФОРМАТИКА, ПРОЦЕССЫ УПРАВЛЕНИЯ, , том , с.

МАТЕМАТИКА

УДК 517.956.225

APERIODICAL ISOPERIMETRIC PLANAR HOMOGENIZATION WITH CRITICAL DIAMETER: UNIVERSAL NON-LOCAL STRANGE TERM FOR A DYNAMICAL UNILATERAL BOUNDARY CONDITION

© г. **J. I. Díaz, T. A. Shaposhnikova, A. V. Podolskiy**

We study the asymptotic behavior of the solution to the diffusion equation in a planar domain, perforated by tiny sets of different shapes with a constant perimeter and a uniformly bounded diameter, when the diameter of a basic cell, ε, goes to 0. This makes the structure of the heterogeneous domain aperiodical. On the boundary of the removed sets (or the exterior to a set of particles, as it arises in chemical engineering), we consider the dynamic unilateral Signorini boundary condition containing a large-growth parameter $\beta(\varepsilon)$. We derive and justify the homogenized model when the problem's parameters take the «critical values». In that case, the homogenized is universal (in the sense that it does not depend on the shape of the perforations or particles) and contains a «strange term» given by a non-linear, non-local in time, monotone operator $\mathbf{H}$ that is defined as the solution to an obstacle problem for an ODE operator. The solution of the limit problem can take negative values even if, for any ε, in the original problem, the solution is non-negative on the boundary of the perforations or particles.

1. INTRODUCTION

The present paper continues the studies started in [5], [7] for dimensions $n \geqslant 3$ to the case of a domain $\Omega \subset \mathbb{R}^2$. Here, we consider the homogenization problem for the Poisson equation in a planar domain that is obtained by the removal of tiny particles of different geometrical shapes with a constant perimeter and a uniformly bounded diameter on the boundary of which the dynamic unilateral Signorini condition is imposed. It contains a term that depends on the diameter of the basic cell, $\varepsilon > 0$. The homogenization of the problems with the Signorini conditions of the form $u_\varepsilon \geqslant 0$, $\partial_\nu u_\varepsilon + \alpha(\varepsilon)\sigma(u_\varepsilon) \geqslant 0$, $u_\varepsilon(\partial_\nu u_\varepsilon + \alpha(\varepsilon)\sigma(u_\varepsilon)) = 0$, (where $\sigma(u)$ is a function of the problem's solution) specified on the boundary of the perforations was studied in many works [3], [4], [8]. In the paper [8], it was proved that for the «critical» values of the problem's parameters, the so-called «strange term» (see, e.g. [2]) appears in the homogenized model. It has the form $\mathcal{A}H(u_0^+) - \mathcal{B}u_0^-$, where $H(u)$ is the solution to some functional equation in the case when the perforations or particles are balls, and in the general case $H(u)$ is defined as a solution to some exterior boundary value problem and it is a new non-linear term in the effective equation, moreover, it depends only on the positive part of a solution to the homogenized problem. In contrast to the aforementioned papers, in the present paper (where we assume $\sigma \equiv 0$ for simplicity), the «strange term» is a non-local in time non-linear monotone operator defined as a solution to an obstacle problem with an ordinary differential operator, moreover, it is applied not only to the positive part of a solution u_0, but also to the negative part u_0^-. Note that the paper [3] studies the homogenization problem in the exterior of the periodically distributed particles of the critical size with the classic Signorini boundary conditions, i.e. with the conditions of the form $u_\varepsilon \geqslant 0$, $(A(x/\varepsilon)\nabla u_\varepsilon, \nu) \geqslant 0$, $u_\varepsilon(A(x/\varepsilon)\nabla u_\varepsilon, \nu) = 0$, for an elliptic operator in the divergence form with rapidly oscillating coefficients. It was shown that the term of the form μu_0^- appears in the homogenized problem, for some measure μ which is not well identified.

The paper [18] is devoted to the homogenization of boundary value problems in domains perforated along $(n-1)$ – dimensional manifold without the condition of periodicity of the location of particles removed from the domain. In the above work, the non-critical values of the problem's parameters are studied.

We also draw the difference between the results of the present paper and the case of «big» particles that have a radius of order ε. The paper [11] studied the homogenization problem for the elliptic operator with the rapidly oscillating coefficients specified in the exterior of the particles of the size of order ε and the classic Signorini boundary conditions. It was shown that the solution to the homogenization problem is always non-negative. Using the results from [12] and [13], it is easy to show that for a domain perforated by sets of the size of the order that equals the period of the structure the solution of the homogenized problem will also be a non-negative function. Moreover, when the big particles have the same shape, the limit problem depends strongly on the shape and so the effectiveness can be optimized by choosing some suitable shapes against others: see, e.g [15].

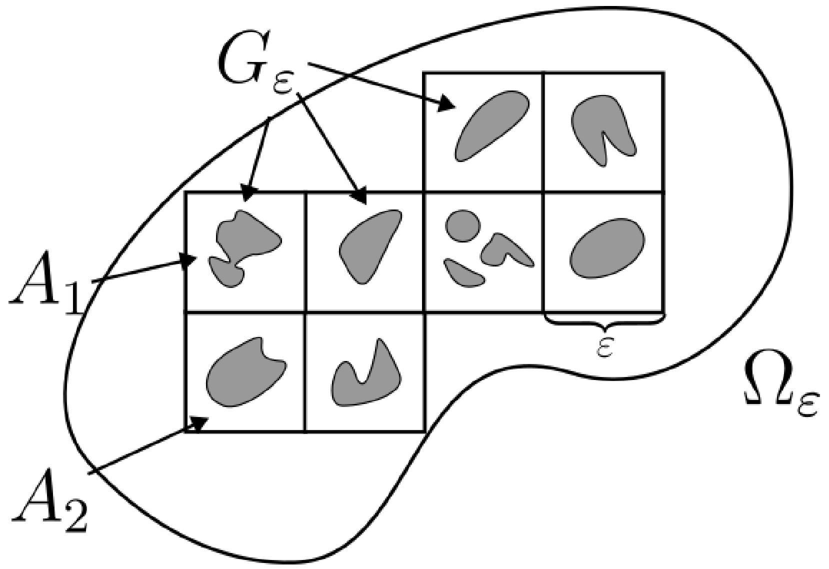

Figure 1. The perforated domain Ω_ε and the set of perforations G_ε.

2. PROBLEM STATEMENT AND SOLUTION ESTIMATES

Let Ω be a bounded domain on the plane $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, $Y = (-1/2, 1/2)^2$ and $A_1, A_2, \ldots$, are different domains with Lipschitz boundary lying in the ball $T^0_{1/4}$ of the radius $1/4$ with the center in the coordinates origin. Suppose that for an arbitrary index $i = 1, 2, \ldots$ the set A_i is diffeomorphic to a ball and $|\partial A_i| = l$, where $l = const > 0$ is independent of i. It is also possible to consider the inclusions composed of a finite number of sets diffeomorphic to a ball with the total boundary length equals to l.

For a set B and a positive number δ, we denote $\delta B = \{x : \delta^{-1}x \in B\}$. Let $\varepsilon > 0$ be a small parameter and $\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}$.

Assume that

$$\lim_{\varepsilon\to 0} \beta(\varepsilon) a_\varepsilon \varepsilon^{-2} = C_1^2, \tag{1}$$

and

$$\lim_{\varepsilon\to 0} \frac{1}{\beta(\varepsilon) a_\varepsilon \ln(\frac{a_\varepsilon}{\varepsilon})} = -C_2^2, \tag{2}$$

where $C_1, C_2 \neq 0$. For example, the conditions (1) and (2) are satisfied if $a_\varepsilon = \varepsilon \exp(-\frac{\alpha^2}{\varepsilon^2})$, $\beta(\varepsilon) = \varepsilon \exp(\frac{\alpha^2}{\varepsilon^2})$, $\alpha \neq 0$.

We define

$$G^j_\varepsilon = a_\varepsilon G^j + \varepsilon j, \quad G_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} G^j_\varepsilon,$$

where G^j coincides with one of the sets $A_i, i = 1, 2, \ldots,$, $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^2 : G^j_\varepsilon \subset Y^j_\varepsilon = \varepsilon Y + \varepsilon j, G^j_\varepsilon \cap \widetilde{\Omega}_\varepsilon \neq \emptyset\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-2}$, $d = const > 0$.

We point out that the choice of G^j as one of the isoperimetrical sets A_i can be made randomly, nevertheless, our treatment is different from the important study made in [16].

Note that

$$\overline{G^j_\varepsilon} \subset T^j_{a_\varepsilon} \subset T^j_{\varepsilon/4} \subset Y^j_\varepsilon,$$

where T^j_r is a ball of radius r with the center in the point $P^j_\varepsilon = \varepsilon j$.

Define the sets

$$\begin{gathered} \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\, S_\varepsilon = \partial G_\varepsilon,\, \partial\Omega_\varepsilon = S_\varepsilon \bigcup \partial\Omega, \\ Q^T_\varepsilon = \Omega_\varepsilon \times (0,T),\, T > 0,\, S^T_\varepsilon = S_\varepsilon \times (0,T),\, \Gamma^T = \partial\Omega \times (0,T). \end{gathered} \tag{3}$$

Note that Ω_ε is a perforated domain that has a non-periodic structure in general. Figure 1 depicts an example of such a domain.

We consider in Q_ε^T the initial boundary value problem with the dynamic Signorini conditions specified on the boundary of the inclusions

$$\begin{cases} -\Delta_x u_\varepsilon = f(x,t), & (x,t)\in Q_\varepsilon^T,\\ u_\varepsilon \geqslant 0, & (x,t)\in S_\varepsilon^T,\\ \beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon \geqslant 0, & (x,t)\in S_\varepsilon^T\\ u_\varepsilon(\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon) = 0, & (x,t)\in S_\varepsilon^T,\\ u_\varepsilon(x,t)=0, & (x,t)\in \Gamma^T,\\ u_\varepsilon(x,0)=0, & x\in S_\varepsilon, \end{cases} \tag{4}$$

where $f\in H^1(0,T;L^2(\Omega))$, ν is outward normal vector to the boundary S_ε^T.
By $H^1(\Omega_\varepsilon,\partial\Omega)$, we denote the closure in $H^1(\Omega_\varepsilon)$ of the set of the infinitely differentiable functions in $\overline{\Omega}_\varepsilon$ vanishing near the boundary $\partial\Omega$. The spaces $L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, $L^2(0,T;L^2(S_\varepsilon))$, $H^1(0,T;L^2(\Omega))$ that are used below, are defined by standard means (see. [19], [20]).
We introduce the convex closed sets

$$\mathscr{K}_\varepsilon = \{v\in H^1(\Omega_\varepsilon,\partial\Omega) : v\geqslant 0 \text{ for a.e. } x\in S_\varepsilon\},$$
$$\mathcal{K}_\varepsilon = \{v\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega)) : v(.,t)\in \mathscr{K}_\varepsilon \text{ for a.e. } t\in(0,T)\}.$$

We say that the function $u_\varepsilon\in\mathcal{K}_\varepsilon$ such that $\partial_t u_\varepsilon\in L^2(0,T;L^2(S_\varepsilon))$ and $u_\varepsilon(x,0)=0$ is a strong solution to the problem (4) if for an arbitrary function $v\in\mathcal{K}_\varepsilon$ it satisfies the inequality

$$\beta(\varepsilon)\int_0^T\int_{S_\varepsilon}\partial_t u_\varepsilon(v-u_\varepsilon)dsdt + \int_{Q_\varepsilon^T}\nabla u_\varepsilon\nabla(v-u_\varepsilon)dxdt \geqslant \int_{Q_\varepsilon^T} f(v-u_\varepsilon)dxdt. \tag{5}$$

Theorem 1. *There exists a unique strong solution $u_\varepsilon(x,t)$ to the problem* (4) *and it satisfies the estimates*

$$\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \sqrt{\beta(\varepsilon)}\|u_\varepsilon\|_{C([0,T];L^2(S_\varepsilon))} \leqslant K\|f\|_{L^2(0,T;L^2(\Omega))}, \tag{6}$$

and

$$\sqrt{\beta(\varepsilon)}\|\partial_t u_\varepsilon\|_{L^2(0,T;L^2(S_\varepsilon))} + \|\nabla u_{\varepsilon,\delta}\|_{C([0,T];L^2(\Omega_\varepsilon))} \leqslant K\|f\|_{H^1(0,T;L^2(\Omega))}, \tag{7}$$

Proof. It closely follows some ideas in [5] and [6]. We consider an auxiliary problem with a penalty term

$$\begin{cases} -\Delta u_{\varepsilon,\delta} = f(x,t), & (x,t)\in Q_\varepsilon^T,\\ \beta(\varepsilon)\partial_t u_{\varepsilon,\delta} + \partial_\nu u_{\varepsilon,\delta} + \beta(\varepsilon)\delta^{-1}(u_{\varepsilon,\delta})^- = 0, & (x,t)\in S_\varepsilon^T,\\ u_{\varepsilon,\delta}(x,t)=0, & (x,t)\in\Gamma^T,\\ u_{\varepsilon,\delta}(x,0)=0, & x\in S_\varepsilon, \end{cases} \tag{8}$$

where $\delta = const > 0$ is a parameter, $u^+=\sup(0,u)$, $u^- = u-u^+$. Note that the function $\sigma(u)=u^-$ is monotone.
We say that a function $u_{\varepsilon,\delta}\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ is a strong solution to the problem (8), if $\partial_t u_{\varepsilon,\delta}\in L^2(0,T;L^2(S_\varepsilon))$, $u_{\varepsilon,\delta}(x,0)=0$ as $x\in S_\varepsilon$, and if it satisfies the integral identity

$$\beta(\varepsilon)\int_0^T\int_{S_\varepsilon}\partial_t u_{\varepsilon,\delta} v dsdt + \int_0^T\int_{\Omega_\varepsilon}\nabla u_{\varepsilon,\delta}\nabla v dxdt +$$
$$+\delta^{-1}\beta(\varepsilon)\int_0^T\int_{S_\varepsilon}(u_{\varepsilon,\delta})^- v dsdt = \int_{Q_\varepsilon^T} f v dxdt, \tag{9}$$

where v is an arbitrary function from $L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$. The results of the paper [6] imply that the problem (8) has a unique solution and the following estimates hold

$$\|u_{\varepsilon,\delta}\|_{L^2(0,T;H^1(\Omega_\varepsilon))} + \sqrt{\beta(\varepsilon)}\operatorname*{ess\,sup}_{t\in[0,T]}\|u_{\varepsilon,\delta}\|_{L^2(S_\varepsilon)} \leqslant K\|f\|_{L^2(0,T;L^2(\Omega))}, \tag{10}$$

$$\sqrt{\beta(\varepsilon)}\|(u_{\varepsilon,\delta})^-\|_{L^2(0,T;L^2(S_\varepsilon))} \leqslant K\sqrt{\delta}\|f\|_{L^2(0,T;L^2(\Omega))}, \tag{11}$$

$$\sqrt{\beta(\varepsilon)}\|\partial_t u_{\varepsilon,\delta}\|_{L^2(0,T;L^2(S_\varepsilon))} + \operatorname*{ess\,sup}_{t\in[0,T]}\|\nabla u_{\varepsilon,\delta}\|_{L^2(\Omega_\varepsilon)} \leqslant K\|f\|_{H^1(0,T;L^2(\Omega))}, \tag{12}$$

where the constant K is independent of ε and δ.

From the estimates (10)-(12), we conclude that there exists a subsequence (we preserve the notation of the original sequence for it) such that, as $\delta \to 0$,

$$\begin{gathered} u_{\varepsilon,\delta} \rightharpoonup u_\varepsilon \text{ weakly in } L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega)),\\ u_{\varepsilon,\delta} \to u_\varepsilon \text{ strongly in } C([0,T];L^2(S_\varepsilon)),\\ \partial_t u_{\varepsilon,\delta} \rightharpoonup \partial_t u_\varepsilon \text{ weakly in } L^2(0,T;L^2(S_\varepsilon)),\\ (u_{\varepsilon,\delta})^- \to 0 \text{ strongly in } L^2(0,T;L^2(S_\varepsilon)). \end{gathered} \tag{13}$$

Now, we show that u_ε is a solution to (4). Taking into account that $(u_{\varepsilon,\delta})^- \to 0$ in $L^2(0,T;L^2(S_\varepsilon))$ as $\delta \to 0$, we get $u_\varepsilon \geqslant 0$ for a.e. $x \in S_\varepsilon$ and $t \in [0,T]$, i.e. $u_\varepsilon \in \mathcal{K}_\varepsilon$.
Let $v \in \mathcal{K}_\varepsilon$. From the integral identity (9), we derive

$$\begin{gathered} \beta(\varepsilon)\int\limits_0^T\int\limits_{S_\varepsilon}\partial_t u_{\varepsilon,\delta}(v-u_{\varepsilon,\delta})dsdt + \int\limits_0^T\int\limits_{\Omega_\varepsilon}\nabla u_{\varepsilon,\delta}\nabla(v-u_{\varepsilon,\delta})dsdt +\\ +\delta^{-1}\beta(\varepsilon)\int\limits_0^T\int\limits_{S_\varepsilon}(u_{\varepsilon,\delta})^-(v-u_{\varepsilon,\delta})dsdt = \int\limits_{Q_\varepsilon^T} f(v-u_{\varepsilon,\delta})dxdt. \end{gathered} \tag{14}$$

According to (13), as $\delta \to 0$, we have

$$\lim_{\delta\to 0}\beta(\varepsilon)\int\limits_0^T\int\limits_{S_\varepsilon}\partial_t u_{\varepsilon,\delta}(v-u_{\varepsilon,\delta})dsdt = \beta(\varepsilon)\int\limits_0^T\int\limits_{S_\varepsilon}\partial_t u_\varepsilon(v-u_\varepsilon)dsdt, \tag{15}$$

$$\lim_{\delta\to 0}\int\limits_0^T\int\limits_{\Omega_\varepsilon}\nabla u_{\varepsilon,\delta}\nabla(v-u_{\varepsilon,\delta})dxdt \leqslant \int\limits_0^T\int\limits_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla(v-u_\varepsilon)dxdt. \tag{16}$$

It is easy to see that

$$\int\limits_0^T\int\limits_{S_\varepsilon}(u_{\varepsilon,\delta})^-(v-u_{\varepsilon,\delta})dsdt = \int\limits_0^T\int\limits_{S_\varepsilon}(u_{\varepsilon,\delta})^- v dsdt - \int\limits_0^T\int\limits_{S_\varepsilon}|(u_{\varepsilon,\delta})^-|^2 dsdt \leqslant 0.$$

Therefore, $u_\varepsilon \in \mathcal{K}_\varepsilon$ satisfies the inequality (5) for an arbitrary function $v \in \mathcal{K}_\varepsilon$.
Now, we prove that a solution to the problem (4) is unique. Suppose that there are two solutions $u_{1,\varepsilon}\, u_{2,\varepsilon} \in \mathcal{K}_\varepsilon$, and both satisfy the variational inequality (5). Taking $v = u_{2,\varepsilon}$ in the inequality for $u_{1,\varepsilon}$, and $v = u_{1,\varepsilon}$ in inequality for $u_{2,\varepsilon}$, we sum the obtained inequalities and get

$$\begin{gathered} \beta(\varepsilon)\int\limits_0^T\int\limits_{S_\varepsilon}\partial_t(u_{1,\varepsilon}-u_{2,\varepsilon})(u_{1,\varepsilon}-u_{2,\varepsilon})dsdt +\\ +\int\limits_0^T\int\limits_{\Omega_\varepsilon}|\nabla(u_{1,\varepsilon}-u_{2,\varepsilon})|^2 dxdt \leqslant 0, \end{gathered} \tag{17}$$

which implies that $u_{1,\varepsilon} = u_{2,\varepsilon}$ a.e. in Q_ε^T.

□

It is well known (see [10]) that there exists a linear extension operator $P_\varepsilon : H^1(\Omega_\varepsilon,\partial\Omega) \to H_0^1(\Omega)$ such that

$$\|(P_\varepsilon u)\|_{L^2(\Omega_\varepsilon)} \leqslant K\|\nabla u\|_{L^2(\Omega_\varepsilon)},\ \ \|P_\varepsilon u\|_{H_0^1(\Omega)} \leqslant K\|u\|_{H^1(\Omega_\varepsilon,\partial\Omega)},$$

where $K > 0$ is a constant independent of ε. The estimates (6), (7) imply

$$\|P_\varepsilon u_\varepsilon\|_{L^2(0,T;H_0^1(\Omega))} \leqslant K, \tag{18}$$

hence, for some subsequence (we preserve the original sequence notation for it), as $\varepsilon \to 0$, we have

$$P_\varepsilon u_\varepsilon \rightharpoonup u_0 \text{ weakly in } L^2(0,T;H_0^1(\Omega)). \tag{19}$$

3. STATEMENT OF THE MAIN RESULT AND TEST FUNCTION CONSTRUCTION

Theorem 2. *Let $a_\varepsilon, \beta(\varepsilon)$ satisfy conditions* (1), (2) *and let u_ε be a solution to the problem* (4). *Then, u_0 is the unique weak solution to the problem*

$$\begin{cases} -\Delta u_0 + 2\pi C_1^2 C_2^2 (u_0 - H_{u_0}) = f(x,t), & (x,t) \in Q^T = \Omega \times (,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ H_{u_0} \geqslant 0,\ \partial_t H_{u_0} + \mathcal{L} H_{u_0} \geqslant \mathcal{L} u_0, & \\ H_{u_0}\Big(\partial_t H_{u_0} + \mathcal{L}(H_{u_0} - u_0)\Big) = 0, & (x,t) \in Q^T, \\ H_{u_0}(x,0) = 0, & x \in \Omega, \end{cases} \tag{20}$$

where $\mathcal{L} = \frac{2\pi C_2^2}{l}$, $l = |\partial A_j|$, $j = 1, 2, \ldots$.

Remark 1. *To construct the «strange term» H_{u_0} (that does not appear in the case of big perforations or particles) in the homogenized problem* (20), *we need to solve the variational inequality for an ordinary differential operator*

$$\begin{cases} \frac{d}{dt} H_\phi + \mathcal{L} H_\phi \geqslant \mathcal{L}\phi,\ H_\phi \geqslant 0, \\ H_\phi(\frac{d}{dt} H_\phi + \mathcal{L} H_\phi - \mathcal{L}\phi) = 0, \quad t \in (0,T), \\ H_\phi(0) = 0. \end{cases} \tag{21}$$

The papers [5] and [7] studied that problem and the properties of the operator $\mathbf{H} : L^2(0,T) \to L^2(0,T)$, $\mathbf{H}(\phi) = H_\phi$, where $\phi \in L^2(0,T)$, H_ϕ is the solution to the problem (21).
It is known (see. [5], [7]) that for every $\phi \in L^2(0,T)$ there exists a unique function $H_\phi \in H^1(0,T)$ satisfying the variational inequality

$$\int\limits_0^T H_\phi'(v - H_\phi)dt + \mathcal{L}\int\limits_0^T H_\phi(v - H_\phi)dt \geqslant \mathcal{L}\int\limits_0^T \phi(v - H_\phi)dt, \tag{22}$$

where v is an arbitrary function from the space $L^2(0,T)$ such that $v \geqslant 0$. Also, $H_\phi(0) = 0$, $H_\phi(t) \geqslant 0$ for every $t \in (0,T)$. The inequality (22) is the variational formulation of the problem (21).
We present the properties of the operator $\mathbf{H}$, that were studied in the paper [7], in the following theorem.

Theorem 3. *Let the operator $\mathbf{H} : L^2(0,T) \to L^2(0,T)$, that maps a function ϕ into the solution of the problem* (21) *H_ϕ. Then, $\mathbf{H}$ is Lipschitz continuous and monotone. In other words, for arbitrary functions ϕ and ψ from $L^2(0,T)$ and related to them solutions of the problem* (21), *H_ϕ and H_ψ, we have the inequalities*

$$\|H_\phi - H_\psi\|_{L^2(0,T)} \leqslant \|\phi - \psi\|_{L^2(0,T)}, \quad \int\limits_0^T (H_\phi - H_\psi)(\phi - \psi)dt \geqslant 0. \tag{23}$$

Remark 2. *Taking into account the properties of the operator $\mathbf{H}$ given in Theorem 3, the homogenized problem* (20) *has a unique solution $u_0 \in L^2(0,T; H_0^1(\Omega))$ understood in the sense of the integral identity*

$$\int\limits_0^T\int\limits_\Omega \nabla u_0 \nabla v dx dt + 2\pi C_1^2 C_2^2 \int\limits_0^T\int\limits_\Omega (u_0 - \mathbf{H}(u_0)) v dx dt = \int\limits_0^T\int\limits_\Omega f v dx dt, \tag{24}$$

where v is an arbitrary function from $L^2(0,T; H_0^1(\Omega))$. The proof is the same that is in the Proposition 2 of [5].
Let $H^j_{\phi,\varepsilon}(t)$, ($j \in \Upsilon_\varepsilon$) be a solution on $(0,T)$ to the problem

$$\begin{cases} \frac{d}{dt} H^j_{\phi,\varepsilon} + \mathcal{L} H^j_{\phi,\varepsilon} \geqslant \mathcal{L}\phi(P^j_\varepsilon, t),\ H^j_{\phi,\varepsilon} \geqslant 0, \\ H^j_{\phi,\varepsilon}\Big(\frac{d}{dt} H^j_{\phi,\varepsilon} + \mathcal{L} H^j_{\phi,\varepsilon} - \mathcal{L}\phi(P^j_\varepsilon, t)\Big) = 0, \\ H^j_{\phi,\varepsilon}(0) = 0, \end{cases} \tag{25}$$

where $\phi(x,t) = \psi(x)\eta(t)$, $\psi \in C_0^\infty(\Omega)$, $\eta \in C^1([0,T])$ are arbitrary, $x \in \Omega$ is a parameter in the problem (25).
To construct the test function, we need to define two auxiliary capacity type problems

$$\Delta q^j_\varepsilon = 0,\ x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon};\ q^j_\varepsilon = 1,\ x \in \partial G^j_\varepsilon;\ q^j_\varepsilon = 0,\ x \in \partial T^j_{\varepsilon/4}, \tag{26}$$

and

$$\Delta w^j_\varepsilon = 0,\ x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}};\ w^j_\varepsilon = 1,\ x \in \partial T^j_{a_\varepsilon};\ w^j_\varepsilon = 0,\ x \in \partial T^j_{\varepsilon/4}. \tag{27}$$

Define the function

$$q_\varepsilon = \begin{cases} q_\varepsilon^j, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}, \end{cases} \tag{28}$$

and

$$w_\varepsilon = \begin{cases} w_\varepsilon^j, & x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in \overline{T^j_{a_\varepsilon}},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}. \end{cases} \tag{29}$$

Note that the solution to the problem (27) is given by

$$w_\varepsilon^j(x) = \frac{\ln\left(\frac{4r}{\varepsilon}\right)}{\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)},$$

where we denote $r = |x|$.

The following result is special for dimension $n = 2$ and it is the key point of several arguments which will be used later.

Theorem 4. *Let $n = 2$. The following inequality is true,*

$$\|w_\varepsilon - q_\varepsilon\|_{H^1(\Omega)} \leqslant K\varepsilon, \tag{30}$$

where the constant K here and below doesn't depend on ε.

The proof of this theorem is given in [14]. It is easy to see that $w_\varepsilon \rightharpoonup 0$ and $q_\varepsilon \rightharpoonup 0$ weakly in $H^1_0(\Omega)$ as $\varepsilon \to 0$. Notice that in the random framework used in the paper [16], it is assumed that $w_\varepsilon = q_\varepsilon$. So, in this sense, our approach is more general since only the perimeter must be the same.

We introduce an auxiliary function

$$W_{\varepsilon,\phi}(x,t) = \begin{cases} q_\varepsilon^j(x)(\phi(x,t) - H^j_{\phi,\varepsilon}(t)), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}. \end{cases} \tag{31}$$

Taking into account properties of the functions w_ε, q_ε and $H^j_{\phi,\varepsilon}$, we get $P_\varepsilon W_{\varepsilon,\phi} \rightharpoonup 0$ weakly in $H^1(Q^T)$ as $\varepsilon \to 0$.

During the proof of the Theorem 2 below, we will use the «oscillating test function»

$$v = \phi(x,t) - W_{\varepsilon,\phi}(x,t) \tag{32}$$

in the variational inequality given by (4). Note that $v \in \mathcal{K}_\varepsilon$. Indeed, for $x \in \partial G^j_\varepsilon$, $t \in [0,T]$, we have

$$v(x,t) = \phi(x,t) - \phi(x,t) + H^j_{\phi,\varepsilon}(t) = H^j_{\phi,\varepsilon}(t) \geqslant 0.$$

4. PROOF OF THE THEOREM 2

Proof. As u_ε satisfies the variational inequality (5), then u_ε also satisfies integral inequality of the following form

$$\int_0^T\int_{\Omega_\varepsilon} \nabla v \nabla(v - u_\varepsilon)dxdt + \beta(\varepsilon)\int_0^T\int_{S_\varepsilon} \partial_t v(v - u_\varepsilon)dsdt \geqslant \tag{33}$$

$$\geqslant \int_0^T\int_{\Omega_\varepsilon} f(v - u_\varepsilon)dxdt - \frac{1}{2}\beta(\varepsilon)\|v(x,0)\|^2_{L^2(S_\varepsilon)},$$

where v is an arbitrary function from $\mathcal{K}_\varepsilon$ such that $\partial_t v \in L^2(0,T;L^2(S_\varepsilon))$.

In this inequality, we take v as the function given by (32). Taking into account that $v(x,0) = H^j_{\phi,\varepsilon}(0) = 0$, we derive

$$\int_0^T\int_{\Omega_\varepsilon} \nabla(\phi - W_{\varepsilon,\phi})\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt +$$

$$+\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{\partial G^j_\varepsilon} \partial_t H^j_{\phi,\varepsilon}(t)(H^j_{\phi,\varepsilon} - u_\varepsilon)dsdt \geqslant \tag{34}$$

$$\geqslant \int_0^T\int_{\Omega_\varepsilon} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt.$$

Using that $W_{\varepsilon,\phi} \rightharpoonup 0$ weakly in $H^1(Q^T)$ as $\varepsilon \to 0$, we have

$$\lim_{\varepsilon\to 0}\int\limits_0^T\int\limits_{\Omega_\varepsilon} f(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dx = \int\limits_0^T\int\limits_{\Omega} f(\phi - u_0)dxdt. \tag{35}$$

Analogously,

$$\lim_{\varepsilon\to 0}\int\limits_0^T\int\limits_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt = \int\limits_0^T\int\limits_{\Omega} \nabla\phi\nabla(\phi - u_0)dxdt. \tag{36}$$

Using the definition of $W_{\varepsilon,\phi}$ and Theorem 4, we get

$$\begin{aligned}
&\int\limits_0^T\int\limits_{\Omega_\varepsilon} \nabla W_{\varepsilon,\phi}\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt = \\
&= \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} (\phi(x,t) - H^j_{\phi,\varepsilon}(t))\nabla q^j_\varepsilon\nabla(\phi(x,t) - W_{\varepsilon,\phi} - u_\varepsilon)dxdt + \\
&+ \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} q^j_\varepsilon\nabla\phi\nabla(\phi - W_{\varepsilon,\phi} - u_\varepsilon)dxdt = \\
&\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla q^j_\varepsilon\nabla((\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon}(t))(\phi - W_{\varepsilon,\phi} - u_\varepsilon))dxdt + \kappa_\varepsilon = \\
&= \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla(q^j_\varepsilon - w^j_\varepsilon)\nabla((\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon})(\phi - W_{\varepsilon,\phi} - u_\varepsilon))dxdt + \\
&+ \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla w^j_\varepsilon\nabla((\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon}(t))(\phi - W_{\varepsilon,\phi} - u_\varepsilon))dxdt + \kappa_\varepsilon = \\
&= \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{T^j_{a_\varepsilon}}} \nabla w^j_\varepsilon\nabla((\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon}(t))(\phi - W_{\varepsilon,\phi} - u_\varepsilon))dxdt + \theta_\varepsilon \equiv \\
&\equiv J_\varepsilon + \theta_\varepsilon,
\end{aligned} \tag{37}$$

where $\theta_\varepsilon \to 0$ as $\varepsilon \to 0$,

$$J_\varepsilon \equiv \sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{T^j_{\varepsilon/4}\setminus\overline{T^j_{a_\varepsilon}}} \nabla w^j_\varepsilon\nabla((\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon}(t))(\phi - W_{\varepsilon,\phi} - u_\varepsilon))dxdt.$$

By using that $\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{\varepsilon/4}} = \frac{4}{\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}$, $\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{a_\varepsilon}} = -\frac{1}{a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}$, we have

$$\begin{aligned}
J_\varepsilon = &\frac{4}{\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}} (\phi(P^j_\varepsilon,t) - H^j_{\phi,\varepsilon}(t))(\phi(x,t) - u_\varepsilon)dsdt - \\
&-\frac{1}{a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{a_\varepsilon}} (\phi(P^j_\varepsilon,x) - H^j_{\phi,\varepsilon}(t))(H^j_{\phi,\varepsilon} - u_\varepsilon)dsdt.
\end{aligned}$$

Next, we use the estimate derived in [14].

$$
\begin{gathered}
\Big|\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G_\varepsilon^j}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon}(t))(\phi(x,t)-u_\varepsilon)dsdt-\\
-\frac{l}{2\pi}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{a_\varepsilon}}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon}(t))(\phi(x,t)-u_\varepsilon)dsdt\Big|\leqslant Ka_\varepsilon\varepsilon^{-1}.
\end{gathered}
\tag{38}
$$

Due to conditions (1) and (2), the right-hand side of the estimate (38) converges to zero as $\varepsilon\to 0$. From the estimate (38), we derive

$$
\begin{gathered}
-\frac{1}{a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{a_\varepsilon}}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon})(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt=\\
=\Big\{-\frac{\beta(\varepsilon)}{\beta(\varepsilon)a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{a_\varepsilon}}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon})(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt+\\
+\frac{2\pi\beta(\varepsilon)}{l\beta(\varepsilon)a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon})(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt\Big\}-\\
-\frac{2\pi\beta(\varepsilon)}{l\beta(\varepsilon)a_\varepsilon\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon})(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt\equiv\\
\equiv B_{1,\varepsilon}+B_{2,\varepsilon}.
\end{gathered}
\tag{39}
$$

According to (38), we have $\lim\limits_{\varepsilon\to 0}B_{1,\varepsilon}=0$. Considering (2), we derive

$$
\begin{gathered}
\lim_{\varepsilon\to 0}B_{2,\varepsilon}=\\
=\lim_{\varepsilon\to 0}\frac{2\pi C_2^2\beta(\varepsilon)}{l}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}(\phi(P_\varepsilon^j,t)-H^j_{\phi,\varepsilon})(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt)=0.
\end{gathered}
\tag{40}
$$

Taking into account (39), (40) and (34), we collect the integrals over S_ε, and get

$$
\begin{gathered}
\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}\partial_t H^j_{\phi,\varepsilon}(t)(H^j_{\phi,\varepsilon}-u_\varepsilon)dsdt+\\
+\frac{2\pi C_2^2\beta(\varepsilon)}{l}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}(H^j_{\phi,\varepsilon}-\phi(P_\varepsilon^j,t))(H^j_{\phi,\varepsilon}(t)-u_\varepsilon)dsdt)=\\
=\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial G^j_\varepsilon}\Big(\partial_t H^j_{\phi,\varepsilon}+\frac{2\pi C_2^2}{l}(H^j_{\phi,\varepsilon}-\phi(P_\varepsilon^j,t))(H^j_{\phi,\varepsilon}-u_\varepsilon)dsdt\leqslant 0.
\end{gathered}
\tag{41}
$$

For the integrals over $\partial T^j_{\varepsilon/4}$ we have the convergence "from length to surface averaging" (see. [5])

$$
\begin{gathered}
\lim_{\varepsilon\to 0}\Big(-\frac{4}{\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})}\Big)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{\partial T^j_{\varepsilon/4}}(H^j_{\phi,\varepsilon}-\phi(P_\varepsilon^j,t))(H^j_{\phi,\varepsilon}-u_\varepsilon)dsdt=\\
=2\pi C_1^2C_2^2\int\limits_0^T\int\limits_\Omega(H_\phi-\phi)(\phi-u_0)dxdt.
\end{gathered}
\tag{42}
$$

The inequality (34) and (35)-(42) imply that u_0 satisfies the inequality

$$\int_0^T\int_\Omega \nabla\phi\nabla(\phi-u_0)dxdt + 2\pi C_1^2C_2^2\int_0^T\int_\Omega(\phi-\mathbf{H}(\phi))(\phi-u_0)dxdt \geqslant \\ \geqslant \int_0^T\int_\Omega f(\phi-u_0)dxdt, \tag{43}$$

where ϕ is an arbitrary function from $L^2(0,T;H_0^1(\Omega))$. Now, using the continuity of the operator $\mathbf{H}$, we can prove that u_0 is a unique weak solution to the problem (20). Indeed, to show this, we substitute $\phi = u_0 \pm \lambda w$, where $w \in L^2(0,T;H_0^1(\Omega))$ is arbitrary, $\lambda > 0$, and pass to the limit as $\lambda \to 0$ in the integral inequality. Thus, we get that the function u_0 exactly satisfies the integral identity (24).

□

FINANCING

The research of J. I. Díaz was partially supported by the project PID2020-112517GB-I00 of the Spanish State Research Agency (AEI).

REFERENCES

[1] *Bekmaganbetov K.A., Chechkin G.A., Chepyzov V.V.* Attractors and a «strange term» in homogenized equation // Comptes Rendus Mecanique. 2020. V. 348. I. 5. P. 351-359.

[2] *Cioranescu D., Murat F.* Un terme etŕange venu dáilleurs // Nonlinear Part. Diff. Eq. Appl. 1982. V. 60 P. 98-138.

[3] *Conca C., Murat F., Timofte C.* A generalized strange term in Signorini's type problems // ESAIM Math. Model. Numer. Anal. 2003. V. 57. I. 3. P. 773-805.

[4] *Díaz J.I., Gómez-Castro D., Shaposhnikova T.A.* Nonlinear Reaction-Diffusion Processes for Nanocomposites. Anomalous improved homogenization. Berlin. De Gruyter. 2021. P. 184. doi: https://doi.org/10.1515/9783110648997

[5] *Díaz J.I., Podolskiy A.V., Shaposhnikova T.A.* Unexpected regionally negative solutions of the homogenization of Poisson equation with dynamic unilateral boundary conditions: critical symmetric particles. // Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat. 2023. V. 118. I. 9. https://doi.org/10.1007/s13398-023-01503-w

[6] *Díaz J.I., Shaposhnikova T.A., Zubova M.N.* A strange non-local monotone operator arising in the homogenization of a diffusion equation with dynamic nonlinear boundary conditions on particles of critical size and arbitrary shape // EJDE. 2022. V. 2022, N. 52, P. 1-32.

[7] *Podolskiy A.V., Shaposhnikova T.A.* Homogenization of a parabolic equation in a perforated domain with a unilateral dynamic boundary condition: critical case // Contemporary Mathematics. Fundamental Directions. 2022. V. 68. N. 4. P. 671-685.

[8] *Jager W., Neuss-Radu M., Shaposhnikova T.A.* Homogenization of a variational inequality for the Laplace operator with nonlinear restriction for the flux on the interior boundary of a perforated domain // Nonlinear Anal. Real World Appl. 2014. V. 15. P. 367-380.

[9] *Sandrakov G.* Homogenization of variational inequalities with the Signorini condition on perforated domains. // Doklady Mathematics. 2004. V. 70 N. 3. P.941-944.

[10] *Oleinik O.A., Shaposhnikova T.A.* On homogenization problem for the Laplace operator in partially perforated domains with Neumann's condition on the boundary of cavities // Rend.Mat. Acc. Lincei. 1995. V. 6. S. 9. P.133-142.

[11] *Pastukhova S.E.* Homogenization of a mixed problem with Signorini condition for an elliptic operator in a perforated domain. // Sbornik Math. 2001. V. 192. N. 2. P. 245-260.

[12] *Anguiano M.* Existence, uniqueness and homogenization of nonlinear parabolic problems with dynamical boundary conditions in perforated media. // Mediterr. J. Math. 2020. V. 17. N. 1. P. 1-22.

[13] *Timofte C.* Parabolic problems with dynamical boundary conditions in perforated media // Math. Model. Anal. 2003. V. 8. P. 337-350.

[14] *Perez E., Shaposhnikova T.A., Zubova M.N.* A homogenization problem in a domain perforated by tiny isoperimetric holes with nonlinear Robin type boundary conditions. // Doklady Mathematics. 2014. V. 90, N. 1, P. 1-6.

[15] *Díaz J. I., Gómez–Castro D., Timofte C.* The effectiveness factor of reaction-diffusion equations: homogenization and existence of optimal pellet shapes // Journal of Elliptic and Parabolic Equations. 2016. V. 2. I. 1. P. 119–129.

[16] *Caffarelli L.A., Mellet A.* Random homogenization of an obstacle problem // Ann. l'Institut Henri Poincare Anal. Non Lineaire. 2009. V. 26. I. 2. P. 375–395.

[17] *Wang W., Duan J.* Homogenized Dynamics of Stochastic Partial Differential Equations with Dynamical Boundary Conditions // Commun. Math. Phys. 2007. V. 275. P. 163–186.

[18] *D. I. Borisov, A.I. Mukhametrakhimova.* Uniform convergence and asymptotics for problems in domains finely perforated along a prescribed manifold in the case of the homogenized Dirichlet condition // Sbornik: Mathematics. 2021. V. 212. I. 8. P. 1068–1121. DOI: 10.1070/sm9435.

[19] *Lions J.-L.* Quelques methodes de resolution des problemes aux limites non lineaires. Paris: Dund. 1968. 554 P.

[20] *Evans L.C.* Partial Differential Equations. AMS, 2010. 749 P.