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ABSENCE OF FINITE-TIME EXTINCTION FOR SOLUTIONS OF THE SEMILINEAR KLEIN-GORDON EQUATION WITH DAMPING*

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To the memory of Haïm Brezis: my adviser, my master, my friend

Abstract

We prove that the solutions of the damped Klein-Gordon equation with a monotone perturbation cannot fulfill the finite extinction time property, even if the perturbation is a non-Lipschitz (or multivalued) function of the unknown u. This contrasts with the case of the nonlinear Schrödinger damped equation (recent results dealing with this same monotone expressions but with a purely imaginary coefficient), and with the case of nonlinear parabolic equations with strong absorption (for which the finite extinction time property is well-known since the middle of the seventies of the last century).

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MSC: 35L71, 81Q05, 37L05, 35B35.

1 Introduction

Non-linear hyperbolic equations, like the Klein-Gordon equation, have been of considerable interest to mathematicians and physicists since the pioneering works in 1950 by L.I. Schiff, K. Jördens and I.E. Segal. Among other

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reasons, the equations arise in Relativistic Quantum Mechanics (see the references in [43]). The main goal of this paper is to analyze the asymptotic behavior of solutions when $t \to +\infty$. The long-time behavior of the semilinear damped Klein-Gordon equation is relevant because it provides insights into the stability, dispersion, and asymptotic dynamics of wave-like phenomena in nonlinear systems. Understanding these aspects is crucial in various physical applications, including field theory, optics, and condensed matter physics. In certain asymptotic regimes, the damped semilinear Klein-Gordon equation exhibits behavior reminiscent of the nonlinear Schrödinger equation (NLS). This connection arises through paraxial approximations in wave propagation, where the Klein-Gordon equation reduces to an NLS in the weakly relativistic regime. Nevertheless, as we will show, the asymptotics of solutions for a long time are very different for these two dispersive equations, and also in comparison with the semilinear parabolic equations with non-Lipschitz terms.

The nonlinear function which appears as a perturbation of the linear damped Klein-Gordon operator is like an additional control of the energy dissipation. In absence of diffusion, in the linear ordinary differential equation it is well-known that the damping term suppresses oscillations and drives monotonically the system towards the asymptotic state ($u = 0$). We will show here that in the presence of a monotone perturbation $\beta(u)$ the solutions present infinitely oscillations, instead of purely exponential decay. In particular, there is no finite time extinction property. We will prove this for the general semilinear damped Klein-Gordon equation

$$\begin{cases} u_{tt} + \gamma u_t - c^2 \Delta u + mu + \beta(u) \ni 0 & \text{in } \Omega \times (0, +\infty), \\ u_\lambda = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u_\lambda(0, x) = u_0(x) & \text{on } \Omega, \\ u_{\lambda t}(0, x) = v_0(x) & \text{on } \Omega. \end{cases} \tag{1}$$

where $c, \gamma, m > 0$ and $\beta(u)$ is a maximal monotone graph of $\mathbb{R}^2$ such that $0 \in \beta(0)$. Our main interest deals with the case $\Omega = \mathbb{R}^N$, but sometimes it is useful to start by the consideration of the case in which Ω is a bounded open set. The two main examples are the equation with saturation (sometimes written in a single-valued way)

$$u_{tt} + \gamma u_t - c^2 \Delta u + mu + a\frac{u}{|u|} = 0 \quad \text{in } \mathbb{R}^N \times (0, +\infty), \tag{2}$$

which corresponds to the special case of

$$\beta(r) = asign(r) = \begin{cases} a & \text{if } r > 0, \\ [-a, a] & \text{if } r = 0, \\ -a & \text{if } r < 0, \end{cases} \tag{3}$$

and the case with a strong absorption term

$$u_{tt} + \gamma u_t - c^2 \Delta u + mu + a\,|u|^{-(1-p)}\,u = f(x,t) \quad \text{in } \mathbb{R}^N \times (0, +\infty), \tag{4}$$

with $p \in (0,1)$, corresponding to

$$\beta(r) = a|u|^{-(1-p)}u. \tag{5}$$

Concerning the existence of solutions, both cases were already considered by J.-L. Lions in [44] (among many other authors), for the case of Ω bounded, $\gamma = m = 0$.

There are many results in the literature showing that under some conditions on the nonlinear term $\beta(u)$ the solutions $u(.,t)$ converge to zero when $t \to +\infty$ (see, e.g, Chapter 10 of [22] and its many references). Here we will prove, in any case, the absence of finite-time extinction for this semilinear Klein-Gordon equation with damping for any choice of the maximal monotone graph $\beta(u)$, once we assume that $0 \in \beta(0)$. We will prove that the time decay is at most exponential. This contrasts with the case of parabolic equations with a strong absorption term (see [2]) or with a maximal monotone graph β multivalued at the origin (see [15], [18] and the survey [28]), and even with the dispersive case of damped Schrödinger equations with single-valued non-Lipschitz perturbation (see [23], [24], [6], [7], [8], [9]) or the damped Schrödinger equation (see [10]).

The organization of this paper is as follows: Section 2 is devoted to proving the existence of solutions. The absence of finite extinction time property for the problem (1) will be presented in Section 3 where we start by considering the case of the second order nonlinear ordinary differential equation. The presentation of the new results for hyperbolic equations is here accompanied by numerous remarks, containing a large number of references, to place the results for these hyperbolic equations in comparison with some results established for the nonlinear Schrödinger equation, parabolic equations with similar nonlinear terms and, even, other hyperbolic nonlinear equations with nonlinearities applied on the damping term u_t instead of the own unknown u.

2 On the existence of solutions

Concerning the existence of solutions, we start by considering the case of a bounded domain Ω and also with a possible forzing term $f(x,t)$

$$\left\{\begin{array}{ll} u_{tt}+\gamma u_t-c^2\Delta u+mu+\beta(u)\ni f(x,t) & \text{in } \Omega\times(0,+\infty), \\ u_\lambda=0 & \text{on } \partial\Omega\times(0,+\infty), \\ u_\lambda(0,x)=u_0(x) & \text{on } \Omega, \\ u_{\lambda t}(0,x)=v_0(x) & \text{on } \Omega. \end{array}\right. \tag{6}$$

Theorem 1. *Assume Ω bounded and let $\beta=\partial j$, with $0\in\beta(0)$, be such that*

$$D(\beta)=\mathbb{R}. \tag{7}$$

Let $T>0$ arbitrary, and consider

$$u_0\in H_0^1(\Omega) \text{ with } j(u_0)\in L^1(\Omega), \tag{8}$$

$$v_0\in L^2(\Omega), \tag{9}$$

$$f\in L^1(0,T;L^2(\Omega)). \tag{10}$$

Then:
1. There exists $u\in L^\infty(0,T;H_0^1(\Omega))$, $u_t\in L^\infty(0,T;L^2(\Omega))$, weak solution of (6), in the sense that there exists a function $g\in L^1(\Omega\times(0,T))$ such that $g(x,t)\in\beta(u(x,t))$ a.e. $\Omega\times(0,T)$ and the equation of (6) is satisfied in the sense that

$$u_{tt}+\gamma u_t-c^2\Delta u+mu+g=f. \tag{11}$$

2. If, in addition

$$u_0\in H^2(\Omega),\ v_0\in H_0^1(\Omega), \text{ and there exists } g_v\in L^2(\Omega)\ g_v\in\beta(v_0) \text{ a.e.on } \Omega, \tag{12}$$

$$f\in L^2(0,T;L^2(\Omega)) \text{ and } f_t\in L^1(0,T;L^2(\Omega)), \tag{13}$$

then $u\in C([0,T];H_0^1(\Omega))\cap L^\infty(0,T;H^2(\Omega))$, $u_t\in L^\infty(0,T;H_0^1(\Omega))$ and $u_{tt}\in L^\infty(0,T;L^2(\Omega))$.

Proof. A proof of the above result, for the special case of $\gamma=0$ and $m=0$, was given in [13, Theorem III.3], and the additional regularity in [12] (Theorem III.3). For completeness, here we will extend their arguments to our formulation. We consider the approximation of β by its Yosida approximation $\beta_\lambda = \partial j_\lambda$ and we approximating the data by more regular data $u_{0,\lambda}$, $v_{0,\lambda}$, and f_λ satisfying the stronger conditions (12) and (13). As a first step, we consider the regularized problem

$$\begin{cases} u_{\lambda tt} + \gamma u_{\lambda t} - c^2\Delta u_\lambda + m u_\lambda + \beta_\lambda(u_\lambda) = f_\lambda(x,t) & \text{in } \Omega\times(0,+\infty), \\ u_\lambda = 0 & \text{on } \partial\Omega\times(0,+\infty), \\ u_\lambda(0,x) = u_{0,\lambda}(x) & \text{on } \Omega, \\ u_{\lambda t}(0,x) = v_{0,\lambda}(x) & \text{on } \Omega. \end{cases} \tag{14}$$

We get the existence and uniqueness of a solution since the operator

$$D(A) = \left(H_0^1(\Omega)\cap H^2(\Omega)\right)\times H_0^1(\Omega), \qquad A = \begin{pmatrix} 0 & -I \\ -c^2\Delta + mI & \gamma I \end{pmatrix},$$

is maximal monotone in the Hilbert space $H = H_0^1(\Omega)\times L^2(\Omega)$ (see [12]) and the operator

$$B = \begin{pmatrix} 0 \\ \beta_\lambda \end{pmatrix}$$

is Lipschitz on H (see Remark 3.14 of [14]). In that case we get a strong solution of (14) since

$$U_0 = \begin{pmatrix} u_{0,\lambda} \\ v_{0,\lambda} \end{pmatrix} \in D(A)$$

and

$$F = \begin{pmatrix} 0 \\ f_\lambda \end{pmatrix} \text{ is such that } F\in L^2(0,T;H), \quad \frac{d}{dt}F \in L^1(0,T;H).$$

Multiplying by $u_{\lambda t}$ and integrating over Ω we get

$$\frac{1}{2}\frac{d}{dt}\left|u_{\lambda t}\right|_{L^2}^2 + \frac{c^2}{2}\frac{d}{dt}(\left|\nabla u_\lambda\right|_{L^2}^2 + \frac{m}{2}\left|u_\lambda\right|_{L^2}^2) + \gamma\left|u_{\lambda t}\right|_{L^2}^2 + \frac{d}{dt}\int_\Omega j_\lambda(u_\lambda)dx \tag{15}$$
$$\leq \left|f_\lambda\right|_{L^2}\left|u_{\lambda t}\right|_{L^2}.$$

From this we conclude that the sequences $|u_{\lambda t}|_{L^2}^2$ and $|\nabla u_\lambda|_{L^2}^2 + |u_\lambda|_{L^2}^2$ are bounded in $L^\infty(0,T)$ as $\lambda\to 0$. On the other hand, multiplying (14) by the resolvent $v_\lambda = (I+\lambda\beta)^{-1}u_\lambda$, using that it is a contraction and the monotonicity of the operator $-c^2\Delta u_\lambda + m u_\lambda$, integrating over $\Omega\times(0,T)$, we obtain

$$\int_0^T\int_\Omega \beta_\lambda(u_\lambda)v_\lambda \leq \int_0^T\int_\Omega |f_\lambda(x,t)|\,|u_\lambda| + \int_0^T\int_\Omega |u_{\lambda t}|^2$$
$$+\int_\Omega |u_{\lambda t}(x,T)|\,|u_\lambda(x,T)| + \int_\Omega |v_{0,\lambda}(x)|\,|u_{0,\lambda}(x)|.$$

Thus, this term is bounded as $\lambda \to 0$ and so, there exists a subsequence $\lambda_n \to 0$ such that

$$\begin{array}{ll} u_{\lambda_n} \to u & \text{weakly in } L^\infty(0,T;H_0^1(\Omega)), \\ u_{\lambda_n t} \to u_t & \text{weakly in } L^\infty(0,T;L^2(\Omega)), \\ u_{\lambda_n} \to u & \text{a.e. on } \Omega \times (0,T), \\ \beta_\lambda(u_\lambda) \to g & \text{weakly in } L^1(\Omega \times (0,T)). \end{array}$$

The last convergence was a consequence of Theorem 18 of [13] (see also Theorem 1.1 of [49]), thanks to the assumption (7). □

For the important case of $\Omega = \mathbb{R}^N$, it is useful to assume some supplementary conditions on the support (supp) of the data.

Theorem 2. *The statement of Theorem 1 remains valid for the case $\Omega = \mathbb{R}^N$ once we assume that*

$$\textit{supp } u_0 \textit{ and supp } v_0 \textit{ are compact sets of } \mathbb{R}^N,$$

and

$$\textit{supp } f(\cdot,t) \textit{ is compact, for a.e. } t \in (0,T).$$

Then, in addition

$$\textit{supp } u(\cdot,t) \textit{ is compact, for a.e. } t \in (0,T),$$

and the function $g \in L^1(\Omega \times (0,T))$, $g(x,t) \in \beta(u(x,t))$ a.e. $\Omega \times (0,T)$, satisfying (11), is such that

$$\textit{supp } g(\cdot,t) \textit{ is compact, for a.e. } t \in (0,T).$$

The proof of this theorem is a consequence of the following basic result on the dependence cone:

Lemma 1. *Let $T_0 > 0$ and $x_0 \in \Omega$. Let β satisfying (7) and let u_0, v_0, $f(x,t)$ as in the first part of Theorem 1. Let's suppose that*

$$B(x_0, cT_0) = \{x \in \mathbb{R}^N : |x - x_0| \le cT_0\} \subset \Omega.$$

Assume that u_0 and v_0 vanish almost everywhere on $B(x_0, cT_0)$ and that $f(\cdot,t)$ vanishes almost everywhere on $B(x_0, c(T_0 - t))$, for a.e. $t \in (0,T_0)$. Then the solution u given in Theorem 1 vanishes on the cone

$$\bigcup_{t\in(0,T_0)} B(x_0, c(T_0-t)) = \big\{(x,t) \in \mathbb{R}^N \times (0,T_0) : |x - x_0| \le c(T_0 - t)\big\}.$$

Proof. Without loss of generality we may assume that the data u_0, v_0, $f(x,t)$ are regular as in the second part of Theorem 1 and vanishing on the indicated sets. In addition, we can also assume that $c = 1$. Indeed, by rescaling the spatial variable

$$X = \mu x,\ \ \mu = 1/c,\ U(X,t) = u(x,t),$$

we find that

$$c^2 \Delta_x u = \Delta_X U.$$

For simplicity in the notation, we keep the old notation (identifying X and U with x and u) but assuming now $c = 1$. Let u_λ be the solution of the approximate problem (14) with $c = 1$. We multiply the equation by $u_{\lambda t}\psi_n(|x - x_0|)$ with

$$\psi_n(r) = \begin{cases} 1 & \text{if } r \in [0, \rho - 1/n], \\ -n(\rho - r) & \text{if } r \in [\rho - 1/n, \rho], \\ 0 & \text{if } r \in [\rho, \rho_0], \end{cases}$$

where $\rho_0 > 0$ is such that $\rho_0 > T_0$. Then we have

$$-\int_\Omega \Delta u_\lambda (u_{\lambda t}\psi_n(|x - x_0|)) = \int_\Omega \nabla u_\lambda \cdot (\nabla u_{\lambda t}\psi_n(|x - x_0|))$$
$$+ \int_\Omega \nabla u_\lambda \cdot (u_{\lambda t}\nabla\psi_n(|x - x_0|)).$$

Using spherical coordinates (r, ω) with center x_0 we have

$$n\int_{\rho-1/n}^{n} u_{\lambda t}\nabla u_\lambda \cdot \frac{x - x_0}{|x - x_0|}dx = n\int_{\rho-1/n}^{n}\int_{S^{N-1}} u_{\lambda t}\nabla u_\lambda \cdot \overrightarrow{\vartheta} r^{N-1} d\omega dr,$$

where $\overrightarrow{\vartheta}$ is the outward normal vector at $x \in S_\rho(x_0) = \partial B(x_0, \rho)$. As in Lemma 2.1 of [36], from Lebesgue's differentiation theorem and the fact that $r \to \int_{S^{N-1}} u_{\lambda t}\nabla u_\lambda \cdot \overrightarrow{\vartheta} r^{N-1} d\omega \in L^1(0, \rho_0)$, we deduce that for almost all $\rho \in (0, \rho_0)$,

$$\lim_{n\to\infty} n\int_{\rho-1/n}^{n} u_{\lambda t}\nabla u_\lambda \cdot \frac{x - x_0}{|x - x_0|}dx = \int_{S_\rho(x_0)} u_{\lambda t}\nabla u_\lambda \cdot \overrightarrow{\vartheta} r^{N-1} ds.$$

Then, going to the limit ($n \to \infty$), as in (15), we deduce (taking $\rho = T_0 - t$)

$$\frac{1}{2}\int_{B(x_0,T_0-t)} \frac{\partial}{\partial t}((u_{\lambda t})^2 + |\nabla u_\lambda|^2 + m\,|u_\lambda|^2) + \gamma\,|u_{\lambda t}|^2_{L^2(B(x_0,T_0-t))}$$
$$= ((T_0 - t))^{N-1}\int_{S^{N-1}} u_{\lambda t}\nabla u_\lambda((T_0 - t)\xi) \cdot \overrightarrow{\vartheta} d\xi$$
$$+ \int_{B(x_0,T_0-t)} (f_\lambda - \beta_\lambda(u_\lambda))u_{\lambda t}.$$

We also point out that if $\phi \in W^{1,1}(0,T:L^1(\Omega))$ we have

$$\frac{d}{dt}\int_{B(x_0,T_0-t)}\phi(x,t)dx = \frac{d}{dt}\int_0^{T_0-t} r^{N-1}dr\int_{S^{N-1}}\phi(r\xi,t)d\xi$$
$$= \int_{B(x_0,T_0-t)}\frac{\partial\phi}{\partial t}(x,t)dx - (T_0-t)^{N-1}\int_{S^{N-1}}\phi((T_0-t)\xi,t)d\xi.$$

Then we have

$$\frac{1}{2}\frac{d}{dt}\int_{B(x_0,T_0-t)}((u_{\lambda t})^2 + |\nabla u_\lambda|^2 + m\,|u_\lambda|^2)$$
$$\leq -(T_0-t)^{N-1}\int_{S^{N-1}}\Big[\Big\{(u_{\lambda t})^2 + |\nabla u_\lambda|^2 + m\,|u_\lambda|^2$$
$$-2u_{\lambda t}\nabla u_\lambda\Big\}((T_0-t)\xi)\cdot\overrightarrow{\upsilon}\Big] + \textstyle\int_{B(x_0,T_0-t)}(f_\lambda - \beta_\lambda(u_\lambda))u_{\lambda t}.$$

By the Cauchy and Young inequalities

$$|2u_{\lambda t}\nabla u_\lambda\cdot\overrightarrow{\upsilon}| \leq 2\,|u_{\lambda t}|\,|\nabla u_\lambda| \leq (u_{\lambda t})^2 + |\nabla u_\lambda|^2.$$

On the other hand, since β_λ is Lipschitz continuous and $|\beta_\lambda(u_\lambda)| \leq C(\lambda)\,|u_\lambda|$, applying that $f_\lambda(.,t)$ vanishes almost everywhere on $B(x_0,T_0-t)$ for a.e. $t\in(0,T_0)$,

$$\int_{B(x_0,T_0-t)}(f_\lambda - \beta_\lambda(u_\lambda))u_{\lambda t} \leq \frac{C(\lambda)}{2\sqrt{m}}\int_{B(x_0,T_0-t)}((u_{\lambda t})^2 + m\,|u_\lambda|^2).$$

Then

$$\frac{1}{2}\frac{d}{dt}\int_{B(x_0,T_0-t)}((u_{\lambda t})^2+|\nabla u_\lambda|^2+m\,|u_\lambda|^2) \leq \frac{C(\lambda)}{2\sqrt{m}}\int_{B(x_0,T_0-t)}((u_{\lambda t})^2+m\,|u_\lambda|^2).$$

Integrating the above inequality

$$\int_{B(x_0,T_0-t)}((u_{\lambda t})^2 + |\nabla u_\lambda|^2 + m\,|u_\lambda|^2)dx$$
$$\leq e^{\frac{C(\lambda)}{2\sqrt{m}}}\int_{B(x_0,T_0)}((v_{0\lambda})^2 + |\nabla u_{0\lambda}|^2 + m\,|u_{0\lambda}|^2)dx = 0,$$

for all $t\in(0,T_0)$ (and not only a.e. $t\in(0,T_0)$, since $u_\lambda \in C([0,T];H_0^1(\Omega))$). This proves the result for u_λ and $g_\lambda = \beta_\lambda(u_\lambda)$. For the more general case, as in the proof of Theorem 1, we know that $u_{\lambda_n} \to u$ a.e. on $\Omega\times(0,T)$. So, u also vanishes on the same cone (notice that it is independent on λ_n). Finally, since $\beta_\lambda(u_\lambda) \to g$ weakly in $L^1(\Omega\times(0,T))$, and we know that $g = f - (u_{tt} + \gamma u_t - c^2\Delta u + mu) \in L^1(\Omega\times(0,T))$, we conclude that g also vanishes on the same cone. □

Proof. (*of Theorem 2*) Given $T > 0$, let $\widetilde{\Omega}$ be a bounded regular set such that the solution $\widetilde{u}$ of the problem (6) on $\widetilde{\Omega} \times (0,T)$, satisfies that support $\widetilde{u}(\cdot,t) \subset \widetilde{\Omega}$, for a.e. $t \in (0,T)$ [as a consequence of Lemma 1 and the assumptions on the data]. Then the function

$$u(x,t) = \begin{cases} \widetilde{u}(x,t) & \text{if } x \in \widetilde{\Omega}, \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \widetilde{\Omega}, \end{cases}$$

satisfies all the requirements (after extending by zero the function $\widetilde{g}$ such that $\widetilde{g} \in \beta(\widetilde{u})$ a.e. on $\widetilde{\Omega} \times (0,T)$). □

Remark 1. *Lemma 1 extends Theorem 1.4.63 of [17] (stated for the case of the linear wave equation). In particular, we have replaced their approximation argument with Lemma 2.1 of [36] and the Yosida approximation of β. Theorem 2 is valid for unbounded domains Ω, once the dependence cone associated with the data is assumed to be contained in $\Omega \times (0,T)$), or when β is a monotone continuous function such that $\beta(0) = 0$ (see Theorem 2 of [49] where the compactnees of the support of the data is not required).*

Remark 2. *Problems like (6) are called by some authors (see, e.g., [12], [43] and [38]) as "variational inequalities of the second type", once they are written in the terms of the primitive j of β:*

$$\begin{gathered} \int_\Omega (u_{tt} + \gamma u_t)(v-u) + \int_\Omega c^2 \nabla u \cdot \nabla(v-u) \\ +m \int_\Omega u(v-u) + \int_\Omega j(v) - \int_\Omega j(v) \\ \geq \int_\Omega f(x,t)(v-u) \text{ for any } v \in H_0^1(\Omega) \text{ a.e. } t \in (0,T). \end{gathered}$$

The uniqueness of solutions to this type of problem is a delicate question when β is not a global Lipschitz continuous function. Alternatively, there are many different criteria in the literature (see many references in [45], [49], [48], [19], [17] , [22], and also [40], where the case of complex-valued solutions was considered).

The uniqueness of solutions for the multivalued hyperbolic equation (2) was mentioned in [44] as an open problem. The following result offers a positive answer in the class of non-degenerate solutions, already introduced in [35] for a different parabolic problem.

Definition 1. *Let $u \in L^\infty(\Omega)$. Given ϵ_0, $0 < \epsilon_0 < 1$, for $\epsilon \in (0, \epsilon_0)$, we say that u is a non-degenerate function if it satisfies the following property: if there exists a constant $C > 0$ such that for any $\epsilon \in (0, \epsilon_0)$*

$$\left|\left\{x \in \Omega : \left|u\right| \leq \epsilon\right\}\right| \leq C\epsilon.$$

The key point is the following technical result:

Lemma 2. *(see [35]) Let $w, \hat{w} \in L^\infty(\Omega)$ and assume that w satisfies the nondegeneracy property. Let β as in (3). Then $\forall q \in [1, \infty)$ there exists $\hat{C} > 0$ such that if $z, \hat{z} \in L^\infty(\Omega)$, with $z(x) \in \beta(w)$, $\hat{z}(x) \in \beta(\hat{w})$ a.e. $x \in \Omega$, we have*

$$\| z - \hat{z} \|_{L^q(\Omega)} \leq a\hat{C} \| w - \hat{w} \|_{L^\infty(\Omega)}^{1/q} .$$

Theorem 3. *Assume $N = 1$, $\Omega = \mathbb{R}$, $m > 0$, and assume the data such that there exists a bounded nondegenerate weak solution u of (6) with β given by (3). Then u is the unique bounded weak solution of the problem.*

Proof. Assume that there exists another bounded weak solution $\widehat{u}$ corresponding to the same data u_0, v_0, $f(x,t)$ and let $\widehat{g}(x,t) \in \beta(\widehat{u}(x,t))$ be the corresponding function appearing in the equation. Let $w = u - \widehat{u}$. Subtracting the two equations and multiplying the difference of them by w_t we get that, if we define the energy $E(t)$ of the difference by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}} \left(\frac{1}{2}|w_t|^2 + \frac{c^2}{2}|w_x|^2 + \frac{m}{2}|w|^2\right) dx,$$

then

$$\frac{d}{dt}E(t) \leq -\gamma \int_{\mathbb{R}} |w_t|^2 dx + \int_{\mathbb{R}} |g - \widehat{g}|\, |w_t|.$$

Since $N = 1$, by the Morrey theorem (see Theorem 9.12 of [16])

$$\frac{c^2}{2}\int_{\mathbb{R}} |w_x|^2 dx + \frac{m}{2}\int_{\mathbb{R}} |w|^2 dx \geq \min(\frac{c^2}{2}, \frac{m}{2}) \left(\int_{\mathbb{R}} |w_x|^2 dx + \int_{\mathbb{R}} |w|^2 dx\right)$$
$$\geq C \| w \|_{L^\infty(\mathbb{R})}^2 .$$

On the other hand, by Lemma 2

$$\begin{aligned}\int_{\mathbb{R}} |g - \widehat{g}|\, |w_t| \quad &\leq \| g - \widehat{g} \|_{L^2(\mathbb{R})} \| w_t \|_{L^2(\mathbb{R})} \leq \tfrac{1}{2} \| g - \widehat{g} \|_{L^2(\mathbb{R})}^2 + \tfrac{1}{2} \| w_t \|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{a\hat{C}}{2} \| w \|_{L^\infty(\mathbb{R})}^2 + \frac{1}{2} \| w_t \|_{L^2(\mathbb{R})}^2 \\ &\leq \frac{a\hat{C}}{2C\min(\frac{c^2}{2}, \frac{m}{2})} \left(\frac{c^2}{2}\int_{\mathbb{R}} |w_x|^2 dx + \frac{m}{2}\int_{\mathbb{R}} |w|^2 dx\right) \\ &\quad + \frac{1}{2} \| w_t \|_{L^2(\mathbb{R})}^2 .\end{aligned}$$

Then, there exists $K > 0$ such that

$$\frac{d}{dt}E(t) \leq KE(t),$$

and then

$$0 \leq E(t) \leq E(0)e^{Kt}.$$

Since $E(0) = 0$, we get that $E(t) \equiv 0$, which proves that $\widehat{u} = u$. □

Remark 3. *Obviously, the existence of bounded, non-degenerate solutions requires additional conditions on the data. For instance for $N = 1$, the use of the D'Alembert formula for the associated linear problem allows to see that if the initial data and $f(.,t)$ are bounded and non-degenerate, then the solution also satisfies both properties. For $N > 1$ the boundedness of the solution can be proved once we assume bounded the data by means of the use of the associate Green function (see [5], [37], [50]) of the associate linear equation We emphasize that the presence of the damping term γu_t and the assumption $D(\beta) = \mathbb{R}$ are, in some way, essential in our treatment. For example, for the obstacle problem (when β is given by $\beta(r) = \{0\}$ if $r > 0$, $\beta(0) = (-\infty, 0]$ and $\beta(r) = \phi$, the empty set, if $r < 0$) the term $g(x,t) \in \beta(u(x,t))$ becomes a singular measure and the behavior of the solution is quite different (see, e.g., [47] and [39]).*

Remark 4. *The case of a saturation nonlinearity can be understood in the framework of Control Theory as a special case of feedback control, of bang-bang type, for the linear wave equation, to get the exact controllability to zero (finite time stabilization) at time T_e: i.e., we can reformulate the equation as*

$$u_{tt} - c^2\Delta u + mu = f(x,t) + y(x,t) \quad in \ \ \mathbb{R}^N \times (0, +\infty),$$

with the control $y(x,t) = \gamma u_t + a\frac{u}{|u|}$ (see some general references on this point of view in [10]). The more difficult case, in which the control is only defined on a small part ω of Ω, requires additional conditions (see, e.g., the presentation made in [41]).

Remark 5. *Concerning the uniqueness of solutions of problem with β given by (5), the result proved in [43] (Théorème 1.2; for $\gamma = m = 0$) remains* *valid for the case $\gamma \geq 0$ and $m \geq 0$ (see also Lemma 3.1 of [49]). It requires the condition*

$$p \leq \frac{N}{N-2},$$

and thus it holds for $p \in (0,1)$ for any $N \geq 2$. Notice that this contrasts with Theorem 3 where it applies to the case $N = 1$.

3 Absence of the finite extinction time property.

It is convenient to start our analysis by the consideration of the nonlinear ordinary differential equation (ODE) with saturation

$$\begin{cases} u'' + \gamma u' + mu + a\frac{u}{|u|} = 0 \quad t \in (0, +\infty), \\ u(0) = u_0, \\ u'(0) = v_0. \end{cases} \tag{16}$$

In the case $a = 0$, the ODE reduces to the linear ODE $u'' + \gamma u' + mu = 0$. This second-order ODE with constant coefficients has as characteristic equation

$$r^2 + \gamma r + m = 0.$$

The roots of the characteristic equation are

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4m}}{2}.$$

In the case of strong damping ($\gamma^2 > 4m$), the roots are real and distinct

$$r_1 = \frac{-\gamma + \sqrt{\gamma^2 - 4m}}{2}, \quad r_2 = \frac{-\gamma - \sqrt{\gamma^2 - 4m}}{2}.$$

Both roots are negative and then the general solution is

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

where C_1 and C_2 are constants determined by the initial conditions. The solution $u(t)$ is a linear combination of two decaying exponential functions, so there are no oscillations because the roots r_1 and r_2 are real. This system is overdamped, and the solution decays to zero without crossing the equilibrium point $u = 0$ more than once.

The situation radically changes when $a > 0$ since the ODE becomes nonlinear. Even in the case of strong damping, the nonlinear term $a\frac{u}{|u|}$

dominates near $u = 0$, causing the solution to oscillate infinitely. Indeed, for $u > 0$, the ODE becomes $u'' + \gamma u' + mu + a = 0$ and the solution in this region is a decaying exponential (due to strong damping) shifted by the constant term $-a/m$

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} - a/m.$$

When $u < 0$, the ODE becomes $u'' + \gamma u' + mu - a = 0$ and the solution, in this region, is a decaying exponential shifted by the constant term a/m.

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + a/m.$$

This switching behavior leads to an infinite number of oscillations, even in the case of strong damping. We can explore the sequence of crossing times T_n. Let T_n be the n-th crossing time (the time at which $u(t)$ crosses zero for the n-th time). Let us explain how to calculate T_2 from T_1 and, in general, T_{n+1} from T_n. The ODE is piecewise linear, with different dynamics in the regions $u > 0$ and $u < 0$. Step 1. Calculation of T_2 from T_1: We solve for $u(t)$ in $t \in [0, T_1)$. Assume, for instance, $u(0) = u_0 > 0$ and $u'(0) = 0$. For $t \in [0, T_1)$, $u(t) > 0$, so the solution is:

$$u(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} - \frac{a}{m}.$$

Using the initial conditions, we get

$$\begin{cases} C_1 + C_2 - \frac{a}{m} = u_0, \\ r_1 C_1 + r_2 C_2 = 0. \end{cases}$$

We can solve this system to find C_1 and C_2. Step 2: Find T_1. At $t = T_1$, $u(T_1) = 0$. i.e.,

$$C_1 e^{r_1 T_1} + C_2 e^{r_2 T_1} - \frac{a}{m} = 0.$$

This is a transcendental equation for T_1, which can be solved numerically. Step 3: Solve for $u(t)$ in $t \in (T_1, T_2)$. For $t \in (T_1, T_2)$, $u(t) < 0$, so the solution is:

$$u(t) = C_1'^{r_1(t-T_1)} + C_2'^{r_2(t-T_1)} + \frac{a}{m}.$$

We must use the conditions $u(T_1) = 0$ and $u'(T_1) = u'(T_1^-)$ (continuity of u') to solve for C_1' and C_2'. Step 4: Find T_2. At $t = T_2$, $u(T_2) = 0$, which leads to

$$C_1'^{r_1(T_2-T_1)} + C_2'^{r_2(T_2-T_1)} + \frac{a}{m} = 0.$$

This is another transcendental equation for T_2, which can be solved numerically. The process can be generalized in an obvious way: i) Solve for $u(t)$ in the current regime ($u > 0$ or $u < 0$) using the appropriate general solution. ii) Use the crossing condition $u(T_n) = 0$ to solve for T_n, iii) Use the solution in the new regime (opposite to the previous one) starting from $t = T_n$. iv) Solve for T_{n+1} using the crossing condition $u(T_{n+1}) = 0$. This recursive process can be continued to compute all crossing times T_n. The transcendental equations for T_n cannot be solved analytically, in general. However, they can be solved numerically using methods such as Newton's method, Bisection method or Fixed-point iteration.

It is easy to see that the difference $T_{n+1} - T_n$ (the time intervals between consecutive zero crossings) increases with n. Indeed, the damping term $\gamma u'$ ensures that the amplitude of the oscillations decreases over time. As the amplitude decreases, the time intervals between consecutive crossings $T_{n+1} - T_n$ increase because the oscillations become slower (see Figure 1, below).

Instead to prove more rigorously that there is no finite-time extinction for this example, we return to the study of the Klein-Gordon equation since the same arguments apply to the nonlinear ordinary differential equation. Notice that the above behavior of the solution is very different when considering some related elliptic problems, which in radial coordinates leads to the equation

$$-u'' - \frac{(N-1)}{t}u' + mu + \beta(u) \ni 0,\ t \in (0, +\infty),$$

with $m \geq 0$ (see, e.g., Section 2.2 of [29]).

Theorem 4. *Let $\beta = \partial j$ with $0 \in \beta(0)$, satisfying (7). Let u_0 and v_0 as in part 1 of Theorem 1 (if Ω is bounded) or as in Theorem 2 (for $\Omega = \mathbb{R}^N$), and let $f \equiv 0$. Let u be any weak solution of (1). We define the energy*

$$E(t) = \int_\Omega \left(\frac{1}{2}|u_t|^2 + \frac{c^2}{2}|\nabla u|^2 + \frac{m}{2}|u|^2 + j(u) \right) dx.$$

Then,

$$E(t) \geq E(0)e^{-2\gamma t} > 0 \text{ for any } t > 0. \tag{17}$$

In particular, u cannot satisfy the finite extinction time property.

The proof of this result will use that if $f \equiv 0$, and if the data are regular (as in Part 2 of Theorem 1) then we can get an identity for the derivative of the energy and not only an inequality (see a related result in Théoreme 1.6 of [43] and also in [48]).

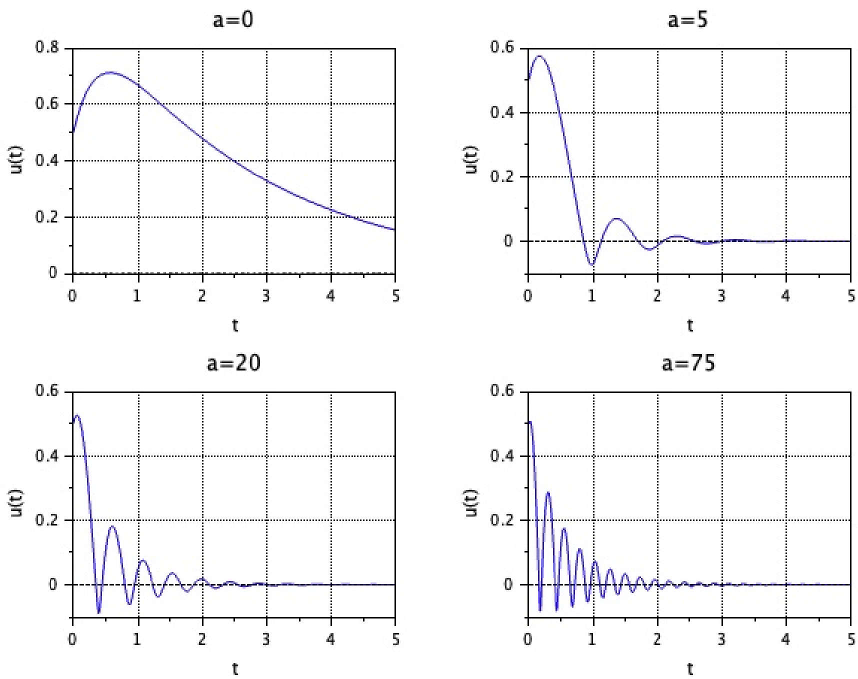

Figure 1: Numerical experiences corresponding to different values of the parameter a, with $m = 1$ $\gamma=\sqrt{4m+5}$ and $u_0 = 0.5$ and $v_0 = 1$. Notice that $u \notin C^2$ and that the oscillations are not due to the presence on any trigonometric function (since the roots r_1, r_2 are negative real numbers).

Lemma 3. *Given $\lambda > 0$, let β_λ be the Yosida approximation $\beta_\lambda = \partial j_\lambda$ of β, and let $u_{0,\lambda}$, $v_{0,\lambda}$, satisfying the stronger conditions (12) and (13). Let u_λ be the unique solution of the regularized problem*

$$\begin{cases} u_{\lambda tt} + \gamma u_{\lambda t} - c^2 \Delta u_\lambda + m u_\lambda + \beta_\lambda(u_\lambda) = 0 & \text{in } \Omega \times (0, +\infty), \\ u_\lambda = 0 & \text{on } \partial\Omega \times (0, +\infty), \\ u_\lambda(0, x) = u_{0,\lambda}(x) & \text{on } \Omega, \\ u_{\lambda t}(0, x) = v_{0,\lambda}(x) & \text{on } \Omega. \end{cases} \tag{18}$$

Define

$$E_\lambda(t) = \frac{1}{2} \int_\Omega \left(\frac{1}{2} |u_{\lambda t}|^2 + \frac{c^2}{2} |\nabla u_\lambda|^2 + \frac{m}{2} |u_\lambda|^2 + j_\lambda(u_\lambda) \right) dx.$$

Then

$$\frac{d}{dt} E_\lambda(t) = -\gamma \int_\Omega |u_{\lambda t}|^2 dx. \tag{19}$$

Proof. We drop the subindex λ for simplicity in the notation. Since by Part 2 of Theorem 1 we know that $u \in C([0,T]; H_0^1(\Omega)) \cap L^\infty(0,T; H^2(\Omega))$, $u_t \in L^\infty(0,T; H_0^1(\Omega))$ and $u_{tt} \in L^\infty(0,T; L^2(\Omega))$, $\beta_\lambda(u_\lambda) \in L^2(0,T; L^2(\Omega))$ and $\beta_\lambda = \partial j_\lambda$ (and the subdifferential is single-valued, since β_λ is a Lipschitz function) we can justify that

$$\frac{d}{dt} E(t) = \int_\Omega \left(\frac{1}{2} \left[u_{tt} u_t + c^2 \nabla u \cdot \nabla u_t + m u u_t \right] + u_t \beta_\lambda(u_\lambda) \right) dx. \tag{20}$$

Multiplying the equation (18) by u_t we get

$$u_{tt} u_t - c^2 \Delta u u_t + m u u_t + \beta_\lambda(u) u_t = -\gamma \left(u_t\right)^2 \tag{21}$$

On the other hand, since $u_t \in L^\infty(0,T; H_0^1(\Omega))$

$$-\int_\Omega \Delta u u_t = \int_\Omega \nabla u \cdot \nabla u_t. \tag{22}$$

Substituting, we see that there are several cancellations and finally we get the identity (19). □

Proof. (*of Theorem 4*) Assume for the moment the data as in Lemma 3. Since

$$\int_\Omega |u_{\lambda t}|^2 dx \le 2E_\lambda(t),$$

(notice that $j_\lambda(u_\lambda) \geq 0$ since $j_\lambda(0) = j'_\lambda(0) = 0$ and j_λ is convex), we conclude, by Lemma 3, that

$$\frac{d}{dt}E_\lambda(t) \geq -2\gamma E_\lambda(t) \text{ a.e. } t \in (0, +\infty).$$

Integrating in t we get

$$E_\lambda(t) \geq E_\lambda(0)e^{-2\gamma t} > 0 \text{ for any } t > 0. \tag{23}$$

As in the proof of Theorem 1, for data $u_0 \in H_0^1(\Omega)$ and $v_0 \in L^2(\Omega)$, we know that $E_\lambda(t) \to E(t)$ as $\lambda \to 0$. Estimate (23) is stable by approximation of the data, and thus we get (17). In the case of $\Omega = \mathbb{R}^N$ the proof is the same since $u_0 \in H_0^1(\mathbb{R}^N)$ and $v_0 \in L^2(\mathbb{R}^N)$ are assumed, both, with compact support and then we apply Theorem 2 which reduces the problem to the case of a bounded domain Ω, large enough. □

Remark 6. *The role of the absorption term for the hyperbolic equation is different with respect to parabolic semilinear equations (see, e.g., the results quoted in [2]) concerning how the condition $p \in (0,1)$ implies the extinction in a finite time of the solution.*

Remark 7. *There is a very extensive literature dealing with the study of the decay of solutions to the Cauchy problem for the damped wave equation with weak absorption (equation (4) with $p > 1$): see, e.g., [46] and its references.*

Remark 8. *The presence of the damping term was fundamental in the proof of Theorem 4. We do not know the asymptotic behavior of solutions of the equation with a strong absorption in the absence of any damping term. It is easy to see that some special solutions may have a finite extinction time. Indeed, consider, for instance, the wave equation with a nonlinear absorption:*

$$u_{tt} - \Delta u + a|u|^{p-1}u = f(x,t) \tag{24}$$

where $f(x,t)$ will be defined later. We search for a possible self-similar solution (of separated variables) in the form:

$$u(x,t) = C(T-t)_+^\alpha v(x), \tag{25}$$

where T is the extinction time, $C > 0$ and α is an exponent to be determined. Computing the derivatives we have $u_t = \alpha C(T-t)_+^{\alpha-1}v(x)$, $u_{tt} = \alpha(\alpha -$

$1)C(T-t)_+^{\alpha-2}v(x)$. *The nonlinear term is* $a|u|^{p-1}u = aC^p(T-t)_+^{\alpha p}|v|^{p-1}v$. *For consistency in the equation, the dominant terms must have the same time dependence, which leads to the condition* $\alpha p = \alpha - 2$, *i.e.,*

$$\alpha = \frac{2}{1-p}$$

Then, if we assume $f(t,x) = \lambda C(T-t)_+^{2/(1-p)}v(x)$, *we get that constant* C *is given as*

$$C = \left[\frac{a}{\lambda - \frac{2(1+p)}{(1-p)^2}}\right]^{1/(1-p)},$$

once we assume $\lambda > \frac{2(1+p)}{(1-p)^2}$, *and then function* $v \in H_0^1(\mathbb{R}^N)$ *must satisfy the nonlinear elliptic equation* $-\Delta v + |v|^{p-1}v = v(x)$. *For a study of this equation see, e.g., [31] and its references.*

Remark 9. *In the case of parabolic problems, there are many papers dealing with the finite extinction time property obtained using the construction of super and subsolutions and the comparison principle (see, e.g., [18], [26], [34]). For the application of some other abstract methods see, e.g., [52], [11], [30]. An early survey on this property can be found in [28]). Curiously, this property also holds for linear parabolic equations in the presence of suitable delayed terms (see [25]).*

Remark 10. *Notice that in the case in which the multivalued maximal monotone graph* β *is applied on the damping term (instead on the unknown* u*)*

$$\begin{cases} u_{tt} - c^2\Delta u + mu + \beta(u_t) \ni f(x,t) & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \\ u_t(x,0) = v_0(x) & \text{on } \Omega, \end{cases} \tag{26}$$

the conclusion, when $f(x,t) \in \overset{\circ}{\beta(0)}$ *a.e. on* $\Omega \times (T_\varepsilon, +\infty)$ *is different to the finite extinction time: in that case, it can be proved that there exists a finite time* $T_e > 0$, *such that*

$$u(x,t) = \varsigma(x) \text{ on } \Omega \text{ for any } t \geq T_e,$$

with $\varsigma \in H_0^1(\Omega)$) *such that* $-c^2\Delta\varsigma + m\varsigma \in -\beta(0)$ *on* Ω. *See [20], [21], [4], [42] and [33]. However, if the friction is given by a power,* $p \in (0,1)$, *of* u_t *(for instance, as it happens with the hyperbolic damped equation* $u_{tt} - \Delta u + |u_t|^{p-1}u_t = 0$*), in general, the extinction does not occur in a finite time. That was rigorously proved in the case of the damped oscillator* $m\ddot{x} + \mu\,|\dot{x}|^{\alpha-1}\,\dot{x} + kx = 0$ *when* $\alpha \in (0,1)$*: see [32], [1] and [51]. It was shown in those references that the generic asymptotic behavior above described for the limit case* $\alpha = 0$ *is only exceptional for the sublinear case* $\alpha \in (0,1)$ *since the generic orbits* $(x(t), \dot{x}(t))$ *decay to* $(0,0)$ *in a infinite time and only two one-parameter families of them decay to* $(0,0)$ *in a finite time: in other words, when* $\alpha \to 0$ *the exceptional behavior becomes generic. The finite extinction time for the wave equation with dry friction equation fails if it is perturbed with a linear term* γu_t *(see [3]).*

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