# Beyond the classical Strong Maximum Principle: sign-changing forcing term and flat solutions 

Jesús Ildefonso Díaz and Jesús Hernández*


#### Abstract

We show that the classical Strong Maximum Principle, concerning positive supersolutions of linear elliptic equations vanishing on the boundary of the domain can be extended, under suitable conditions, to the case in which the forcing term is sign-changing. In addition, for the case of solutions the normal derivative on the boundary may also vanish on the boundary (definition of flat solution). This leads to examples in which the unique continuation property fails. As a first application, we show the existence of positive solutions for a sublinear semilinear elliptic problem of indefinite sign. A second application, concerning the positivity of solutions of the linear heat equation, for some large values of time, with forcing and/or initial datum changing sign is also given.


## 1 Introduction

In a pioneering paper, on 1910, S. Zaremba 49 established the well-known Strong Maximum Principle saying, in a simple formulation, that if $\Omega$ is a smooth bounded domain in $\mathbb{R}^{N}$ and a function $u$ verifies

$$
\begin{cases}-\Delta u \geq f(x) & \text { in } \Omega  \tag{1}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

then

$$
\begin{equation*}
u(x)>0 \text { in } \Omega \tag{u}
\end{equation*}
$$

assumed that

$$
\begin{equation*}
f(x) \geq 0 \text { in } \Omega, \quad f \neq 0 \tag{f}
\end{equation*}
$$

The extension to a more general second order elliptic operator was due to E. Hopf, on 1927, in his famous paper [38. Moreover, some years later, on 1952, E. Hopf 39 and O.A. Oleinik [42], independently, proved that under the above conditions the normal derivative of $u$ satisfies the following sign condition

$$
\begin{equation*}
\frac{\partial u}{\partial n}<0 \text { on } \partial \Omega \tag{2}
\end{equation*}
$$

(see, the survey [2] for many other historical informations).
A "quantitative strong maximum" principle was obtained since 1987 (41, [34, [50], [8], [5],...). In order to be more precise, we will work in the class of very weak supersolutions: the theory of very weak solutions of below was introduced in an unpublished paper by Haïm Brezis on 1971, later reproduced in (9]) (see a regularity extension in 35]). This theory applies to the more general class of data $f \in L_{l o c}^{1}(\Omega)$ for which it is possible to give a meaning to the notion of solution of the corresponding problem. We assume

$$
f \in L^{1}(\Omega: \delta)=\left\{g \in L_{l o c}^{1}(\Omega) \text { such that } \int_{\Omega}|g(x)| \delta(x) d x<\infty\right\}
$$

[^0]where
$$
\delta(x)=d(x, \partial \Omega)
$$

Then, the above mentioned "quantitative strong maximum principle" says that if $f$ satisfies $\left.\mathrm{P}_{f}\right]$, and thus

$$
\begin{equation*}
\int_{\Omega} f(x) \delta(x) d x>0 \tag{3}
\end{equation*}
$$

then we get the following estimate, called in [5] as the Uniform Hopf Inequality (UHI),

$$
\begin{equation*}
u(x) \geq C \delta(x) \int_{\Omega} f(x) \delta(x) d x \text { a.e. } x \in \Omega \tag{4}
\end{equation*}
$$

for some $C>0$ only dependent on $\Omega$. Notice that, if for instance $u \in W^{1,1}(\Omega)$, this implies $\frac{\partial u}{\partial n}<0$ on $\partial \Omega$.

The main goal of this paper is to show that the sign assumption $\left(\mathrm{P}_{f}\right)$ can be removed so that, under suitable conditions, any supersolution $u$ satisfying (1) for suitable sign-changing functions $f(x)$ is again strictly positive on $\Omega$. Moreover, under suitable conditions, this strictly positive supersolution for some changing sign datum $f(x)$ does not satisfy the condition (2). In some cases this kind of sign-changing datum $f(x)$ still may satisfy the condition (3) (see, Remark 2.2) but the conclusion (4) may fail (see the notion of flat solution given below).

As far as we know, curiously enough, such type of extension of the classical strong maximum principle was not presented in the previous literature on the subject (see, e.g., 44, [45, 48, [10] and [12], among many other papers and books: the survey [2] contains more than 230 references on the subject, until 2022, but it seems that none of them deals with the case in which $f(x)$ changes sign). There are some papers in the literature which could lead to some related conclusion but their statements are not presented in the same form that in this paper (see, e.g., Remark 3.11 below).

We will pay a special attention to the case in which $f(x)<0$ in some neighborhood of $\partial \Omega$, but many other cases can be also considered (see Remark 2.10). In order to present our results we will use the decomposition

$$
f(x)=f^{+}(x)-f^{-}(x)
$$

with

$$
f^{+}(x)=\max (f(x), 0), f^{-}(x)=-\min (f(x), 0)
$$

(notice that $f^{-}(x) \geq 0$ ). We assume in that paper that the region where $f(x)$ is negative has at least a part which is touching $\partial \Omega$, nevertheles different cases can be also considered by similar arguments (see Remark 3.9 below). We assume that there exists an open subset $\Omega^{+} \subset \Omega$ such that

$$
\begin{cases}f(x) \geq 0 & \text { a.e. } x \in \Omega^{+}  \tag{5}\\ f(x) \leq 0 & \text { a.e. } x \in \Omega \backslash \Omega^{+} \\ \partial \Omega \text { satisfies the interior sphere condition, } & \\ \sup _{x \in \Omega^{+}} d(x, \partial \Omega)>0 & \end{cases}
$$

The kind of "new assumptions" on $f(x)$ giving positive solutions are of the following type:
$\left(\mathrm{H}_{1}\right)$ A suitable balance expressing that the negative zone of $f(x)$ takes place near the boundary $\partial \Omega$ : condition (5) holds and there exists a compact set $K \subset \Omega^{+}$where $f \neq 0$ on $K$ and

$$
\begin{equation*}
\int_{\Omega} f^{+} \delta>\frac{C_{K}^{*}}{c_{* K}} \int_{\Omega} f^{-} \delta \tag{6}
\end{equation*}
$$

with $c_{K}^{*}<C_{K}^{*}$, some positive constants (depending on $K$ ) which will be defined later (see Lemma 3.1 below).
[This will allow us to conclude that $u \geq C^{+}$on $\partial K$, for a suitable $C^{+}>0$ : see expression (52) below],
$\left(\mathrm{H}_{2}\right)$ A suitable decay of $f(x)$ near the boundary $\partial \Omega$ : there exists $\alpha>1$ such that

$$
\left\{\begin{array}{c}
\min _{\overline{\Omega-K}}\left((\alpha-1)\left|\nabla \varphi_{1}\right|^{2}-\lambda_{1} \varphi_{1}^{2}\right)>0 \quad \text { and }  \tag{7}\\
f(x) \geq-M \varphi_{1}(x)^{\alpha-2} \text { a.e. } x \in \Omega
\end{array}\right.
$$

with $K$ the compact mentioned in $\left(\mathrm{H}_{1}\right)$ and

$$
\begin{equation*}
M=\frac{\alpha C^{+} \min _{\overline{\Omega-K}}\left((\alpha-1)\left|\nabla \varphi_{1}\right|^{2}-\lambda_{1} \varphi_{1}^{2}\right)}{\max _{\partial K} \varphi_{1}^{\alpha}} \tag{8}
\end{equation*}
$$

Here $\varphi_{1}$ denotes the first eigenfunction of the Laplacian operator in $\Omega$, given by

$$
\begin{cases}-\Delta \varphi_{1}=\lambda_{1} \varphi_{1} & \text { in } \Omega  \tag{9}\\ \varphi_{1}=0 & \text { on } \partial \Omega\end{cases}
$$

with $\varphi_{1}>0$ normalized by $\left\|\varphi_{1}\right\|_{L^{\infty}(\Omega)}=1$. We recall that by well-known results $\varphi_{1} \sim \delta, \delta(x)=$ $d(x, \partial \Omega)$. Notice that the first condition in $\left(\mathrm{H}_{2}\right)$ has a geometrical meaning (see Remark 3.5).

Under such type of "new assumptions" on $f$ we will prove:
(A) the positivity of $u$, property $\left(\mathrm{P}_{u}\right)$, still holds. In addition, if, for instance, $u \in W^{1,1}(\Omega)$ then $\frac{\partial u}{\partial n} \leq 0$ on $\partial \Omega$,
(B) under additional conditions on $f(x)$, the positive solution of the linear problem

$$
\begin{cases}-\Delta u=f(x) & \text { in } \Omega  \tag{10}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

[i.e., now with the equality symbol $=$, instead $\geq$ ] does not satisfy 2 but $\frac{\partial u}{\partial n}=0$ on $\partial \Omega$.
Property (B) corresponds to the notion of flat solution already considered by different authors in the framework of some nonlinear problems (see, e.g., [17], 40, 30, etc.). The existence of flat solutions shows that assumption $\left(\overline{\mathrm{P}_{f}}\right)$ is necessary to conclude $(4)$. Notice also that a flat solution $u$ on a problem 10 on the domain $\Omega$ can be extended by zero to get the unique solution $\widetilde{u}$ of a similar problem associated to an extended domain $\widetilde{\Omega} \supsetneq \Omega$ with the right hand side given by

$$
\widetilde{f}(x)= \begin{cases}f(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \widetilde{\Omega}-\Omega\end{cases}
$$

In this way we can construct solutions with compact support for data with compact support becoming negative near the boundary of its support. This proves that the version of the strong maximum principle obtained in 12 (ensuring that the solution $u \geq 0$ of a linear problem 10 corresponding to a datum $f \geq 0$, cannot vanish on some positively measured subset of $\Omega$ except if $u \equiv 0$ on $\Omega$ ) has optimal conditions on $f(x)$.

It is a curious fact that the above considerations are motivated, in some sense, after the long experience in the study in different semilinear equations with a non-Lipschitz perturbation in the last fifty years. For instance, in the study of semilinear problems

$$
\begin{cases}-\Delta u=g(x)-\beta(u) & \text { in } \Omega  \tag{11}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

with $g \geq 0$ and $\beta$ a continuous function, for instance, such that $\beta(0)=0$, it is well known the existence of a flat solution under suitable conditions on $\beta$ and $g$. For the case of $\beta$ non-decreasing, subdifferential of the convex function $j, \beta=\partial j$, such that

$$
\int_{0} \frac{d s}{\sqrt{j(s)}}<+\infty
$$

and $g \neq 0$ we send the reader to Theorem 1.16 of [17] (the so called "non-diffusion of the support property": see also the generalization presented in [7] and [1] for the case of $\beta$ a multivalued
maximal monotone graph). For the autonomous case $g \equiv 0$ and $\beta$ non monotone see, e.g., [27], [31, 32] and its references. This means that if we take

$$
\begin{equation*}
f(x)=g(x)-\beta(u(x)) \tag{12}
\end{equation*}
$$

with $u$ the flat solution of 11 then $u$ is also a flat solution of the corresponding linear problem (10). Note that, necessarily, such $f(x)$ becomes negative near the boundary $\partial \Omega$.

The above extension of the strong maximum principle admits many generalizations which will be indicated in form of a series of remarks (Schrödinger equation [Remark 3.7], operators with a first order term [Remark 3.9], linear non-local operators [Remark 3.8], nonlinear elliptic operators and obstacle problem [Remark 3.10, etc.).

The organization of this paper is the following: we start by proving, in Section 2, a version, as simple as possible, for the one-dimensional case in which assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$ can be easily formulated in an optimal way. The $N$-dimensional case and without symmetry conditions is presented in Section 3. An application to some sublinear indefinite semilinear equations (see, e.g., [37]) and [33])) will be given in Section 4. Finally, in Section 5 we will consider the linear parabolic problem

$$
\begin{cases}u_{t}-\Delta u=f(x, t) & \text { in } \Omega \times(0,+\infty) \\ u=0 & \text { on } \partial \Omega \times(0,+\infty) \\ u(x, 0)=u_{0}(x) & \text { on } \Omega\end{cases}
$$

We will show that the above arguments for stationary equations, jointly to some related results (23]), allow to prove that, under suitably changing sign conditions on $u_{0}(x)$ and or on $f(x, t)$ (even if it occurs for any $t>0$ ), we get the global positivity of $u(x, t)$ on $\Omega$, for large values of time $\left(\exists t_{0} \geq 0\right.$ such that $u(x, t)>0$ a.e. $x \in \Omega$, for any $\left.t>t_{0}\right)$.

## 2 The symmetric one-dimensional linear problem

For the sake of the exposition, here we consider supersolutions $u(x)$ of the symmetric one-dimensional linear problem on the domain $\Omega=(-R, R)$

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x) \geq f(x) \quad \text { in }(-R, R)  \tag{13}\\
u( \pm R)=0 .
\end{array}\right.
$$

We assume the symmetry condition

$$
f(x)=f(-x), f=f^{+}-f^{-}
$$

and we will work in the framework of the space $L^{1}(\Omega: \delta)$, with $\delta(x)=d(x, \partial \Omega)$ (and then $\delta(r)=R-r$ if $r \in(0, R))$. We assume

$$
\begin{equation*}
f \in L^{1}(\Omega: \delta), \text { i.e. } \int_{0}^{R}|f(s)|(R-s) d s<\infty \tag{14}
\end{equation*}
$$

and we consider very weak supersolutions, i.e., functions such that $u \in L^{1}(-R, R)$, with $u^{\prime \prime} \in$ $L^{1}(\Omega: \delta)$, satisfying

$$
\begin{equation*}
-\int_{\Omega} u \psi^{\prime \prime} \geq \int_{\Omega} f \psi \tag{15}
\end{equation*}
$$

for any $\psi \in W^{2, \infty}(\Omega) \cap W_{0}^{1, \infty}(\Omega)$ such that $\psi \geq 0$. Notice that since any function $\psi \in W^{2, \infty}(\Omega) \cap$ $W_{0}^{1, \infty}(\Omega)$ satisfies that $|\psi(x)| \leq C \delta(x)$ for any $x \in \bar{\Omega}$, for some $C>0$, then the expressions in 15 make sense. The notion of very weak solution is similar but replacing the symbol $\geq$ by $=$. In some parts of our exposition we will refer to symmetric solutions (and not merely supersolutions). By well-known results (see, e.g. [9], 35]) we have that $u \in C([0, R]) \cap C^{1}[0, R), u=u(r), r=|x|$

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(r)=f(r)  \tag{16}\\
u(R)=0, u^{\prime}(0)=0
\end{array} \quad \text { in }(0, R),\right.
$$

Let us see how the type of assumptions $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$, mentioned in the Introduction, can be easily formulated, and even in an optimal way.

Theorem 2.1 We assume that $f(x)$ becomes negative near the boundary in the following sense: there exists

$$
\begin{equation*}
r_{0} \in(0, R) \tag{17}
\end{equation*}
$$

such that

$$
\left\{\begin{array}{c}
f(x)=f^{+}(x) \geq 0 \text { if } x \in\left(0, r_{0}\right), f^{+} \neq 0 \text { on }\left(0, r_{0}\right),  \tag{18}\\
f(x)=-f^{-}(x) \leq 0 \text { if } x \in\left(r_{0}, R\right),
\end{array}\right.
$$

(A) Assume the "balance condition"

$$
\begin{equation*}
\int_{0}^{r_{0}} f^{+}(s)\left(R-r_{0}\right) d s>\int_{r_{0}}^{R} f^{-}(s)(R-s) d s \tag{19}
\end{equation*}
$$

and the "decay condition"

$$
\begin{equation*}
\int_{r}^{R}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t<(R-r) \int_{0}^{r_{0}} f^{+}(s) d s, \text { for any } r \in\left(r_{0}, R\right) \tag{20}
\end{equation*}
$$

Then any symmetric supersolution $u$ satisfies

$$
\begin{equation*}
u>0 \text { in }(-R, R) \tag{21}
\end{equation*}
$$

Moreover, if for instance $u \in C^{1}[0, R]$, then we have

$$
\begin{equation*}
u^{\prime}(R) \leq 0 \text { and } u^{\prime}(-R) \geq 0 \tag{22}
\end{equation*}
$$

In addition, assumed (19), if $u$ is a solution then $u>0$ if and only if the decay condition (20) holds.
(B) Assume now (19), (20) and

$$
\begin{equation*}
f^{-} \in L^{1}(\Omega), \text { i.e., } \int_{r_{0}}^{R} f^{-}(s) d s<\infty \tag{23}
\end{equation*}
$$

Let $u$ be the unique solution $u$ of (16). Then $u$ is flat $\left(u^{\prime}( \pm R)=0\right)$ if and only if the following condition holds

$$
\begin{equation*}
\int_{0}^{r_{0}} f^{+}(s) d s=\int_{r_{0}}^{R} f^{-}(s) d s, \text { i.e. } \int_{0}^{R} f(s) d s=0 \tag{24}
\end{equation*}
$$

Remark 2.2 It is easy to see that assumptions (18) and 19) imply that $f(x)$ satisfies the positivity of the weighted integral (3). Indeed,

$$
\int_{0}^{r_{0}} f(s)(R-s) d s \geq \int_{0}^{r_{0}} f^{+}(s)\left(R-r_{0}\right) d s>\int_{r_{0}}^{R} f^{-}(s)(R-s) d s
$$

As a matter of fact, one of the main goals of the present paper is to prove a conjecture raised by the first author and communicated to Jean Michel Morel in 1985 (when preparing the joint paper [34]), concerning the possibility for sign-changing data $f(x)$ to keep the positivity of the weighted integral (3) and guaranteeing also the positivity of the supersolution.

Example 2.3 Before to give the proof, let us see the different kinds of behaviour of solutions for a simple example with a changing sign right hand side term. Consider

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=f(x) \quad \text { in }(-a, a)  \tag{25}\\
u( \pm a)=0
\end{array}\right.
$$

for different values of $a \geq 1$ with

$$
f(x)= \begin{cases}1 & \text { if } x \in(-1,1)  \tag{26}\\ -1 & \text { if } x \in(-a,-1) \cup(1, a)\end{cases}
$$



Figure 1: Representation of the exact solution of 25, and the value of its normal derivative at the boundary, when the forcing is given by 26), for different values of $a$.

The Figure 1 below shows different cases for the values of $a=1, a=1.8, a=2$ and $a=2.2$.
In the first case of Figure $1 a=1$, the forcing $f(x)$ is strictly positive and the classical strong maximum principle applies. In the case $a=1.8$ we see that the solution is strictly positive and its normal derivative at the boundary is again strictly negative. For $a=2$ the conditions of part $B$ of Theorem 2.1 hold and we see that the solution is flat. Finally, for $a=2.2$ the assumptions of part A of Theorem 2.1 fail and the solution becomes negative near the boundary.

Proof of Theorem 2.1. By well-known results (see, e.g., 9], 8, [35]) we may assume that $u$ is symmetric and has some regularity properties, i.e. $u$ is such that $u \in C([0, R]) \cap C^{1}([0, R))$ with $u^{\prime \prime} \in L^{1}((0, R): R-r), u=u(r), r=|x|$ and

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(r) \geq f(r) \quad \text { in }(0, R),  \tag{27}\\
u(R)=0, u^{\prime}(0)=0 .
\end{array}\right.
$$

To prove part (A) let us start by proving that

$$
\begin{equation*}
\text { (19) implies that } u\left(r_{0}\right)>0 \text {. } \tag{28}
\end{equation*}
$$

Multiplying the equation by $(R-r)$, with $r \in\left(r_{0}, R\right)$, and using we have

$$
-\int_{r}^{R} u^{\prime \prime}(s)(R-s) d s \geq-\int_{r}^{R} f^{-}(s)(R-s) d s
$$

Integrating by parts, and using the boundary condition we get

$$
-u^{\prime}(R)(R-R)+u^{\prime}(r)(R-r)-\int_{r}^{R} u^{\prime}(s) d s \geq-\int_{r}^{R} f^{-}(s)(R-s) d s
$$

i.e., since $u(R)=0$, for any $r \in\left(r_{0}, R\right)$ we get

$$
\begin{equation*}
u^{\prime}(r)(R-r)+u(r) \geq-\int_{r}^{R} f^{-}(s)(R-s) d s \tag{29}
\end{equation*}
$$

On the other hand, from (14) and the structure assumptions we have that in fact $f^{+} \in$ $L^{1}\left(0, r_{0}\right)$. Then, for any $r \in\left[0, r_{0}\right]$ we have

$$
\begin{equation*}
\int_{0}^{r} u^{\prime \prime}(s) d s=u^{\prime}(r) \leq-\int_{0}^{r} f^{+}(s) d s \leq 0 \tag{30}
\end{equation*}
$$

(which clearly implies that the maximum of $u$ is taken at $r=0$ ). Thus, from 18)

$$
\begin{equation*}
u^{\prime}\left(r_{0}\right) \leq-\int_{0}^{r_{0}} f^{+}(s) d s<0 \tag{31}
\end{equation*}
$$

Substituting in 29, since $u \in C([0, R]) \cap C^{1}(0, R)$, we get that

$$
-\left(R-r_{0}\right) \int_{0}^{r_{0}} f^{+}(s) d s+u\left(r_{0}\right) \geq-\int_{r_{0}}^{R} f^{-}(s)(R-s) d s
$$

which proves that 19 implies property 28, i.e., $u\left(r_{0}\right)>0$. [In fact, it is easy to see that if $u$ is a solution (and not merely a supersolution) then assumption $\sqrt{19}$ is also a necessary condition to have $\left.u\left(r_{0}\right)>0\right]$. Notice also that, from 18 we always have

$$
\int_{0}^{r_{0}} f^{+}(s)(R-s) d s>\left(R-r_{0}\right) \int_{0}^{r_{0}} f^{+}(s) d s
$$

Then we get

$$
\left\{\begin{array}{l}
u^{\prime \prime}(r) \leq-f^{+}(r) \leq 0 \quad \text { in }\left(0, r_{0}\right),  \tag{32}\\
u\left(r_{0}\right)>0, u^{\prime}(0)=0,
\end{array}\right.
$$

which implies, from (30), that

$$
u(r) \geq u\left(r_{0}\right)>0 \text { for any } r \in\left[0, r_{0}\right]
$$

To complete the proof of (A), we see that from the structure conditions (18), for any $r \in\left[r_{0}, R\right.$ ) we have

$$
\begin{equation*}
\int_{r_{0}}^{r} u^{\prime \prime}(s) d s \leq \int_{r_{0}}^{r} f^{-}(s) d s \tag{33}
\end{equation*}
$$

Then, integrating again, for any $\widehat{r} \in(r, R)$ we have

$$
-\int_{r}^{\widehat{r}} u^{\prime}(t) d t+u^{\prime}\left(r_{0}\right)(\widehat{r}-r) \geq-\int_{r}^{\widehat{r}}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t
$$

i.e., from (31)

$$
\begin{equation*}
-u(\widehat{r})+u(r)-(\widehat{r}-r) \int_{0}^{r_{0}} f^{+}(s) d s \geq-\int_{r}^{\widehat{r}}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t \tag{34}
\end{equation*}
$$

Assumption 20) implies that,

$$
\begin{equation*}
\int_{0}^{R}\left(\int_{r_{0}}^{r} f^{-}(s) d s\right) d r<+\infty \tag{35}
\end{equation*}
$$

and thus, for any $r \in\left[r_{0}, R\right)$

$$
\lim _{\widehat{r} \nearrow R} \int_{r}^{\widehat{r}}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t=\int_{r}^{R}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t<+\infty
$$

Then, making $\widehat{r} \nearrow R$, in (34) we get

$$
\begin{equation*}
u(r) \geq(R-r) \int_{0}^{r_{0}} f^{+}(s) d s-\int_{r}^{R}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t \tag{36}
\end{equation*}
$$

which leads to the positivity conclusion 21 .
Obviously, if for instance $u \in C^{1}[0, R]$, since $u(R)=0$ we conclude that $u^{\prime}( \pm R) \leq 0$.
To prove (B), we observe that assumed (19) and (20), then if $u$ is a solution of (16) the inequality (33) becomes an equality. Making $r=R$ and using (31), since $u \in W^{2,1}(0, R)$ we get that $u \in C^{1}[0, R]$ and

$$
u^{\prime}(R)=\int_{r_{0}}^{R} f^{-}(s) d s-\int_{0}^{r_{0}} f^{+}(s) d s
$$

This proves the necessary and sufficient condition in order to have flat solutions. $\square$
The existence of nonnegative solutions with compact support in a larger domain $\widetilde{\Omega}=(-\widetilde{R}, \widetilde{R})$, with $\widetilde{R}>R$ is a simple consequence of the existence of flat solutions on the small domain $\Omega=$ $(-R, R)$. Notice that this proves a failure of the unique continuation property under the below conditions.

Corollary 2.4 Let $\widetilde{f} \in L^{1}(\widetilde{\Omega})$ be the extension of a given function $f \in L^{1}(\Omega)$, i.e. such that

$$
\widetilde{f}(x)= \begin{cases}f(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \widetilde{\Omega} \backslash \Omega\end{cases}
$$

Assume that $f$ satisfies the conditions (19), (20) and (24). Let $u$ be the unique solution of (16) and let $\widetilde{u}$ be the extension of $u$ defined as

$$
\widetilde{u}(x)= \begin{cases}u(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \widetilde{\Omega} \backslash \Omega\end{cases}
$$

Then $\widetilde{u} \in C([0, \widetilde{R}]) \cap C^{1}(0, \widetilde{R})$ and it is the unique weak solution of the problem

$$
\left\{\begin{array}{l}
-\widetilde{u}^{\prime \prime}(x)=\widetilde{f}(x) \quad \text { in }(-\widetilde{R}, \widetilde{R})  \tag{37}\\
\widetilde{u}( \pm \widetilde{R})=0
\end{array}\right.
$$

Proof. By Theorem 2.1 we know that $u(x)$ is a flat solution of the problem (16) associated to $f \in L^{1}(\Omega)$ on $\Omega$ and then the extension $\widetilde{u}$ is a weak solution of the extended problem (37). By the uniqueness of solutions for such problem, $\widetilde{u}(x)$ is the unique function satisfying (37).

Remark 2.5 No flat solution may satisfy the Hopf conclusion (2) nor the decay estimate (4). This proves that the non-negative condition assumed on $f(x)$ in the corresponding results is necessary. Analogously, the Corollary 2.4 proves that the version of the strong maximum principle obtained in [12] (ensuring that the solution $u \geq 0$ of a linear problem 10) corresponding to a datum $f \geq 0$, cannot vanish on some positively measured subset of $\Omega$ except if $u \equiv 0$ on $\Omega$ ) fails for positive solutions corresponding to changing sign data $f(x)$.

Remark 2.6 It is possible to give an alternative proof of Theorem 2.1 by using the Green function associated to the Dirichlet problem when the datum $f^{-}(x)$ is symmetric. Indeed, as before we can
argue only for very weak solutions of the problem 25). Let us assume $a=1$. Then the Green function $G(x, y)$ is given by

$$
G(x, y)=\left\{\begin{array}{lc}
\frac{1}{2}(1-y)(1+x) & \text { if }-1<x<y  \tag{38}\\
\frac{1}{2}(1+y)(1-x) & \text { if } y \leq x<1
\end{array}\right.
$$

(see, e.g., 47], p.54) and since $f \in L^{1}((-1,+1): \delta)$ we know that

$$
\begin{equation*}
u(x)=\int_{-1}^{+1} G(x, y) f(y) d y, \quad x \in[-1,+1] \tag{39}
\end{equation*}
$$

Recall that in this special case $\delta(x)=1-|x|$. Then, some straightforward computations lead to the explicit formula

$$
\begin{equation*}
u(x)=(1-|x|) \int_{0}^{|x|} f(s) d s+\int_{|x|}^{1} f(s)(1-s) d s, \quad x \in[-1,+1] \tag{40}
\end{equation*}
$$

From this formula, by assuming (18), (19) and (20), we can deduce again the expressions (36) and (34), and the proof follows. Once again, we see that $u>0$ if and only if (18), (19) and (20) hold.

Remark 2.7 Clearly, condition (14) is weaker than (23). On the other hand, we point out that the decay assumption (20) indicates that the indefinite integral $\int_{r}^{R}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t$ should decay to zero, as $r \nearrow R$, with a rate less than a linear decay (and in fact on the whole interval $\left(r_{0}, R\right)$ ). Notice that this does not require a pointwise decay of the type $f(r) \rightarrow 0$ as $r \nearrow R$. Indeed, if, for instance, $f^{-}(r)=C(R-r)^{-\alpha}$ for some $C>0$ and $\alpha>0$ then condition 14) holds if and only if $\alpha \in(0,2)$. Moreover

$$
\int_{r_{0}}^{t} f^{-}(s) d s=\frac{C}{(1-\alpha)}\left[\left(R-r_{0}\right)^{1-\alpha}-(R-t)^{1-\alpha}\right]
$$

and

$$
\begin{gathered}
\int_{r}^{R}\left(\int_{r_{0}}^{t} f^{-}(s) d s\right) d t=\frac{C\left(R-r_{0}\right)^{1-\alpha}(R-r)}{(1-\alpha)}-\frac{C}{(1-\alpha)} \int_{r}^{R}(R-t)^{1-\alpha} d t \\
=\frac{C\left(R-r_{0}\right)^{1-\alpha}(R-r)}{(1-\alpha)}+\frac{C}{(1-\alpha)(2-\alpha)}(R-r)^{2-\alpha}
\end{gathered}
$$

Then, the decay condition (20) holds if we assume $C$ small and $\alpha \in(0,1)$. Notice that it fails for $\alpha \in(1,2)$ (similar computations can be carried out for $\alpha=1$ ).

Remark 2.8 Concerning the optimality of the decay condition, as indicated in Theorem 2.1. in the class of solutions of (16), the condition is optimal. It is not difficult to build explicit examples of functions $f(r)$ satisfying (14) and such that $f(r)>0$ on a very large zone on $r \in\left(0, r_{0}\right)$ and only negative in a very small region $r \in\left(r_{0}, R\right)$, for which the unique solution $u$ of (16) is negative near the boundary $r=R$. This is the case, for instance, if we consider

$$
f(r)= \begin{cases}F & \text { if } r \in\left(0, r_{0}\right) \\ -C_{f}(R-r)^{-3 / 2} & \text { if } r \in\left(r_{0}, R\right)\end{cases}
$$

for some positive constants $F$ and $C_{f}$. It is clear that $f^{-} \in L^{1}(\Omega: \delta)$ and thus there exists a unique solution $u$ of the corresponding problem (16). Moreover, by using formula (36) (which becomes an equality for the case of solutions, and not merely supersolutions) it is easy to check that, even if $\varepsilon=R-r_{0}$ is small, if $F / C_{f}$ is large enough then there exists $r^{*} \in\left(r_{0}, R\right)$ such that $u(r)>0$ if $r \in\left[0, r^{*}\right), u\left(r^{*}\right)=0, u(r)<0$ if $r \in\left(r^{*}, R\right)$ and $u(R)=0$.

Remark 2.9 Notice that, under the structure condition 18) on $f(x)$ we have

$$
\int_{0}^{r_{0}} f^{+}(s) d s=\int_{\Omega} f^{+}(x) d x, \int_{r_{0}}^{R} f^{-}(s) d s=\int_{\Omega} f^{-}(x) d x
$$

and thus condition (24) is equivalent to

$$
\int_{0}^{R} f(s) d s=0
$$

Remark 2.10 The case in which $f>0$ on a large part of the domain but with $f<0$ in an interior subset of $\Omega$ can be also considered by this type of techniques. In that case the positive solutions satisfy that $u^{\prime}(R)<0$ and $u^{\prime}(-R)>0$ but they may generate interior points $x_{0} \in \Omega$ where $u\left(x_{0}\right)=u^{\prime}\left(x_{0}\right)=0$ and also the formation of an internal "dead core". Notice that, again, this proves a failure of the unique continuation property under such conditions. Here is an example with a similar structure to Example 1 but now reversing the positive and negative subsets of $f(x)$.


Figure 2: Representation of the solution of problem 25 when the forcing is like 26) but with the oposite sign on $f(x)$.

In the first case of Figure $2 a=4$, the forcing $f(x)$ is now positive close to the boundary and negative on $(-1,1)$ : the solution is positive and a local minimum is created at $x=0$. The dependence on a of the local minimum $u(0)$ is also included in Figure 2. For $a=3.4142$ we have $u(0)=0$. Finally, In Figure 2 it is also represented the solution u corresponding to a forcing term
$f(x)$ vanishing on an interval $(-b, b)$ of $x=0$, i.e.,

$$
f(x)=\left\{\begin{array}{cl}
1 & \text { if } x \in(-a-b,-1-b) \cup(1+b, a+b)  \tag{41}\\
0 & \text { if } x \in(-b, b) \\
-1 & \text { if } x \in(-1-b,-b) \cup(b, 1+b)
\end{array}\right.
$$

We see the formation of an internal "dead core" (the interval $(-0.5,0.5)$ ) where $u=0$.
Remark 2.11 In fact, the property $\frac{\partial u}{\partial n}=0$ is a local property and it may occur only on a part of $\partial \Omega$. A mixed situation, exhibiting this fact, is given in the following example (see also (14]). Consider the problem

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=f(x) \quad \text { in }(0,1),  \tag{42}\\
u(0)=u(1)=0,
\end{array}\right.
$$

with

$$
f(x)=a x-1
$$

Notice that $f(x)$ changes sign if $a>1$. It is easy to see that the unique solution of (42) is

$$
u(x)=\frac{x^{2}}{2}-\frac{a x^{3}}{6}+\frac{(a-3) x}{6}
$$

On the other hand, since the roots of $u(x)=0$ are $x_{-}=1$ and $x_{+}=-1+\frac{3}{a}$ we see that $u(x)>0$ if $a \geq 3$. In fact, for $a=3$ we have $u^{\prime}(0)=0$ and $u^{\prime}(0)>0$ if $a>3$.
Remark 2.12 Notice that a necessary condition in order to have the positivity of the solutions of problem (42) is that

$$
\begin{equation*}
\int_{0}^{1} f(s) \varphi_{1}(s) d s>0 \tag{43}
\end{equation*}
$$

(it suffices to multiply the equation by $\varphi_{1}$ and integrate twice by parts). The above example shows that this necessary condition is not sufficient in order to have the positivity of the solution. Indeed, we can study the values of a for which $\int_{\Omega} f(x) \varphi_{1}>0$ : we have

$$
\int_{\Omega} f(x) \varphi_{1}(x) d x=\int_{0}^{1}(a x-1) \sin \pi x d x=\frac{a-2}{\pi}
$$

and then $\int_{\Omega} f \varphi_{1}>0$ for $a>2$ and, otherwise, the positivity of $u$ requires $a \geq 3$ (i.e., for $a \in(2,3]$ we have that $\int_{\Omega} f(x) \varphi_{1}>0$ but $\left.u \ngtr 0\right)$. Notice also that if we take $\Omega=(-1,1)$,

$$
f(x)= \begin{cases}3 x-1 & \text { if } x \in(0,1) \\ -3 x-1 & \text { if } x \in(-1,0)\end{cases}
$$

then the solution of

$$
\left\{\begin{array}{l}
-u^{\prime \prime}(x)=f(x) \quad \text { in }(-1,1),  \tag{44}\\
u(-1)=u(1)=0,
\end{array}\right.
$$

is

$$
u(x)= \begin{cases}\frac{1}{2} x^{2}(1-x) & \text { if } x \in(0,1) \\ \frac{1}{2} x^{2}(1+x) & \text { if } x \in(-1,0)\end{cases}
$$

and thus, $u(x)>0$ for any $x \in(-1,0) \cup(0,1)$ and $u(0)=0$. In addition, if we take

$$
f(x)= \begin{cases}3(x-b)-1 & \text { if } x \in(b, 1+b) \\ 0 & \text { if } x \in(-b, b) \\ -3(x+b)-1 & \text { if } x \in(-1-b,-b)\end{cases}
$$

for some $b>0$ then the solution has a dead core in $(-b, b)$,

$$
u(x)= \begin{cases}\frac{1}{2}(x-b)^{2}(1+b-x) & \text { if } x \in(b, 1+b) \\ 0 & \text { if } x \in(-b, b) \\ \frac{1}{2}(x+b)^{2}(1+b+x) & \text { if } x \in(-1-b,-b)\end{cases}
$$

## 3 The N -dimensional case and a general bounded domain $\Omega$

Now we consider the $N$-dimensional case and a general bounded regular domain $\Omega$. Our strategy will be quite closed to the main idea of the proof of Theorem 2.1 for the one-dimensional case. We will assume $f(x)$ such that there exists an open subset $\Omega^{+} \subset \Omega$ verifying the hypothesis (5).

In a first step, we will prove something weaker that in the first step of the proof of Theorem 2.1 , we will not prove the positivity of $u$ on $\Omega^{+}$(in Theorem 2.1 the interval $\left[0, r_{0}\right]$ ) but on a regular compact $K$ contained in $\Omega^{+}$: we will show that there exists a positive constant $C^{+}$such that any supersolution $u$ of (1) satisfies

$$
\begin{equation*}
u \geq C^{+} \text {on } K \tag{45}
\end{equation*}
$$

see the expression 52 below.
For the proof, we will need to work with some auxiliary problems of the type

$$
\begin{cases}-\Delta \varsigma_{y}=\chi_{B_{\varrho}(y)} & \text { in } \Omega  \tag{46}\\ \varsigma_{y}=0 & \text { on } \partial \Omega\end{cases}
$$

where $y \in \partial K, \chi_{A}$ denotes the characteristic function of $A$ and $\varrho>0$ is small enough such that

$$
B_{\varrho}(y) \subset \Omega^{+}
$$

Due to the compactness of $K$ and the classical strong maximum principle (Hopf-Oleinik boundary lemma, since $\partial \Omega$ satisfies the interior sphere condition), it is well-known that for any $y \in \partial K$ there exist two positive constants $c_{K}(y)<C_{K}(y)$ such that

$$
\begin{equation*}
c_{K}(y) \delta(x) \leq \varsigma_{y}(x) \leq C_{K}(y) \delta(x) \text { a.e. } x \in \Omega \tag{47}
\end{equation*}
$$

where $\delta(x)=d(x, \partial \Omega)$. Using the compactness of $K$, it is possible to get the following result (that will be proved later) giving some uniform estimates:

Lemma 3.1 There exists two positive constants $0<c_{K}^{*}<C_{K}^{*}$ such that

$$
\begin{equation*}
c_{K}^{*} \delta(x) \leq \varsigma_{y}(x) \leq C_{K}^{*} \delta(x) \text { a.e. } x \in \Omega, \text { for any } y \in \partial K . \tag{48}
\end{equation*}
$$

As in the one-dimensional case, there will be a second step in the proof of the main result of this section, where we prove that, under the balance and decay conditions mentioned in the Introduction, the unique solution $v$ of the problem on the ring $\Omega \backslash K$

$$
\begin{cases}-\Delta v=-f^{-}(x) & \text { in } \Omega \backslash K  \tag{49}\\ v=0 & \text { on } \partial \Omega \\ v=C^{+} & \text {on } \partial K\end{cases}
$$

is a positive subsolution and thus

$$
0<v(x) \leq u(x) \text { a.e. } x \in \Omega \backslash K
$$

For simplicity in the exposition (since there are other different options) this subsolution $v(x)$ will be constructed in terms of a suitable power of $\varphi_{1}$, the normalized first eigenfunction of the Laplacian operator on $\Omega$. We recall that $u$ is a very weak supersolution and that this means that $u \in L^{1}(\Omega)$, and

$$
\begin{equation*}
-\int_{\Omega} u \Delta \psi \geq \int_{\Omega} f \psi \tag{50}
\end{equation*}
$$

for any $\psi \in W^{2, \infty}(\Omega) \cap W_{0}^{1, \infty}(\Omega)$ such that $\psi \geq 0$. Again, since any function $\psi \in W^{2, \infty}(\Omega) \cap$ $W_{0}^{1, \infty}(\Omega)$ satisfies that $|\psi(x)| \leq C \delta(x)$ for any $x \in \bar{\Omega}$, for some $C>0$, then the expressions in 50 make sense. The notion of very weak solution is similar but replacing the symbol $\geq$ by $=$. We have

Theorem 3.2 Let $f \in L^{1}(\Omega: \delta)$. Assume ( $H_{1}$ ) and $\left(H_{2}\right)$. Then any supersolution $u$ of (1) satisfies that

$$
u(x)>0 \text { a.e. } x \in \Omega .
$$

If in addition $f \in L^{1}(\Omega)$ and

$$
\begin{equation*}
\int_{\Omega} f(x) d x=0 \tag{51}
\end{equation*}
$$

then the unique weak solution $u \in W_{0}^{1,1}(\Omega)$ of the linear problem (10) is a flat solution.
Proof. First step. As in the proof of the first part of Lemma 3.2 of [8] (in which the authors offer an alternative proof to the main result of 41) we will use the mean value theorem (in our case on $\Omega^{+}$) but in a different way. As mentioned before, let $K$ be a regular compact contained in $\Omega^{+}$. Since $K$ is compact and $\Omega^{+}$is an open subset of $\Omega$, $\operatorname{dist}\left(K, \partial \Omega^{+}\right)>0$. Let $\varrho<\operatorname{dist}\left(K, \partial \Omega^{+}\right)$.Then,

$$
B_{\varrho}(y) \subset \Omega^{+} \text {for any } y \in \partial K, \text { and } d\left(\partial B_{\rho}(y), \partial \Omega^{+}\right)>0 .
$$

Let $\varsigma_{y}(x)$ be the solution of the auxiliary problem 46. Let us assume, for the moment, that $u \in C^{0}\left(\Omega^{+}\right)$. Then, since $-\Delta u \geq 0$ in $\Omega^{+}$we conclude that there exists a positive constant $\widehat{c}$ such that, for any $y \in \partial K$

$$
u(y) \geq \frac{1}{\left|B_{\varrho}(y)\right|} \int_{B_{\varrho}(y)} u=\widehat{c} \int_{\Omega} u\left(-\Delta \varsigma_{y}\right)
$$

Using 50 , i.e. integrating twice by parts (since $\varsigma_{y} \in W^{2, \infty}(\Omega) \cap W_{0}^{1, \infty}(\Omega)$ and $\varsigma_{y} \geq 0$ ), and using the uniform estimates (48) we have

$$
\int_{\Omega} u\left(-\Delta \varsigma_{y}\right) \geq \int_{\Omega} f \varsigma_{y} \geq c_{K}^{*} \int_{\Omega} f^{+} \delta-C_{K}^{*} \int_{\Omega} f^{-} \delta
$$

Then, thanks to the balance condition $\left(\mathrm{H}_{1}\right)$, we get that

$$
u \geq C^{+} \text {on } \partial K
$$

with

$$
\begin{equation*}
C^{+}=\widehat{c}\left[c_{K}^{*} \int_{\Omega} f^{+} \delta-C_{K}^{*} \int_{\Omega} f^{-} \delta\right]>0 \tag{52}
\end{equation*}
$$

Moreover, since

$$
\left\{\begin{array}{cc}
-\Delta u \geq f \geq 0 & \text { in } \stackrel{\circ}{K}, \\
u \geq C^{+} & \text {on } \partial K,
\end{array}\right.
$$

we get 45 when $u \in C^{0}\left(\Omega^{+}\right)$.
In the general case it is enough to approximate $f$ and $\Delta u$ by a sequence of regular functions such that $f_{n} \in L^{\infty}(\Omega), u_{n} \in C^{0}(\Omega)$, satisfying the assumptions of the statement for each $n \in \mathbb{N}$. We know that the corresponding sequences of functions $f_{n}$ and $u_{n}$ converge in $L^{1}(\Omega: \delta)$ and $L^{1}(\Omega)$, respectively (see [9], 35). Then we arrive to the conclusion since the inequality (45) is stable for the strong convergence in $L^{1}(\Omega)$.

Second step. Let us prove that, under $\left(\mathrm{H}_{1}\right)$ and $\left(\mathrm{H}_{2}\right)$, the function

$$
w(x)=k \varphi_{1}(x)^{\alpha}
$$

is a subsolution to problem $(49)$, for some positive constant $k$ and for some $\alpha>1$, where $\varphi_{1}$ denotes the first eigenfunction of the Laplacian operator in $\Omega$ (see (9)). We have

$$
\begin{gathered}
\nabla w=k \alpha \varphi_{1}^{\alpha-1} \nabla \varphi_{1} \\
\Delta w=k \alpha(\alpha-1) \varphi_{1}^{\alpha-2}\left|\nabla \varphi_{1}\right|^{2}-\lambda_{1} k \alpha \varphi_{1}^{\alpha} .
\end{gathered}
$$

Thus, by assumption $\left(\mathrm{H}_{2}\right)$, over the ring $\overline{\Omega-K}$ there exists $\varepsilon>0$ such that

$$
\varepsilon=\frac{\min }{\Omega-K}\left((\alpha-1)\left|\nabla \varphi_{1}\right|^{2}-\lambda_{1} \varphi_{1}^{2}\right)
$$

Then we have

$$
-\Delta w-f(x) \leq-\varepsilon k \alpha \varphi_{1}^{\alpha-2}+M \varphi_{1}^{\alpha-2}=[M-\varepsilon k \alpha] \varphi_{1}^{\alpha-2}
$$

On the other hand, the inequality

$$
w(x) \leq C^{+} \text {on } \partial K
$$

holds if we take, for instance,

$$
k=\frac{C^{+}}{\max _{x \in \partial K}\left\{\varphi_{1}(x)^{\alpha}\right\}}
$$

since $\max _{\partial K} \varphi_{1}{ }^{\alpha}>0$. Then, from the definition of $M$ given in (8) we have that $M-\varepsilon k \alpha=0$ and thus

$$
\left\{\begin{array}{lr}
-\Delta w(x)-f(x) \leq 0 \leq-\Delta u(x)-f(x) & \text { in } \Omega \backslash K \\
w \leq u & \text { on } \partial K \\
w=u=0 & \text { on } \partial \Omega
\end{array}\right.
$$

(the trace, on $\partial K$, of the very weak solution $u$ is well defined by the regularity results of [35]). Then, since $\Delta w \in L^{1}(\Omega: \delta)$, we can apply the comparison principle on the ring $\Omega \backslash K$ (see, e.g., [9]) and we get that

$$
0<w(x) \leq u(x), \text { a.e. } x \in \Omega \backslash K
$$

which ends the proof of (A).
The proof of the second conclusion will follows again from the Green's formula. We recall that now $u \in W_{0}^{1,1}(\Omega)$ and $\Delta u \in L^{1}(\Omega)$, which gives a meaning to the Green's formula since $\frac{\partial u}{\partial n} \in L^{1}(\partial \Omega)$. From (51), integrating in (10), we get

$$
\int_{\partial \Omega} \frac{\partial u}{\partial n}(\sigma) d \sigma=-\int_{\Omega} \Delta u(x) d x=\int_{\Omega} f(x) d x=0
$$

and since we already know that $\frac{\partial u}{\partial n}(\sigma) \leq 0$ on $\partial \Omega$ we conclude that $\frac{\partial u}{\partial n}=0$ on $\partial \Omega$.
Proof of Lemma 3.1. The conclusion can be obtained in some different ways. We will give here some direct arguments. By applying the Uniform Hopf Inequality to problem (46) we get

$$
\varsigma_{y}(x) \geq C(\Omega)\left(\int_{B_{\varrho}(y)} \delta(x) d x\right) \delta(x) \text { a.e. } x \in \Omega
$$

The function

$$
\mu(y)=\int_{B_{\varrho}(y)} \delta(x) d x
$$

is a continuous function on the compact $\partial K$. Thus, there exists $\underline{\mu}>0$ such that $\mu(y) \geq \underline{\mu}$ for any $y \in \partial K$ and then we get

$$
c_{K}^{*} \delta(x) \leq \varsigma_{y}(x) \text { a.e. } x \in \Omega, \text { for any } y \in \partial K
$$

with

$$
c_{K}^{*}=C(\Omega) \underline{\mu} .
$$

On the other hand, let $\eta(x)$ satisfying

$$
\begin{cases}-\Delta \eta=\chi_{\Omega^{+}} & \text {in } \Omega  \tag{53}\\ \eta=0 & \text { on } \partial \Omega\end{cases}
$$

Then, by the comparison principle and well-known regularity results we get

$$
\varsigma_{y}(x) \leq \eta(x) \leq \bar{C} \delta(x), \text { a.e. } x \in \Omega
$$

for some $\bar{C}>0$ and for any $y \in \partial K$. Thus $\sup _{y \in \partial K} C_{K}(y) \leq \bar{C}$.
As in the one-dimensional problem, the existence of nonnegative solutions with compact support in a larger domain $\widetilde{\Omega} \supseteq \Omega$, is a simple consequence of the existence of flat solutions on the small domain.

Corollary 3.3 Let $\widetilde{f} \in L^{1}(\widetilde{\Omega})$ be the extension of a given function $f \in L^{1}(\Omega)$, i.e. such that

$$
\widetilde{f}(x)= \begin{cases}f(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \widetilde{\Omega} \backslash \Omega\end{cases}
$$

Assume that $f$ satisfies the conditions $\left(H_{1}\right),\left(H_{2}\right)$, and (51). Let $u$ be the unique solution of (10) and let $\widetilde{u}$ be the extension of $u$ defined as

$$
\widetilde{u}(x)= \begin{cases}u(x) & \text { if } x \in \Omega \\ 0 & \text { if } x \in \widetilde{\Omega} \backslash \Omega .\end{cases}
$$

Then $\widetilde{u}$ is the unique weak solution of the problem

$$
\begin{cases}-\Delta \widetilde{u}=\widetilde{f}(x) & \text { in } \widetilde{\Omega},  \tag{54}\\ \widetilde{u}=0 & \text { on } \partial \widetilde{\Omega} .\end{cases}
$$

Remark 3.4 The comments on the optimality of the non-negative character of $f(x)$ in previous statements of the strong maximum principle made in the previous section apply also for N dimensional linear problems. Nevertheless, the extension to other second order elliptic operators is a delicate point. The case of Lipschitz coefficients can be treated (see Remarks 3.9 and 3.10) but, even if $f(x)$ is non-negative, there are several counterexamples in the literature showing that the strong maximum principle may fail for merely bounded coefficients (see, e.g., [5], [2] and its references). It would be interesting to see if the application of the Green's function allows to get sharp conditions on the datum $f(x)$ as in the one-dimensional case (Remark 2.6).

Remark 3.5 The first condition in ( $H_{2}$ ) has a geometrical meaning. It requires that if $x_{M} \in \Omega$ is such $\varphi_{1}\left(x_{M}\right)=1$ then $x_{M} \in K \subset \Omega^{+}$. When $\Omega$ is a ball, $\Omega=B_{R}(0)$, this condition requires that $0 \in K \subset \Omega^{+}$. This holds in the case of Theorem 2.1. Notice also that the value of $\alpha>1$ which makes possible this condition must increase if $d\left(\Omega \backslash \Omega^{+}, 0\right)$ decreases.

Remark 3.6 Arguing as in [1], it seems possible to get a "local" condition on $f(x)$ on some neighborhood of some points $x_{0} \in \partial \Omega$ in order to construct suitable local supersolutions implying that $\frac{\partial u}{\partial n}\left(x_{0}\right)=0$ (see also (14]).

Remark 3.7 It is quite easy to adapt the above proof to the case of the Schrödinger operator with an absorption potential $V(x)$

$$
\begin{cases}-\Delta u+V(x) u \geq f(x) & \text { in } \Omega  \tag{55}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

The easy case concerns bounded absorption potentials

$$
0 \leq V(x) \leq M_{V} \text { on } \Omega
$$

Indeed, the second part of the proof consists in finding a subsolution on the region $\Omega-K$ and using again $w(x)=k \varphi_{1}(x)^{\alpha_{V}}$ we see that

$$
-\Delta w+V(x) w \geq-k \alpha_{V} \varphi_{1}^{\alpha_{V}-2}\left(\left(\alpha_{V}-1\right)\left(\left|\nabla \varphi_{1}\right|^{2}-\varphi_{1}^{2}\left[\lambda_{1}+V(x)\right]\right)\right.
$$

It is not difficult to see that we can take $\alpha_{V} \geq \alpha$ (the exponent when there is no absorption term). Then, $w_{V}(x)=k \varphi_{1}(x)^{\alpha_{V}} \leq k \varphi_{1}(x)^{\alpha}$, i.e., the subsolution in the case with absorption is smaller than the one for the case without absorption.

The case of an unbounded potential requires additional approximation arguments. For some related results see [18], [19], [20], [43], [25], [24], [28] and [29].

Remark 3.8 The case of the Schrödinger equation for a nonlocal diffusion (the fractional Laplace operator) and an unbounded absorption potential can be also considered with the techniques of the paper [26] (exposition made in [22]).

Remark 3.9 The study of the presence of linear transport terms in the equation can be also carried out (see some similar techniques in [25], [24] and [6]). When the negative part of $f(x)$ takes place on an interior subregion of $\Omega$ (as indicated in Remark 2.10) and $f^{-}(x)$ satisfies some suitable conditions, it is possible to prove the positivity of the solution of

$$
\begin{cases}-\Delta u+\mathbf{b}(x) \cdot \nabla u \geq f(x) & \text { in } \Omega  \tag{56}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

by means of probabilistic methods (see [16]), which also allow to consider the case of a general elliptic operator with Lipschitz coefficients.

Remark 3.10 Theorem 3.2 can be extended also to the case of some nonlinear diffusion operators of the type of the p-Laplace operator and the one arising in the study Bingham fluids and double phase operators (see [14] and the presentation made in [17] and [13]). Some related papers applying other properties of the distance function are [3] and [36]. The positivity of solutions is also very relevant in the class of the obstacle problems. The unilateral condition $u \geq 0$ leads to the existence of solutions with compact support once that the right hand side is assumed to be negative in a neighborhood of the boundary (see, e.g., [7] and [17]). Theorem 3.2 allows to get the optimality of the assumptions made on $f(x)$ in order to get solutions with compact support. Finally, it seems possible to modify the proof of Theorem 3.2 (specially its first step) as to be applied to suitable fully nonlinear equations of Monge-Ampère type (see [15]) and also to the case of a general elliptic operator with Lipschitz coefficients. A paper, now in preparation, will develop all these extensions.

Remark 3.11 After the publication of a first version of this paper in Arxiv, Boyan Sirakov communicated to the authors that the positivity of solutions and the non-vanishing of the normal derivative could be proved once the right-hand side has a positive part that is sufficiently larger than its negative part, in a suitable integral sense. This follows from Theorem 1.1 of his paper [46], dealing with viscosity solutions of a general class of non-divergent second order elliptic equations. Such an application is not explicitly mentioned and existence of flat solutions is not considered in his paper.

Remark 3.12 Also, after the publication of a first version of this paper in Arxiv, Romeo Leylekian communicated to the authors that the results of this section can be applied to get some new properties of solutions of higher order elliptic equations involving the bi-Laplacian operator.

## 4 Application to some sublinear indefinite semilinear equations

Let us give a short application of the results in previous sections to some sublinear indefinite semilinear equations (see, e.g. [4], [37]) and [33]) for many references and additional results): consider nonnegative solutions of the problem

$$
\begin{cases}-\Delta u=\lambda u+m(x) u^{\alpha} & \text { in } \Omega  \tag{57}\\ u=0 & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a smooth bounded domain in $\mathbb{R}^{N}, 0<\alpha<1, m \in L^{\infty}(\Omega)$ changes $\operatorname{sign}$ on $\Omega$ and $\lambda$ is a real parameter. Problems of this type arise in the study of problems with nonlinear diffusion in mathematical biology and porous media. For simplicity we will assume

$$
\left\{\begin{array}{c}
m \in L^{\infty}(\Omega), \text { and }\left|\Omega^{+}\right|,\left|\Omega^{-}\right|>0, \text { with }  \tag{58}\\
\Omega^{+}=\{x \in \Omega \mid m(x)>0\} \text { and } \Omega^{-}=\{x \in \Omega \mid m(x)<0\}
\end{array}\right.
$$

We have:
Theorem 4.1 Assume (58), $m(x)$ satisfies $\left(H_{1}\right)$, ( $H_{2}$ ) and let $0 \leq \lambda<\lambda_{1}$. Then there is a solution $u_{\lambda}>0$ to (57).

Proof. We will follow an idea already presented in Lemma 4.2 in 37. Let now $U$ be the unique solution of the linear problem

$$
\begin{cases}-\Delta U=m(x) & \text { in } \Omega  \tag{59}\\ U=0 & \text { on } \partial \Omega\end{cases}
$$

By Theorem 3.2 we know that

$$
\begin{equation*}
U>0 \text { in } \Omega \tag{60}
\end{equation*}
$$

Let us check that function $u_{0}=[(1-\alpha) U]^{1 /(1-\alpha)}$ is a subsolution of problem 57). Indeed

$$
\nabla u_{0}=[(1-\alpha) U]^{\alpha /(1-\alpha)} \nabla U
$$

and

$$
\operatorname{div}\left(\nabla u_{0}\right)=\alpha[(1-\alpha) U]^{(2 \alpha-1) /(1-\alpha)}|\nabla U|^{2}-m(x)[(1-\alpha) U]^{\alpha /(1-\alpha)}
$$

which gives

$$
\begin{aligned}
-\Delta u_{0}-\lambda u_{0}-m(x)\left(u_{0}\right)^{\alpha} & =-\alpha[(1-\alpha) U]^{\frac{(2 \alpha-1)}{(1-\alpha)}}|\nabla U|^{2} \\
-\lambda[(1-\alpha) U]^{1 /(1-\alpha)} & \leq 0
\end{aligned}
$$

if $\lambda \geq 0$.
On the other hand, as a supersolution $u^{0}=C \psi$ we pick the function $\psi>0$ as the solution of the problem

$$
\begin{cases}-\Delta \psi=\lambda \psi+1 & \text { in } \Omega  \tag{61}\\ \psi=0 & \text { on } \partial \Omega\end{cases}
$$

(recall that $0 \leq \lambda<\lambda_{1}$ ). Then we have

$$
-\Delta u^{0}-\lambda u^{0}-m(x)\left(u^{0}\right)^{\alpha}=C-C^{\alpha} m(x) \psi^{\alpha}=C^{\alpha}\left(C^{1-\alpha}-m(x) \psi^{\alpha}\right)>0
$$

if $C>\left(\|m\|_{L^{\infty}}\|\psi\|_{L^{\infty}}^{\alpha}\right)^{1 /(1-\alpha)}$. On the other hand, we know that $U, \psi \in C_{0}^{1}(\bar{\Omega})$, with

$$
\frac{\partial U}{\partial n} \leq 0 \text { and } \frac{\partial \psi}{\partial n}<0 \text { on } \partial \Omega
$$

Then, for $C$ large enough we know that $u_{0} \leq u^{0}$ on $\Omega$. Then the existence of a solution $u_{\lambda}>0$ to (57) follows from the super and subsolution method in 37.

Remark 4.2 Many other results on the problem (57) can be found in [37]. For instance, if in Theorem 4.1 we assume, in addition, that the unique weak solution $U$ of the linear problem (10) satisfies $\frac{d U}{\partial n}<0$ on $\partial \Omega$ then there is uniqueness of positive solutions to problem (57). Indeed, the uniqueness of positive solutions can be obtained by means of a suitable change of unknowns (see Theorem 4.4 of [37] and the extension presented in [33]), or by means of some hidden convexity arguments (see, e.g., [21]).

Remark 4.3 It is easy to see that the above subsolution becomes, in fact, an exact solution, if we replace the coefficient $m(x)$ by the function

$$
\widehat{m}(x)=m(x)+\alpha(1-\alpha)^{\frac{(2 \alpha-1)}{(1-\alpha)}} \frac{|\nabla U|^{2}}{U}+\lambda(1-\alpha)^{1 /(1-\alpha)} U
$$

In that case, when $\alpha \in\left(\frac{1}{2}, 1\right)$ the positive solution becomes a "flat solution" since in that case the exponent of $U$ in the expression of $\nabla u_{0}$ is $\alpha /(1-\alpha)>1$.

## 5 An application to the linear parabolic problem

Another application of Theorems 2.1 and 3.2 deals with the question of the positivity of solutions of the linear parabolic problem

$$
\begin{cases}u_{t}-\Delta u=f(x, t) & \text { in } \Omega \times(0,+\infty)  \tag{62}\\ u=0 & \text { on } \partial \Omega \times(0,+\infty) \\ u(x, 0)=u_{0}(x) & \text { on } \Omega\end{cases}
$$

By using the arguments for stationary equations of Sections 2 and 3, jointly to some related results (Díaz-Fleckinger [23]), we can prove that, under suitably changing sign conditions on $u_{0}(x)$ and/or on $f(x, t)$ (even if it occurs for any $t>0$ ), we get the global positivity of $u(x, t)$ on $\Omega$, for large values of time. Here is a special statement of this kind of results:

Theorem 5.1 Let $\Omega$ as Theorem 3.2 and let $f \in L_{l o c}^{1}\left(0,+\infty: L^{1}(\Omega: \delta)\right)$ with

$$
f(x, t) \geq g(x) \text { a.e. } x \in \Omega, \text { a.e. } t>0
$$

with $g(x)$ changing sign and satisfying the conditions of Theorem 3.2. Assume $u_{0} \in L^{1}(\Omega: \delta)$ such that there exists a $\widehat{u_{0}} \in L^{2}(\Omega)$, with $u_{0}(x) \geq \widehat{u_{0}}(x)$ a.e. $x \in \Omega$ such that

$$
\int_{\Omega} \widehat{u_{0}} \varphi_{1}=0
$$

Let $u \in C\left([0,+\infty): L^{1}(\Omega: \delta)\right)$ be the unique mild solution of problem (62). Then, there exists $t_{0} \geq 0$ such that $u(x, t)>0$ a.e. $x \in \Omega$, for any $t>t_{0}$.

Proof. Let $v(x)$ be the solution of the stationary problem

$$
\begin{cases}-\Delta v=g(x) & \text { in } \Omega  \tag{63}\\ v=0 & \text { on } \partial \Omega\end{cases}
$$

and let $w \in C\left([0,+\infty): L^{2}(\Omega)\right)$ be the unique solution of the homogeneous parabolic problem

$$
\begin{cases}w_{t}-\Delta w=0 & \text { in } \Omega \times(0,+\infty)  \tag{64}\\ w=0 & \text { on } \partial \Omega \times(0,+\infty) \\ w(x, 0)=\widehat{u_{0}}(x) & \text { on } \Omega\end{cases}
$$

Then, it is clear that the function

$$
\underline{u}(t, x)=v(x)+w(t, x)
$$

is a subsolution to problem (62). Moreover, by Proposition 2.3 of [23] (originally, an unpublished result due to the first author and J. M. Morel)

$$
\begin{equation*}
|w(t, x)| \leq C\left\|\widehat{u_{0}}\right\|_{L^{2}(\Omega)} e^{-\lambda_{2} t} \varphi_{1}(x), \text { a.e. } x \in \Omega, \text { for any } t>t_{0} \tag{65}
\end{equation*}
$$

where $\lambda_{2}$ is the second eigenvalue of the Laplacian operator on $\Omega$. Then, the conclusion follows by Theorem 3.2 and 65 since we have the comparison principle in the class of mild solutions $u \in C\left([0,+\infty): L^{1}(\Omega: \delta)\right)$ on the Banach space $X=L^{1}(\Omega: \delta)$ (see, e.g. [9]).

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J.I. Díaz

Instituto Matemático Interdisciplinar (IMI)
Dept. Análisis Matemático y Matemática Aplicada
U. Complutense de Madrid

Parque de las Ciencias
28040-Madrid. Spain
jidiaz@ucm.es
J. Hernández

Instituto Matemático Interdisciplinar (IMI)
U. Complutense de Madrid

Parque de las Ciencias
28040-Madrid. Spain
jesus.hernande@telefonica.net


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