On the corrector term in the homogenization of the nonlinear Poisson-Robin problem giving rise to a strange term: Application to an optimal control problem

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Abstract. We consider the homogenization process corresponding to some heterogeneous problems that are given by the Poisson equation with a nonlinear Robin-type boundary conditions on the interior boundary of some small perforations (or the boundary of some small particles) in the so-called critical case, giving rise to the appearance of a strange term in the limit semilinear equation. We prove the strong convergence, in the corresponding Sobolev space, of the solutions with a suitable corrector term. In contrast with other previous results in the literature, we do not assume any additional regularity on the solution of the limit equation: we prove that when the spatial dimension is $n = 3$ or $n = 2$, then the inherent H^2 regularity is enough to get such a strong convergence. As an application we consider an optimal control problem in which the cost functional involves the gradient of the state solutions, being independent of the nonlinear term arising in the Robin boundary conditions. By working with the corresponding adjoint problem, we show that the limit of the optimal controls are given as a suitable optimal control associated with the new cost functional in which the strange term and other related terms arise in some unexpected way.

Keywords: homogenization, perforated domain, critical case, nonlinear Poisson-Robin problem, "strange" term, strong convergence, corrector term, optimal control.

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1. Introduction

The main goal of this paper is twofold. In the first part, we consider the question of the convergence of $u_\varepsilon \in H^1(\Omega_\varepsilon, \partial\Omega)$ (the notations will be presented in Section 2), as $\varepsilon \to 0$, in the case of the spatial dimension $n = 3$ (and also $n = 2$), of solutions of the Poisson equation with a nonlinear Robin-type boundary condition

$$\begin{cases} -\Delta u_\varepsilon = f, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma}\sigma(x, u_\varepsilon) = 0, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{1}$$

where $f \in L^2(\Omega)$, in a ε-periodically perforated domain Ω_ε (or formed by the exterior domain to a set of ε-particles) with internal boundary S_ε. We assume that $\sigma(x, \cdot)$ is a regular monotone function for a.e. $x \in \overline{\Omega}$. Here $\gamma = 3$ is a dimensional parameter (for other values of γ the problem is not too relevant if $n = 3$: see [4] and Section 2.5 of [3]). We assume that the perforations (or particles) G_ε are homothetic to a contracted unit ball G_0 by the relation $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$ and that the radius $a_\varepsilon = C_0\varepsilon^\alpha$ is *critical*, i.e. $\alpha = 3$ and C_0 is an arbitrarily given positive constant.

In this critical case, the more general convergence result was given in [5] for the case of $\sigma(u)$ being an arbitrary maximal monotone graph, generalizing many previous papers in the literature dealing with more specific functions $\sigma(x, u_\varepsilon)$, after the pioneering results, around the eighties of the last century, by V. Marchenko and E. Hruslov, D. Cioranescu and F.Murat, among many others (see references in [13], [14], [11] and the expanded presentation made in the monograph [3]). It was shown that, for the critical exponent given in general by $\alpha = n/(n-2)$ for dimensions $n > 2$, u_ε converges weakly in $H^1(\Omega_\varepsilon, \partial\Omega)$, with

$$H^1(\Omega_\varepsilon, \partial\Omega) = \overline{\{u \in \mathcal{C}^\infty(\Omega_\varepsilon) : u \text{ vanishes in the neighborhood of } \partial\Omega\}}^{H^1(\Omega_\varepsilon)},$$

to a homogenized function $u_0 \in H_0^1(\Omega)$ which is identified by the emergence of an unexpected new term (the so-called *strange term)* in the effective equation

$$\begin{cases} -\Delta u_0 + \mathcal{A}_3 H(x, u_0) = f, & x \in \Omega \\ u_0 = 0 & x \in \partial\Omega, \end{cases} \tag{2}$$

where $\mathcal{A}_3 = 4\pi C_0$. Here, the *strange term* $H(x, u)$ is given as the unique solution to the functional equation

$$C_0^{-1} H(x, u) = \sigma(x, u - H(x, u)). \tag{3}$$

Nothing similar arises in the case of *big particles*, $1 < \alpha < n/(n-2)$ and $\gamma = \alpha(n-1) - n$ (see [13], [14], [11] and [3]) where up to a constant $H(x, u) = \sigma(x, u)$ and the convergence $u_\varepsilon \to u_0$ takes place strongly in $H^1(\Omega_\varepsilon, \partial\Omega)$ (see also [21]). On the contrary, we will show here that for the *critical* case, i.e. $\gamma = \alpha = n/(n-2)$ ($\gamma = \alpha = 3$ if $n = 3$) the convergence $u_\varepsilon \to u_0$ in $H^1(\Omega_\varepsilon, \partial\Omega)$ can never take place strongly in $H^1(\Omega_\varepsilon, \partial\Omega)$. Then, an important natural question arises: what is the limit of the gradient term ∇u_ε (and of the associated *total power* $\mathcal{P}_\varepsilon$ defined below), as $\varepsilon \to 0$? Can we identify a *corrector term* $C_\varepsilon(x, u_0)$, such that $u_\varepsilon - u_0 + C_\varepsilon(x, u_0)$ converges strongly in $H^1(\Omega_\varepsilon, \partial\Omega)$ to zero?

The study of the above question is always a central subject in the homogenization theory, starting with the famous paper by De Giorgi and Spagnolo [2] for the case $\alpha = 1$. In that case, there is an important change in the diffusion operator for the effective equation satisfied by u_0, and again the convergence is merely in the weak topology. We also recall that for the case of big particles $1 < \alpha < n/(n-2)$, $C_\varepsilon(x, u_0) \equiv 0$; see, e.g., [13], [14], [11] and [21].

The previous results in the literature on the construction of a corrector term in the critical case usually required some additional regularity on the limit solution u_0: $u_0 \in C^1(\overline{\Omega})$ in [21] (devoted to the case of a regular function $\sigma(x, u)$ being strictly increasing on u), or $u_0 \in W^{1,\infty}(\Omega)$ (in the case of the big particles [13], [14] and for multivalued Signorini maximal monotone graph in [11] and [10]). It is well-known that to get such regularity on u_0 requires that the datum $f(x)$ is also Lipschitz-continuous, $f \in W^{1,\infty}(\Omega)$. This regularity is quite restrictive when in the modeling (for instance in Chemical Engineering)

$f(x)$ represents the coupling with some other different magnitude (for instance a thermic source).

In this paper, we will present the first proof in the literature, as far as we know, in which the corrector term and the strong convergence are obtained without any additional regularity on u_0 (and so for a general datum $f \in L^2(\Omega)$). Our proof will be peculiar to low dimensions of the space: $1 < n \leq 3$. We recall that for $n = 1$ no *strange term* can be formed in the homogenized equation at a critical scale (see, e.g., [4] and [3]). In addition, we are interested in proving the convergence of the *power of the internal forces.* Following the usual notations in Continuum Mechanics (see, e.g. Theorem 4.5 of [19]), we define the *power of the internal forces for the problem* (1) by

$$\mathcal{P}_\varepsilon^{int} = \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, u_\varepsilon) u_\varepsilon ds$$

and the respective *power of the internal forces for the problem (2)* by

$$\mathcal{P}_0^{int} = \int_{\Omega} |\nabla u_0|^2 dx + \mathcal{A}_3 \int_{\Omega} H(x, u_0) u_0 dx.$$

Notice that the convergence of the *power of the external forces*

$$\int_{\Omega_\varepsilon} f u_\varepsilon dx \to \int_{\Omega} f u_0 dx,$$

is a consequence of the above-mentioned weak convergence results.

The second main goal of this paper is to apply the above type of strong convergence results to the study of the limit case of some optimal control problems with distributed controls $v \in L^2(\Omega_\varepsilon)$. So, we consider the state equation

$$\begin{cases} -\Delta u_\varepsilon = f + v, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} \sigma(x, u_\varepsilon) = 0, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

with a solution $u_\varepsilon(v)$, and consider the cost functional to be minimized, $J_\varepsilon : L^2(\Omega_\varepsilon) \to \mathbb{R}$, of the following form

$$J_\varepsilon(v) = \frac{1}{2} \|\nabla u_\varepsilon(v)\|^2_{L^2(\Omega_\varepsilon)} + \frac{\delta}{2} \|v\|^2_{L^2(\Omega_\varepsilon)}, \tag{5}$$

for a given parameter $\delta > 0$. Our goal is to identify the limit of the optimal controls v_ε and its respective optimal states $u_\varepsilon(v_\varepsilon)$.

We point out that the previous results in the literature on the asymptotic behavior of some optimal control heterogeneous problems by means of homogenization techniques were devoted to *linear* boundary conditions (see, e.g., [16], [18], [15], [6] and [7]) except the paper [17] which deals with an obstacle problem on the boundary but where the control is its own obstacle function and so it is very different from our control formulation.

By using some special forms of the Pontryagin Maximum Principle it can be shown that if v_ε is the optimal control (which we know that it exists thanks to well-known results, see, e.g. [12], [20])

$$J_\varepsilon(v_\varepsilon) = \min_{v\in L^2(\Omega_\varepsilon)} J_\varepsilon(v), \tag{6}$$

then, necessarily, $v_\varepsilon = -\delta^{-1}P_\varepsilon$, with P_ε be the adjoint state of the optimal state $u_\varepsilon(v_\varepsilon)$ given as the solution of the nonlinear boundary value problem

$$\begin{cases} \Delta P_\varepsilon = \Delta u_\varepsilon, & x \in \Omega_\varepsilon, \\ \partial_\nu(P_\varepsilon - u_\varepsilon) + \varepsilon^{-\gamma}\sigma_u(x,u_\varepsilon)P_\varepsilon = 0, & x \in S_\varepsilon, \\ P_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{7}$$

Notice the presence of the term $\sigma_u(x,u_\varepsilon)$ is well justified once we make suitable assumptions as, for instance,

$$\begin{cases} \sigma \in C^2(\overline{\Omega}\times\mathbb{R}) \text{ and} \\ 0 < k_1 \le \frac{\partial\sigma(x,u)}{\partial u} \le k_2, \text{ for some positive constants } k_1,\ k_2. \end{cases} \tag{8}$$

In this paper, we will prove the convergence of the pair $(u_\varepsilon, P_\varepsilon)$ to the pair (u_0, P_0) defined as a solution to the nonlinear coupled system

$$\begin{cases} -\Delta u_0 + \mathcal{A}_3 H(x,u_0) = f - \delta^{-1}P_0, & x \in \Omega \\ -\Delta P_0 + \mathcal{A}_3 H_u(x,u_0)P_0 = -\Delta u_0 + \frac{\mathcal{A}_3}{2}\frac{\partial}{\partial u}(H^2)(x,u_0), & x \in \Omega, \\ u_0 = P_0 = 0 & x \in \partial\Omega, \end{cases} \tag{9}$$

where $\mathcal{A}_3 = 4\pi C_0$. We will prove that this system is associated to the optimal problem corresponding to the limit cost functional

$$J_0(v) = \frac{1}{2}\int_\Omega |\nabla u(v)|^2 dx + \frac{\mathcal{A}_3}{2}\int_\Omega H^2(x,u(v))dx + \frac{\delta}{2}\int_\Omega v^2 dx. \tag{10}$$

Notice that the adjoint optimal problem is rather different from the original adjoint problem (7) and that a related "new strange term", $\frac{\partial}{\partial u}(H^2)(x,u_0)$, arises in the limit formulation of the optimality system (9): see also Remark 4 for the case of "big particles". We point out that from (3), we get that

$$H_u(x,u) = \frac{\sigma_u(x,u-H)}{C_0^{-1} + \sigma_u(x,u-H)} \tag{11}$$

and thus $\frac{\partial}{\partial u}(H^2)(x,u_0) \in L^2(\Omega)$.

The statements of the main results of this paper are the following:

Theorem 1. *Let $n = 3$, $f \in L^2(\Omega)$ and assume (8). Let u_ε be the solution to (1) and let u_0 be the unique solution of (2). Then*

$$\begin{gathered} \|u_\varepsilon - u_0 + W_\varepsilon(x)H(x,u_0)\|_{H^1(\Omega_\varepsilon,\partial\Omega)} \to 0 \textit{ and} \\ \varepsilon^{-\gamma/2}\|u_\varepsilon - u_0 + H(x,u_0)\|_{L^2(S_\varepsilon)} \to 0, \textit{ as } \varepsilon \to 0. \end{gathered} \tag{12}$$

In addition,

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_\Omega |\nabla u_0|^2 dx + \mathcal{A}_3\int_\Omega H^2(x,u_0)dx \tag{13}$$

and

$$\mathcal{P}_\varepsilon^{int} \to \mathcal{P}_0^{int} \ \textit{as}\ \varepsilon \to 0, \tag{14}$$

i.e.

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,u_\varepsilon)u_\varepsilon ds = \mathcal{A}_3 \int_\Omega H(x,u_0)u_0 dx - \mathcal{A}_3 \int_\Omega H^2(x,u_0)dx, \ \textit{as}\ \varepsilon \to 0. \tag{15}$$

Notice that (13) implies that the convergence of u_ε to u_0 cannot be strong in $H^1(\Omega_\varepsilon, \partial\Omega)$ (as mentioned before, this contrasts with the strong convergence in any $W^{1,p}(\Omega)$ when $1 \le p < 2$). It seems possible to extend the final conclusion of the above convergence result to the case of more general nonlinear terms $\sigma(x,u_\varepsilon)$ (see Remark 3) but we have preferred to maintain the kind of assumptions (8) which are needed to justify the limit problem in the control problem (9) in order to maintain an unity of exposition in this paper.

Concerning the control problem, we will start by constructing the associated adjoint problem (in the sense of [12] and [20]) for the adjoint state P_ε. This will require a stronger regularity on the nonlinear function $\sigma(x,u)$: we will assume also

$$\left|\frac{\partial^2\sigma(x,u)}{\partial u^2}\right| \le k_3, \text{ for some } k_3 > 0. \tag{16}$$

Theorem 2. *Let $n = 3$. Assume (8) and (16). Let $f \in L^2(\Omega)$ and let $(u_\varepsilon, P_\varepsilon)$ be the solution of the coupled nonlinear system (4) and (7). Then:*
i) $(u_\varepsilon, P_\varepsilon)$ converges strongly in $L^2(\Omega)^2$ to the pair (u_0, P_0), solution to the nonlinear system (9).
ii) $v_0 = -\delta^{-1}P_0$ is the optimal control, and $(u(v_0), v_0)$ is the optimal pair, for the optimization problem of finding a minimum in $L^2(\Omega)$ of the functional $J_0(v)$ given by (10), where in this definition of $J_0(v)$ the state $u = u(v)$ is the weak solution of the semilinear problem

$$\begin{cases} -\Delta u + \mathcal{A}_3 H(x,u) = f + v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{17}$$

Moreover, we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0). \tag{18}$$

Notice that although J_ε is independent of the nonlinear term σ, the limit cost functional J_0 depends on the corresponding nonlinear efficient term $H(x,u)$ (which depends on σ through its definition by (3).

The key tool for the proof of Theorem 2 is the use of the strong convergence stated in Theorem 1. We will prove that the corrector is given through a different application of an auxiliary function already used in the proof of the weak convergence $u_\varepsilon \rightharpoonup u_0$ in $H^1(\Omega_\varepsilon, \partial\Omega)$: instead of using it to adapt any test function v of the limit problem (2) to be used as a test function in the original heterogeneous problem (1), it will be used to adapt the own limit solution u_0 to be inserted as a test function in the framework of the original heterogeneous problem (1). In the critical case, the corrector function will be given by

$$C_\varepsilon(x,u_0) = W_\varepsilon(x)H(x,u_0),$$

where $W_\varepsilon(x)$ is an *oscillating function* which allows to use the good properties of the *subcrtical case* (arising in the study of *big particles*: see [11], [10] and Section 5.5 of [3])

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon, \\ 1, & x \in G^j_\varepsilon,\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \overline{\cup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}, \end{cases} \tag{19}$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^3 : (a_\varepsilon G_0 + \varepsilon j) \bigcap \widetilde{\Omega_\varepsilon} \neq \emptyset\}$, $T^j_{\varepsilon/4}$ is the ball of radii $\varepsilon/4$ with the center in the point $P^j_\varepsilon = \varepsilon j$, $G^j_\varepsilon = a_\varepsilon G_0 + \varepsilon j$, and w^j_ε is a *capacity type solution* to the cell problem which allows to measure the relevance of the ε^α-contracting perforation (or particle) G^j_ε in terms of the ε-contracting basic cell

$$\begin{cases} \Delta w^j_\varepsilon = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ w^j_\varepsilon = 1, & x \in \partial G^j_\varepsilon, \\ w^j_\varepsilon = 0, & x \in \partial T^j_{\varepsilon/4}, \end{cases} \tag{20}$$

(a more detailed explanation of the notations will be given later).

We also present, in the last Section, the corresponding homogenization results for the case $n = 2$. That needs only slight modifications in the proofs after modifying the definition of the state problem (for instance, the adsorption term and the radius of perforations (or particles) depend exponentially on ε).

Since the proof of the homogenization of the control problem (4), (6) generalizes the proof of the convergence in Theorem 1, we will organize the paper giving priority to the study of the control problem. In Section 2, we recall the main notations and prove Theorems 1 and 2. The case of dimension $n = 2$ is considered in the final Section 3.

2. Notations and proof of the convergence results for $n = 3$

Let Ω be a bounded domain in $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$. We denote by G_0 the ball of unit radius in $\mathbb{R}^3$ centered at the origin of coordinates. For a domain B and $\delta > 0$, we define the set $\delta B = \{x \in \mathbb{R}^3$ such that $\delta^{-1}x \in B\}$. For $\varepsilon > 0$, we consider the domain

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\}.$$

We set

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G^j_\varepsilon,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^3 : (a_\varepsilon G_0 + \varepsilon j) \bigcap \widetilde{\Omega}_\varepsilon \neq \emptyset\}$ of cardinality $|\Upsilon_\varepsilon| \cong d\varepsilon^{-3}$, $d = const > 0$: here $\mathbb{Z}^3$ is the set of vectors in $\mathbb{R}^3$ with integer coordinates. Define $Y^j_\varepsilon = \varepsilon Y + \varepsilon j$, $P^j_\varepsilon = \varepsilon j$, where $Y = (-1/2, 1/2)^3$. Note that $\overline{G^j_\varepsilon} \subset \overline{Y^j_\varepsilon}$ and the center of the ball $G^j_\varepsilon = a_\varepsilon G_0 + \varepsilon j$ coincides with the center of the cube Y^j_ε. We consider the so-called *critical case,* i.e. we assume that $a_\varepsilon = C_0\varepsilon^3$, for some $C_0 = const > 0$.

Next, we define the perforated domain (or the external set to the set of particles)

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},$$

with the boundary

$$\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon,\ \ S_\varepsilon = \partial G_\varepsilon.$$

2.1. **Uniform in ε estimates of u_ε and v_ε in Theorem 2.** First, we note that the problem (4) has a unique weak solution $u_\varepsilon(v)$ for any fixed $v \in L^2(\Omega_\varepsilon)$ and the following estimate holds

$$\|\nabla u_\varepsilon(v)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon(v)\|^2_{L^2(S_\varepsilon)} \le K(\|f\|^2_{L^2(\Omega_\varepsilon)} + \|v\|^2_{L^2(\Omega_\varepsilon)}). \tag{21}$$

From here, we get that $\|\nabla u_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} \le C$. As v_ε is the optimal control, we derive $J_\varepsilon(v_\varepsilon) \le J_\varepsilon(0)$. But, we have $J_\varepsilon(0) = \frac{1}{2}\|\nabla u_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} \le C$. Therefore, we obtain the estimate

$$\frac{1}{2}\|\nabla u_\varepsilon(v_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|v_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} = J_\varepsilon(v_\varepsilon) \le J_\varepsilon(0) \le C,$$

where the constant is independent of ε. Hence, we get

$$\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \le K,$$

and immediately have that

$$\|\nabla u_\varepsilon(v_\varepsilon)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon(v_\varepsilon)\|^2_{L^2(S_\varepsilon)} \le K$$

From the definition of weak solutions of the problem (7) for function P_ε, by taking the P_ε itself as a test function, we get

$$\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla P_\varepsilon|^2 dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma_u(x, u_\varepsilon) P_\varepsilon^2 ds = \int_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla P_\varepsilon dx \\
&\le \Big(\int_{\Omega_\varepsilon} |\nabla P_\varepsilon|^2 dx\Big)^{1/2} \Big(\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx\Big)^{1/2} \le \frac{1}{2}\|\nabla P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \frac{1}{2}\|\nabla u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{22}$$

Using (8), we conclude

$$\|\nabla P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|P_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \le K.$$

Let $\widetilde{u_\varepsilon}$ and $\widetilde{P_\varepsilon}$ be the extensions of functions u_ε and P_ε on Ω, such that $\widetilde{u_\varepsilon}, \widetilde{P_\varepsilon} \in H^1_0(\Omega)$ (see, e.g., [21] and [3]). Then, the following estimates hold

$$\begin{aligned}
&\|\widetilde{u_\varepsilon}\|_{H^1_0(\Omega)} \le K\|u_\varepsilon\|_{H^1(\Omega_\varepsilon, \partial\Omega)},\ \ \|\nabla \widetilde{u_\varepsilon}\|_{L^2(\Omega)} \le K\|\nabla u_\varepsilon\|_{L^2(\Omega_\varepsilon)}, \\
&\|\widetilde{P_\varepsilon}\|_{H^1_0(\Omega)} \le K\|P_\varepsilon\|_{H^1(\Omega_\varepsilon, \partial\Omega)},\ \ \|\nabla \widetilde{P_\varepsilon}\|_{L^2(\Omega)} \le K\|\nabla P_\varepsilon\|_{L^2(\Omega_\varepsilon)}.
\end{aligned} \tag{23}$$

Notice that we can extend function v_ε with zero on $\Omega \setminus \Omega_\varepsilon$. From the estimates (23), it follows that there is a sub-sequence (still denoted by ε) such that, as $\varepsilon \to 0$,

$$\begin{aligned}
&\widetilde{u_\varepsilon} \to u_0 \text{ strongly in } L^2(\Omega),\ \ \widetilde{u_\varepsilon} \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega), \\
&\widetilde{P_\varepsilon} \to P_0 \text{ strongly in } L^2(\Omega),\ \ \widetilde{P_\varepsilon} \rightharpoonup P_0 \text{ weakly in } H^1_0(\Omega).
\end{aligned} \tag{24}$$

Remark 1. *From* (8) *and* (11)*, it is easy to see that*

$$\tilde{k}_1 = \frac{k_1}{k_2 + C_0^{-1}} \leq H_u \leq \frac{k_2}{k_1 + C_0^{-1}} = \tilde{k}_2$$

From here, we derive

$$\tilde{k}_1 u^2 + H(x,0)u \leq H(x,u)u \leq \tilde{k}_2 u^2 + H(x,0)u$$

Remark 2. *In the linear case, when* $\sigma(x,u) \equiv a(x)u$*, we have* $H(x,u) = \frac{a(x)}{C_0^{-1}+a(x)}u$ *and* $H_u(x,u) = \frac{a(x)}{C_0^{-1}+a(x)}$*. Then,* $\frac{\partial}{\partial u}(H^2) = 2\big(\frac{a(x)}{C_0^{-1}+a(x)}\big)^2 u$*. Substituting these expressions into* (9)*, we derive the result presented in the paper* [17].

2.2. **Limit problem for u_0.** We split the proof of Theorems 1 and 2 into several parts. We start by identifying the limit problem for u_0 and the occurrence of the "strange term" $H(x,u)$: a question which is common to the proof of both theorems.

Proposition 1. *Let* $f \in L^2(\Omega)$*, and let* u_0*,* $P_0 \in H_0^1(\Omega)$ *given by* (24)*. Then* u_0 *satisfies the partial differential equation indicated in (9).*

Proof. First, by monotonicity arguments (see, e.g. [5] and Section 2.2.1 of [3]), it is easy to see that the function u_ε satisfies the integral inequality

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla\phi \nabla(\phi - u_\varepsilon)dx + \varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x,\phi)(\phi - u_\varepsilon)ds \\ &\qquad \geq \int_{\Omega_\varepsilon} (f - \delta^{-1}P_\varepsilon)(\phi - u_\varepsilon)dx, \end{aligned} \tag{25}$$

for any test function $\phi \in H^1(\Omega_\varepsilon, \partial\Omega)$. We recall the definition of the auxiliary function W_ε (already mentioned in (19) of the Introduction). Now, given $\phi \in H_0^1(\Omega)$, we take $\phi - W_\varepsilon H(x,\phi)$ as a test function in the integral inequality (25), and derive

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla(\phi - W_\varepsilon H(x,\phi))\nabla(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx \\ &+\varepsilon^{-\gamma} \int_{S_\varepsilon} \sigma(x, \phi - H(x,\phi))(\phi - H(x,\phi) - u_\varepsilon)ds \\ &\quad \geq \int_{\Omega_\varepsilon} (f - \delta^{-1}P_\varepsilon)(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx. \end{aligned}$$

For the right-hand side, we have

$$\int_{\Omega_\varepsilon} (f - \delta^{-1}P_\varepsilon)(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx \to \int_{\Omega} (f - \delta^{-1}P_0)(\phi - u_0)dx$$

as $\varepsilon \to 0$. On the other hand, we decompose the first integral in the left-hand side in the following way

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - W_\varepsilon H - u_\varepsilon)dx - \int_{\Omega_\varepsilon} \nabla(W_\varepsilon H)\nabla(\phi - W_\varepsilon H - u_\varepsilon)dx$$

For the first integral, we have

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\nabla\phi\nabla(\phi-W_\varepsilon H-u_\varepsilon)dx=\int\limits_{\Omega}\nabla\phi\nabla(\phi-u_0)dx$$

We transform the second integral in the following way

$$\int\limits_{\Omega_\varepsilon}\nabla(W_\varepsilon H)\nabla(\phi-W_\varepsilon H-u_\varepsilon)dx$$

$$=\int\limits_{\Omega_\varepsilon}\nabla W_\varepsilon\nabla(H(\phi-W_\varepsilon H-u_\varepsilon))dx+\alpha_\varepsilon$$

$$=\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G_\varepsilon^j}\partial_\nu w_\varepsilon^j H(\phi-H-u_\varepsilon)ds$$

$$+\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}\partial_\nu w_\varepsilon^j H(\phi-u_\varepsilon)ds+\alpha_\varepsilon=\mathcal{I}_\varepsilon+\alpha_\varepsilon,$$

where $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$. Using the definition of functions w_ε^j, we compute (see, e.g., [5] or Section 3.1.5.1 of [3])

$$\mathcal{I}_\varepsilon=\frac{C_0^{-1}\varepsilon^{-\gamma}}{1-4a_\varepsilon\varepsilon^{-1}}\int\limits_{S_\varepsilon}H(\phi-H-u_\varepsilon)ds$$

$$-\frac{4^2C_0\varepsilon}{1-4a_\varepsilon\varepsilon^{-1}}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}H(\phi-u_\varepsilon)ds=\mathcal{I}_{1,\varepsilon}-\mathcal{I}_{2,\varepsilon}.$$

To find the limit of the second term, we use the following "from surface to volume" lemma (see [21] or Theorem 4.5 of [3]):

Lemma 1. *Let $h_\varepsilon\in H_0^1(\Omega)$ and $h_\varepsilon\rightharpoonup h$ weakly in $H_0^1(\Omega)$ as $\varepsilon\to 0$. Then*

$$4^2\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}h_\varepsilon ds-4\pi\int\limits_{\Omega}hdx\to 0,\ \ \varepsilon\to 0.$$

According to the Lemma 1, we get

$$\lim_{\varepsilon\to 0}\mathcal{I}_{2,\varepsilon}=\mathcal{A}_3\int\limits_{\Omega}H(x,\phi)(\phi-u_0)dx.$$

The second integral, $\mathcal{I}_{1,\varepsilon}$, disappears in the limit when we combine it with the integral over S_ε in the inequality (25). Indeed, we have

$$\lim_{\varepsilon\to 0}\mathcal{I}_{1,\varepsilon}=\lim_{\varepsilon\to 0}C_0^{-1}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}H(x,\phi)(\phi-H-u_\varepsilon)ds,$$

hence, the equality (3) implies that

$$\lim_{\varepsilon\to 0}(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,\phi-H)(\phi-H-u_\varepsilon)ds-\mathcal{I}_{1,\varepsilon})=0.$$

Therefore, for any $\phi \in H_0^1(\Omega)$, the limit function u_0 satisfies the integral inequality

$$\int\limits_\Omega \nabla\phi\nabla(\phi - u_0)dx + \mathcal{A}_3 \int\limits_\Omega H(x,\phi)(\phi - u_0)dx \geq \int\limits_\Omega (f - \delta^{-1}P_0)(\phi - u_0)dx.$$

Now, we take $\phi = u_0 \pm \lambda\psi$, where $\lambda > 0$ and $\psi \in H_0^1(\Omega)$ and pass to the limit as $\lambda \to 0$. Finally, we get that u_0 satisfies integral identity

$$\int\limits_\Omega \nabla u_0 \nabla\psi dx + \mathcal{A}_3 \int\limits_\Omega H(x,u_0)\psi dx = \int\limits_\Omega (f - \delta^{-1}P_0)\psi dx, \tag{26}$$

where ψ is an arbitrary function from $H_0^1(\Omega)$. Hence, u_0 satisfies the partial differential equation indicated in (9). ∎

2.3. **End of the proofs of Theorems 1 and 2.**

2.3.1. *The corrector term.* The following result proves the role of the corrector term in the case of the control problem and, at the same time, it proves the first part of the statement of Theorem 1 (take $v_\varepsilon \equiv 0$ in this case).

Proposition 2. *Let $f \in L^2(\Omega)$, $v_\varepsilon \rightharpoonup v_0$ weakly in $L^2(\Omega)$ as $\varepsilon \to 0$ and let u_ε be the solution to (4) with optimal control v_ε. Let u_0 be given by* (24). *Then, the strong convergences (12) hold. In addition*

$$\lim_{\varepsilon\to 0} \int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int\limits_\Omega |\nabla u_0|^2 dx + \mathcal{A}_3 \int\limits_\Omega H^2(x,u_0)dx. \tag{27}$$

Proof of Proposition 2: *convergence part* . First of all, we recall that since $H(x,u)$ is a Lipschitz function then u_0 is actually in the space $H^2(\Omega)$ (see [1] and [9]). Due to the Sobolev embedding theorem in dimension $n = 3$, $H^2(\Omega)$ is continuously embedded into $W^{1,6}(\Omega)$ and $C(\overline{\Omega})$.
We set $\phi = u_\varepsilon - u_0 + W_\varepsilon H(x,u_0)$ as a test function in the integral identity for the function u_ε and $\psi = \tilde{u}_\varepsilon - u_0 + W_\varepsilon H(x,u_0)$ as a test function in the problem (26). We recall that $W_\varepsilon \to 0$ strongly in $L^2(\Omega)$ (see, e.g. [5] or Section 3.1.5.1 of [3]). Next, we substitute the second identity from the first and get

$$\begin{aligned}
&\int\limits_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_0 + W_\varepsilon H)|^2 dx - \int\limits_{\Omega_\varepsilon} \nabla(W_\varepsilon H)\nabla(u_\varepsilon - u_0 + W_\varepsilon H)dx \\
&- \int\limits_{G_\varepsilon} \nabla u_0 \nabla(\tilde{u}_\varepsilon - u_0 + H)dx - \mathcal{A}_3 \int\limits_\Omega H(\tilde{u}_\varepsilon - u_0 + W_\varepsilon H)dx \\
&+\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} (\sigma(x,u_\varepsilon) - \sigma(x,u_0 - H))(u_\varepsilon - u_0 + H)ds \\
&+\varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma(x,u_0 - H)(u_\varepsilon - u_0 + H)ds = - \int\limits_{G_\varepsilon} f(\tilde{u}_\varepsilon - u_0 + H)dx \\
&+ \int\limits_{\Omega_\varepsilon} (v_\varepsilon - v_0)(\tilde{u}_\varepsilon - u_0 + W_\varepsilon H)dx - \int\limits_{G_\varepsilon} v_0(\tilde{u}_\varepsilon - u_0 + H)dx.
\end{aligned} \tag{28}$$

For the integrals on the right-hand side, we have

$$\Big|\int_{G_\varepsilon} f(\tilde{u}_\varepsilon - u_0 + H)dx\Big| \le \|f\|_{L^2(G_\varepsilon)}\|\tilde{u}_\varepsilon - u_0 + H\|_{L^6(G_\varepsilon)}|G_\varepsilon|^{1/3}$$
$$\le K\|f\|_{L^2(\Omega)}\|\tilde{u}_\varepsilon - u_0 + H\|_{H^1(\Omega)}(a_\varepsilon\varepsilon^{-1}) \le K a_\varepsilon \varepsilon^{-1},$$

$$\Big|\int_{G_\varepsilon} v_0(\tilde{u}_\varepsilon - u_0 + H)dx\Big| \le \|v_0\|_{L^2(G_\varepsilon)}\|\tilde{u}_\varepsilon - u_0 + H\|_{L^6(G_\varepsilon)}|G_\varepsilon|^{1/3}$$
$$\le K\|v_0\|_{L^2(\Omega)}\|\tilde{u}_\varepsilon - u_0 + H\|_{H^1(\Omega)}(a_\varepsilon\varepsilon^{-1}) \le K a_\varepsilon \varepsilon^{-1}.$$

As $a_\varepsilon \ll \varepsilon$, the two integrals converge to zero. For the last integral on the right-hand side, we use the fact that $u_\varepsilon \to u_0$ and $W_\varepsilon \to 0$ strongly in $L^2(\Omega)$ and $v_\varepsilon \rightharpoonup v_0$ weakly in $L^2(\Omega)$. Hence, we derive

$$\int_{\Omega_\varepsilon} (v_\varepsilon - v_0)(u_\varepsilon - u_0 + W_\varepsilon H)dx \to 0 \text{ as } \varepsilon \to 0.$$

Therefore, all the integrals on the right-hand side of (28) converge to zero as $\varepsilon \to 0$. Hence, we transform (28) into

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - u_0 + W_\varepsilon H)|^2 dx + \varepsilon^{-\gamma}\int_{S_\varepsilon}(\sigma(x,u_\varepsilon) - \sigma(x,u_0 - H))(u_\varepsilon - u_0 + H)ds$$
$$= \int_{\Omega_\varepsilon} \nabla(W_\varepsilon H)(u_\varepsilon - u_0 + W_\varepsilon H)dx + \int_{G_\varepsilon} \nabla u_0 \nabla(\tilde{u}_\varepsilon - u_0 + H)dx \tag{29}$$
$$+\mathcal{A}_3\int_{\Omega} H(\tilde{u}_\varepsilon - u_0 + W_\varepsilon H)dx - \varepsilon^{-\gamma}\int_{S_\varepsilon}\sigma(x,u_0 - H)(u_\varepsilon - u_0 + H)ds + \kappa_\varepsilon,$$

were $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$. We will show that the right-hand side of (29) converges to zero as $\varepsilon \to 0$. First, as the volume of G_ε tends to zero as $\varepsilon \to 0$, we have

$$\int_{G_\varepsilon} \nabla u_0 \nabla(\tilde{u}_\varepsilon - u_0 + H)dx \to 0, \text{ as } \varepsilon \to 0.$$

Now, we transform the integral with the W_ε on the right-hand side in the following way

$$\int_{\Omega_\varepsilon} \nabla(W_\varepsilon H)\nabla(u_\varepsilon - u_0 + W_\varepsilon H)dx = \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla(H(u_\varepsilon - u_0 + W_\varepsilon H))dx$$
$$- \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla H(u_\varepsilon - u_0 + W_\varepsilon H)dx + \int_{\Omega_\varepsilon} W_\varepsilon \nabla H \nabla(u_\varepsilon - u_0 - W_\varepsilon H)dx \tag{30}$$

Using Holder's inequality, we estimate the second integral in the right-hand side of the identity (30)

$$\Big|\int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla H(u_\varepsilon - u_0 - W_\varepsilon H)dx\Big|$$
$$\le \|\nabla W_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|\nabla H\|_{L^6(\Omega_\varepsilon)}\|u_\varepsilon - u_0 + W_\varepsilon H\|_{L^3(\Omega_\varepsilon)}$$
$$\le K\|\tilde{u}_\varepsilon - u_0 + W_\varepsilon H\|_{L^3(\Omega)}.$$

As above, we estimate the third integral in the right-hand side of the (30)

$$\Big|\int\limits_{\Omega_\varepsilon} W_\varepsilon \nabla H \nabla(u_\varepsilon - u_0 - W_\varepsilon H)dx\Big|$$
$$\le \|\nabla H\|_{L^6(\Omega_\varepsilon)}\|W_\varepsilon\|_{L^3(\Omega_\varepsilon)}\|\nabla(u_\varepsilon - u_0 + W_\varepsilon H)\|_{L^2(\Omega_\varepsilon)}$$
$$\le K\|u_0\|_{H^2(\Omega)}\|\tilde{u}_\varepsilon - u_0 + W_\varepsilon H\|_{H^1(\Omega)}\|W_\varepsilon\|_{L^3(\Omega_\varepsilon)} \le K\|W_\varepsilon\|_{L^3(\Omega_\varepsilon)}.$$

From the weak convergence of $\tilde{u}_\varepsilon - u_0 - W_\varepsilon H$ to zero in $H^1(\Omega)$ and the Sobolev embedding theorem, we conclude that $\tilde{u}_\varepsilon - u_0 + W_\varepsilon H \to 0$ strongly in $L^3(\Omega)$. Also, we have the weak convergence of W_ε to zero in $H^1(\Omega)$, hence, W_ε converges strongly to zero in $L^3(\Omega_\varepsilon)$. Therefore, the two integrals converge to zero as $\varepsilon \to 0$. The last integral from (30) is decomposed in the following way

$$\int\limits_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla(H(u_\varepsilon - u_0 + W_\varepsilon H))dx$$
$$= \sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} \partial_\nu w^j_\varepsilon H(u_\varepsilon - u_0)ds + \sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial G^j_\varepsilon} \partial_\nu w^j_\varepsilon H(u_\varepsilon - u_0 + H)ds$$
$$= -\frac{4^2 C_0\varepsilon}{1-4C_0\varepsilon^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u_0)ds + \frac{\varepsilon^{-\gamma}}{1-4C_0\varepsilon^2}\int\limits_{S_\varepsilon} C_0^{-1}H(u_\varepsilon - u_0 + H)ds$$
$$= -4^2C_0\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u_0)ds + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon} C_0^{-1}H(u_\varepsilon - u_0 + H)ds$$
$$+\varepsilon^{-\gamma}\frac{4\varepsilon^2 C_0^{-1}}{(1-4C_0\varepsilon^2)}\int\limits_{S_\varepsilon} H(u_\varepsilon - u_0 + H)ds - \frac{4^3C_0^2\varepsilon^3}{1-4\varepsilon^2 C_0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u)ds$$
$$= J^1_\varepsilon + J^2_\varepsilon + J^3_\varepsilon + J^4_\varepsilon.$$

First, we estimate the last integrals in the expression above. We have

$$|J^3_\varepsilon| = \Big|\varepsilon^{-\gamma}\frac{4\varepsilon^2 C_0^{-1}}{1-4\varepsilon^2 C_0}\int\limits_{S_\varepsilon} C_0^{-1}H(u_\varepsilon - u_0 + H)ds\Big|$$
$$\le K\varepsilon^2\varepsilon^{-\gamma}|S_\varepsilon|^{1/2}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)} \le K\varepsilon^2\varepsilon^{-\gamma/2}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)}$$
$$\le K\varepsilon^2\varepsilon^{-\gamma/2}(\|u_\varepsilon\|_{L^2(S_\varepsilon)} + \|u_0 - H\|_{L^2(S_\varepsilon)}) \le K\varepsilon^2.$$

Here, the last inequality is derived by using the estimate (21) for u_ε, and, as $u_0 \in C(\overline{\Omega})$ and H is Lipschitz continuous, we have

$$\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(u_0 - H(x,u_0))^2 ds \le K\varepsilon^{-3}|S_\varepsilon| \le K\varepsilon^{-3}\varepsilon^6\varepsilon^{-3} \le K.$$

Next, using the Lemma 1 (the sum of the integrals in J_ε^4 multiplied by ε converges as $\varepsilon \to 0$), we derive the following estimate

$$|J_\varepsilon^4| = \Big|\frac{4^3 C_0^2 \varepsilon^3}{1-4\varepsilon^2 C_0}\sum_{j\in\Upsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u_0)ds\Big| \le K\varepsilon^2.$$

Hence, both J_ε^3 and J_ε^4 converge to zero as $\varepsilon \to 0$.

To get the limit of the last two integrals, we combine them with the last two integrals on the left-hand side of (29). First, we have

$$\Big|J_\varepsilon^1 + \mathcal{A}_3\int\limits_\Omega H(\tilde{u}_\varepsilon - u_0 + W_\varepsilon H)dx\Big| \le \Big|\int\limits_\Omega H^2 W_\varepsilon dx\Big|$$
$$+C_0\Big|4^2\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u_0)ds - 4\pi\int\limits_\Omega H(\tilde{u}_\varepsilon - u_0)dx\Big|.$$

For the first expression, it is easy to see that

$$\Big|\int\limits_\Omega H^2 W_\varepsilon dx\Big| \le K\|W_\varepsilon\|_{L^2(\Omega)} \le K\varepsilon^2.$$

For the second expression, Lemma 1 implies

$$\Big|4^2\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}} H(u_\varepsilon - u_0)ds - 4\pi\int\limits_\Omega H(\tilde{u}_\varepsilon - u_0)dx\Big| \to 0 \text{ as } \varepsilon \to 0.$$

Hence, we get

$$\Big|J_\varepsilon^1 + \mathcal{A}_3\int\limits_\Omega H(\tilde{u}_\varepsilon - u_0 + W_\varepsilon H)dx\Big| \to 0 \text{ as } \varepsilon \to 0.$$

Then, expression (3) implies

$$J_\varepsilon^2 - \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x, u_0 - H)(u_\varepsilon - u_0 + H)ds$$
$$= \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(C_0^{-1}H - \sigma(x, u_0 - H))(u_\varepsilon - u_0 + H)ds = 0.$$

Combining the obtained estimates, we derive from (28) the inequality

$$\|\nabla(u_\varepsilon - u_0 + W_\varepsilon H)\|^2_{L^2(\Omega_\varepsilon)}$$
$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma(x, u_\varepsilon) - \sigma(x, u_0 - H))(u_\varepsilon - u_0 + H)ds \le \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Using the properties of the function σ, we finally get

$$\|\nabla(u_\varepsilon - u_0 + W_\varepsilon H)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\|u_\varepsilon - u_0 + H\|^2_{L^2(S_\varepsilon)} \to 0$$

as $\varepsilon \to 0$. (Notice that the proof of the first part of the statement of Theorem 1 holds once we make $v_\varepsilon = v_0 = 0$ in the above arguments).

Proof of Proposition 2: final part. Let us now prove the identification of the limit of ∇u_ε. We have

$$\int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx = \int_{\Omega_\varepsilon} |\nabla u_0|^2 dx + \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^2 dx$$

$$+ \Bigg[\int_{\Omega_\varepsilon} |\nabla (u_\varepsilon - u_0 + W_\varepsilon H)|^2 dx + 2 \int_{\Omega_\varepsilon} \nabla (u_\varepsilon - u_0 + W_\varepsilon H) \nabla u_0 dx$$

$$- 2 \int_{\Omega_\varepsilon} \nabla (u_\varepsilon - u_0 + W_\varepsilon H) \nabla (W_\varepsilon H) dx - 2 \int_{\Omega_\varepsilon} \nabla u_0 \nabla (W_\varepsilon H) dx \Bigg].$$

Using that $\|u_\varepsilon - u_0 + W_\varepsilon H\|_{H^1(\Omega_\varepsilon, \partial\Omega)} \to 0$ and $W_\varepsilon \rightharpoonup 0$ in $H_0^1(\Omega)$, we conclude that integrals in the square brackets converge to zero as $\varepsilon \to 0$. Then, we transform the integral

$$\int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^2 dx = \int_{\Omega_\varepsilon} H^2 |\nabla W_\varepsilon|^2 dx$$

$$+2 \int_{\Omega_\varepsilon} H W_\varepsilon \nabla W_\varepsilon \nabla H dx + \int_{\Omega_\varepsilon} W_\varepsilon^2 |\nabla H|^2 dx$$

$$= \int_{\Omega_\varepsilon} \nabla W_\varepsilon \nabla (H^2 W_\varepsilon) dx + \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Again, we have used the weak convergence to zero of W_ε in $H_0^1(\Omega_\varepsilon)$. The definition of W_ε further implies that

$$\int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^2 dx = \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_\nu w_\varepsilon^j H^2 ds + \alpha_\varepsilon = \varepsilon^{-\gamma} C_0^{-1} \sum_{j \in \Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} H^2 ds + \alpha_{1,\varepsilon},$$

where $\alpha_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$. Again, we use the continuity of the functions u_0 and H, and conclude that according to the mean value theorem there exists $\xi_\varepsilon^j \in \partial G_\varepsilon^j$ such that

$$\int_{\partial G_\varepsilon^j} H^2 ds = H^2(\xi_\varepsilon^j, u_0(\xi_\varepsilon^j)) |\partial G_\varepsilon^j| = H^2(\xi_\varepsilon^j, u_0(\xi_\varepsilon^j)) 4\pi \varepsilon^6 C_0^2.$$

Summing over all of the cells, we get

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^2 dx &= \varepsilon^{-3} C_0^{-1} 4\pi \varepsilon^6 C_0^2 \sum_{j \in \Upsilon_\varepsilon} H^2(\xi_\varepsilon^j, u_0(\xi_\varepsilon^j)) \\ &= \mathcal{A}_3 \sum_{j \in \Upsilon_\varepsilon} \varepsilon^3 H^2(\xi_\varepsilon^j, u_0(\xi_\varepsilon^j)). \end{aligned} \tag{31}$$

The last expression converges to the integral of H^2 over Ω as $\varepsilon \to 0$. Then we get

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla (W_\varepsilon H)|^2 dx = \mathcal{A}_3 \int_{\Omega} H^2(x, u_0) dx.$$

The obtained convergence implies the identification result (27).■

Remark 3. *It seems possible to apply this technique avoiding to assume more regularity on the limit solution u_0 to the case of more general nonlinear terms $\sigma(x, u_\varepsilon)$ as for instance the Signorini unilateral boundary conditions (as in* [11], [10] *and* [3]*) but we have preferred to maintain the kind of assumptions (8) which are needed to justify the limit problem in the control problem in order to maintain a unity of exposition in this paper.*

2.3.2. *Limit problem for P_0.*

Proposition 3. *Let P_ε be a solution of the adjoint problem. Then the limit function P_0 defined in* (24) *satisfies the partial differential equation given in (9).*

Proof. Define

$$I_\varepsilon \equiv \int\limits_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla(\phi W_\varepsilon)dx,$$

where $\phi \in C_0^\infty(\Omega)$.
From the integral identity for P_ε, it follows that

$$I_\varepsilon + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma_u(x, u_\varepsilon)P_\varepsilon \phi ds = \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi W_\varepsilon)dx.$$

From Proposition 2, we have

$$\begin{aligned} &\varepsilon^{-\gamma}\Big|\int\limits_{S_\varepsilon}(\sigma_u(x, u_\varepsilon) - \sigma_u(x, u_0 - H))P_\varepsilon \phi ds\Big| \\ &\le K \max_{x\in\overline{\Omega}, u\in\mathbb{R}} |\sigma_{uu}|\varepsilon^{-\gamma}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)}\|P_\varepsilon\|_{L^2(S_\varepsilon)} \\ &\le K\varepsilon^{-\gamma/2}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Hence, we re-write the integral identity in the following way

$$I_\varepsilon + \varepsilon^{-\gamma} \int\limits_{S_\varepsilon} \sigma_u(x, u_0 - H)P_\varepsilon \phi ds + \kappa_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi W_\varepsilon)dx, \tag{32}$$

where $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$.
Using the integral identity of the notion of the weak solution for u_ε, we get

$$\int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi W_\varepsilon)dx = -\varepsilon^{-\gamma}\int\limits_{S_\varepsilon} \sigma(x, u_\varepsilon)\phi ds + \int\limits_{\Omega_\varepsilon} f\phi W_\varepsilon dx - \int\limits_{\Omega_\varepsilon} P_\varepsilon \phi W_\varepsilon dx.$$

As $W_\varepsilon \to 0$ strongly in $L^2(\Omega_\varepsilon)$, we conclude that the last two integrals in the right-hand side of the above expression converge to zero as $\varepsilon \to 0$. Using Proposition 1, we have

$$\begin{aligned} &\varepsilon^{-\gamma}\Big|\int\limits_{S_\varepsilon}(\sigma(x, u_\varepsilon) - \sigma(x, u_0 - H(x, u_0)))\phi ds\Big| \\ &\le K\varepsilon^{-\gamma}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)}\|\phi\|_{L^2(S_\varepsilon)} \\ &\le K\varepsilon^{-\gamma/2}\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)} \to 0 \text{ as } \varepsilon \to 0. \end{aligned}$$

Then, using that u_0 is continuous and H is Lipschitz continuous, we derive

$$\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,u_0-H)\phi ds=\varepsilon^{-\gamma}C_0^{-1}\int\limits_{S_\varepsilon}H(x,u_0)\phi ds=\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)\phi dx+\kappa_\varepsilon$$

where $\kappa_\varepsilon\to 0$ as $\varepsilon\to 0$. Hence, combining the above convergences and estimates, we conclude

$$\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla(\phi W_\varepsilon)dx=-\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)\phi dx.$$

Therefore, from the expression (32), we get

$$I_\varepsilon=-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma_u(x,u_0-H)P_\varepsilon\phi ds-\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)\phi dx+\kappa_{1,\varepsilon},\tag{33}$$

where $\kappa_{1,\varepsilon}\to 0$ as $\varepsilon\to 0$.

On the other hand, since w_ε^j is a weak solution to the problem (20), we obtain

$$\begin{aligned}I_\varepsilon&=\int\limits_{\Omega_\varepsilon}\nabla W_\varepsilon\nabla(P_\varepsilon\phi)dx+\alpha_\varepsilon\\&=\varepsilon^{-\gamma}C_0^{-1}\int\limits_{S_\varepsilon}P_\varepsilon\phi ds-4^2C_0\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}P_\varepsilon\phi ds+\beta_\varepsilon,\end{aligned}\tag{34}$$

where $\alpha_\varepsilon,\beta_\varepsilon\to 0$ as $\varepsilon\to 0$.

Comparing (33) and (34), we derive

$$\begin{aligned}&\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma_u(x,u_0-H)+C_0^{-1})P_\varepsilon\phi ds\\&=-\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)\phi dx+4^2C_0\varepsilon\sum_{j\in\Upsilon_\varepsilon}\int\limits_{\partial T^j_{\varepsilon/4}}P_\varepsilon\phi ds+\alpha_{1,\varepsilon},\end{aligned}\tag{35}$$

where $\alpha_{1,\varepsilon}\to 0$ as $\varepsilon\to 0$.

Now, we set $\phi(x)=H_u(x,u_0)\psi(x)$ as a test function in (35), where $\psi\in C_0^\infty(\Omega)$ is an arbitrary function. Passing to the limit as $\varepsilon\to 0$, we get

$$\begin{aligned}&\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma_u(x,u_0-H)P_\varepsilon\psi ds\\&=-\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)H_u(x,u_0)\psi dx+\mathcal{A}_3\int\limits_{\Omega}H_u(x,u_0)P_0\psi dx.\end{aligned}\tag{36}$$

Consequently, taking into account (36), we get that the function P_0 satisfies the following integral identity

$$\begin{aligned}&\int\limits_{\Omega}\nabla P_0\nabla\psi dx+\mathcal{A}_3\int\limits_{\Omega}H_u(x,u_0)P_0\psi dx\\&=\int\limits_{\Omega}\nabla u_0\nabla\psi dx+\mathcal{A}_3\int\limits_{\Omega}H(x,u_0)H_u(x,u_0)\psi dx,\end{aligned}$$

where ψ is an arbitrary function from $C_0^\infty(\Omega)$. This implies that (u_0, P_0) is a weak solution to the system (9). Hence, the Proposition is proved. ∎

2.3.3. *Proof of Theorem 1: Final part (proof of the convergence (15)).* We have the convergence

$$\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(u_\varepsilon - u_0 + H(x,u_0))^2 ds \to 0,\ \varepsilon \to 0. \tag{37}$$

So, we derive

$$I_\varepsilon \equiv \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,u_\varepsilon)u_\varepsilon ds = \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}\sigma(x,u_0 - H(x,u_0))u_\varepsilon ds + J_{1,\varepsilon}, \tag{38}$$

where

$$J_{1,\varepsilon} = \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma(x,u_\varepsilon) - \sigma(x,u_0 - H(x,u_0)))u_\varepsilon ds.$$

For $J_{1,\varepsilon}$ we have

$$|J_{1,\varepsilon}| \le \Big(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(\sigma(x,u_\varepsilon) - \sigma(x,u_0 - H(x,u_0)))^2 ds\Big)^{1/2}\Big(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}u_\varepsilon^2 ds\Big)^{1/2}$$

$$\le K\Big(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}(u_\varepsilon - u_0 + H(x,u_0))^2 ds\Big)^{1/2}\Big(\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}u_\varepsilon^2 ds\Big)^{1/2} \to 0,\ \varepsilon \to 0.$$

Taking into account that $\sigma(x,u_0 - H(x,u_0)) = C_0^{-1}H(x,u_0)$, from (38), we conclude

$$I_\varepsilon = \varepsilon^{-\gamma}C_0^{-1}\int\limits_{S_\varepsilon}H(x,u_0)u_0 ds - \varepsilon^{-\gamma}C_0^{-1}\int\limits_{S_\varepsilon}H^2(x,u_0)ds + J_{1,\varepsilon} + J_{2,\varepsilon}, \tag{39}$$

where

$$J_{2,\varepsilon} = \varepsilon^{-\gamma}\int\limits_{S_\varepsilon}C_0^{-1}H(x,u_0)(u_\varepsilon - u_0 + H(x,u_0))ds. \tag{40}$$

Using again the convergence (37), we have

$$J_{2,\varepsilon} \to 0,\ \varepsilon \to 0.$$

Applying that $u_0 \in C(\overline{\Omega})$, we get the convergence (15) from (39) arguing as in (31) and this concludes the proof of Theorem 1∎

2.3.4. *Proof of Theorem 2: Final part (the limit of the cost functional).* Now, we will show (18). For the function $v_\varepsilon = -\delta^{-1}P_\varepsilon$, we have

$$J_\varepsilon(\delta^{-1}P_\varepsilon) = \frac{1}{2}\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \frac{1}{2\delta}\int\limits_{\Omega_\varepsilon} P_\varepsilon^2 dx.$$

Using the identification of the limit of ∇u_ε in (27) we derive

$$\lim_{\varepsilon\to 0} J_\varepsilon(-\delta^{-1}P_\varepsilon) = \lim_{\varepsilon\to 0}\frac{1}{2}\int\limits_{\Omega_\varepsilon} |\nabla u_\varepsilon|^2 dx + \frac{1}{2\delta}\int\limits_{\Omega_\varepsilon} P_\varepsilon^2 dx$$

$$= \frac{1}{2}\int\limits_{\Omega} |\nabla u_0|^2 dx + \frac{\mathcal{A}_3}{2}\int\limits_{\Omega} H^2(x,u_0)dx + \frac{1}{2\delta}\int\limits_{\Omega} P_0^2 dx = J_0(-\delta^1 P_0).$$

This concludes the proof of Theorem 2.■

Remark 4. *In the case of big particles,* $1 < \alpha < n/(n-2)$ *and* $\gamma = \alpha(n-1) - n$, *the previous homogenization results (see, e.g.,* [13], [14], [11] *and* [3]*) and the control techniques (*[20]*), jointly with the arguments as in the proof of Theorem 2, allow to see that the associated limit optimality system would be*

$$\begin{cases} -\Delta u_0 + \mathcal{A}\sigma(x,u_0) = f - \delta^{-1}P_0, & x \in \Omega \\ -\Delta P_0 + \mathcal{A}\sigma_u(x,u_0)P_0 = -\Delta u_0, & x \in \Omega, \\ u_0 = P_0 = 0 & x \in \partial\Omega, \end{cases} \tag{41}$$

with the associated cost functional limit

$$J_0(v) = \frac{1}{2}\int\limits_{\Omega} |\nabla u(v)|^2 dx + \frac{\delta}{2}\int\limits_{\Omega} v^2 dx, \tag{42}$$

for a suitable positive constant $\mathcal{A}$.

3. The control problem for $n = 2$

In this section, we describe the optimal control problem and its homogenization in the case $n = 2$. We consider a state problem of the form

$$\begin{cases} -\Delta u_\varepsilon = f + v, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(x,u_\varepsilon) = 0, & x \in S_\varepsilon, \\ u_\varepsilon = 0, & x \in \partial\Omega, \end{cases} \tag{43}$$

where $v \in L^2(\Omega_\varepsilon)$, $f \in L^2(\Omega)$, ν is the unit outward normal vector to S_ε, and now $a_\varepsilon = \varepsilon\exp(-\alpha^2/\varepsilon^2)$, $\beta(\varepsilon) = \varepsilon\exp(\alpha^2/\varepsilon^2)$, for some $\alpha \neq 0$. We assume again that the function $\sigma(x,u)$ is a smooth function in $x \in \overline{\Omega}$ and $u \in \mathbb{R}$ satisfying (8) and (16).

We consider a cost functional $J_\varepsilon : L^2(\Omega_\varepsilon) \to \mathbb{R}$ given by

$$J_\varepsilon(v) = \frac{1}{2}\|\nabla u_\varepsilon(v)\|^2_{L^2(\Omega_\varepsilon)} + \frac{\delta}{2}\|v\|^2_{L^2(\Omega_\varepsilon)}, \tag{44}$$

for some $\delta > 0$. It is well known that there exists a pair $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ (see, e.g. [12]), called optimal, such that the function v_ε is the optimal control satisfying

$$J_\varepsilon(v_\varepsilon) = \min_{v \in L^2(\Omega_\varepsilon)} J_\varepsilon(v). \tag{45}$$

As in the case $n = 3$, the adjoint problem, related to the state problem (43), takes the following form

$$\begin{cases} \Delta P_\varepsilon = \Delta u_\varepsilon, & x \in \Omega_\varepsilon, \\ \partial_\nu (P_\varepsilon - u_\varepsilon) + \beta(\varepsilon)\sigma_u(x, u_\varepsilon) P_\varepsilon = 0, & x \in S_\varepsilon, \\ P_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{46}$$

Moreover, the optimal control v_ε is characterized by the variational inequality

$$\int_{\Omega_\varepsilon} (P_\varepsilon + \delta v_\varepsilon)(v - v_\varepsilon) dx \geq 0, \tag{47}$$

where v is an arbitrary element of $L^2(\Omega_\varepsilon)$, and hence

$$v_\varepsilon = -\delta^{-1} P_\varepsilon. \tag{48}$$

Following the method described in the previous Sections, it is possible to get some uniform estimates on u_ε, P_ε, in $H^1(\Omega_\varepsilon, \partial\Omega)$, and on v_ε in $L^2(\Omega_\varepsilon)$. Then, using the properties of the extension operator, we obtain the uniform boundedness of the norms of the H^1-extensions of u_ε and P_ε in $H_0^1(\Omega)$. Hence, a sub-sequence exists for which convergences (24) are valid. The next theorem gives the description of limit functions u_0, v_0 defined in (24)

Theorem 3. *Let $n = 2$, $f \in L^2(\Omega)$ and let $(u_\varepsilon, P_\varepsilon)$ be a solution to the nonlinear system*

$$\begin{cases} -\Delta u_\varepsilon = f - \delta^{-1} P_\varepsilon, & x \in \Omega_\varepsilon, \\ \Delta P_\varepsilon = \Delta u_\varepsilon, & x \in \Omega_\varepsilon, \\ \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(x, u_\varepsilon) = 0, & x \in S_\varepsilon, \\ \partial_\nu (P_\varepsilon - u_\varepsilon) + \beta(\varepsilon)\sigma_u(x, u_\varepsilon) P_\varepsilon = 0, & x \in S_\varepsilon, \\ u_\varepsilon = P_\varepsilon = 0, & x \in \partial\Omega. \end{cases} \tag{49}$$

Then the pair (u_0, P_0) is a solution to the semilinear system

$$\begin{cases} -\Delta u_0 + \mathcal{A}_2 H(x, u_0) = f - \delta^{-1} P_0, & x \in \Omega, \\ -\Delta P_0 + \mathcal{A}_2 H_u(x, u_0) P_0 = -\Delta u_0 + \frac{\mathcal{A}_2}{2}\frac{\partial}{\partial u}(H^2)(x, u_0), & x \in \Omega, \\ u_0 = P_0 = 0, & x \in \partial\Omega, \end{cases} \tag{50}$$

where $\mathcal{A}_2 = 2\pi/\alpha^2$ and H is the unique solution to the functional equation

$$\frac{1}{\alpha^2} H = \sigma(x, u - H).$$

Remark 5. *Note that the function $v_0 = -\delta^{-1} P_0$ is the optimal control, and $(u(v_0), v_0)$ is the optimal pair, of the optimization problem of finding a minimum in $L^2(\Omega)$ of the functional*

$$J_0(v) = \frac{1}{2}\int_\Omega |\nabla u(v)|^2 dx + \frac{\mathcal{A}_2}{2}\int_\Omega H^2(x, u(v)) dx + \frac{\delta}{2}\int_\Omega v^2 dx, \tag{51}$$

where $u = u(v)$ is the unique weak solution of the problem

$$\begin{cases} -\Delta u + \mathcal{A}_2 H(x,u) = f + v, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{52}$$

Moreover, following the same arguments given in the previous Sections, we have that

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0). \tag{53}$$

To prove the Theorem 3, we will adapt the proof of the Theorem 2 to the case $n = 2$. We will divide the proof into three Propositions. First, we get the limit problem for the function u_0.

Proposition 4. *Let $n = 2$, $f \in L^2(\Omega)$, and let $u_0, P_0 \in H_0^1(\Omega)$ be the limit of $\tilde{u}_\varepsilon$ and $\tilde{P}_\varepsilon$ respectively. Then, the limit function u_0 is a solution to*

$$\begin{cases} -\Delta u_0 + \mathcal{A}_2 H(x,u_0) = f - \delta^{-1} P_0, & x \in \Omega, \\ u_0 = 0, & x \in \partial\Omega. \end{cases} \tag{54}$$

Proof. Using the monotonicity of the function σ, it is easy to see that the function u_ε satisfies the integral inequality

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla\phi \nabla(\phi - u_\varepsilon)dx + \beta(\varepsilon) \int_{S_\varepsilon} \sigma(x,\phi)(\phi - u_\varepsilon)ds \\ \geq \int_{\Omega_\varepsilon} (f - \delta^{-1}P_\varepsilon)(\phi - u_\varepsilon)dx \end{aligned} \tag{55}$$

for an arbitrary function ϕ from $C_0^\infty(\Omega)$.
Next, we take $\zeta_\varepsilon = \phi - W_\varepsilon H(x,\phi)$ as the test function in the (55), where W_ε is defined again by (19). Then, we get

$$\begin{aligned} &\int_{\Omega_\varepsilon} \nabla(\phi - W_\varepsilon H(x,\phi))\nabla(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx \\ &+\beta(\varepsilon) \int_{S_\varepsilon} \sigma(x, \phi - H(x,\phi))(\phi - H(x,\phi) - u_\varepsilon)ds \\ &\geq \int_{\Omega_\varepsilon} (f - \delta^{-1}P_\varepsilon)(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx. \end{aligned} \tag{56}$$

Using the definition of W_ε, we transform the integral over Ω_ε in the right-hand side of (56) in the following way

$$\begin{aligned} \mathcal{I}_\varepsilon &= \int_{\Omega_\varepsilon} \nabla(\phi - W_\varepsilon H(x,\phi))\nabla(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx \\ &= \int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx - \sum_{j\in\Upsilon_\varepsilon} \int_{\partial G_\varepsilon^j} \partial_\nu w_\varepsilon^j H(\phi - H - u_\varepsilon)ds \\ &\quad - \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w_\varepsilon^j H(\phi - u_\varepsilon)ds + \alpha_{1,\varepsilon}, \end{aligned}$$

where $\alpha_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$.
Then, by using the explicit expression of w_ε^j and modifying the right-hand side of the previous expression

$$\int_{\Omega_\varepsilon} \nabla\phi\nabla(\phi - W_\varepsilon H(x,\phi) - u_\varepsilon)dx + \frac{\beta(\varepsilon)}{\alpha^2}\int_{S_\varepsilon} H(\phi - H - u_\varepsilon)ds$$
$$-\frac{4}{\varepsilon \ln\frac{4a_\varepsilon}{\varepsilon}} \sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} H(\phi - u_\varepsilon)ds + \alpha_{2,\varepsilon}$$

where $\alpha_{2,\varepsilon} \to 0$ as $\varepsilon \to 0$. Next, we use the lemma proved in [21] (see also Theorem 4.5 of [3]) but now for the special case of $n = 2$.

Lemma 2. *Let $u_\varepsilon \in H_0^1(\Omega)$ and $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$ as $\varepsilon \to 0$. Then,*

$$\Big|\sum_{j\in\Upsilon_\varepsilon} \int_{\partial T^j_{\varepsilon/4}} \frac{4}{\varepsilon\ln(\frac{4a_\varepsilon}{\varepsilon})} H(\phi - u_\varepsilon)ds + \mathcal{A}_2 \int_\Omega H(\phi - u_0)dx\Big| \to 0, \ \ as\ \varepsilon \to 0.$$

Then, we get the convergence

$$\lim_{\varepsilon\to 0}\left(\mathcal{I}_\varepsilon + \beta(\varepsilon)\int_{S_\varepsilon} \sigma(x,\phi - H)(\phi - H - u_\varepsilon)ds\right)$$
$$= \int_\Omega \nabla\phi\nabla(\phi - u_0)dx + \mathcal{A}_2\int_\Omega H(\phi - u_0)dx$$

Hence, passing to the limit in as $\varepsilon \to 0$, we get that u_0 satisfies

$$\int_\Omega \nabla\phi\nabla(\phi - u_0)dx + \mathcal{A}_2 \int_\Omega H(\phi - u_0)dx \ge \int_\Omega f(\phi - u_0)dx,$$

for an arbitrary test function $\phi \in C_0^\infty(\Omega)$.From here, we derive that u_0 is the weak solution to the problem (54). ∎

The next theorem gives some regularity results of the solution u_0.

Proposition 5. *Let $n = 2$, $f \in L^2(\Omega)$, $v_\varepsilon \rightharpoonup v_0$ weakly in $L^2(\Omega)$ and u_ε is the solution to (43). Then*

$$\begin{gathered}\|u_\varepsilon - u_0 + W_\varepsilon H(x,u_0)\|_{H^1(\Omega,\partial\Omega)} \to 0,\\ \beta^{1/2}(\varepsilon)\|u_\varepsilon - u_0 + H(x,u_0)\|_{L^2(S_\varepsilon)} \to 0, \ \ as\ \varepsilon \to 0.\end{gathered} \tag{57}$$

Proof. Again, from the results for semilinear elliptic equations (see, e.g. [1]), we know that $u_0 \in H^2(\Omega)$. The Sobolev embedding theorem implies that $H^2(\Omega)$ is continuously embedded into $C(\overline{\Omega})$. Therefore, we can use the same arguments as in the proof of Theorem 2 and get the convergence statement. ∎

Next, we derive the limit problem for P_0.

Proposition 6. *Let P_ε be a solution of the adjoint problem. Then the limit function P_0 is a weak solution to the following boundary value problem*

$$\begin{cases} -\Delta P_0 + \mathcal{A}_2 H_u(x,u_0)P_0 = -\Delta u_0 + \mathcal{A}_2 H_u(x,u_0)H(x,u_0), & x \in \Omega, \\ P_0 = 0, & x \in \partial\Omega, \end{cases}$$

where u_0 is the solution to (54).

Proof. We follow again the same ideas of the proof of the Proposition 3. According to the Proposition 5, we have

$$\beta(\varepsilon)\Big|\int\limits_{S_\varepsilon} \sigma_u(x,u_\varepsilon) - \sigma_u(x,u_0 - H)P_\varepsilon \phi ds\Big|$$
$$\leq K \max_{x\in\overline{\Omega},u\in\mathbb{R}} |\sigma_{uu}|\beta(\varepsilon)\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)}\|P_\varepsilon\|_{L^2(S_\varepsilon)}$$
$$\leq K\beta^{1/2}(\varepsilon)\|u_\varepsilon - u_0 + H\|_{L^2(S_\varepsilon)} \to 0,$$

as $\varepsilon \to 0$, where $\phi \in C_0^\infty(\Omega)$. Hence, we can re-write the integral identity for the function P_ε in the following way

$$\int\limits_{\Omega_\varepsilon} \nabla P_\varepsilon \nabla(\phi W_\varepsilon)dx + \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma_u(x,u_0 - H)P_\varepsilon \phi ds + \kappa_\varepsilon = \int\limits_{\Omega_\varepsilon} \nabla u_\varepsilon \nabla(\phi W_\varepsilon)dx,$$

where $\kappa_\varepsilon \to 0$ as $\varepsilon \to 0$.

Using the similar arguments as in Proposition 3, we derive

$$\beta(\varepsilon)\int\limits_{S_\varepsilon} (\sigma_u(x,u_0 - H) + \alpha^{-2})P_\varepsilon \phi ds$$
$$= \mathcal{A}_2 \int\limits_{\Omega} H(x,u_0)\phi dx - \frac{4}{\varepsilon \ln \frac{4a_\varepsilon}{\varepsilon}} \sum_{j\in\Upsilon_\varepsilon} \int\limits_{\partial T^j_{\varepsilon/4}} P_\varepsilon \phi ds + \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Now, we set $\phi = H_u(x,u_0)\psi$ as a test function in the expression above, where $\psi \in C_0^\infty(\Omega)$, and passing to the limit as $\varepsilon \to 0$

$$\lim_{\varepsilon\to 0} \beta(\varepsilon)\int\limits_{S_\varepsilon} \sigma_u(x,u_0 - H)P_\varepsilon \psi ds$$
$$= -\mathcal{A}_2 \int\limits_{\Omega} H(x,u_0)H_u(x,u_0)\psi dx + \mathcal{A}_2 \int\limits_{\Omega} H_u(x,u_0)P_0\psi dx.$$

Note, that we have used Lemma 2 to derive the limit.

Using the obtained convergence, we derive that the function P_0 satisfies the integral identity

$$\int\limits_{\Omega} \nabla P_0 \nabla\psi dx + \mathcal{A}_2 \int\limits_{\Omega} H_u(x,u_0)P_0\psi dx$$
$$= \int\limits_{\Omega} \nabla u_0 \nabla\psi dx + \mathcal{A}_2 \int\limits_{\Omega} H(x,u_0)H_u(x,u_0)\psi dx,$$

where ψ is an arbitrary function from $C_0^\infty(\Omega)$. This completes the proof. ∎

Combining results of Proposition 4 with Proposition 6, we derive Theorem 3.

Remark 6. *It would be interesting to extend the different results of this paper to the case of particles of arbitrary shape (for instance with the techniques introduced in the paper* [8]*), to the study of more general terms $\sigma(x, u_\varepsilon)$ in the Robin boundary condition and, also, to the case of quasilinear Poisson type equations in the line of previous convergence results as, for instance,* [11] *and* [3]*. The difficulties are of many different types (for instance, the Pontryagin maximum principle for p-Laplacian type operators in the control problem is not completely well-known in the present dates).*

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References

[1] Brezis, H., Monotonicity methods in Hilbert spaces and some applications to nonlinear partial differential equations, In: Contributions to Nonlinear Funct. Analysis, Madison, 1971, E. Zarantonello (ed.), Acad. Press, NY, 1971, 101-156.

[2] De Giorgi, E. and Spagnolo S., Sulla convergenza degli integrals dell'energia, Boll. Un. Mat. It. 8. (1973) 391-411.

[3] Díaz J.I., Gómez-Castro D. and Shaposhnikova T.A., Nonlinear Reaction-Diffusion Processes for Nanocomposites. Anomalous improved homogenization. De Gruyter, Berlin, 2021.

[4] Díaz J. I., Gómez-Castro D., Podolskii A. V. and Shaposhnikova T. A. Non-existence of critical scales in the homogenization of the problem with p-Laplace diffusion and nonlinear reaction in the boundary of periodically distributed particles in n-dimensional domains when $p > n$. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A.Matemáticas 112 (2018), 331–340.

[5] Díaz J.I., Gómez-Castro D., Podol'skiy A.V. and Shaposhnikova T.A., Characterizing the strange term in critical size homogenization: quasilinear equation with a nonlinear boundary condition involving a general maximal monotone graph. Advances in Nonlinear Analysis 8 (2019), 679-693,

[6] Díaz J.I., Podolskiy A.V., Shaposhnikova T.A. On the convergence of controls and cost functionals in some optimal control heterogeneous problems when the homogenization process gives rise to some strange term // J. Math. Anal. Appl. 2021. DOI: 10.1016/j.jmaa.2021.125559

[7] Díaz J. I., Podolskiy A. V. and Shaposhnikova T. A. Boundary control and homogenization: optimal climatization through smart double skin boundaries. Differential and Integral Equations. 35, 3-4 (2022), 191–210.

[8] Díaz J. I., Podolskii A. V. and Shaposhnikova T. A. On the strange nonlocal homogenized problem arising as the limit of a Poisson equation with dynamical unilateral conditions on the boundary of perforations of critical size and arbitrary shape. Electron. J. Differential Equations, Vol. 2024 (2024), No. 03, 1-29.

[9] Evans L. C. Partial Differential Equations. AMS, Providence, Rhode Island.

[10] Gómez D., Lobo M., Pérez E., Podolskii A. V. and Shaposhnikova T. A., Unilateral problems for the p-Laplace operator in perforated media involving large parameters, ESAIM Control, Optim. Calc. Var. 24 (2018), 921-964.

[11] Jäger W., Neuss-Radu M., and Shaposhnikova T. A.. Homogenization of a variational inequality for the Laplace operator with nonlinear restriction for the flux on the interior boundary of a perforated domain. Nonlinear Analysis: Real World Applications 15.0 (2014), 367–380.

[12] Lions J. -L. Controle optimal de systemes gouvernes par des equations aux derivees partielles. Etudes Mathematiques Collection dirigee par P. Lelong. Dunod Gauthier-Villars, Paris 1968.

[13] Kaizu S. The Poisson equation with semilinear boundary conditions in domains with many tiny holes, J. Fac. Sci. Univ. Tokyo Sect. IA Math. 36 (1989), 43-86.

[14] Kaizu S. The Poisson equation with nonautonomous semilinear boundary conditions in domains with many time holes, SIAM J. Math. Anal. 22 (1991), 1222-1245.

[15] Strömqvist M.H. Optimal Control of the obstacle Problem in a Perforated Domain. Appl. Math. Optim. (2012), V. 66. PP. 239-255.

[16] Saint Jean Paulin J., Zoubairi H. Optimal control and "strange term" for a Stokes problem in perforated domains. Portugaliae Mathematica, V. 59. Fasc. 2-2002. Nova Serie.

[17] Podolskiy, A.V. and Shaposhnikova, T.A., Optimal Control and Strange Term Arising from Homogenization of the Poisson Equation in the Perforated Domain with the Robin-type Boundary Condition in the Critical Case // Doklady Mathematics, 2020, V. 102. pp. 497-501.

[18] Rajesh M. Convergence of some energies for the Dirichlet problem in perforated domains. Rendiconti di Matematica, Ser. VII, V. 21, Roma (2001), PP. 259-274.

[19] Temam, R. and Miranville, A.M., Mathematical Modeling in Continuum Mechanics, Second Edition, Cambridge Univ.Press, 2005.

[20] Troltzsch, F., Optimal control of partial differential equations, American Mathematical Society, Graduate Studies in Mathematics Vol. 112, Providence, Rhode Island, 2010.

[21] Zubova M.N., Shaposhnikova T.A. Homogenization of boundary value problems in perforated domains with the third boundary condition and the resulting change in the character of the nonlinearity in the problem. Diff. Eq. 2011, V. 47. N 1. PP. 78-90.

[22] Zubova M.N., Shaposhnikova T.A. Homogenization of the boundary value problem for the Laplace Operator in a perforated domain with rapidly oscillating nonhomogeneous Robin-type condition on the boundary of holes in critical case. Doklady Mathematics, 2017. V. 96. N 1. PP. 344-347.

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