A Note on the Farkas' Lemma and the Maximum Principle for Elliptic PDEs

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Dedicated to my friend Manuel López Pellicer on the occasion of his 81st anniversary

Abstract. Motivated by a lecture by Richard Aron in the Complutense University of Madrid, we present an extension of the 1902 Farkas' Lemma on combination of nonnegative elements to the framework of the Maximum Principle for higher order elliptic partial differential equations.

Key words: Farkas' Lemma, higher order elliptic partial differential equations, maximum principle.

MSC: 35G15, 46G25, 47A07, 15A69, 90C60.

1 Introduction

In a nice lecture at the UCM ([2]), Richard Aron explained some recent results on a variant of the pioneering Farkas's Lemma ([8]) concerning bilinear forms on $\mathbb{R}^n$ ([3]) and commented that the case of linear functions $f_1, f_2, ..., f_n : \mathbb{R}^n \to \mathbb{R}$ received the attention of more than 800 papers, with generalizations to the case of Hilbert spaces instead of $\mathbb{R}^n$. Let us recall, simply that one of the statements of this Lemma says that if we assume that the property $f_1(x), f_2(x), ..., f_n(x) \geq 0$ for any $x \in \mathbb{R}^n$ implies that $g(x) \geq 0$ for some non trivial linear function $g : \mathbb{R}^n \to \mathbb{R}$ then, necessarily, $g = \sum_{i=1}^n \alpha_i f_i$ for some $\alpha_i \in \mathbb{R}$ with $\alpha_i \geq 0$ for any $i = 1, ..., n$.

A revision of most of the citations of the paper [8] allows to see that many of the generalizations, and applications, of the Lemma deal with Optimization Theory (see, e.g. [5]). The purpose of this note is to present a different application which seems to be unadvertised before in the literature.

Consider the Hilbert space $H = L^2(\Omega)$. For simplicity, let us assume that $\Omega = (0, L)$, for some $L > 0$. Take

$$G_1 : H \to H,$$

the Green operator associated to the usual Laplacian

$$(P_1) \begin{cases} -\frac{d^2u}{dx^2} = h(x) & x \in \Omega = (0, L), \\ u(0) = u(L) = 0 \end{cases}$$

i.e. $G_1(h) = u$. We know that $G_1 \geq 0$ in the sense that $h \geq 0$ on Ω implies that $u \geq 0$ on Ω. As a matter of facts, we know that $u > 0$ in Ω and that in terms of Sobolev spaces

$G_1(h) \in H^2(\Omega) \cap H_0^1(\Omega)$ (see, e.g. [4]). Motivated by the case of higher spatial dimensions, sometimes in the literature it is used the alternative notation $u = G_1(h) = (-\Delta)^{-1}h$.

It is well-known that this property ("the maximum principle") fails, in general, for the case of higher order elliptic equations. Nevertheless, there are some reduced classes of forth order operators (with suitable boundary conditions) for which still the maximum principle holds. This is the case, for instance, of the one-dimensional simple *beam equation* (see, e.g. [6] and its references). Let us define $G_2 : H \to H$, for instance, as the Green operator associated to the biLaplacian problem

$$(P_2) \begin{cases} -\frac{d^4u}{dx^2} = h(x) & x \in (0, L), \\ u(0) = u(L) = 0 \\ u''(0) = u''(L) = 0, \end{cases}$$

i.e. $G_2(h) = u$ (which sometimes it is denoted as $u = G_2(h) = \left[(-\Delta)^{-1}\right]^2 h$). Again, we know that $G_2 \geq 0$ (since $h \geq 0$ on $(0, L)$ implies that $u \geq 0$ on $(0, L)$: see, e.g., [6]). We also know (see, e.g., [4] and [6]) that $G_2(h) \in H^4(\Omega) \cap H_0^2(\Omega)$ and that, in fact, $h \geq 0$ implies that $u > 0$ in Ω.

Consider now $x_0 \in \Omega$ and define the *evaluation functions*

$$f_1^{x_0}, f_2^{x_0} : H \to \mathbb{R}$$

given by

$$f_i^{x_0}(h) = G_i(h)(x_0) \text{ for any } h \in H,$$

(observe that, in fact $G_i(h) \in C(\overline{\Omega})$).

Following the same assumption than in Farkas' Lemma, suppose that there exists an operator $G : H \to H$ ($G \neq 0$) such that it defines (given $h \in H$ and $x_0 \in \Omega$) a function $g^{x_0} : H \to \mathbb{R}$ by means of $g^{x_0}(h) = G(h)(x_0)$. This function is associated to $f_i^{x_0}(h)$ and has the property that $g^{x_0}(h) \geq 0$. Then, the application of the generalization of Farkas' Lemma to Hilbert spaces H implies that

$$g^{x_0} = \alpha_1(x_0) f_1^{x_0} + \alpha_2(x_0) f_2^{x_0}, \text{ for some } \alpha_i(x_0) \geq 0 \text{ for } i = 1, 2.$$

What is specially interesting here is the necessity part ensuring that $\alpha_i(x_0)$ can not take negative values. The main result of this note (Proposition 1 below) gives a characterization of the operator G.

It seems possible to give also some similar applications for the case of very special systems of linear equations (now $f_1, f_2, ..., f_n$ must be understood as bilinear forms). Nevertheless, it is well-known that very strong conditions must be assumed on the coupling terms in order to get the maximum principle for systems of equations. This kind of arguments could be understood in the spirit of the extra conditions on $f_1, f_2, ..., f_n$ assumed in [2], [3] in order to extend the Farkas' Lemma to the case of bilinear forms on $\mathbb{R}^n$.

2 Main result

The following result gives a characterization of the operator G mentioned in the Introduction.

Proposition 1 *i)* $\alpha_1(x_0) + \alpha_2(x_0) > 0$ *for a.e.* $x_0 \in I$ *with* $I \subset (0, L)$, $meas(I) > 0$.
ii) If $\alpha_1(x_0) > 0$ *for a.e.* $x_0 \in \Omega$ *then, if we define* $v = G(h)$, *for* $h \in L^2(\Omega)$, *the function* $\frac{v - \alpha_2 G_2(h)}{\alpha_1}$ *belongs to* $H^2(\Omega) \cap H_0^1(\Omega)$ *and we have*

$$(P_G^1) \begin{cases} -\frac{d^2}{dx^2}\left(\frac{v}{\alpha_1(x)}\right) = h(x) - \frac{d^2}{dx^2}\left(\frac{\alpha_2(\cdot)}{\alpha_1(\cdot)}\left[(-\Delta)^{-1}\right]^2 h\right)(x) & x \in (0, L), \\ v(0) = v(L) = 0 \end{cases}$$

In addition, $g^{x_0}(h) = G(h)(x_0) > 0$ *for any* $x_0 \in \Omega$.

iii) If $\alpha_2(x_0) > 0$ for a.e. $x_0 \in \Omega$ then, if we define $v = G(h)$ for $h \in L^2(\Omega)$, the function $\frac{v-\alpha_1 G_1(h)}{\alpha_2}$ belongs to $H^4(\Omega) \cap H_0^2(\Omega)$ and we have

$$(P_G^2)\begin{cases} -\frac{d^4}{dx^2}(\frac{v}{\alpha_2(x)}) = h(x) + \frac{d^4}{dx^4}\left(\frac{\alpha_1(\cdot)}{\alpha_2(\cdot)}(-\Delta)^{-1}h\right)(x) \quad \textit{in } (0,L), \\ v(0) = v(L) = 0 \\ v''(0) = v''(L) = 0 \end{cases}$$

In addition, $g^{x_0}(h) = G(h)(x_0) > 0$ for any $x_0 \in \Omega$.
iv) Assume that for every $x_0 \in \Omega$ we have $g^{x_0} = \alpha_1 f_1^{x_0} + \alpha_2 f_2^{x_0}$ for some constants $\alpha_1, \alpha_2 \in \mathbb{R}$. Then the Green operator G is a non-negative operator if and only if $\alpha_1 \geq 0$ and $\alpha_2 \geq 0$.

Proof. i) From Farkas' Lemma $G(h)(x_0) = \alpha_1(x_0)G_1(h)(x_0) + \alpha_2(x_0)G_2(h)(x_0)$. But $G(h) \neq 0$, $\alpha_i(x_0) \geq 0$ and $G_1(h)(x_0) > 0$ which implies the conclusion.
ii) It is enough to use that

$$\frac{v(x) - \alpha_2(x)G_2(h)(x)}{\alpha_1(x)} = G_1(h)(x) \text{ for any } x \in \Omega.$$

iii) The conclusion follows from the identity

$$\frac{v(x) - \alpha_1(x)G_1(h)(x)}{\alpha_2(x)} = G_2(h)(x) \text{ for any } x \in \Omega.$$

iv) It is easy to check that, due to the special definition of operator G_2, we have

$$G_2(h) = G_1(G_1(h)), \text{ for every } h \in L^2(\Omega),$$

(see, e.g., [1] and [7]). Then

$$G(h)(x_0) = g^{x_0}(h) = \alpha_1 f_1^{x_0}(h) + \alpha_2 f_2^{x_0}(h) = \alpha_1 G_1(h)(x_0) + \alpha_2 G_2(h)(x_0) \text{ for any } h \in H,$$

and in consequence

$$G(h) = [\alpha_1 I + \alpha_2 G_1](G_1(h)) \text{ for any } h \in H,$$

where I is the identity operator on H. Then, the Farkas' Lemma ensures the conclusion.■

Remark 2 *The so-called "domination inequalities" of Gagliardo-Nirenberg type (see, e.g., [4]) show that operator $(-\Delta)^2$ dominates over the operator $(-\Delta)$. Thus, it is natural to expect that the positivity of operator $\alpha_2 G_2$, when $\alpha_2 > 0$, could allow to have the positivity of $G(h) = \alpha_1 G_1(h) + \alpha_2 G_2(h)$ for suitable $\alpha_2 < 0$, with $|\alpha_2|$ small enough. The application of the Farkas' Lemma given in iv) makes it clear: necessarily α_2 must be nonnegative in order to have the positivity of operator G.*

3 Conclusion

In this short note we have presented an extension of the 1902 Farkas' Lemma on combination of nonnegative elements to the framework of the Maximum Principle for higher order elliptic partial differential equations, by first time in the literature. Many other further results seem available in the framework of higher order elliptic or parabolic partial differential equations.

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