Anisotropic partial symmetrization for some quasilinear equations in comparison with a p-Laplace realization on some of the coordinates

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Abstract

In this note we prove a comparison result for a class of homogeneous Dirichlet boundary problems for anisotropic operators of the form $-\operatorname{div}(a(|\nabla_x u|)\nabla_x u) - u_{yy}$, by using Steiner symmetrization. We show that the solution to the problem whose data are symmetrizated in sense of Steiner and the operator is a $p-$laplacian type operator, i.e. $\Delta_{p,x} u - u_{yy}$ has maximum mass concentration. The proof uses the technique of finite-differences discretization in y introduced in the previous paper by the authors jointly to some co-authors [7], where a comparison result with respect to Steiner symmetrization in the nonlinear framework has been proved for the first time.

1 Introduction

In pioonering papers [22, 23] Talenti developed symmetrization techniques (see also [24, 19]) that allow to obtain a priori estimates for weak solutions to problems of the type

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^n$ and $a : (0, +\infty) \to (0, +\infty)$ is a monotone function satisfying the ellipticity condition

$$a(\xi)\xi \cdot \xi \geq |\xi|^p, \qquad \forall \xi \in \mathbb{R}^n.$$

Any L^p or any Orlicz norms of the solution u to problem (1.1) can be estimated by the same norm of the solution v to the spherically symmetric problem

$$\begin{cases} -\Delta_p v = f^\star & \text{in } \Omega^\star \\ v = 0 & \text{on } \partial\Omega^\star \end{cases} \tag{1.2}$$

where $\Omega^\star$ is the ball of $\mathbb{R}^n$ centered at zero having the same Lebesgue measure of Ω, $f^\star$ is the Schwarz rearrangement of f, that is the the spherically symmetric function, decreasing with respect to $|x|$, whose level sets $\{x \in \Omega : |f^\star(x)| > t\}$ have the same measure of the corresponding level sets of f, $\{x \in \Omega : |f(x)| > t\}$.

Talenti's approach has been successfully extended for example, by introducing lower order terms, by weakening the ellipticity condition, by considering parabolic equations or different boundary value problems (see, [2, 14, 23] and references therein). In all these papers referred above a comparison results of the type

$$u^*(s) \leq v^*(s)$$

or the weaker inequality

$$\int_0^s u^*(s)\, ds \leq \int_0^s v^*(s)\, ds$$

has been proved, where u^* is the *decreasing rearrangement of* u defined by

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\} \tag{1.3}$$

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and $\mu_u(t)$ is the *distribution function of* u defined as

$$\mu_u(t) = \mathcal{L}^n\{x \in \Omega : |u(x)| > t\}|,. \tag{1.4}$$

Here, and in the following, by $\mathcal{L}^k(E)$ we intend the Lebesgue measure of the set $E \subset \mathbb{R}^k$.
In any case the two inequalities implies L^p or Orlicz norms of the solution u.

This type of results simplifies the problem to derive apriori estimates for solutions to problem (1.1) since it reduces to the study of a one-dimensional problem. In this process of global symmetrization, information about possible symmetry properties satisfied by some subgroup of spatial variables is lost, for example due to the presence of convection terms in certain directions that do not affect some other spatial variables, etc. In these cases an approach based on Steiner symmetrization is useful.

To state the main results of this paper, we need to introduce a few definitions (cfr. Section 2). If u is a function defined in $\Omega \subset \mathbb{R}^{n+k} \equiv \mathbb{R}^n \times \mathbb{R}^k$: for any $y \in \mathbb{R}^k$, denote by Ω_y the $y-$section defined as $\Omega_y = \{x \in \mathbb{R}^n : (x,y) \in \Omega\}$. Let $\mu_u(\cdot, y)$ be the distribution funcion of

$$x \in \Omega_y \mapsto u(x,y) \in \mathbb{R}\,. \tag{1.5}$$

The *decreasing rearrangement (in codimension n)* of this function (1.5) is

$$u^*(s,y) = \sup\{t \geq 0 : \mu_u(t,y) > s\}\,, \quad (s,y) \in \Omega_y^* \times \mathbb{R}^k\,, \tag{1.6}$$

where $\Omega_y^* = (0, \mathcal{L}^n(\Omega_y))$. If ω_n denotes the measure of the unit ball of $\mathbb{R}^n$, we define the *Steiner rearrangement of* u as

$$u^\#(x,y) = u^*(\omega_n|x|^n, y), \qquad (x,y) \in \Omega^\#\,, \tag{1.7}$$

The *Steiner rearrangement of* Ω denoted here by $\Omega^\#$ is the set whose indicator function is $(\chi_\Omega)^\#$.

A comparison result with respect to Steiner symmetrisation is proved in [1, 4, 6, 9] for linear elliptic operators. In these cases a mass comparison result has been proved

$$\int_0^s u^*(\sigma,y)\,d\sigma \leq \int_0^s v^*(\sigma,y)\,d\sigma\,,$$

for a.e. $s \in (0, |\Omega|)$ and a.e. in $y \in \mathbb{R}^k$, which easily imply the a priori estimate on u in L^p or Orlicz norms. Similar results have also been proven in the paper [6] by using a simpler approach; Neumann boundary value problems have been studied in [16] (see also [9]). In all these papers quoted above only linear elliptic operatos have been considered.
A new approach that cover a class of nonlinear operators has been introduced in the more recent paper [7]; it allows to obtain the first mass comparison result in the nonlinear framework. Such a technique is applied to a class of anisotropic quasilinear operators and it is based on the discretization with respect to the variable y of the operators and a fine approximization process.
In this paper we use a similar approach introduced in [7] for proving a mass comparison result of the same type for a restricted class of operators. To be more specific we consider the class of homogeneous Dirichlet problems of the type

$$\begin{cases} -\operatorname{div}_x\Big(a(|\nabla_x u|)\nabla_x u\Big) - u_{yy} = f & \text{in } \Omega_1 \times \Omega_2 \\ u = 0 & \text{on } \partial(\Omega_1 \times \Omega_2)\,. \end{cases} \tag{P}$$

where, for sake of simplicity,

$$\Omega = \Omega_1 \times \Omega_2\,,\ \Omega_1 \subset \mathbb{R}^n\,,\ \text{ is open bounded Lipschitz domain} \quad \text{and} \quad \Omega_2 = (0,1)\,. \tag{1.8}$$

The function $a : (0,+\infty) \to (0,+\infty)$ is assumed to be in $\mathcal{C}^1(0,+\infty)$, such that, for some constant $C > 1$ and $p > 1$

$$t^{p-2} \leq a(t) \leq Ct^{p-2}\,, \tag{1.9}$$

and

$$-1 < i_a \leq s_a < \infty\,, \tag{1.10}$$

where

$$i_a = \inf_{t>0} \frac{ta'(t)}{a(t)}, \qquad s_a = \sup_{t>0} \frac{ta'(t)}{a(t)}\,.$$

Moreover we assume that

$$f \in L^{\max\{2,p'\}}(\Omega) \tag{1.11}$$

For every $q \in [1,\infty]$ we denote by $q' := \frac{q}{q-1}$ its conjugate exponent.

Some comments on assumptions (1.9) and (1.10) are in order. The standard $p-$Laplace operator corresponds to the choice $a(t) = t^{p-2}$, with $p > 1$ and $i_a = s_a = p-2$ in this case. Moreover, in [11] (see 2.29), the following growth condition is proved

$$a(1)t^{i_a} \le a(t) \le a(1)t^{s_a}\,, \qquad t > 0 \tag{1.12}$$

Therefore (1.9) and (1.12) hold at the same time, or, equivalently, the operator $a(t)$ satisfies the following condition

$$\max\{t^{p-2}, a(1)t^{i_a}\} \le a(t) \le \min\{Ct^{p-2}, a(1)t^{s_a}\}\,, \qquad t > 0\,. \tag{1.13}$$

Further consequences of such condition are in Section 2.

The natural space in which we consider weak solution to problem (P) is the anisotropic Sobolev space $W_0^{1,\mathbf{m}}(\Omega)$, which we define in Section 2. This space takes in account the growth of the operator with respect to the partial derivatives of u in x and in y governed by different powers.

The aim of this paper is to prove the following result

Theorem 1.1. *Let a satisfy* (1.9), (1.10) *and let $0 \le f, g \in L^{\max\{2,p'\}}(\Omega)$, with $g = g^{\#}$, $u \in W_0^{1,\mathbf{m}}(\Omega)$ be the weak solution of the problem* (P) *and $v \in W_0^{1,\mathbf{m}}(\Omega^{\#})$ be the weak solution of*

$$\begin{cases} -\Delta_{p,x} v - v_{yy} = g & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}\,. \end{cases} \tag{P$^{\#}$}$$

If

$$\int_0^s f^*(\sigma, y)d\sigma \le \int_0^s g^*(\sigma, y)d\sigma\,, \qquad \text{for all } s \in [0, |\Omega_1|] \text{ and for a.e. } y \in \Omega_2\,, \tag{1.14}$$

then, for the decreasing rearrangements u^* and v^* we have the following mass comparison:

$$\int_0^s u^*(\sigma, y)d\sigma \le \int_0^s v^*(\sigma, y)d\sigma\,, \qquad \text{for all } s \in [0, |\Omega_1|] \text{ and for a.e. } y \in \Omega_2\,. \tag{1.15}$$

Note that, if $g = f^{\#}$, problem (P$^{\#}$) reduces to the symmetrized problem

$$\begin{cases} -\Delta_{p,x} v - v_{yy} = f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega^{\#}\,, \end{cases} \tag{P$^{\#}$}$$

and the mass comparison (1.15) is a Talenti's type result, pointed out by first time in the paper [10] where an equation with zero order terms have been considered.

Inequality (1.15) can be used to estimate any Orlicz norm of $u(\cdot, y)$ by the same norm of $v(\cdot, y)$.
As the comparison result proved in [7], Theorem 1.1 generalizes a result of [1] to the class of anisotropic problems (P). The main difference with comparison result proved in [7] consists in the choice of symmetrizated problem. Actually a more general class of operators a are considered in [7] which satisfy the classical growth condition, without regularity assumption, and an argument of *fine approximation* allows to reduce to the case of approximated smooth operator a_ε satisfying (1.10). This setting allows to compare problem (P) with the problem having the same operator a and Steiner rearrangement of the datum f and of the domain Ω. The present result consider a smaller class of operators a which are smooth and satisfy (1.10), but it allows to compare all problems of the considered class with the symmetrizated problem corresponding to the operator $-\Delta_{p,x} v - v_{yy}$ whose data are rearranged with respect to Steiner symmetrization and the proof is simpler than the one in [7]. In addition we present here a new quantitative estimate for the discretized problems.

The approach used in this paper is based on the techniques used in [7]. As in [7], we discretize in the y-derivative, to obtain a family of problems

$$\begin{cases} -\operatorname{div}_x \Big(a(|\nabla_x u_j|)\nabla_x u_j \Big) - \dfrac{u_{j+1} - 2u_j + u_{j-1}}{h^2} = f_j & \text{in } \ \Omega_1, \\ u_j = 0 & \text{on } \partial\Omega_1, \quad j = 1, \cdots, N, \\ u_0 = u_{N+1} \equiv 0 & \text{in } \ \Omega_1, \end{cases} \tag{P$_h$}$$

where

$$f_j(x) = f(x, jh)\,, \qquad (N+1)h = 1. \tag{1.16}$$

Then we prove that we recover solutions of (P) as $h \to 0$.

In this setting we easily prove a comparison results in a very classical manner: we rearrange each equation and apply a comparison argument for the system (P_h). We define

$$U_j(s) = \int_0^s u_j^*(\sigma)d\sigma, \qquad V_j(s) = \int_0^s v_j^*(\sigma)d\sigma, \qquad F_j(s) = \int_0^s f_j^*(\sigma)d\sigma$$

and, for $j \in \{1, \cdots, N\}$, and we have that U_j is a weak solution of

$$\begin{cases} \kappa_n(s)\left(-\kappa_n(s)\dfrac{d^2U_j}{ds^2}\right)^{p-1} - \dfrac{U_{j+1}-2U_j+U_{j-1}}{h^2} \le F_j & \text{in } \Omega_1^*, \\ \dfrac{dU_j}{ds}(|\Omega_1|) = U_j(0) = 0, \end{cases} \tag{$\underline{P_h^*}$}$$

where $\kappa_n(s)$ denotes the perimeter of a ball of measure s, that is

$$\kappa_n(s) = n\omega_n^{1/n} s^{1/n'} \tag{1.17}$$

and V_j solves the same problem, except that the above differential inequalities become equalities (see (3.12) below). Then as in [7] we recover some regularity of U_j. This regularity is sufficient to apply accretivity results for ($\underline{P_h^*}$) in L^∞, from which we deduce $U_j \le V_j$ for every j. We devote Section 4 to showing that we can pass to the limit as $h \to 0$, and recover solutions of the original problem.

2 Preliminary results

In this section we recall a few properties of Schwarz or Steiner rearrangements and results concerning the solution to problem (P) which will be used in the paper.

2.1 Schwarz rearrangement

In this subsection we introduce some notation and recall classical properties of Schwarz rearrangement. Consider a non-negative measurable $u : \Omega \to \mathbb{R}$. We define the *Schwarz rearrangement* of u as

$$u^\star(x) = u^*(\omega_n|x|^n), \qquad \text{for } x \in \Omega^\star. \tag{2.1}$$

The relation between u^* and μ is the following

$$\mu(u^*(s)) = |\{x \in \Omega : u(x) > u^*(s)\}| \le s \le |\{x \in \Omega : u(x) \ge u^*(s)\}| = \mu(u^*(s)^-)$$

and equalities hold if and only if μ is continuous or, equivalently, if u^* has no flat zone. Since μ is monotone, the set of discontinuities is, at most, countable, hence has measure zero.

The rearrangement of u is constructed so that, for any $E \subset \Omega$,

$$\int_E u(x)dx \le \int_0^{\mathcal{L}^n(E)} u^*(\sigma)d\sigma, \tag{2.2}$$

and, for a.e. $s \in \Omega^*$

$$\int_{u>u^*(s)} u\, dx = \int_0^s u^*(\sigma)d\sigma. \tag{2.3}$$

It is well known that if $u \in W_0^{1,p}(\Omega)$, for some $1 \le p \le \infty$, then also $u^\star \in W_0^{1,p}(\Omega^\star)$, and, by the classical Pólya-Szegö inequality, the $W^{1,p}$ norm is not increased (see for example [3, 5, 8] and the references therein), in the sense that

$$\int_{u=t} |\nabla u|^{p-1} d\mathcal{H}^{n-1} \ge \int_{u^\star=t} |\nabla u^\star|^{p-1} d\mathcal{H}^{n-1}\,. \tag{2.4}$$

By definition, we easily deduce that

$$|\nabla u^\star(x)| = \left[\kappa_n(s)\left(-\frac{du^*}{ds}(s)\right)\right]\Bigg|_{s=\omega_n|x|^n} \qquad \text{for a.e. } x \in \Omega^\star\,. \tag{2.5}$$

The following result is well-known (see [7] and references therein). We repeat its proof for sake of completeness.

Lemma 2.1. *Let $u \in W^{1,\infty}(\Omega_1)$. Then, u^* is locally absolutely continuous and*

$$0 \le -\kappa_n(s)\frac{du^*}{ds} \in L^\infty(\Omega_1^*).$$

If, furthermore, $a(|\nabla u|)\nabla u \in H^1(\Omega_1)$ and (1.9) is in force, then for a.e. $s \in \Omega_1^*$, we have

$$-\int_{u>u^*(s)} \operatorname{div}\left(a(|\nabla u|)\nabla u\right)\, dx \ge \kappa_n(s)\left(-\kappa_n(s)\frac{du^*}{ds}(s)\right)^{p-1}. \tag{2.6}$$

Proof. We split the proof in several steps.

Step 1. $u \in \mathcal{C}_c^\infty(\Omega)$. Denote, for a.e. $s \in \Omega_1^*$ the outer normal to $\{x : u(x) > u^*(s)\}$ by

$$\nu(x) = -\frac{\nabla u(x)}{|\nabla u(x)|} \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \{u = u^*(s)\}\,.$$

By the divergence theorem and (1.9) we get

$$-\int_{u>u^*(s)} \operatorname{div}\left(a(|\nabla u|)\nabla u(x)\right)\, dx = \tag{2.7}$$

$$\tag{2.8}$$

$$= \int_{u=u^*(s)} a(|\nabla u|)\nabla u(x)\cdot \nu(x)\, d\mathcal{H}^{n-1} = \int_{u=u^*(s)} a(|\nabla u|)|\nabla u(x)|\, d\mathcal{H}^{n-1} \ge \int_{u=u^*(s)} |\nabla u|^{p-1}\, d\mathcal{H}^{n-1}\,. \tag{2.9}$$

Taking (2.4) and (2.5) into account, we prove the result.

Step 2. General case. Let u be as in the statement. Since u is Lipschitz continuous and vanishes on the boundary, then, by [18], $u^\star$ is Lipschitz continuous. In particular $\kappa_n(s)du^*/ds \in L^\infty(\Omega_1^*)$. Then, by using the density of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ for any $p \in [1, +\infty)$ and the above regularity on $u^\star$ we have that there exits a sequence $u_k \in \mathcal{C}_c^\infty(\Omega)$ such that

$$u_k \to u \qquad \text{in } L^1(\Omega_1)$$
$$\nabla u_k \overset{*}{\rightharpoonup} \nabla u \qquad \text{in } L^\infty(\Omega_1)^n.$$

Moreover, by the classical Minty argument for monotone quasilinear operators ([21]) we also have that

$$a(|\nabla u_k|)\nabla u_k \rightharpoonup a(|\nabla u|)\nabla u \quad \text{ in } H^1(\Omega_1)^n.$$

Step 2a. Convergence of the rearranged term. We prove that, up to a subsequence, for any $0 \le \varphi \in L^\infty(0, |\Omega_1|)$ we have that

$$\liminf_k \int_0^{|\Omega_1|}\left(-\kappa_n(s)\frac{du_k^*}{ds}(s)\right)^p \varphi(s)ds \ge \int_0^{|\Omega_1|}\left(-\kappa_n(s)\frac{du^*}{ds}(s)\right)^p \varphi(s)ds\,. \tag{2.10}$$

It is clear that $\|\kappa_n(s)du_k^*/ds\|_{L^\infty} \le C$, hence, up to a subsequence (still denoted by u_k)

$$\kappa_n(s)\frac{du_k^*}{ds} \overset{\star}{\rightharpoonup} \xi \qquad \text{in } L^\infty(\Omega^*).$$

Since $u_k \to u$ in $L^1(\Omega)$ we have $u_k^* \to u^*$ in $L^1(\Omega^*)$. Hence, for φ such that $\kappa_n(s)\varphi \in W^{1,\infty}(\Omega^*)$ we have

$$\int_{\Omega_1^*} \xi\varphi = \lim_k \int_{\Omega_1^*} \kappa_n(s)\frac{du_k^*}{ds}\varphi = -\lim_k \int_{\Omega_1^*} u_k^*\frac{d}{ds}(\kappa_n(s)\varphi) = -\int_{\Omega_1^*} u^*\frac{d}{ds}(\kappa_n(s)\varphi) = \int_{\Omega_1^*} \kappa_n(s)\frac{du^*}{ds}\varphi.$$

Hence,

$$\xi = \kappa_n(s)\frac{du^*}{ds}.$$

Fix $0 \le \varphi \in L^\infty(0, |\Omega_1|)$. Since $A(t) = t^p$ is convex and continuous when $p \ge 1$, the map

$$g \mapsto \int_0^{|\Omega_1|} g(s)^p \varphi(s)ds$$

is convex and lower semicontinuous in the topology of $L^r(\Omega)$ for any $r \ge 1$. Therefore, it is also weak-lower semicontinuous in $L^r(\Omega)$. Thus,

$$\liminf_k \int_0^{|\Omega_1|}\left(-\kappa_n(s)\frac{du_k^*}{ds}(s)\right)^p \varphi(s)ds \ge \int_0^{|\Omega_1|}\left(-\kappa_n(s)\frac{du^*}{ds}(s)\right)^p \varphi(s)ds$$

Step 2b. Convergence of the divergence term Let us prove that

$$-\int_{\{u_k>u_k^*(\cdot)\}} \operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)\,dx \longrightarrow -\int_{\{u>u^*(\cdot)\}} \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big)\,dx, \qquad \text{in } L^1(\Omega^*). \tag{2.11}$$

Consider the map

$$s\in\Omega_1^* \mapsto \Phi_k(s) = -\int_{\{u_k>u_k^*(s)\}} \operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)\,dx = -\int_\Omega \operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)\,\chi_{\{u_k>u_k^*(s)\}}\,dx.$$

We have that $\operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)$ converges weakly in L^2. Let us prove that, for a.e. $s\in\Omega^*$

$$\chi_{\{u_k>u_k^*(s)\}} \longrightarrow \chi_{\{u>u^*(s)\}} \text{ in } L^2(\Omega). \tag{2.12}$$

First, let us prove the convergence a.e. $x\in\Omega$: if, s is such that

$$\lim_k u_k^*(s) = u^*(s) \tag{2.13}$$

then

$$\left\{x\in\Omega : \lim_k \chi_{\{u_k>u_k^*(s)\}}(x) \neq \chi_{\{u>u^*(s)\}}(x)\right\} \subset \{x\in\Omega : u(x)\neq u^*(s)\}.$$

Indeed, let $s\in\Omega^*$ and $x\in\Omega$ be such that $u(x)<u^*(s)$. Take $\varepsilon = (u^*(s)-u(x))/4$. For $k\ge k_\varepsilon$ large enough $|u_k^*(s)-u^*(s)|\le\varepsilon$ and (since u_k converges in $\mathcal{C}(\overline{\Omega})$), $|u_k(x)-u(x)|\le\varepsilon$. But then $u_k(x)<u_k^*(s)$. Hence $\chi_{\{u_k>u_k^*(s)\}}(x) = \chi_{\{u>u^*(s)\}}(x)$. The same holds for the limit. We can repeat the same argument if $u(x)>u^*(s)$.
Since $u_k^*\to u^*$ in $L^1(\Omega^*)$, up to a subsequence, $u_k^*\to u^*$ a.e. Hence, (2.13) holds a.e. On the other hand,

$$\mathcal{L}^n\{x\in\Omega : u(x)\neq u^*(s)\} = \mu(u^*(s)^-) - \mu(u^*(s)).$$

Since u^* and μ are monotone functions, the set of s such that $\mu(u^*(s))$ is discontinuous at s is countable. Hence, the set of s such that (2.12) does not hold has measure 0.
Since the sequence is pointwise bounded by 1, due the Dominated Convergence Theorem we have (2.12).
Hence, as $k\to+\infty$,

$$\Phi_k(s) = -\int_\Omega \operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)\chi_{\{u_k>u_k^*(s)\}}\,dx \longrightarrow -\int_\Omega \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big)\chi_{\{u>u^*(s)\}}\,dx, \qquad \text{a.e. } s\in\Omega^*.$$

Step 2c. Comparison of the limits Apply Step 1 to this final subsequence to deduce that (2.6) holds with u substitute by u_k. Multiplying both sides by $-\frac{du_k^*}{ds}\varphi(s)$, integrating in s and passing to the limit we deduce that

$$\int_0^{|\Omega_1|}\left\{-\int_{\{u>u^*(s)\}} \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big)\,dx\right\}\left(-\frac{du^*}{ds}\right)\varphi(s)ds \ge \int_0^{|\Omega_1|}\left(-\kappa_n(s)\frac{du^*}{ds}(s)\right)^p\varphi(s)ds\,.$$

Since this holds for any φ, we have that for a.e. $s\in[0,|\Omega_1|]$

$$\left\{-\int_{\{u>u^*(s)\}} \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big)\,dx\right\}\left(-\frac{du^*}{ds}\right) \ge \left(-\kappa_n(s)\frac{du^*}{ds}(s)\right)^p.$$

Taking into account (2.7) we have that

$$-\int_{\{u>u^*(s)\}} \operatorname{div}\Big(a(|\nabla u|)\nabla u\Big) = -\lim_k \int_{\{u_k>u_k^*(s)\}} \operatorname{div}\Big(a(|\nabla u_k|)\nabla u_k\Big)\,dx \ge 0.$$

Hence, (2.6) holds when $du^*/ds=0$. Everywhere else $-du^*/ds>0$ so we can divide an recover the result. □

2.2 Steiner rearrangement

If u is a function defined in $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, for any $y \in \mathbb{R}$, denote bu $u^{\#}$ the Steiner symmetrization of u defined in (1.7). Notice that it is spherically symmetric in x, and radially non-increasing in this variable. When there is no y variable (i.e. $m = 0$), it coincides with *Schwarz symmetrised of* u and all the result exposed in the previous paragraph still holds for a.e.y fixed.

2.3 Weak solutions to problem (P)

Denote $\mathbf{m} = (p, ...p, 2) \in \mathbb{R}^{n+1}$, denote by $W_0^{1,\mathbf{m}}(\Omega)$ the weak clousure of $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1,\mathbf{m}} = \|\nabla_x u\|_p + \|\nabla_y u\|_2 .$$

Embeddings of the kind $W_0^{1,\mathbf{m}}(\Omega) \subset L^q(\Omega)$ can be founf in [17].
As far as the existence of a solution to problem (P) concerns, note that problem (P) is the Eulero equation of the strictly convex functional

$$J(u) = \int_\Omega \Big(B(|\nabla_x u|) + |\nabla_y u|^2 - fu \Big) dx\, dy \,, \tag{2.14}$$

in $W_0^{1,\mathbf{m}}(\Omega)$, where the function $B : [0, +\infty) \to (0, +\infty)$ is given by

$$B(t) = \int_0^t \beta(s) ds \,, \quad t \geq 0 \,, \tag{2.15}$$

and $\beta : [0, +\infty) \to (0, +\infty)$ is defined as

$$\beta(t) = \begin{cases} a(t)t & t > 0 \\ 0 & t = 0 \end{cases} \tag{H$_1$}$$

Moreover by the first inequality in (1.10),

the function β is strictly increasing for $t > 0$

and hence B is strictly convex (see [12]). This implies that J is strictly convex. Therefore, a unique minimizer exists and it is also a weak solution of (P).

3 Comparison result for discrete problem (P$_h$)

For sake of completeness in this Section we repeate the same arguments used in [7] for proving the existence of a weak solution to the discrete problems (P$_h$), its uniqueness and a mass comparison result. We prove that (P$_h$) has a unique solution $\mathbf{u}_h = (u_j^h)$ in the space $X_N^p(\Omega)$, defined as

$$X_N^p(\Omega_1) = \{ \mathbf{u} \in L^2(\Omega_1)^{N+2} : \nabla u_j \in L^p(\Omega_1) \,, j = 1, \ldots, N \,, \quad u_0 = u_{N+1} = 0 \} \,. \tag{3.1}$$

Moreover it is a minimizer of the functional

$$J_h(\mathbf{u}) = \sum_{j=1}^N \int_{\Omega_1} B(|\nabla_x u_j|)\, dx + \sum_{j=0}^N \int_{\Omega_1} \left(\frac{u_{j+1} - u_j}{h} \right)^2 dx - \sum_{j=1}^N \int_{\Omega_1} f_j u_j \, dx \,. \tag{3.2}$$

We explicitly remark that the data $f_j(x)$ defined in (1.16) depend only on the variable $x \in \mathbb{R}^n$ and the variable y is not present. Therefore we denote ∇_x simply by ∇ in the whole section.

3.1 Existence, uniqueness and regularity of solutions of the discrete problem (P$_h$)

Let us begin by giving the definition of solution to the discrete problem (P$_h$).

Definition 3.1. We say that a function $\mathbf{u} \in X_N^p(\Omega_1)$ is a *weak solution* of (P$_h$) if

$$\int_{\Omega_1} a(\nabla u_j) \nabla u_j \cdot \nabla \varphi_j - \int_{\Omega_1} \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \varphi_j = \int_\Omega f_j \varphi_j, \quad \forall j \in \{1, \cdots, N\} \ \ \forall \boldsymbol{\varphi} \in X_N^p(\Omega_1). \tag{3.3}$$

Remark 3.1. Notice that, for $\mathbf{u}, \boldsymbol{\varphi} \in X^p_N(\Omega_1)$ we have that

$$\sum_{j=1}^{N} \frac{-u_{j+1}+2u_j-u_{j-1}}{h^2}\varphi_j = \sum_{j=0}^{N} \frac{u_{j+1}-u_j}{h}\frac{\varphi_{j+1}-\varphi_j}{h} = \sum_{j=1}^{N} u_j \frac{-\varphi_{j+1}+2\varphi_j-\varphi_{j-1}}{h^2}. \tag{3.4}$$

Hence, it is easy to see that the weak formulation (3.3) is equivalently to the following weak formulations

$$\int_{\Omega_1}\sum_{j=1}^{N} a(\nabla u_j)\nabla u_j\cdot\nabla\varphi_j + \int_{\Omega_1}\sum_{j=0}^{N}\frac{u_{j+1}-u_j}{h}\frac{\varphi_{j+1}-\varphi_j}{h} = \int_{\Omega_1}\sum_{j=1}^{N} f_j\varphi_j, \tag{3.5}$$

and

$$\int_{\Omega_1}\sum_{j=1}^{N} a(\nabla u_j)\nabla u_j\cdot\nabla\varphi_j + \int_{\Omega_1}\sum_{j=1}^{N} u_j\frac{-\varphi_{j+1}+2\varphi_j-\varphi_{j-1}}{h^2} = \int_{\Omega_1}\sum_{j=1}^{N} f_j\varphi_j. \tag{3.6}$$

The following result holds true

Proposition 3.1. *Assume* (1.9), (1.10) *and let* $\mathbf{f} = (f_j) \in L^{\max\{2,p'\}}(\Omega)^N$ *where* $f_j \geq 0$. *Then, according to Definition 3.1, there exists a unique weak solution* $\mathbf{u} = (u_j) \in X^p_N(\Omega_1)$ *to the discrete problem* (P_h) *where* $u_j \geq 0$. *Moreover it is the global minimiser in* $X^p_N(\Omega_1)$ *of* J_h *given by* (3.2).

Proof. Since $B'' = \beta' > 0$, B is strictly convex. Hence J_h is strictly convex and it has a unique minimiser. Applying (3.4) and reproducing the proof in [7], we deduce that the Euler-Lagrange equations for J_h are precisely (P_h). To check that $u_j \geq 0$ we use $\varphi_j = (u_j)_-$ as a test function, to deduce $(u_j)_- = 0$. □

The weak solution $\mathbf{u} = (u_j) \in X^p_N(\Omega_1)$ to the discrete problem (P_h) verifies some regularity properties given by the following result

Theorem 3.1. *Let* $\mathbf{f} \in L^\infty(\Omega_1)^N$. *Then, the unique weak solution of* (P_h) *is in* $W^{1,\infty}_0(\Omega_1)^{N+2}$ *and*

$$a(|\nabla u_j|)\nabla u_j \in H^1(\Omega_1). \tag{3.7}$$

Proof The discrete problem can be equivalently written as a diagonal system of equations, i.e.

$$-\operatorname{div}\Big(a(|\nabla u_j|)|\nabla u_j|\Big) = H_j(\mathbf{u}) = f_j + \frac{u_{j+1}-2u_j+u_{j-1}}{h} \qquad \text{in } \Omega_1.$$

It is proven in [20, Theorem 2] that, if $\mathbf{f} \in L^\infty(\Omega_1)^N$ then $\mathbf{u} \in L^\infty(\Omega_1)^N$.
Moreover by [12], since $u_j \in L^2(\Omega_1)$ by the minimisation argument, we deduce

$$\|\nabla u_j\|_{L^\infty(\Omega_1)} \leq C\beta^{-1}\left(\|H_j\|_{L^{n,1}(\Omega_1)}\right), \tag{3.8}$$

and by [13],

$$a(|\nabla u_j|)\nabla u_j \in W^{1,2}(\Omega_1). \tag{3.9}$$

Now let us consider the rearranged problem of the discrete problem (P_h)

$$\begin{cases} -\Delta_{p,x} v - \dfrac{v_{j+1}-2v_j+v_{j-1}}{h^2} = g_j & \text{in } \Omega_1^\star, \\ v_j = 0 & \text{on } \partial\Omega_1^\star,\ j=1,\cdots,N, \\ v_0 = v_{N+1} = 0 & \text{in } \Omega_1^\star \end{cases} \tag{3.10}$$

Our aim is to compare the weak solution of (P_h) with the weak solution of the rearranged problem (3.10). Arguing as before, we deduce that it has a unique solution $\mathbf{v} \in X^p_N(\Omega_1^\star)$.

Proposition 3.2. *Let* $\mathbf{f} \in \mathcal{C}_c(\Omega_1)^N$ *and let* $\mathbf{u} \in X^p_N(\Omega_1)$ *and* $\mathbf{v} \in X^p_N(\Omega_1^\sharp)$ *be the unique solutions of* (P_h) *and* (3.10) *respectively. Define, for every* $j \in \{0,\cdots,N+1\}$

$$U_j(s) = \int_0^s u_j^*(\sigma)\,d\sigma,\ V_j(s) = \int_0^s v_j^*(\sigma)\,d\sigma,\ F_j(s) = \int_0^s f_j^*(\sigma)d\sigma,\ G_j(s) = \int_0^s g_j(\sigma)d\sigma.$$

Then, for every $j \in \{1,\cdots,N\}$, U_j *and* V_j *are in* $\mathcal{C}(\overline{\Omega_1^*})$ *and satisfy*

$$\kappa_n(s)\,\frac{d^2U_j}{ds^2},\quad \kappa_n(s)\,\frac{d^2V_j}{ds^2} \in L^\infty(\Omega_1^*). \tag{3.11}$$

Moreover $\mathbf{U} = (U_j)$ *is a solution of* $(\underline{\mathrm{P}_h^*})$ *and* $\mathbf{V} = (V_j)$ *is a solution of*

$$\kappa_n(s)\left(-\kappa_n(s)\,\frac{d^2V_j}{ds^2}\right)^{p-1} - \frac{V_{j+1}-2V_j+V_{j-1}}{h^2} = G_j \tag{3.12}$$

Also, $U_0 = U_{N+1} = V_0 = V_{N+1} = 0$.

Proof We proceed as in [23], and using standard inequalities for the rest. By Lemma 2.1 and Theorem 3.1, we have (3.11). To check that the inequality of $(\underline{\mathrm{P}_h^*})$ is satisfied, for $s \in [0, |\Omega_1|]$ we can integrate over the level set of u_j

$$-\int_{u_j>u_j^*(s)} \mathrm{div}\left(a(|\nabla u_j|)\nabla u_j(x)\right)dx + \int_{u_j>u_j^*(s)} \frac{-u_{j+1}+2u_j-u_{j-1}}{h^2}dx = \int_{u_j>u_j^*(s)} f_j(x)dx\,. \tag{3.13}$$

Notice that, due to (3.7) is, (3.13) is well defined. Let us consider separately the three quantities which appear above. As regards the first term in (3.13), we apply Lemma 2.1, and hence, for a.e. $s \in \Omega_1^*$

$$-\int_{u_j>u_j^*(s)} \left(\mathrm{div}_x\left(a(|\nabla u_j|)\nabla u_j\right)\right)dx \geq \kappa_n(s)\left(-\kappa_n(s)\frac{\partial u_j^*}{\partial s}(s)\right)^{p-1}. \tag{3.14}$$

As regards the other two terms in (3.13), by (2.2) and (2.3) we get

$$\int_{u_j>u_j^*(s)} \frac{-u_{j+1}+2u_j-u_{j-1}}{h^2}dx \geq \int_0^s \frac{-u_{j+1}^*+2u_j^*-u_{j-1}^*}{h^2}dx \tag{3.15}$$

and

$$\int_{u_j>u_j^*(s)} f_j(x)\,dx \leq \int_0^s f_j^*(\sigma)\,d\sigma\,. \tag{3.16}$$

Collecting (3.14)-(3.16) we get that the function U_j is a weak solution of $(\underline{\mathrm{P}_h^*})$ with $\frac{d^2U_j}{ds^2} \in L^\infty$. This completes the proof for U_j.
Analogously, the same arguments apply to the equation in (3.10) : since the solution v_j equals $v_j^\#$, then all the inequalities in (3.14)-(3.16) hold as equalities. □

3.2 Mass comparison result for discrete problem (P_h)

The aim of this section is to prove the comparison result given by Proposition 3.3 below. Its proof is a modified version of the analogous result proved in [7] (see also [14, Theorem 1]). We repeat here a sketch of the proof.

Proposition 3.3. *Let* $\mathbf{U}$ *and* $\mathbf{V}$ *be as in Proposition 3.2.Then there exists a constant* $C_N > 0$ *(depending on* N*) such that*

$$\|(U_j - V_j)_+\|_{L^\infty(\Omega_1^*)} \leq C_N \|(F_j - G_j)_+\|_{L^\infty(\Omega_1^*)}. \tag{3.17}$$

In particular, if $F_j \leq G_j$ *then* $U_j \leq V_j$ *for all* j*, and hence*

$$\int_0^s u_j^* \leq \int_0^s v_j^* \qquad \forall j,\ \text{a.e. } s \in [0, |\Omega_1|]. \tag{3.18}$$

Proof Let us consider the operator

$$AU = \kappa_n(s)\left(-\kappa_n(s)\,\frac{d^2U}{ds^2}\right)^{p-1},$$

defined in the domain

$$D(A) = \left\{U \in L^\infty(\Omega_1^*) : \kappa_n(s)\,\frac{d^2U}{ds^2} \in L^\infty(\Omega_1^*),\ \frac{dU}{ds}(|\Omega_1|) = 0,\ U(0) = 0\right\}.$$

In [7] it is proven that the operator A is T-accretive in L^∞, for all $U, V \in D(A)$ and $\lambda > 0$, that is

$$\left\|(U-V)_+\right\|_{L^\infty} \leq \left\|\Big(U - V + \lambda(AU - AV)\Big)_+\right\|_{L^\infty}. \tag{3.19}$$

Due to ($\underline{\mathrm{P}_h^*}$) and (3.10) we have

$$\frac{h^2}{2}(AU_j - AV_j) + (U_j - V_j) \leq \frac{1}{2}(U_{j+1} - V_{j+1}) + \frac{1}{2}(U_{j-1} - V_{j-1}) + F_j - G_j.$$

Applying (3.19) with $\lambda = \frac{h^2}{2}$, we get

$$\|(U_j - V_j)_+\|_{L^\infty} \leq \frac{1}{2}\|(U_{j+1} - V_{j+1})_+\|_{L^\infty} + \frac{1}{2}\|(U_{j-1} - V_{j-1})_+\|_{L^\infty} + \|(F_j - G_j)_+\|_{L^\infty}.$$

We can rewrite this as

$$\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} \begin{pmatrix} \|(U_1 - V_1)_+\|_{L^\infty} \\ \vdots \\ \|(U_N - V_N)_+\|_{L^\infty} \end{pmatrix} \leq \frac{h^2}{2} \begin{pmatrix} \|(F_1 - G_1)_+\|_{L^\infty} \\ \vdots \\ \|(F_N - G_N)_+\|_{L^\infty} \end{pmatrix} \tag{3.20}$$

where the inequality holds coordinate by coordinate. Let us call the matrix D_2 and denote the vector by $\mathbb{X}$ in (3.20) and by $\mathbb{Y}$ the vector in the right hand side of (3.20). Notice that the vector components of $\mathbb{X}$ are non-negative. We have the Cholesky decomposition

$$D_2 = C^t C, \qquad \text{where } C = \begin{pmatrix} 1 & -1 & & & \\ 0 & 1 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & -1 \\ & & & 0 & 1 \end{pmatrix}.$$

Multiplying (3.20) by x, we obtain

$$0 \leq \|C\mathbb{X}\|_2^2 = \mathbb{X}^t C^t C \mathbb{X} = \mathbb{X}^t A \mathbb{X} \leq \left\|C^{-1}\right\| \|C\mathbb{X}\|_2 \|\mathbb{Y}\|_2 .$$

Therefore

$$\|C\mathbb{X}\|_2 \leq \left\|C^{-1}\right\| \|\mathbb{Y}\|_2$$

and using that C is a coercive matrix we get (3.17). In particular, if $F_j \leq G_j$ then $\mathbb{Y} = \mathbf{0}$ and we get $\|(U_j - V_j)_+\|_{L^\infty} = 0$. Hence $U_j \leq V_j$. □

4 Proof of main result Theorem 1.1

In order to prove Theorem 1.1, we begin by discretise with respect to y

We make use of the *floor function*:

$$\lfloor z \rfloor = \min\{k \in \mathbb{Z} : k \geq z\}$$

i.e. $\lfloor z \rfloor = k$ means $k \leq z < k + 1$.
Let $f \in \mathcal{C}_c^\infty(\Omega_1 \times \Omega_2)$. Let $N \in \mathbb{N}$ and let $h = 1/(N + 1)$, and consider the constant interpolation

$$f_j^h(x, y) = f(x, jh), \qquad j = \left\lfloor \frac{y}{h} \right\rfloor.$$

Define $\mathbf{u}^h = (u_j^h)$ the unique solution of (P_h) with data $\mathbf{f}^h = (f_j^h)$.

Let us consider the linear interpolation

$$u^h(x, y) = u_j^h(x) + \frac{u_{j+1}^h(x) - u_j^h(x)}{h}(y - jh), \qquad j = \left\lfloor \frac{y}{h} \right\rfloor.$$

or equivalently

$$u^h(x, y) = \left(\frac{y}{h} - j\right) u_j^h(x) + \left(\frac{y}{h} - j\right) u_{j+1}^h(x), \qquad j = \left\lfloor \frac{y}{h} \right\rfloor.$$

The weak solution $\mathbf{u}^h = (u_j^h)$ to discrete problem satisfies the mass comparison result, according to Proposition 3.3. Therefore we prove an a priori estimate of u^h in $W_0^{1,\mathbf{m}}(\Omega)$, since this apriori estimate allows to identify a limit function u which we prove to be the weak solution to problem (P). Our aim is prove that we can pass to the limit $u^h \to u$ at least in $L^1(\Omega_1 \times \Omega_2)$. This will be sufficient to show that the comparison of masses is preserved and the passage to the limit yields the mass comparison result for problem (P).
Finally a crucial tool in proving that the limit function u is the solution to problem (P) is the well-known Minty's trick. In the following remark we repeat the description of this tool given in [7].

Remark 4.1. We will apply the old trick of Minty [21] (see also [15, §5.1.3]): if A is a monotone operator and $Au = f$, then for all test functions φ we have $0 \le (Au - A\varphi, u - \varphi) = (f - A\varphi, u - \varphi)$ hence

$$(A\varphi, \varphi - u) \ge (f, \varphi - u).$$

One then recovers the equation by letting $\varphi = u + \lambda\psi$, so $\lambda(A(u + \lambda\psi), \psi) \ge \lambda(f, \psi)$. As $\lambda \to 0^+$ one has $(Au, \psi) \ge (f, \psi)$, while as $\lambda \to 0^-$ one has $(Au, \psi) \le (f, \psi)$. Hence $(Au, \psi) = (f, \psi)$, or $Au = f$. In particular, this trick applies if

$$(Au, v) = \int_\Omega E(\nabla u)\nabla v$$

Since $u^h(x, y)$ depends on the two variables x, y, from now on we use again the notation ∇_x.

Step 1. u^h **is a bounded sequence in** $W_0^{1,\mathbf{m}}(\Omega)$**.** Let us check that u^h is a bounded sequence in $W_0^{1,\mathbf{m}}(\Omega)$. We compute

$$\begin{aligned}\int_{\Omega_2}\int_{\Omega_1} |\nabla_x u^h(x,y)|^p dxdy &= \sum_{j=1}^N \int_{jh}^{(j+1)h} \int_{\Omega_1} \left| \nabla_x \left(u_j^h(x) + \frac{u_{j+1}^h(x) - u_j^h(x)}{h}(y - jh) \right) \right|^p dx \\ &\le Ch \sum_{j=1}^N \int_{\Omega_1} \left|\nabla_x u_j^h(x)\right|^p dxdy.\end{aligned}$$

On the other hand

$$\frac{\partial u^h}{\partial y}(x, y) = \frac{u_{j+1}^h(x) - u_j^h(x)}{h} \qquad j = \left\lfloor \frac{y}{h} \right\rfloor.$$

From (3.5) we deduce that

$$\int_{\Omega_1} \sum_{j=1}^N a(|\nabla u_j^h|)|\nabla u_j^h|^2 dx + \int_{\Omega_1} \sum_{j=0}^N \left(\frac{\partial u_j^h}{\partial y}\right)^2 dx = \int_{\Omega_1} \sum_{j=1}^N f_j^h u_j^h dx,$$

Since a satisfies growth conditions (1.9), $C > 1$ and the previous estimate, we deduce that

$$\int_{\Omega_1}\int_{\Omega_2} |\nabla_x u^h|^p dydx + \int_{\Omega_1}\int_{\Omega_2} \left(\frac{\partial u^h}{\partial y}\right)^2 dydx \le C \int_{\Omega_2}\int_{\Omega_1} f^h u^h dx\,. \tag{4.1}$$

In [17] it has been proven that, if $u \in W_0^{1,\mathbf{m}}(\Omega)$ there exist positive constants C_1 and C_2 such that

$$\|u\|_p \le C_1 \|\nabla_x u\|_p \qquad \text{and} \qquad \|u\|_2 \le C_2 \left\|\frac{\partial u}{\partial y}\right\|_2 . \tag{4.2}$$

Collecting (4.1) and (4.2), we deduce that for some positive constant C we have

$$\|\nabla_x u^h\|_p^p \le C\|f^h\|_{p'}^{p'} \qquad \text{and} \qquad \left\|\frac{\partial u^h}{\partial y}\right\|_2^2 \le C\|f^h\|_2^2\,,$$

which means that $\{u^h\}$ is bounded in $W_0^{1,\mathbf{m}}(\Omega)$, since, by assumption, $f^h \in L^{\max\{2,p'\}}$.

Step 2. $u^h \rightharpoonup$ **u weakly in** $W_0^{1,\mathbf{m}}(\Omega)$ **as** $h = \frac{1}{N+1} \to 0$**.** By Step 1 we deduce there exists a subsequence, which we still denote by $\{u^h\}$, and a function $u \in W_0^{1,\mathbf{m}}(\Omega)$ such that

$$u^h \rightharpoonup u \text{ in } W_0^{1,\mathbf{m}}(\Omega)\,.$$

Step 3. u is a solution of (P). It remains to prove that u is a solution of (P). Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$. Define

$$D_y^h\varphi(x,y) = \frac{\varphi(x,(j+1)h) - \varphi(x,jh)}{h}, \qquad j = \left\lfloor \frac{y}{h} \right\rfloor.$$

Going back to the weak formulation (3.5) with $\varphi_j(x) = \varphi(x, jh)$ we have

$$\int_{\Omega_1} \sum_{j=1}^N a(|\nabla_x u^h(x,jh)|)\nabla_x u^h(x,jh) \cdot \nabla_x \varphi(x,jh)dx + \int_{\Omega_1} \sum_{j=1}^N \frac{\partial u^h}{\partial y}(x,jh) D_y^h \varphi(x,y)dx$$
$$= \int_{\Omega_1} \sum_{j=1}^N f(x,jh)\varphi(x,jh)dx.. \tag{4.3}$$

Since the derivatives with respect to y are piecewise constant

$$\int_{\Omega_1} h\sum_{j=1}^N a(|\nabla_x u^h(x,jh)|)\nabla_x u^h(x,jh) \cdot \nabla_x \varphi(x,jh)dx + \int_{\Omega} \frac{\partial u^h}{\partial y}(x,y) D_y^h \varphi(x,y)dy\,dx$$
$$= \int_{\Omega_1} h\sum_{j=1}^N f(x,jh)\varphi(x,jh)dx.$$

By the Taylor expansion, we know that

$$\left\| \frac{\partial \varphi}{\partial y} - D_y^h \varphi \right\|_{L^\infty} \le h \left\| \frac{\partial^2 \varphi}{\partial y^2} \right\|_{L^\infty}$$

Thus

$$\int_{\Omega_1} h\sum_{j=1}^N a(|\nabla_x u^h(x,jh)|)\nabla_x u^h(x,jh) \cdot \nabla_x \varphi(x,jh)dx + \int_{\Omega} \frac{\partial u^h}{\partial y}\frac{\partial \varphi}{\partial y}dy\,dx = \int_{\Omega_1} h\sum_{j=1}^N f(x,jh)\varphi(x,jh)dx + R(h)$$

where

$$|R(h)| \le Ch \left\| \frac{\partial u^h}{\partial y} \right\|_{L^2} \left\| \frac{\partial^2 \varphi}{\partial y^2} \right\|_{L^2} \le Ch.$$

Since $a(|\xi|)\xi$ is monotone we can apply Minty's trick (see Remark 4.1). We can write

$$\int_{\Omega_1} h\sum_{j=1}^N a(|\nabla_x \varphi(x,jh)|)\nabla_x \varphi(x,jh) \cdot \nabla_x(\varphi(x,jh) - u^h(x,jh))dx + \int_{\Omega} \frac{\partial \varphi}{\partial y}(x,y)\left(\frac{\partial \varphi}{\partial y}(x,y) - \frac{\partial u^h}{\partial y}(x,y)\right)dy\,dx$$
$$\ge \int_{\Omega_1} h\sum_{j=1}^N f(x,jh)(\varphi(x,jh) - u^h(x,jh))dx + R(h).$$

We can apply the regularity of φ and f to deduce

$$\int_{\Omega} a(|\nabla_x \varphi(x,y)|)\nabla_x \varphi(x,y) \cdot \nabla_x(\varphi(x,y) - u^h(x,y))dxdy + \int_{\Omega} \frac{\partial \varphi}{\partial y}(x,y)\left(\frac{\partial \varphi}{\partial y}(x,y) - \frac{\partial u^h}{\partial y}(x,y)\right)dy\,dx$$
$$\ge \int_{\Omega} f(x,y)(\varphi(x,y) - u^h(x,y))dx + \bar{R}(h).$$

where $\bar{R}(h)$ also tends to zero. So we can pass to the limit for $h \to 0$, to deduce

$$\int_{\Omega} a(|\nabla_x \varphi(x,y)|)\nabla_x \varphi(x,y) \cdot \nabla_x(\varphi(x,y) - u(x,y))dxdy + \int_{\Omega} \frac{\partial \varphi}{\partial y}(x,y)\left(\frac{\partial \varphi}{\partial y}(x,y) - \frac{\partial u}{\partial y}(x,y)\right)dy\,dx$$
$$\ge \int_{\Omega} f(x,y)(\varphi(x,y) - u(x,y))dx. \tag{4.4}$$

By density, we deduce that the formula also holds for $\varphi \in W_0^{1,\mathbf{m}}(\Omega)$. We take $\varphi = u + \lambda w$ with $w \in W_0^{1,\mathbf{m}}(\Omega)$. As $\lambda \to 0^\pm$ we recover the equation (P). This completes the proof.

Step 4. Passage to the limit We have that $u^h \to u$ in $L^1(\Omega)$ as $h \to 0$. Analogously for $\Omega^\#$ we have that $v^h \to v$ in $L^1(\Omega^\#)$. Therefore $(u^h)^* \to u^*$ and $(v^h)^* \to v^*$ in $L^1(\Omega_1^* \times \Omega_2)$. Due to Proposition 3.3 we have that

$$\int_0^s (u^h)^*(\sigma, y)d\sigma \leq \int_0^s (v^h)^*(\sigma, y)d\sigma, \qquad \forall s \in (0, |\Omega_1|), y \in \Omega_2.$$

Passing to the limit we recover the result. Finally, arguing by density, we recover the result also when f belongs to $L^{\max\{2,p'\}}$.

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