UNIQUENESS AND QUALITATIVE BEHAVIOR OF SOLUTIONS TO A POROUS MEDIUM EQUATION WITH NEWTONIAN POTENTIAL PRESSURE OF NONLINEAR DENSITY FUNCTION

NGUYEN ANH DAO AND JESUS ILDEFONSO DÍAZ

ABSTRACT. In this paper, we would like to study nonnegative solutions of the following problem:

$$\begin{cases} u_t = \operatorname{div}\left(u\nabla(-\Delta)^{-1}u^m\right) & \text{in } \mathbb{R}^N \times (0,T)\,, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,, \end{cases}$$

with $m \geq 1$. We establish the well-posedness theory for densities $u_0(x)$ in $C^\gamma(\mathbb{R}^N)$, $\gamma \in (0,1)$; or in $H^s(\mathbb{R}^N)$, $s > \frac{N}{2}$ with compact support respectively.
Concerning the qualitative behavior of solutions, we show that the L^p-estimates of solutions, $1 < p \leq \infty$ are decreasing in time. Moreover, we demonstrate that the solutions satisfy the following universal bound

$$u(x,t) \leq (mt)^{-\frac{1}{m}}, \quad \text{for } (x,t) \in \mathbb{R}^N \times (0,\infty).$$

In addition, we investigate the asymptotic profile of u when $t \to \infty$. Precisely, for any $q \in [1,\infty)$ we have

$$\left\|u(t) - W(t)\right\|_{L^q(\mathbb{R}^N)} \leq Ct^{-\frac{q-1+2^{1-N}}{qm}}, \quad t > 0,$$

where $W(x,t)$ is the vortex patch solution. Hence, we extend the known results of the case $q = m = 1$ in the literature.
We end the paper with a section devoted to the study of symmetrization solutions of the above problem. In particular, we obtain some comparison results in a suitable sense for the symmetrization solutions.

CONTENTS

Date: April 25, 2025.
2010 *Mathematics Subject Classification.* 26D10, 46E35, 35A23.
Key words and phrases. Nonlinear diffusion, hydrodynamic limit, superconductivity, finite speed of propagation, Riesz transforms, commutator.

1. Introduction

Our main purpose of this paper is to study nonnegative solutions of the following equation

$$\begin{cases} \partial_t u = \operatorname{div}(u\nabla p) & \text{in } \mathbb{R}^N\times(0,T), \\ p_u = (-\Delta)^{-1}u^m, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with $m\geq 1$, and space dimension $N\geq 2$, and u_0 is a nonnegative density function. The above problem is a special case of a family of problems of the form

$$\begin{cases} \partial_t u = \operatorname{div}(u^n\nabla p_u) & \text{in } \mathbb{R}^N\times(0,T), \\ p_u = (-\Delta)^{-1}u^m, \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.2}$$

which can be also formulated in terms of the hyperbolic-elliptic nonlinear system

$$\begin{cases} \partial_t u + \nabla u^n\cdot\nabla p_u + u^{m+n} = 0 & \text{in } \mathbb{R}^N\times(0,T), \\ -\Delta p_u = u^m & \text{in } \mathbb{R}^N\times(0,T) \\ u(x,0)=u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.3}$$

Let us discuss several important works dealing with special cases of the above doubly parametric system (1.2). We also mention here that a different (but quite related) formulation was proposed in [5, 6, 7, 8, 9, 10, 11, 12, 17, 22, 31, 37, 38] in which $p_u=(-\Delta)^{-s}u^m$ with $0<s<1$. Note that the limit case $s=0$ leads to the usual degenerate parabolic porous media equation:

$$\partial_t u = \operatorname{div}(u^n\nabla u^m),$$

for a gas with the constitutive law between the pressure and the density given by $p_u=u^m$. The other limit case, $s=1$, corresponds to Eq (1.1), when $n=1$, in which the constitutive law between the pressure and the density is given through the Newtonian potential pressure $p_u=(-\Delta)^{-1}u^m$ (long-range interactions). Notice that now the problem loses its parabolic nature and becomes a hyperbolic-elliptic nonlinear system.

When $m=1$, Eq (1.1) reads as

$$\partial_t u = \operatorname{div}\left(u\nabla(-\Delta)^{-1}u\right). \tag{1.4}$$

Such an equation of this type has been studied by the authors in [3, 13, 14, 27, 29, 36, 43, 44], and the references therein. In two-dimension ($N=2$), Eq (1.4) is known as a Chapman–Rubinstein–Schatzman mean field model of superconductivity (see [16]). Note that Eq (1.4) is directly related to the following equation

$$\partial_t u = \operatorname{div}\left(|u|\nabla(-\Delta)^{-1}u\right), \tag{1.5}$$

which is a mean field model for the motion of vortices in a superconductor in the Ginzburg–Landau theory, see [23, 29, 30, 43]. There, u represents the local vortex-density, and $p=(-\Delta)^{-1}u$ represents the induced magnetic field in the sample. Obviously, Eq (1.5) coincides with Eq (1.4) when $u\geq 0$.

The well-posedness theory of Eq (1.4) for densities $u_0\in L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ was established by the authors in [3, 27, 28, 32, 36, 42, 44] and the references cited therein (see also the book by Majda–Bertozzi, [29]). One of the most interesting properties of solutions to Eq (1.4) is the finite speed of propagation. That is the support of $u(.,t)$ is compact in $\mathbb{R}^N$ and is spreading for every $t>0$ if the initial datum is compactly supported in $\mathbb{R}^N$. This result is known in the literature since vector velocity field ∇p is uniformly bounded for all $(x,t)\in\mathbb{R}^N\times(0,\infty)$ according to the theory of Calderón–Zygmund (see, e.g., [3, 27, 29, 30, 32]).

Furthermore, solutions of Eq (1.4) in $\mathbb{R}^N\times(0,\infty)$ satisfies the following universal bound

$$u(x,t)\leq t^{-1}, \quad \text{for } (x,t)\in\mathbb{R}^N\times(0,\infty), \tag{1.6}$$

see, e.g., [3, 27, 44]. This bound can be obtained by using the characteristic equation associated to u:

$$\begin{cases} \frac{d\Phi_t(x)}{dt} = -\nabla p_u(\Phi_t(x), t), & \Phi_0(x) = x, \\ \frac{du(\Phi_t(x),t)}{dt} = -u^2(\Phi_t(x), t), & u(\Phi_0(x), 0) = u_0(x). \end{cases} \tag{1.7}$$

We emphasize that the universal bound plays a role of barrier in order to prove an existence of solutions to Eq (1.4) with densities u_0 in the space of Radon measures (see [27, 36]).
In [3], Bertozzi et al. established the well-posedness theory of mixed sign solutions to Eq (1.4) for densities $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. We note that the authors considered the Newtonian pressure $p = (\Delta)^{-1}u$ instead of $p = (-\Delta)^{-1}u$ as in Eq (1.4). Thus, their spreading case (the same our study in this paper) is corresponding to the nonpositive densities u_0. Besides, they obtained the asymptotic behavior of solution u as $t \to \infty$. Precisely, if $u_0 \in L^\infty_c(\mathbb{R}^N)$, then the unique solution u of Eq (1.4) satisfies

$$\lim_{t\to\infty} \|u(t) - W(t)\|_{L^1(\mathbb{R}^N)} = 0\,,$$

where

$$W(x,t) = \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}}{1 + \|u_0\|_{L^\infty(\mathbb{R}^N)}t}\mathbf{1}_{B(0,R(t))}, \quad R(t) = R_0\left(1 + \|u_0\|_{L^\infty(\mathbb{R}^N)}t\right)^{\frac{1}{N}}\,,$$

and R_0 depends on u_0, N. Note that $W(x,t)$ is called the vortex patch of solution. Here and through the paper, we denote $\mathbf{1}_\Omega$ by the characteristic function on Ω for every set Ω in $\mathbb{R}^N$.
On the other hand, if $u_0(x)$ is strictly negative at some point in $\mathbb{R}^N$, then the solution is blowing-up in a finite time $T^* = \sup\limits_{x\in\mathbb{R}^N} \dfrac{1}{-u_0(x)}$, see [3, page 2].
We would like to mention that Nieto–Poupaud–Soler [32] also studied Eq (1.4), derived from the Vlasov–Poisson–Fokker–Planck system.
Note that this equation is also a transport equation. Such an equation of this type has been studied by several authors in [1, 2, 15, 22, 28, 36], and the references therein, by using the gradient flow approach. For example, using the 2-Wasserstein distance Loeper [28] obtained a uniqueness of bounded solutions to the Vlasov–Poisson system, which can be derived to Eq (1.4).

Eq (1.4) with fractional potential $p = (-\Delta)^{-s}u$, $s \in (0,1)$ reads as

$$\partial_t u = \operatorname{div}\left(u\nabla(-\Delta)^{-s}u\right)\,. \tag{1.8}$$

The pioneering study of Eq (1.8) has been made by Caffarelli–Vázquez, [10]. They obtained an existence result of nonnegative solutions, and the L^∞-estimate which is decreasing in time. Moreover, the constructed solution satisfies the finite speed of propagation, and the $L^\infty - L^1$ decay estimate

$$\|u(t)\|_{L^\infty} \leq Ct^{-\frac{N}{N+2-2s}}\|u_0\|_{L^1}^{\frac{2-2s}{N+2-2s}}, \quad \text{for } t > 0\,, \tag{1.9}$$

with constant $C = C(N,s)$.
We emphasize that constant $C(N,s)$ in (1.9) stays bounded for all s near 1. Thus, universal bound (1.6) can be derived from (1.9) by letting $s \to 1$. This observation is due to Serfaty–Vázquez, [36]. Some generalizations of Eq (1.8) have been studied by the authors in [5, 17, 38] and the references cited therein.

Concerning the nonlinear mobility cases, Carrillo et al. [13, 14], studied nonnegative solutions of the following equation

$$\begin{cases} \partial_t u = \operatorname{div}\left(u^n\nabla(-\Delta)^{-1}u\right) & \text{in } \mathbb{R}^N\times(0,T)\,, \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N\,. \end{cases} \tag{1.10}$$

When $0 < n < 1$, they proved an existence and uniqueness of solutions in the sense of viscosity solutions for densities $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ with the radial mass. In addition, the authors proved an existence and uniqueness of radial solutions when $n > 1$. We want to mention that the existence and uniqueness of solutions to Eq (1.10) is open for general densities $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

In our knowledge, Eq (1.1) has not been studied when $m > 1$. Then, we would like to study the well-posedness theory for densities u_0 in $C^\gamma(\mathbb{R}^N)$, $\gamma \in (0, 1)$, or in $H^s(\mathbb{R}^N)$, for $s > N/2$ respectively. Moreover, we also study the regularity of solutions in those case.
On the other hand, we investigate the qualitative behavior of solutions to Eq (1.1) such as the L^q-estimates, $q \in (1, \infty]$, the conservation of mass, the universal bound, and the asymptotic profile of solutions via the vortex patch of solutions.
Finally, we study the symmetrization solutions and derive some comparison results in $L^q(\mathbb{R}^N)$-norm, and in the size of support of solutions. Note that the pointwise comparison result is not true for solutions of Eq (1.1), see [36].

Notations. Through the paper, we denote:
- constant by C, which may change from line to line. Moreover, the notation $C = C(\alpha, p, N)$ means that C merely depends on α, p, N.
- ω_N by the volume of the unit ball in $\mathbb{R}^N$.
- $X = L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, equipped by the norm $\|\cdot\|_X = \|\cdot\|_{L^1(\mathbb{R}^N)} + \|\cdot\|_{L^\infty(\mathbb{R}^N)}$.
- $\mathbf{1}_\Omega$ by the characteristic function on $\Omega \subset \mathbb{R}^N$.
- $Q_T = \mathbb{R}^N \times (0, T)$, for $T > 0$.

Next, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \leq CB$. Furthermore, we write $A \approx B$ iff $A \lesssim B \lesssim A$.

Main results. First of all, let us point out the definition of weak solutions to Eq (1.1).

Definition 1.1. *A function $u \in L^\infty(0, T; X)$, $T > 0$ is called a weak solution of Eq (1.1) if u satisfies*
$$\int_{Q_T} (u\varphi_t - u\nabla p_u \cdot \nabla\varphi)\, dxdt = 0, \quad \forall \varphi \in C_c^\infty(Q_T)\,.$$

In the following, we always assume that $u_0 \in X$ is nonnegative. Then, our main results are as follows.

Theorem 1.1. *Let $m > 1, N \geq 2$. Suppose that $u_0 \in H^s(\mathbb{R}^N)$, $s > \frac{N}{2}$. Then, Eq (1.1) has a weak solution u in $L^\infty(0, T; X) \cap L^\infty(0, T; H^s(\mathbb{R}^N))$ for $T > 0$. Moreover, u has the following properties:*
- *(Conservation of mass)*
$$\int_{\mathbb{R}^N} u(x, t)\, dx = \int_{\mathbb{R}^N} u_0(x)\, dx, \quad \forall t \in (0, T)\,. \tag{1.11}$$
- *(L^q-estimate) For any $1 < q \leq \infty$, we have*
$$\|u(., t)\|_{L^q(\mathbb{R}^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)}, \quad \forall t \in (0, T)\,. \tag{1.12}$$
- *(Universal bound)*
$$u(x, t) \leq (mt)^{-\frac{1}{m}}, \quad \textit{for a.e. } (x, t) \in Q_T\,. \tag{1.13}$$

Beside, if u_0 has compact support in $\mathbb{R}^N$, then we obtain a uniqueness of weak solutions of Eq (1.1).

Remark 1.1. *When $m = 1$, the uniqueness of bounded solutions of Eq (1.1) is known in the literature. The proof is based an energy estimate in terms of H^{-1} inner product for a comparison of two densities (see, e.g., [3, 32, 35, 42]). However, such argument cannot be applied to the case of nonlinearity $u^m, m > 1$ due to a technical problem.*
In [28], Loeper obtained the uniqueness result by using the 2-Wasserstein distance in terms of flow Φ_t satisfying (1.7) (see the same argument in [36]). Note that his proof is based on the argument of the optimal transportation, and the facts that $p = (-\Delta)^{-1}u$, and u is of the conservation of mass. Obviously, this argument is not able to apply to $p = (-\Delta)^{-1}u^m, m > 1$ since the mass of $u^m(t)$ is decreasing in time according to (1.13).

Remark 1.2. *One cannot expect that all bounded weak solutions of Eq (1.1) satisfy the Hölder regularity since the vortex patch of solutions, such as W, are even not continuous in Q_T when $m = 1$. This is in contrast to the bounded energy solutions of Eq (1.8) satisfying Hölder regularity, see [11, 12].*

Next, we would like to study the γ-Hölder estimate of solutions to Eq (1.1) for initial data in $C^\gamma(\mathbb{R}^N)$, $\gamma \in (0, 1)$. In fact, one cannot expect that all bounded weak solutions of Eq (1.1) satisfy the Hölder regularity since the vortex patch of solutions, such as W, are even not continuous in Q_T when $m = 1$. This is in contrast to the bounded energy solutions of Eq (1.8) satisfying Hölder regularity, see [11, 12].

Theorem 1.2. *Let $u_0 \in C_0^\gamma(\mathbb{R}^N)$, $\gamma \in (0, 1)$ (the space of γ-Hölder continuous functions with compact support). Then, there exists a unique weak solution u of Eq (1.1) satisfying (1.11)–(1.13). Moreover, $u(x, t) \in L^\infty(0, T; C_0^\gamma(\mathbb{R}^N))$, $T > 0$.*

Our next result is the asymptotic profile of u as $t \to \infty$.

Theorem 1.3. *Assume hypotheses as in Theorem 1.1 (resp. Theorem 1.2). Let u_0 satisfy* $\mathrm{supp}(u_0) \subset \overline{B(0, r_0)}$*, $r_0 > 1$, $\|u_0\|_{L^1(\mathbb{R}^N)} = \omega_N$, and $\|u_0\|_{L^\infty(\mathbb{R}^N)} = 1$. Then, we have the following asymptotic profile of u via the vortex patch solution in L^q-norm, $1 \le q < \infty$:*

$$\left\|u(t) - W(t)\right\|_{L^q(\mathbb{R}^N)} \le C(q, m, N)\frac{(r_0^N - 1)^{\frac{1}{q}}}{t^{\frac{q-1+N2^{1-N}}{qm}}}, \quad \textit{for } t > 0\,, \tag{1.14}$$

with

$$W(x, t) = \frac{\mathbf{1}_{B(0,R(t))}}{(1 + mt)^{\frac{1}{m}}}, \quad R(t) = (1 + mt)^{\frac{1}{mN}}\,. \tag{1.15}$$

Remark 1.3. (1.14) *is known when $q = m = 1$, see, e.g.,* [3, 36].

Finally, we present some estimates on the radial symmetrization solutions.
Let $u, U \in L^1(\mathbb{R}^N)$ be radially symmetric. We say that u is *less concentrated* than U, denoted by $u \prec U$ if

$$\int_{B(0,r)} u(x)\, dx \le \int_{B(0,r)} U(x)\, dx, \quad \text{for all } r > 0. \tag{1.16}$$

Obviously, (1.16) implies that

$$\int_0^s u_*(\sigma)\, d\sigma \le \int_0^s U_*(\sigma)\, d\sigma, \quad \text{for all } s \ge 0\,,$$

where u_*, U_* are the rearrangements of u, U respectively (see the definition and properties of rearrangement in [34, 40]).
Then, we have the following result.

Theorem 1.4. *Assume hypotheses as in Theorem 1.1 (resp. Theorem 1.2). Let $U_0 \in L^\infty_c(\mathbb{R}^N)$ be nonnegative, such that $u_0 \prec U_0$. Let u be the unique bounded weak solution of Eq (1.1) as in Theorem 1.1 (resp. Theorem 1.2). Suppose that U is the unique bounded weak solution of the following problem*

$$\begin{cases} \partial_t U = \operatorname{div}(U\nabla P) \text{ in } Q_T\,, \\ -\Delta P = M_0 U\,, \\ U(x,0) = U_0(x) \text{ in } \mathbb{R}^N\,, \end{cases} \tag{1.17}$$

with $M_0 = \|U_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}$. Note that the existence and uniqueness of solution U to Eq (1.17) for given densities $U_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is known in the literature. Then, we have

$$u(t) \prec U(t), \text{ and } p(t) \prec P(t), \quad \text{for } t > 0\,. \tag{1.18}$$

And, for any $q \in [1, +\infty]$, there holds true

$$\|u(t)\|_{L^q(\mathbb{R}^N)} \le \|U(t)\|_{L^q(\mathbb{R}^N)}\,,\ \|p(t)\|_{L^q(\mathbb{R}^N)} \le \|P(t)\|_{L^q(\mathbb{R}^N)}\,, \quad \text{for } t > 0\,. \tag{1.19}$$

In addition, if

$$\int_{\mathbb{R}^N} u_0(x)\,dx = \int_{\mathbb{R}^N} U_0(x)\,dx, \tag{1.20}$$

then

$$\left|\operatorname{supp}(U(t))\right| \le \left|\operatorname{supp}(u(t))\right|, \text{ for } t \in [0,T]\,. \tag{1.21}$$

Remark 1.4. *The isoperimetric type estimate (1.21) is very useful for non-symmetric patch of vortices, i.e., when condition (1.20) holds since $U_{0*} = u_{0*}$.*

As a consequence of Theorem 1.4, we have the following corollary.

Corollary 1.1. *Assume hypotheses as in Theorem 1.4. Suppose that U_0 is radially symmetric, and $U_{0*} = u_{0*}$ on $[0, +\infty)$. Then, we have $u(t) \prec U(t)$, and*

$$\left|\operatorname{supp}(U(t))\right| \le \left|\operatorname{supp}(u(t))\right|, \text{ for any } t \in [0,T]\,.$$

The paper is organized as follows. The next section is devoted to the preliminary results. In Section 3, we establish the well-posedness theory to a regularizing equation of Eq (1.1) for densities $u_0 \in X$. After that we derive some a priori estimates in the L^q-spaces and the $H^s(\mathbb{R}^N)$-spaces for the approximating solutions. We prove Theorems 1.1, 1.2 in Section 4. Section 5 is devoted to the study of asymptotic behavior of bounded solutions to Eq (1.1) via the vortex patch solutions. Finally, the last section is devoted to the study of radially symmetric solutions.

2. Preliminary results

At the beginning, we introduce the spaces $C^\gamma(\mathbb{R}^N)$, and $H^s(\mathbb{R}^N)$ alternatively.
Given $\gamma \in (0,1)$, the γ-Hölder continuous space is denoted by

$$C^\gamma(\mathbb{R}^N) = \left\{ u : \mathbb{R}^N \to \mathbb{R} \text{ such that } \|u\|_{C^\gamma(\mathbb{R}^N)} := \|u\|_{L^\infty(\mathbb{R}^N)} + \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma} < \infty \right\}.$$

And, the homogeneous γ-Hölder space is denoted by $\dot{C}^\gamma(\mathbb{R}^N)$, equipped with the semi-norm

$$|u|_{C^\gamma(\mathbb{R}^N)} := \sup_{x \neq y} \frac{|u(x)-u(y)|}{|x-y|^\gamma}\,.$$

Next, the inhomogeneous Sobolev space $H^s(\mathbb{R}^N)$, $s \in \mathbb{R}$ is defined as the space of all tempered distributions u in $\mathcal{S}'(\mathbb{R}^N)$ such that

$$\|u\|_{H^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} (1+|\xi|^2)^s |\widehat{u}(\xi)|^2\,d\xi \right)^{1/2},$$

where $\widehat{u}(\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\,dx$.
For any $0 < s < N$, it is known that $(-\Delta)^{-s/2} = \mathcal{I}_s$, the Riesz potential is defined by

$$\mathcal{I}_s(f)(x) = C(N,s)\int_{\mathbb{R}^N} \frac{f(y)}{|x-y|^{N-s}}\,dy\,,$$

where $C(N,s) = \frac{\pi^{N/2}2^s\Gamma(\frac{s}{2})}{\Gamma(\frac{N-s}{2})}$ (see, e.g., [39, Chapter 5]). It is clear that one can write $\mathcal{I}_s(f)(x) = K*f(x)$, with $K(x) = C(N,s)|x|^{s-N}$. In particular, if $s = 2$, then $\mathcal{I}_2$ is the inverse of the Laplacian operator. Note that $K(x) = \frac{1}{2\pi}\log|x|$ when $N = s = 2$.
It is known that

$$\|\mathcal{I}_s(f)\|_{L^q(\mathbb{R}^N)} \lesssim \|f\|_{L^{\frac{qN}{N+sq}}(\mathbb{R}^N)}, \quad \forall f \in L^{\frac{qN}{N+sq}}(\mathbb{R}^N), \tag{2.1}$$

provided that $\frac{qN}{N+sq} > 1$.
Next, we denote the Riesz transforms by

$$\mathcal{R} = (\mathcal{R}_j)_{j=1,\dots,N}, \quad \text{with } \mathcal{R}_j = \partial_{x_j}\mathcal{I}_1,\ j = 1,\dots,N\,.$$

Since $\mathcal{R}_j$, $j = 1,\dots,N$ are the standard Calderón–Zygmund operators, then $\mathcal{R}_j$ map $L^p(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, $1 < p < \infty$ (see [39, Chapter 3]). In this paper, we often use the L^q-estimate of vector velocity ∇p. Thus, we have to study the L^q-estimate of operator $\nabla(-\Delta)^{-1}$. Then, one has

$$\|\nabla(-\Delta)^{-1}f\|_{L^q(\mathbb{R}^N)} = \|\mathcal{R}\mathcal{I}_1(f)\|_{L^q(\mathbb{R}^N)} \lesssim \|\mathcal{I}_1(f)\|_{L^q(\mathbb{R}^N)} \lesssim \|f\|_{L^{\frac{qN}{N+q}}(\mathbb{R}^N)},$$

for all $f \in L^{\frac{qN}{N+q}}(\mathbb{R}^N)$ provided that $\frac{qN}{N+q} > 1 \Leftrightarrow N > p' = \frac{p}{p-1}$. That explains why we restrict our study to $N \geq 2$. Consequently, $\nabla(-\Delta)^{-1}f$ is well-defined for all $f \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.
On the other hand, one has that $\mathcal{R}$ maps $\dot{C}^\gamma(\mathbb{R}^N) \to \dot{C}^\gamma(\mathbb{R}^N)$ (see [39, 29]). Precisely, we have

$$\left|\mathcal{R}(f)\right|_{C^\gamma} \lesssim \left|f\right|_{C^\gamma}, \quad \forall f \in C^\gamma(\mathbb{R}^N)\,. \tag{2.2}$$

Next, the following inequalities are useful for our proof below.

Lemma 2.1 (see [27]). *Let $s > \frac{N}{2}$. Then, there holds true*

$$\left\|\nabla^2(-\Delta)^{-1}f\right\|_{L^\infty} = \|\mathcal{R}\mathcal{R}(f)\|_{L^\infty} \lesssim \|f\|_{L^\infty}\ln\left(1 + \frac{\|f\|_{H^s}}{\|f\|_{L^\infty}}\right) + \|f\|_{L^2}$$

for all $f \in H^s(\mathbb{R}^N)$.

We prove the following interpolation inequality via the Hölder spaces.

Lemma 2.2. *Let $\alpha \in (0,1)$, and $1 \leq p < \infty$. If $u \in \dot{C}^\alpha(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, then we have $u \in L^\infty(\mathbb{R}^N)$, and*

$$\|u\|_{L^\infty} \leq C|u|_{C^\alpha}^{\frac{N/p}{\alpha+N/p}}\|u\|_{L^p}^{\frac{\alpha}{\alpha+N/p}}\,,$$

where $C > 0$ depends on N, p, α.

Proof of Lemma 2.2. Let $\{\varrho_\varepsilon\}_{\varepsilon>0}$ be a sequence of mollifiers. Then, we write

$$\begin{aligned}|u(x)| &= |u(x) - u*\varrho_\varepsilon(x) + u*\varrho_\varepsilon(x)| \leq |u(x) - u*\varrho_\varepsilon(x)| + |u*\varrho_\varepsilon(x)| \\ &\leq \int_{\mathbb{R}^N} |u(x) - u(x-y)|\varrho_\varepsilon(y)\,dy + \|u*\varrho_\varepsilon\|_{L^\infty} \\ &\leq \int_{|y|<\varepsilon} |u|_{C^\alpha}|y|^\alpha\varrho_\varepsilon(y)\,dy + \|u*\varrho_\varepsilon\|_{L^\infty} \\ &\leq \varepsilon^\alpha|u|_{C^\alpha} + \|u*\varrho_\varepsilon\|_{L^\infty}\,.\end{aligned}$$

Thanks to Young's inequality, we deduce from the last inequality that

$$|u(x)| \leq \varepsilon^\alpha|u|_{C^\alpha} + \|u\|_{L^p}\|\varrho_\varepsilon\|_{L^{p'}} \leq \varepsilon^\alpha|u|_{C^\alpha} + C\varepsilon^{-N/p}\|u\|_{L^p}\,.$$

(Here, we accept the notation $p' = \infty$ whenever $p = 1$.)
By minimizing the last inequality with respect to α, one gets

$$|u(x)| \leq C|u|_{C^\alpha}^{\frac{N/p}{\alpha+N/p}} \|u\|_{L^p}^{\frac{\alpha}{\alpha+N/p}} .$$

This yields the proof of Lemma 2.2. □

It is convenient to recall the Grönwall inequality here.

Lemma 2.3 (Grönwall's inequality, see [29]). *If u, q, and $c \geq 0$ are continuous on $[0, t]$, c is differentiable, and*

$$q(t) \leq c(t) + \int_0^t u(s) q(s)\, ds,$$

then

$$q(t) \leq c(0) \exp\left\{\int_0^t u(s)\, ds\right\} + \int_0^t c'(s) \left(\exp\left\{\int_s^t u(\tau)\, d\tau\right\}\right) ds .$$

3. A regularizing problem

To obtain an existence of weak solutions of Eq (1.1) for densities $u_0 \in X$, we study the following regularizing problem of Eq (1.1):

$$\begin{cases} \partial_t u - \varepsilon \Delta u = \operatorname{div}(u \nabla p_u) & \text{in } \mathbb{R}^N \times (0, T)\,, \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N\,. \end{cases} \tag{3.1}$$

Then, we have the following result.

Theorem 3.1. *Let $u_0 \in X$. Then, Eq (3.1) has a unique mild solution $u_\varepsilon \in C([0, T]; X)$. Moreover, for any $q \in [1, \infty]$ we have*

$$\|u_\varepsilon(., t)\|_{L^q(\mathbb{R}^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)}, \quad \forall t \in (0, T)\,. \tag{3.2}$$

And,

$$\varepsilon \|\nabla u_\varepsilon\|^2_{L^2(\mathbb{R}^N)} \leq \|u_0\|_{L^2(\mathbb{R}^N)}, \quad \forall \varepsilon > 0\,. \tag{3.3}$$

As a consequence of (3.2), u_ε exists globally in time ($T = \infty$).

Proof of Theorem 3.1 . For a brief notation, let us drop the dependence on the parameter ε of u_ε, and denote $u = u_\varepsilon$. Now, we look for a mild solution $u \in C([0, T]; X)$ as a fixed point of the map

$$\mathcal{T} : u \mapsto e^{t\varepsilon\Delta} u_0 + \int_0^t \nabla \cdot e^{(t-\tau)\varepsilon\Delta} (u \nabla p_u)(\tau)\, d\tau\,, \tag{3.4}$$

where $e^{t\Delta}$ is the semigroup corresponding to the heat kernel $(4\pi t)^{-\frac{N}{2}} \exp(-\frac{|x|^2}{4t})$.
The following estimate for $e^{t\Delta}$ is fundamental (see [41, Proposition 1.2, Ch. 15]).

Proposition 3.1. *For every $1 \leq q \leq r \leq \infty$, and for $k \in \mathbf{N}$, there holds*

$$\left\|\nabla^k e^{t\varepsilon\Delta} u(t)\right\|_{L^r(\mathbb{R}^N)} \leq C t^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{k}{2}} \|u(t)\|_{L^q(\mathbb{R}^N)}, \quad \forall t > 0,$$

where the constant $C > 0$ depends on the parameters involved.

Let $\overline{B_X(0, R)} \subset C([0, T]; X)$ be the closed ball with center at 0, and radius R. Then, we show that $\mathcal{T}$ is a contraction mapping from $\overline{B_X(0, R)} \to \overline{B_X(0, R)}$ for a suitable number $R > 0$.
By Proposition 3.1, we get

$$\begin{aligned} \|\mathcal{T}(u)(t) - \mathcal{T}(v)(t)\|_{L^r(\mathbb{R}^N)} &= \left\| \int_0^t \nabla \cdot e^{(t-\tau)\varepsilon\Delta} \left(u \nabla p_u - v \nabla p_v\right)(\tau)\, d\tau \right\|_{L^r(\mathbb{R}^N)} \\ &\leq C \int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}} \|u \nabla p_u(\tau) - v \nabla p_v(\tau)\|_{L^q(\mathbb{R}^N)}\, d\tau \end{aligned}$$

$$\leq C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,\|(u-v)\nabla p_u(\tau)\|_{L^q(\mathbb{R}^N)}\,d\tau$$
$$+\,C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,\|v(\nabla p_u-\nabla p_v)(\tau)\|_{L^q(\mathbb{R}^N)}\,d\tau$$
$$:= \mathbf{I}_1+\mathbf{I}_2\,. \tag{3.5}$$

Then, we will study estimate (3.5) for $r=1$ and $r=\infty$ alternatively. To do that, we need to obtain the L^∞-bound for ∇p_u.

Lemma 3.1. *There exists a constant $C=C(N,m)>0$ such that for $u,v\in C([0,T];X)$ we have*

$$\|\nabla p_u(t)-\nabla p_v(t)\|_{L^\infty(\mathbb{R}^N)}\leq C\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}\|u(t)-v(t)\|_X \tag{3.6}$$

for $t\in(0,T)$.

Proof of Lemma (3.1). For $\lambda_0>0$, we have

$$\Big|\nabla p_u(x,t)-\nabla p_v(x,t)\Big| = \frac{1}{\omega_N}\left|-\int_{\mathbb{R}^N}\frac{u^m(y,t)(x-y)}{|x-y|^N}\,dy+\int_{\mathbb{R}^N}\frac{v^m(y,t)(x-y)}{|x-y|^N}\,dy\right|$$
$$\leq \frac{1}{\omega_N}\int_{|x-y|<\lambda_0}\frac{|u^m(y,t)-v^m(y,t)|}{|x-y|^{N-1}}\,dy+\frac{1}{\omega_N}\int_{|x-y|\geq\lambda_0}\frac{|u^m(y,t)-v^m(y,t)|}{|x-y|^{N-1}}\,dy$$
$$= I_1+I_2\,.$$

By the mean value theorem, we obtain

$$I_1\leq m\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}\|u(t)-v(t)\|_{L^\infty(\mathbb{R}^N)}\int_{|x-y|<\lambda_0}|x-y|^{1-N}\,dy$$
$$\leq C(N,m)\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}\|u(t)-v(t)\|_{L^\infty(\mathbb{R}^N)}\lambda_0\,.$$

And

$$I_2\leq C(N,m)\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}\|u(t)-v(t)\|_{L^1(\mathbb{R}^N)}\lambda_0^{1-N}$$

Combining the indicated inequalities yields

$$|\nabla p_u(x,t)-\nabla p_v(x,t)|\leq C\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}\Big(\|u(t)-v(t)\|_{L^\infty(\mathbb{R}^N)}\lambda_0$$
$$+\|u(t)-v(t)\|_{L^1(\mathbb{R}^N)}\lambda_0^{1-N}\Big)\,.$$

By minimizing the right hand side of the last inequality, we get

$$|\nabla p_u(x,t)-\nabla p_v(x,t)|\leq C\max\{\|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)},\|v(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\}$$
$$\times\|u(t)-v(t)\|^{1-\frac{1}{N}}_{L^\infty(\mathbb{R}^N)}\|u(t)-v(t)\|^{\frac{1}{N}}_{L^1(\mathbb{R}^N)}\,. \tag{3.7}$$

Therefore, the proof of Lemma 3.1 follows. □

Remark 3.1. *By letting $v=0$ in (3.7), for every $u\in C([0,T];X)$ we obtain*

$$\|\nabla p_u(t)\|_{L^\infty(\mathbb{R}^N)}\leq C(N,m)\|u(t)\|^{m-\frac{1}{N}}_{L^\infty(\mathbb{R}^N)}\|u(t)\|^{\frac{1}{N}}_{L^1(\mathbb{R}^N)},\quad \textit{for } t\in(0,T). \tag{3.8}$$

Now, we can treat $\mathbf{I}_1$. For any $q\in[1,\infty]$, using the interpolation inequality yields

$$\|u(\tau)\|_{L^q(\mathbb{R}^N)}\leq\|u(\tau)\|^{\frac{q-1}{q}}_{L^\infty(\mathbb{R}^N)}\|u(\tau)\|^{\frac{1}{q}}_{L^1(\mathbb{R}^N)}\leq\|u\|_{C([0,T];X)} \tag{3.9}$$

for $\tau\in(0,T)$.

Then, for $u,v\in\overline{B_X(0,R)}$, it follows from Remark 3.1, and (3.9) that

$$\mathbf{I}_1\leq C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|(u-v)(\tau)\|_{L^q(\mathbb{R}^N)}\|\nabla p_u(\tau)\|_{L^\infty(\mathbb{R}^N)}\,d\tau$$

$$\leq C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|u-v\|_{C([0,T];X)}R^m\,d\tau$$
$$\leq CR^m\|u-v\|_{C([0,T];X)}\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,d\tau\,. \tag{3.10}$$

Concerning $\mathbf{I}_2$, by Lemma 3.1 and (3.9), we obtain

$$\mathbf{I}_2 \leq C\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\|v(\tau)\|_{L^q(\mathbb{R}^N)}\|\nabla p_u(\tau)-\nabla p_v(\tau)\|_{L^\infty(\mathbb{R}^N)}\,d\tau$$
$$\leq C\|v\|_{C([0,T];X)}R^{m-1}\|u-v\|_{C([0,T];X)}\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,d\tau$$
$$\leq CR^m\|u-v\|_{C([0,T];X)}\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,d\tau\,. \tag{3.11}$$

Combining (3.5), (3.10), and (3.11) yields

$$\|\mathcal{T}(u)(t)-\mathcal{T}(v)(t)\|_{L^r} \leq CR^m\|u-v\|_{C([0,T];X)}\int_0^t (t-\tau)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\,d\tau\,. \tag{3.12}$$

By taking $r=q=1$, and $r=q=\infty$ in (3.12) alternatively, we deduce

$$\|\mathcal{T}(u)(t)-\mathcal{T}(v)(t)\|_X \leq CR^m\|u-v\|_{C([0,T];X)}\int_0^t (t-\tau)^{-\frac{1}{2}}\,d\tau$$
$$\leq CR^m\sqrt{T}\|u-v\|_{C([0,T];X)}\,. \tag{3.13}$$

If we take $T>0$ small enough such that $CR^m\sqrt{T}<\frac{1}{2}$, then it follows from (3.13) that

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{C([0,T];X)} \leq \frac{1}{2}\|u-v\|_{C([0,T];X)}, \quad \forall t\in(0,T)\,. \tag{3.14}$$

Therefore, it remains to prove that $\mathcal{T}$ maps $\overline{B_X(0,R)}$ into $\overline{B_X(0,R)}$ if we take $R=2C_\varepsilon\|u_0\|_X$. In fact, applying (3.14) to $v=0$ yields

$$\|\mathcal{T}(u)\|_{C([0,T];X)} \leq \|\mathcal{T}(0)\|_{C([0,T];X)}+\frac{1}{2}\|u\|_{C([0,T];X)}\,. \tag{3.15}$$

Observe that $\mathcal{T}(0)=e^{t\varepsilon\Delta}u_0$. By applying Proposition 3.1 to u_0 with $q=r=1$, and $q=r=\infty$ respectively, we get

$$\|\mathcal{T}(0)(t)\|_{L^1(\mathbb{R}^N)} \leq C_\varepsilon\|u_0\|_{L^1(\mathbb{R}^N)}\,,$$

and

$$\|\mathcal{T}(0)(t)\|_{L^\infty(\mathbb{R}^N)} \leq C_\varepsilon\|u_0\|_{L^\infty(\mathbb{R}^N)}\,.$$

Therefore,

$$\|\mathcal{T}(0)\|_{C([0,T];X)} \leq C_\varepsilon\|u_0\|_X\,. \tag{3.16}$$

A combination of (3.15) and (3.16) implies that

$$\|\mathcal{T}(u)\|_{C([0,T];X)} \leq C_\varepsilon\|u_0\|_X+\frac{1}{2}\|u\|_{C([0,T];X)}\,.$$

Since $R=2C_\varepsilon\|u_0\|_X$, then $\|\mathcal{T}(u)\|_{C([0,T];X)}\leq R$ whenever $u\in\overline{B_X(0,R)}$. As a result, $\mathcal{T}$ maps $\overline{B_X(0,R)}$ into $\overline{B_X(0,R)}$.

In conclusion, there exists a unique mild solution $u_\varepsilon\in C([0,T];X)$ to Eq. (3.1) for some $T>0$.

Next, we prove the a priori L^q-estimates for solution u_ε. In the following, we denote $p_\varepsilon=p_{u_\varepsilon}$ for short.

For every $q > 1$ and for $t \in (0, T)$, we multiply both sides of Eq (3.1) with u_ε^{q-1}, and integrate the resulting equation on $\mathbb{R}^N$. Then, we have

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^N} u_\varepsilon^q(x,t)\,dx + \varepsilon(q-1)\int_{\mathbb{R}^N} u_\varepsilon^{q-2}|\nabla u_\varepsilon(x,t)|^2\,dx + \int_{\mathbb{R}^N} u_\varepsilon \nabla p_\varepsilon \cdot \nabla u_\varepsilon^{q-1}(x,t)\,dx = 0\,. \tag{3.17}$$

Note that

$$\int_{\mathbb{R}^N} \nabla p_\varepsilon \cdot \nabla u_\varepsilon^q(x,t)\,dx = \int_{\mathbb{R}^N} -\Delta p_\varepsilon u_\varepsilon^q(x,t)\,dx = \int_{\mathbb{R}^N} u_\varepsilon^{m+q}(x,t)\,dx\,.$$

Combining the two identities above yields

$$\frac{1}{q}\frac{d}{dt}\int_{\mathbb{R}^N} u_\varepsilon^q(x,t)\,dx + \varepsilon(q-1)\int_{\mathbb{R}^N} u_\varepsilon^{q-2}|\nabla u_\varepsilon(x,t)|^2\,dx + \int_{\mathbb{R}^N} u_\varepsilon^{m+q}(x,t)\,dx \leq 0\,. \tag{3.18}$$

This implies (3.2) for $q \in (1, \infty)$.
Obviously, (3.3) follows by taking $q = 2$ in (3.18).
Next, since (3.2) holds true for $q > 1$, then passing to the limit as $q \to \infty$ yields

$$\|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \leq \|u_0\|_{L^\infty(\mathbb{R}^N)}\,.$$

Concerning the L^1-estimate, for any $\eta > 0$ let us put

$$\chi_\eta(r) = \begin{cases} \operatorname{sign}(r) & \text{if } |r| > \eta \\ \frac{r}{\eta} & \text{if } |r| \leq \eta \end{cases}, \text{ and } S_\eta(l) = \int_0^l \chi_\eta(r)\,dr\,.$$

Then, multiplying equation (3.1) by $\chi_\eta(u_\varepsilon(x,t))$ and integrating the indicated equation yield

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} S_\eta(u_\varepsilon(x,t))\,dx &= -\varepsilon\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \chi'_\eta(u_\varepsilon)(x,t)\,dx - \int_{\mathbb{R}^N} u_\varepsilon \nabla p_\varepsilon \cdot \chi'_\eta(u_\varepsilon)\nabla u_\varepsilon(x,t)\,dx \\ &\leq -\int_{\mathbb{R}^N} u_\varepsilon \nabla p_\varepsilon \cdot \chi'_\eta(u_\varepsilon)\nabla u_\varepsilon(x,t)\,dx\,.\end{aligned}$$

Since $\chi'_\eta(l) = \eta\chi'_{\eta^2}(l^2)$, then we get

$$u_\varepsilon \chi'_\eta(u_\varepsilon)\nabla u_\varepsilon = \frac{1}{2}\chi'_\eta(u_\varepsilon)\nabla(u_\varepsilon^2) = \frac{\eta}{2}\nabla\chi_{\eta^2}(u_\varepsilon^2)\,.$$

With this fact noted, we deduce from the last inequality that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^N} S_\eta(u_\varepsilon(x,t))\,dx &\leq -\frac{\eta}{2}\int_{\mathbb{R}^N} \nabla p_\varepsilon \cdot \nabla\chi_{\eta^2}(u_\varepsilon^2)(x,t)\,dx \\ &= \frac{\eta}{2}\int_{\mathbb{R}^N} \Delta p_\varepsilon \chi_{\eta^2}(u_\varepsilon^2)(x,t)\,dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^N} u_\varepsilon^m \chi_{\eta^2}(u_\varepsilon^2)(x,t)\,dx \leq 0\,. \end{aligned} \tag{3.19}$$

Thanks to the fact $0 \leq S_\eta(l) \leq l$, $\forall l \geq 0$, we deduce from (3.19) that

$$\int_{\mathbb{R}^N} S_\eta(u_\varepsilon(x,t))\,dx \leq \int_{\mathbb{R}^N} S_\eta(u_0)\,dx \leq \|u_0\|_{L^1(\mathbb{R}^N)}\,. \tag{3.20}$$

On the other hand, observe that $S_\eta(u_\varepsilon(x,t)) \to u_\varepsilon(x,t)$ as $\eta \to 0$ for $(x,t) \in Q_T$.
By the dominated convergence theorem, we obtain (3.2) for $q = 1$.

Finally, we show the global existence of $u_\varepsilon(t)$ in time. Indeed, it follows from (3.2) that

$$\|u_\varepsilon(t)\|_X \leq \|u_0(t)\|_X, \quad \forall\varepsilon > 0\,.$$

Fix $\varepsilon > 0$. One can repeat the above argument with initial data $u_\varepsilon(T)$ in order to get the local existence in time of $u_\varepsilon(t)$ in $[T, 2T]$, $[2T, 3T]$, ... As a result, $u_\varepsilon(t)$ exists globally in time. This completes the proof of Theorem 3.1. □

Remark 3.2. *As a consequence of Remark 3.1 and (3.2), we get a uniform L^∞-bound for ∇p_ε in $\varepsilon > 0$. That is*

$$\|\nabla p_\varepsilon(.,t)\|_{L^\infty(\mathbb{R}^N)} \leq C(N,m)\|u_0\|^{m-\frac{1}{N}}_{L^\infty(\mathbb{R}^N)}\|u_0\|^{\frac{1}{N}}_{L^1(\mathbb{R}^N)} \leq C(N,m)\|u_0\|^m_X, \quad \forall t > 0\,, \tag{3.21}$$

where $C(N,m)$ is as in Remark 3.1.

Remark 3.3. *Note that solution u_ε is presented in terms of the Gaussian kernel. According to* ([26, Chapter 5, page 273], *see also* [36, page 18]), *we have*

$$\|u_\varepsilon\|_{C^{0,\beta}(\mathbb{R}^N\times(t_1,t_2))} \leq C\left(\|u_\varepsilon\|_{L^\infty(\mathbb{R}^N\times(t_1,t_2))} + \|u_\varepsilon\|_{L^1(\mathbb{R}^N\times(t_1,t_2))}\right) \leq C'\|u_0\|_X$$

for some $\beta \in (0,1)$. By a bootstrap argument, we deduce that u_ε is a classical solution in $\mathbb{R}^N \times (t_1,t_2)$ for $0 < t_1 < t_2 < \infty$.

Next, we drive the priori $H^s(\mathbb{R}^N)$-estimate for u_ε when $u_0 \in H^s(\mathbb{R}^N)$, $s > 2 + \frac{N}{2}$.

Proposition 3.2. *Let $u_0 \in L^1(\mathbb{R}^N) \cap H^s(\mathbb{R}^N)$, $s > 2 + \frac{N}{2}$. Let u_ε be the unique solution to Eq* (3.1). *Then, there exists a constant $C > 0$ independent of ε, such that*

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{R}^N)} \leq C, \quad \forall t \in (0,T)\,.$$

Proof of Proposition 3.2. Let us denote $\Lambda^s(D)$ the pseudo-differential operator with symbol $(1+|\xi|^2)^{\frac{s}{2}}$, $\|f\|_{H^s(\mathbb{R}^N)} = \|\Lambda^s(D)f\|_{L^2(\mathbb{R}^N)}$, and denote the commutator

$$[\mathcal{T}, u](v) = u\mathcal{T}(v) - \mathcal{T}(uv)\,.$$

For more details on the commutator see, for instance, Kato–Ponce [24].
For short, we drop the notation ε on u_ε. Now, we apply the pseudo-differential operator $\Lambda^s(D)$ to both sides of Eq (3.1), and take the inner product with $\Lambda^s(D)u$ in order to get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s(\mathbb{R}^N)} &= \varepsilon\langle\Delta\Lambda^s(D)u, \Lambda^s(D)u\rangle + \langle\Lambda^s(D)\,\mathrm{div}(u\nabla p_u), \Lambda^s(D)u\rangle \\ &= -\varepsilon\left\|\nabla\Lambda^s(D)u\right\|^2_{L^2(\mathbb{R}^N)} + \langle\Lambda^s(D)\,\mathrm{div}(u\nabla p_u), \Lambda^s(D)u\rangle\,.\end{aligned} \tag{3.22}$$

From the definition of commutator, we find that

$$[\Lambda^s(D)\nabla, \nabla p](u) = \nabla p \cdot \Lambda^s(D)(\nabla u) - \Lambda^s(D)\nabla\cdot(u\nabla p)\,. \tag{3.23}$$

Combining (3.22) and (3.23) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^s(\mathbb{R}^N)} &\leq \langle\nabla p \cdot \nabla\Lambda^s(D)(u), \Lambda^s(D)u\rangle - \langle[\Lambda^s(D)\nabla, \nabla p](u), \Lambda^s(D)u\rangle \\ &:= \mathbf{B}_1 + \mathbf{B}_2\,.\end{aligned} \tag{3.24}$$

In fact, we shall estimate $\mathbf{B}_j$, $j = 1,2$ in terms of $\|u(t)\|^2_{H^s(\mathbb{R}^N)}$.
For $\mathbf{B}_1$, observe that

$$\mathbf{B}_1 = \frac{1}{2}\langle\nabla p, \nabla|\Lambda^s(D)u|^2\rangle = \frac{1}{2}\langle(-\Delta)p, |\Lambda^s(D)u|^2\rangle = \frac{1}{2}\langle u^m, |\Lambda^s(D)u|^2\rangle\,.$$

By (3.2) and Hölder's inequality, we obtain

$$|\mathbf{B}_1| \leq \frac{1}{2}\|u^m(t)\|_{L^\infty(\mathbb{R}^N)}\|u(t)\|^2_{H^s(\mathbb{R}^N)} \leq \frac{1}{2}\|u_0\|^m_{L^\infty(\mathbb{R}^N)}\|u(t)\|^2_{H^s(\mathbb{R}^N)}\,. \tag{3.25}$$

To estimate $\mathbf{B}_2$, we use the commutator estimate by Kato–Ponce [24, Lemma X1].
Put $\mathcal{T}^{s+1} = \Lambda^s(D)\nabla$, $s \in \mathbb{R}$. Then, we have

$$\left\|[\Lambda^s(D)\nabla, \nabla p](u)(t)\right\|_{L^2(\mathbb{R}^N)} = \left\|[\mathcal{T}^{s+1}, \nabla p](u)(t)\right\|_{L^2(\mathbb{R}^N)}$$

$$\lesssim \|\nabla^2 p(t)\|_{L^\infty(\mathbb{R}^N)}\|\mathcal{T}^s u(t)\|_{L^2(\mathbb{R}^N)} + \|u(t)\|_{L^\infty(\mathbb{R}^N)}\|\mathcal{T}^{s+1}\nabla p(t)\|_{L^2(\mathbb{R}^N)}\,. \tag{3.26}$$

In addition, it is not difficult to verify that

$$\|\mathcal{T}^s u\|_{L^2(\mathbb{R}^N)} \le \|\Lambda^s(D)u\|_{L^2(\mathbb{R}^N)} = \|u(t)\|_{H^s(\mathbb{R}^N)}\,, \tag{3.27}$$

and

$$\begin{aligned}\|\mathcal{T}^{s+1}\nabla p\|_{L^2(\mathbb{R}^N)} \lesssim \|\Lambda^s(D)u^m\|_{L^2(\mathbb{R}^N)} &\lesssim \|u(t)\|^{m-1}_{L^\infty(\mathbb{R}^N)}\|u(t)\|_{H^s(\mathbb{R}^N)}\\ &\le \|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}\|u(t)\|_{H^s(\mathbb{R}^N)}\,.\end{aligned} \tag{3.28}$$

Inserting (3.28) and (3.27) into (3.26) yields

$$\left\|[\Lambda^s(D)\nabla,\nabla p](u)\right\|_{L^2(\mathbb{R}^N)} \le C_0\left(\|\nabla^2 p\|_{L^\infty(\mathbb{R}^N)} + \|u_0\|^m_{L^\infty(\mathbb{R}^N)}\right)\|u(t)\|_{H^s(\mathbb{R}^N)}\,, \tag{3.29}$$

where C_0 only depends on N, s.
Since $s > \frac{N}{2}$, then we can apply Lemma 2.1 to $v = u^m$ to get

$$\begin{aligned}\|\nabla^2 p\|_{L^\infty(\mathbb{R}^N)} &\lesssim \|u^m(.,t)\|_{L^\infty}\left(1 + \log\frac{\|u^m(.,t)\|_{H^s(\mathbb{R}^N)}}{\|u^m(.,t)\|_{L^\infty}}\right) + \|u^m(.,t)\|_{L^2}\\ &\lesssim \log\|u^m(.,t)\|_{H^s(\mathbb{R}^N)} + \|u_0\|^m_{L^\infty} + \|u_0\|^m_{L^{2m}}\\ &\lesssim \log\|u(.,t)\|_{H^s(\mathbb{R}^N)} + \log m\|u_0\|^{m-1}_{L^\infty} + \|u_0\|^m_{L^\infty} + \|u_0\|^m_{L^{2m}}\,.\end{aligned} \tag{3.30}$$

By inserting (3.30) into (3.29), we find

$$\left\|[\Lambda^s(D)\nabla,\nabla p](u)\right\|_{L^2(\mathbb{R}^N)} \le C_1\left(\log\|u(.,t)\|_{H^s(\mathbb{R}^N)} + C_2\right)\|u(t)\|_{H^s(\mathbb{R}^N)}\,, \tag{3.31}$$

where $C_1 = C_1(N,s,m)$, and $C_2 = C_2(u_0)$.
By (3.31), applying the Hölder inequality yields

$$|\mathbf{B}_2| \le C_1\left(\log\|u(.,t)\|_{H^s(\mathbb{R}^N)} + C_2\right)\|u(t)\|^2_{H^s(\mathbb{R}^N)}\,. \tag{3.32}$$

A combination of (3.24), (3.25), and (3.32) leads to

$$\frac{d}{dt}\|u(t)\|^2_{H^s(\mathbb{R}^N)} \le C_1\left(\log\|u(.,t)\|_{H^s(\mathbb{R}^N)} + C_2'\right)\|u(t)\|^2_{H^s(\mathbb{R}^N)}\,.$$

By solving the ODE, we obtain

$$\|u(t)\|_{H^s(\mathbb{R}^N)} \le C_3(T)\|u_0\|^{C_4(T)}_{H^s(\mathbb{R}^N)}, \quad \forall t \in (0,T)\,. \tag{3.33}$$

Hence, we obtain the proof of Proposition 3.2. □

Remark 3.4. *Since we only use Lemma 2.1 in the proof of Proposition 3.2, then all the H^s-estimates above are true for $s > \frac{N}{2}$. This observation enable us to prove the H^s-estimate, $s > \frac{N}{2}$ in the proof of Theorem 1.2 below. In addition, from (3.30) and (3.33), we observe that $\|\nabla^2 p_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} + \|u_\varepsilon(t)\|_{H^s(\mathbb{R}^N)}$ is uniformly bounded in $\varepsilon > 0$ for all $t \in (0,T)$.*

4. Well-posedness theory, and regularity of solutions

4.1. **Proof of Theorem 1.1.** To obtain an existence of solutions to Eq (1.1), we first pass to the limit as $\varepsilon \to$ in Eq (3.1) with regular initial data. Thanks to Remark 3.4, we can get an existence of solutions to Eq (1.1) for $u_0 \in H^s$.

- **Step 1: Passing to the limit as $\varepsilon \to 0$.** Suppose that $u_0 \in C_c^\infty(\mathbb{R}^N)$. Let us fix $s \in (2+\frac{N}{2}, 3+\frac{N}{2})$.

It follows from the Sobolev embedding and (3.33) that

$$\|u_\varepsilon(t)\|_{C^{2,\alpha}(\mathbb{R}^N)} \le C_3(T)\|u_0\|^{C_4(T)}_{H^s(\mathbb{R}^N)}, \quad \forall \varepsilon > 0, \tag{4.1}$$

for all $t \in [0,T]$, with $\alpha = s - 2 - \frac{N}{2} \in (0,1)$.
Next, observe that

$$\left\|\operatorname{div}(u \cdot \nabla p_\varepsilon)(t)\right\|_{L^\infty(\mathbb{R}^N)} = \left\|\nabla u_\varepsilon \cdot \nabla p_\varepsilon - u_\varepsilon^{m+1}(t)\right\|_{L^\infty(\mathbb{R}^N)}$$

$$\leq \|\nabla p_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} \|\nabla u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} + \left\|u_\varepsilon^{m+1}(t)\right\|_{L^\infty(\mathbb{R}^N)}$$
$$\lesssim \|u_0\|_X^m \|u_\varepsilon(t)\|_{H^s(\mathbb{R}^N)} + \|u_0\|^{m+1}_{L^\infty(\mathbb{R}^N)}$$
$$\lesssim \|u_0\|_X^m \|u_0\|^{C_4(T)}_{H^s(\mathbb{R}^N)} + \|u_0\|^{m+1}_{L^\infty(\mathbb{R}^N)} .$$

Combining the two inequality yields

$$\|\partial_t u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} = \|\varepsilon\Delta u_\varepsilon(t) + \operatorname{div}(u\cdot\nabla p_\varepsilon)(t)\|_{L^\infty(\mathbb{R}^N)} \leq C, \quad \forall\varepsilon>0, \forall t\in(0,T),$$

where $C = C(T, u_0, m, N) > 0$.
As a consequence, and by the diagonal argument, there exists a subsequence of $\{u_\varepsilon\}$ (still denoted as $\{u_\varepsilon\}$) such that u_ε converges to u in $C(\overline{B_R}\times[0,T])$, for $R, T > 0$.
Now, we show that $p_\varepsilon(x,t) \to p_u := (-\Delta)^{-1}u^m(x,t)$ in distributions sense. That is

$$\int_{Q_T} p_\varepsilon\psi(x,t)\,dxdt \to \int_{Q_T} p_u\psi(x,t)\,dxdt, \quad \forall\psi\in C_c^\infty(Q_T) \tag{4.2}$$

as $\varepsilon\to 0$.
Observe that

$$\int_{Q_T}(p_\varepsilon - p_u)\psi(x,t)\,dxdt = \int_{Q_T}(-\Delta)^{-1}[u_\varepsilon^m - u^m]\psi(x,t)\,dxdt$$
$$= \int_{Q_T}(-\Delta)^{-1/2}[u_\varepsilon^m - u^m](-\Delta)^{-1/2}\psi(x,t)\,dxdt . \tag{4.3}$$

To get (4.2), it is enough to show that the term in (4.3) converges to 0 as $\varepsilon\to 0$.
Fix test function $\psi\in C_c^\infty(Q_T)$. By applying the Riesz potential estimate, one has

$$\|\mathcal{I}_1[\psi](t)\|_{L^q(\mathbb{R}^N)} \lesssim \|\psi(t)\|_{L^{\frac{qN}{N+q}}(\mathbb{R}^N)}$$

provided that $q > \frac{N}{N-1}$. In fact, we shall take $q\in\left(\frac{N}{N-1}, N\right)$ by technical problem.
Thus, we deduce that

$$\|\mathcal{I}_1[\psi]\|^q_{L^q(Q_T)} \lesssim \int_0^T \|\psi(t)\|^q_{L^{\frac{qN}{N+q}}(\mathbb{R}^N)}\,dt = C(\psi) . \tag{4.4}$$

By the same analogue, we also obtain

$$\left\|\mathcal{I}_1[u_\varepsilon^m](t)\right\|_{L^{q'}(\mathbb{R}^N)} \lesssim \left\|u_\varepsilon^m(t)\right\|_{L^{\frac{q'N}{N+q'}}(\mathbb{R}^N)} = \|u_\varepsilon(t)\|^m_{L^{\frac{mq'N}{N+q'}}(\mathbb{R}^N)} \leq \|u_0\|^m_{L^{\frac{mq'N}{N+q'}}(\mathbb{R}^N)},$$

provided that $q' > \frac{N}{N-1} \Longleftrightarrow q\in(\frac{N}{N-1}, N)$. Remind that $q' = \frac{q}{q-1}$.
Then,

$$\left\|\mathcal{I}_1[u_\varepsilon^m](t)\right\|^{q'}_{L^{q'}(Q_T)} \lesssim T\,\|u_0\|^m_{L^{\frac{mq'N}{N+q'}}(\mathbb{R}^N)}, \quad \forall\varepsilon>0 . \tag{4.5}$$

As a consequence, $\{\mathcal{I}_1[u_\varepsilon^m]\}_{\varepsilon>0}$ is uniformly bounded in $L^{q'}(Q_T)$. Hence, there exists a subsequence of $\{\mathcal{I}_1[u_\varepsilon^m]\}_{\varepsilon>0}$ (not labeled) such that $\mathcal{I}_1[u_\varepsilon^m]$ converges weakly to u^* in $L^{q'}(Q_T)$ as $\varepsilon\to 0$.
It remains to show that $u^* = \mathcal{I}_1[u^m]$. In fact, it follows from (3.2) that u_ε^m converges weakly to u^m in $L^{q'}(Q_T)$ as $\varepsilon\to 0$ (up to a subsequence).
For every test function $\phi\in C_c^\infty(Q_T)$ one has from the convolution property that

$$\int_{Q_T}\mathcal{I}_1[u_\varepsilon^m]\phi(x,t)\,dxdt = \int_{Q_T}u_\varepsilon^m\mathcal{I}_1[\phi](x,t)\,dxdt . \tag{4.6}$$

Passing to the limit as $\varepsilon\to 0$ in (4.6) yields

$$\int_{Q_T}u^*\phi(x,t)\,dxdt = \int_{Q_T}u^m\mathcal{I}_1[\phi](x,t)\,dxdt = \int_{Q_T}\mathcal{I}_1[u^m]\phi(x,t)\,dxdt,$$

which implies $u^* = \mathcal{I}_1[u^m]$ in distributions sense. Thus, we conclude that $\mathcal{I}_1[u_\varepsilon^m] \to \mathcal{I}_1[u^m]$ in distributions sense as $\varepsilon \to 0$, likewise (4.3) follows.
Next, from (3.21) we find that $\|\nabla p_\varepsilon\|_{L^\infty(Q_T)}$ is uniformly bounded in $\varepsilon > 0$. As a result, for any $r \in (1, \infty)$, ∇p_ε converges weakly to ∇p_u in $L^r(B_R \times (0, T))$ for $R, T > 0$ (up to a subsequence). It is enough for us to pass to the limit as $\varepsilon \to 0$ in Eq (3.1). Indeed, we write the equation satisfied by u_ε in distributions sense:

$$\int_{Q_T} (-u_\varepsilon \varphi_t - \varepsilon u_\varepsilon \Delta\varphi + u_\varepsilon \nabla p_\varepsilon \cdot \nabla\varphi)\, dxdt = 0, \quad \forall \varphi \in C_c^\infty(Q_T)\,. \tag{4.7}$$

For convenience, we summarize the above limiting results here. For any $R, T > 0$, we have

$$\begin{cases} u_\varepsilon \to u \text{ in } C(\overline{B_R} \times [0, T]), \\ \nabla p_\varepsilon \rightharpoonup \nabla p_u \text{ weakly in } L^q(B_R \times (0, T)), \text{ for } q > 1. \end{cases} \tag{4.8}$$

With the help of (4.8), we find easily that

$$\int_{Q_T} (-u_\varepsilon \varphi_t - \varepsilon u_\varepsilon \Delta\varphi + u_\varepsilon \nabla p_\varepsilon \cdot \nabla\varphi)\, dxdt \to \int_{Q_T} (-u\varphi_t + u\nabla p_u \cdot \nabla\varphi)\, dxdt\,.$$

This implies that u is a weak solution of Eq (1.1).
Moreover, we can verify easily that u satisfies (1.12) since u_ε satisfies (3.2).

• *Conservation of mass.* We show that u conserves the mass.
Let $\varphi_0(x)$ be a smooth function such that $0 \le \varphi_0 \le 1$; $\varphi_0(x) = 0$ if $|x| > 2$; and $\varphi_0(x) = 1$ if $|x| < 1$. For $R > 0$ using $\varphi_0(x/R)$ as a test function to Eq (1.1) yields

$$\begin{aligned} \int_{\mathbb{R}^N} u(x, t)\varphi_0(x/R)\, dx &- \int_{\mathbb{R}^N} u_0(x)\varphi_0(x/R)\, dx \\ &= -\int_0^t \int_{\mathbb{R}^N} u\nabla p_u(x, \tau) \cdot \nabla\varphi_0(x/R)\, dxd\tau\,. \end{aligned} \tag{4.9}$$

From Remark 3.2, for any $t > 0$ we have

$$\begin{aligned} \left| \int_0^t \int_{\mathbb{R}^N} u\nabla p_u(x, \tau) \cdot \nabla\varphi_0(x/R)\, dxd\tau \right| &\le \frac{1}{R} \int_0^t \|u_0\|_{L^1(\mathbb{R}^N)} \|\nabla p_u(t)\|_{L^\infty(\mathbb{R}^N)} \|\varphi_0'\|_{L^\infty(\mathbb{R}^N)}\, d\tau \\ &\le C\frac{t}{R} \|u_0\|_X^{m+1}\,, \end{aligned}$$

where $C = C(N, m, \varphi_0)$.
Thus, we get

$$\lim_{R\to\infty} \left| \int_0^t \int_{\mathbb{R}^N} u\nabla p_u(x, \tau) \cdot \nabla\varphi_0(x/R)\, dxd\tau \right| = 0\,.$$

With the fact noted, and by the monotone convergence theorem, one can pass to the limit as $R \to \infty$ in (4.9) in order to obtain

$$\int_{\mathbb{R}^N} u(x, t)\, dx = \int_{\mathbb{R}^N} u_0(x)\, dx, \quad t > 0\,.$$

Concerning the universal bound in (1.13), and the uniqueness result, we skip the details since their proofs will be given later on for densities $u_0 \in L_c^\infty(\mathbb{R}^N)$.
In conclusion, we obtain the proof of Theorem 1.1 for $u_0 \in H^s(\mathbb{R}^N)$, $s > 2 + \frac{N}{2}$.

Remark 4.1. *We obtain the L^∞-bound of ∇p_u as in (3.21). Moreover, we deduce from (3.33) that $u \in L^\infty(0, T; H^s(\mathbb{R}^N))$.*

• **Step 2: The existence and uniqueness result for $u_0 \in H^s(\mathbb{R}^N)$, $s > \frac{N}{2}$ with compact support.**
Let $\{u_{0,k}\}_{k\geq 1} \subset C_c^\infty(\mathbb{R}^N)$ be such that $u_{0,k}$ converges to u_0 in $L^q(\mathbb{R}^N)$, $q \in [1,\infty)$ and satisfies

$$\|u_{0,k}\|_{L^q(\mathbb{R}^N)} \leq \|u_0\|_{L^q(\mathbb{R}^N)}, \quad \forall q \in [1,\infty)\,.$$

Thanks to the result in **Step 1**, we see that there exists a weak solution $u_k \in L^\infty(0,T;H^s(\mathbb{R}^N))$ to Eq (1.1). In addition, u_k satisfies (1.12) for all $k \geq 1$. This implies that sequence $\{u_k(t)\}_{k\geq 1}$ is uniformly bounded in $L^q(\mathbb{R}^N)$, for $q \in [1,\infty]$.
Now, we claim that $u_k(t) \to u(t)$ strongly in $L^1_{\text{loc}}(\mathbb{R}^N)$ as $k \to \infty$ (up to a subsequence). Indeed, it follows from Remark 3.4 that

$$\|u_\varepsilon(t)\|_{H^s(\mathbb{R}^N)} \leq C, \quad t \in (0,T),$$

with $C = C(T, \|u_0\|_X, \|u_0\|_{H^s(\mathbb{R}^N)})$, $s > \frac{N}{2}$.
This implies that $\partial_t u_k = \text{div}\,(u_k \nabla p_k)$ is bounded in $L^2(0,T;H^{-s}(B_R))$, for any $R > 0$. Here, we denote $H^{-s}(\Omega)$ by the dual space of $H^s_0(\Omega)$, for any bounded set $\Omega \subset \mathbb{R}^N$. Thanks to the compactness result by Aubin–Simon, we obtain $u_k \in C([0,T];L^1(B_R))$, and the above claim follows.
As a consequence, u_k converges strongly to u in $C([0,T];L^p(B_R))$ for $p \in [1,\infty)$ due to the L^∞-boundedness of u_k, u in (1.12). So, u_k^m converges strongly to u^m in $C([0,T];L^p(B_R))$. Repeat the proof of (4.2), we have that

$$p_k = (-\Delta)^{-1} u_k^m \to p_u = (-\Delta)^{-1} u^m \tag{4.10}$$

in distributions sense.
Next, we show that for any $R, T > 0$,

$$\nabla p_k \to \nabla p_u, \quad \text{in } C([0,T];L^r(B_R)) \tag{4.11}$$

as $k \to \infty$ up to a subsequence, for $\frac{N}{N-1} < r < \infty$.
According to the fact $(-\Delta)p_k = u_k^m$, $k \geq 1$ we have

$$\begin{aligned}
\partial_t p_k &= (-\Delta)^{-1}\partial_t u_k^m = (-\Delta)^{-1}\left[m u_k^{m-1}\partial_t u_k\right] = (-\Delta)^{-1}\left[m u_k^{m-1}\,\text{div}(u_k\nabla p_k)\right] \\
&= (-\Delta)^{-1}\left[\text{div}(u_k^m \nabla p_k) - (m-1)u_k^{2m}\right] \\
&= (-\Delta)^{-1}\left[\text{div}(u_k^m \nabla p_k)\right] - (m-1)(-\Delta)^{-1}\left[u_k^{2m}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\partial_t \nabla p_k &= \nabla(-\Delta)^{-1}\left[\text{div}(u_k^m\nabla p_k)\right] - (m-1)\nabla(-\Delta)^{-1}\left[u_k^{2m}\right] \\
&= \sum_{j=1}^N \mathcal{R}\mathcal{R}_j\left[u_k^m \nabla p_k\right] - (m-1)\mathcal{R}\mathcal{I}_1\left[u_k^{2m}\right].
\end{aligned} \tag{4.12}$$

We first treat the second term in (4.12). By (1.12), applying the fundamental estimates of the Riesz transform and the Riesz potential yields

$$\begin{aligned}
\left\|\mathcal{R}\mathcal{I}_1\left[u^{2m}\right](t)\right\|_{L^r(\mathbb{R}^N)} &\lesssim \left\|\mathcal{I}_1\left[u_k^{2m}\right](t)\right\|_{L^r(\mathbb{R}^N)} \\
&\lesssim \left\|u_k^{2m}(t)\right\|_{L^{\frac{rN}{r+N}}(\mathbb{R}^N)} \leq \|u_0\|^{2m}_{L^{\frac{2mrN}{r+N}}(\mathbb{R}^N)}, \quad \forall t > 0,
\end{aligned} \tag{4.13}$$

provided that $r > \frac{N}{N-1}$.
Thus, we deduce that

$$\sup_{t>0}\left\|\mathcal{R}\mathcal{I}_1\left[u^{2m}\right](t)\right\|_{L^r(\mathbb{R}^N)} \lesssim \|u_0\|^{2m}_{L^{\frac{2mrN}{r+N}}(\mathbb{R}^N)}. \tag{4.14}$$

For the first term in (4.12), we apply the estimate of the Riesz transforms twice in order to get

$$\begin{aligned}\left\|\sum_{j=1}^{N}\mathcal{R}\mathcal{R}_j\left[u_k^m\nabla p_k\right](t)\right\|_{L^r(\mathbb{R}^N)} &\lesssim \sum_{j=1}^{N}\left\|\mathcal{R}_j\left[u_k^m\nabla p_k\right](t)\right\|_{L^r(\mathbb{R}^N)}\\ &\lesssim \left\|u_k^m\nabla p_k(t)\right\|_{L^r(\mathbb{R}^N)} \leq \|\nabla p_k\|_{L^\infty(\mathbb{R}^N)}\left\|u_k^m(t)\right\|_{L^r(\mathbb{R}^N)} \lesssim \|u_0\|^m_{L^{mr}(\mathbb{R}^N)}\,.\end{aligned}$$

Note that the last inequality follows from (1.12) and the L^∞-bound of ∇p_k in Remark 4.1. Therefore,

$$\sup_{t>0}\left\|\sum_{j=1}^{N}\mathcal{R}\mathcal{R}_j\left[u_k^m\nabla p_k\right](t)\right\|_{L^r(\mathbb{R}^N)} \lesssim \|u_0\|^m_{L^{mr}(\mathbb{R}^N)}\,. \tag{4.15}$$

From (4.14) and (4.15), we see that $\{\partial_t\nabla p_k\}_{k\geq 1}$ is uniformly bounded in $L^\infty(0,T;L^r(B_R))$ for all $R,T>0$.

On the other hand, we observe that

$$\begin{aligned}\left\|\nabla^2 p_k(t)\right\|_{L^r(\mathbb{R}^N)} &= \left\|\nabla^2(-\Delta)^{-1}[u_k^m](t)\right\|_{L^r(\mathbb{R}^N)} = \left\|\mathcal{R}\mathcal{R}[u_k^m](t)\right\|_{L^r(\mathbb{R}^N)}\\ &\lesssim \left\|u_k^m(t)\right\|_{L^r(\mathbb{R}^N)} \leq \|u_0\|^m_{L^{mr}(\mathbb{R}^N)}\,.\end{aligned} \tag{4.16}$$

This implies that $\left\{\nabla^2 p_k\right\}_{k\geq 1}$ is uniformly bounded in $L^\infty(0,T;L^r(B_R))$ for all $R,T>0$.

Thanks to Aubin–Simon's result, $\{\nabla p_k\}_{k\geq 1}$ is compact in $C([0,T];L^r(B_R))$ for $R,T>0$. As a result, there exists a subsequence of $\{\nabla p_k\}_{k\geq 1}$ (not labeled) such that ∇p_k converges to P in $C([0,T];L^r(B_R))$. With the result noted, (4.11) follows immediately from (4.10) with $P=\nabla p_u$ in distributions sense.

After that, we mimic the proof of (4.7) to show that u is a weak solution to Eq (1.1).

Now, we prove universal bound (1.13).

4.1.1. *Universal bound.* Put $\vec{v}(x,t) = -\nabla p_u(x,t)$. Then, we study the ordinary differential equation

$$\frac{d}{dt}\Phi_t(x) = \vec{v}(\Phi_t(x),t), \qquad \Phi_t(x)_{|_{t=0}} = x\,. \tag{4.17}$$

Note that $\Phi_t(x)$ is a vector field corresponding to density $u(x,t)$. Since u satisfies (1.12), and ∇p_u satisfies (3.21), then it is not difficult to verify that for any $T>0$,

$$\vec{v}\in L^1(0,T;W^{1,1}_{\mathrm{loc}}(\mathbb{R}^N)), \quad \mathrm{div}(\vec{v}) = u^m \in L^1(0,T;L^\infty(\mathbb{R}^N)), \quad \frac{\vec{v}(x,t)}{1+|x|}\in L^1(0,T;X).$$

Thanks to the DiPerna–Lions theorem (see [21, Theorem III.2]), Eq (4.17) has a unique flow $\Phi_t(x)\in C([0,T];\mathcal{M}(\mathbb{R}^N))$. Along the characteristic flow Φ_t, we find that

$$\frac{du(\Phi_t(x),t)}{dt} = -u^{m+1}(\Phi_t(x),t)\,. \tag{4.18}$$

Solving the ODE yields

$$u(\Phi_t(x),t) = \frac{u_0(x)}{\left[1+mtu_0^m(x)\right]^{\frac{1}{m}}} \leq (mt)^{-\frac{1}{m}}\,. \tag{4.19}$$

If we can show that $\Phi_t(x)$ is continuous, and $1-1$ onto on $\mathbb{R}^N$, then the inverse $\Phi_t^{-1}(x)$ is continuous. So, we get

$$u(x,t) = \frac{u_0(\Phi_t^{-1}(x))}{\left[1+mtu_0^m(\Phi_t^{-1}(x))\right]^{\frac{1}{m}}},$$

which implies (1.13).

In fact, we show below that if u_0 has compact support in $\mathbb{R}^N$, then $\Phi_t(x)$ is continuous, and $1-1$

onto on $\mathbb{R}^N$. Thus, we obtain (1.13) for u_0 with compact support. The case where $u_0 \in X$ is not compactly supported can be done by using the smoothing effect to u_0.
Then, we have the following lemma.

Lemma 4.1. *Let $u_0 \in L^\infty_c(\mathbb{R}^N)$, and let $\Phi_t(x)$ satisfy (4.17). Then, $\Phi_t(x)$ is continuous and with a continuous inverse mapping $\Phi_t^{-1}(x)$. Moreover, there exist constants C_j, C'_j, $j = 1, 2$ depending on N, m, T, u_0 such that*

$$C'_1|x_1 - x_2|^{e^{C'_2 t}} \leq |\Phi_t(x_1) - \Phi_t(x_2)| \leq C_1|x_1 - x_2|^{e^{-C_2 t}}, \quad \textit{for } t \in (0, T)\,.$$

Proof of Lemma 4.1. It is known that support of $u(t)$ is spreading and compact in $\mathbb{R}^N$ for every $t > 0$. Precisely, if $\mathrm{supp}(u_0) \subset B(0, r_0)$ for some $r_0 > 0$, then

$$\mathrm{supp}(u(.,t)) \subset B(0, r(t))\,, \tag{4.20}$$

for $t > 0$, where $r(t) = r_0 \left(1 + m\|u_0\|^m_{L^\infty(\mathbb{R}^N)} t\right)^{\frac{1}{Nm}}$.
The proof of (4.20) can be found in [36, Theorem 7.1] (see also [3, Theorem 3.1]) for the case $m = 1$. This proof is still true for the case $m > 1$ since we only need to estimate velocity $\vec{v} = -\nabla(-\Delta)^{-1}u^m$ and utilize universal bound (1.13). So, we skip the detail and refer to the reader. As a result, one has

$$\mathrm{supp}(u(.,t)) \subset B(0, r(T)), \quad \forall t \in (0, T).$$

Next, let us fix $T > 0$. For any $t \in (0, T)$ and for $x_1, x_2 \in \mathbb{R}^N$, $x_1 \neq x_2$, we set

$$l(t) = \left|\Phi_t(x_1) - \Phi_t(x_2)\right|.$$

From the fact

$$\Phi_t(x) = x + \int_0^t \vec{v}(x, s)\, ds, \tag{4.21}$$

we observe that if $x_1 \neq x_2$, then there exists a time $\tau_0 > 0$ such that $l(t) > 0$ for all $t \in (0, \tau_0)$.
By (4.20), we have that

$$\begin{aligned}
\left|\vec{v}(\Phi_t(x_1), t) - \vec{v}(\Phi_t(x_2), t)\right| &= \frac{1}{\omega_N}\left|\int_{\mathbb{R}^N}\left(\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right)u^m(y, t)\, dy\right| \\
&\leq \frac{\|u_0\|^m_{L^\infty(\mathbb{R}^N)}}{\omega_N}\int_{\{|y| < r(T)\}}\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| dy \\
&:= \frac{\|u_0\|^m_{L^\infty(\mathbb{R}^N)}}{\omega_N}\left(\mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3\right),
\end{aligned}$$

where

$$\begin{cases}
\mathbf{A}_1 = \displaystyle\int_{|\Phi_t(x_1) - y| \leq 2l(t)}\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| dy, \\
\mathbf{A}_2 = \displaystyle\int_{|\Phi_t(x_2) - y| \leq 2l(t)}\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| dy, \\
\mathbf{A}_3 = \displaystyle\int_{|\Phi_t(x_1) - y| > 2l(t), |\Phi_t(x_2) - y| > 2l(t)}\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| dy.
\end{cases}$$

We first estimate $\mathbf{A}_1$. From the triangle inequality, we have

$$|\Phi_t(x_2) - y| \leq |\Phi_t(x_1) - y| + |\Phi_t(x_1) - \Phi_t(x_2)| \leq 3l(t)$$

whenever $|\Phi_t(x_1) - y| \leq 2l(t)$.
Therefore,

$$\mathbf{A}_1 \leq \int_{|\Phi_t(x_1) - y| \leq 2l(t)} \frac{dy}{|\Phi_t(x_1) - y|^{N-1}} + \int_{|\Phi_t(x_2) - y| \leq 3l(t)} \frac{dy}{|\Phi_t(x_2) - y|^{N-1}} \leq C(N) l(t)\,.$$

By the same analogue as in the proof of $\mathbf{A}_1$, we also obtain

$$\mathbf{A}_2 \le C(N)l(t)\,.$$

It remains to study $\mathbf{A}_3$. Since $|\Phi_t(x_1) - y|, |\Phi_t(x_2) - y| > 2l(t)$, then it follows from the mean value theorem that

$$\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| \le C(N) \max\left\{\frac{|\Phi_t(x_1) - \Phi_t(x_2)|}{|\Phi_t(x_1) - y|^N}, \frac{|\Phi_t(x_1) - \Phi_t(x_2)|}{|\Phi_t(x_2) - y|^N}\right\}.$$

Assume without loss of generality that

$$\left|\frac{\Phi_t(x_1) - y}{|\Phi_t(x_1) - y|^N} - \frac{\Phi_t(x_2) - y}{|\Phi_t(x_2) - y|^N}\right| \le \frac{C(N)l(t)}{|\Phi_t(x_1) - y|^N}.$$

By (4.21), (3.21), and the fact $|y| < r(T)$, we deduce from the triangle inequality that

$$|\Phi_t(x_1) - y| \le |\Phi_t(x_1)| + |y| \le |x_1| + t\|\vec{v}\|_{L^\infty(Q_T)} + r(T) \le R^*(T)\,,$$

for all $t \in (0, T)$, with $R^*(T) := |x_1| + C(N, m)T\|u_0\|_X^m + r(T)$.
Then,

$$\begin{aligned}\mathbf{A}_3 &\le C(N)l(t) \int_{\{2l(t)<|\Phi_t(x_1)-y|<R^*(T)\}} \frac{dy}{|\Phi_t(x_1) - y|^N} \\ &\le Cl(t)[\,\ln R^*(T) - \ln l(t)]\,,\end{aligned}$$

Combining the above estimates of $\mathbf{A}_j$, $j = 1, 2, 3$ yields

$$\left|\vec{v}(\Phi_t(x_1), t) - \vec{v}(\Phi_t(x_2), t)\right| \le Cl(t) + Cl(t)[\,\ln R^*(T) - \ln l(t)], \quad \text{for } t \in (0, T)$$

With the last inequality noted, we deduce from (4.17) that

$$\begin{cases} -Cl(t)[\,\ln(eR^*(T)) - \ln l(t)] \le l'(t) \le Cl(t)[\,\ln(eR^*(T)) - \ln l(t)]\,, \\ l(0) = |x_1 - x_2|\,. \end{cases}$$

It is clear that $y(t) = \ln l(t)$ satisfies the ordinary differential inequalities

$$\begin{cases} -C[\,\ln(eR^*(T)) - y(t)] \le y'(t) \le C[\,\ln(eR^*(T)) - y(t)]\,, \\ y(0) = \ln|x_1 - x_2|\,. \end{cases} \tag{4.22}$$

Solving (4.22), we obtain

$$C_1'|x_1 - x_2|^{e^{C_2' t}} \le l(t) \le C_1|x_1 - x_2|^{e^{-C_2 t}}, \quad \text{for } t \in (0, T)\,, \tag{4.23}$$

where constants C_j, C_j', $j = 1, 2$ depend on T, N, m, u_0, x_1.
Hence, we get the proof of Lemma 4.1. □

Thanks to Lemma 4.1, we obtain the universal bound (1.13) for $u_0 \in H^s(\mathbb{R}^N)$ with compact support. As mentioned above, (1.13) holds for the constructed solutions to Eq (1.1) with densities $u_0 \in H^s(\mathbb{R}^N)$ by the smoothing effect.

Next, we prove the uniqueness result of weak solutions to Eq (1.1) for $u_0 \in H^s(\mathbb{R}^N)$ with compact support.

4.1.2. *Uniqueness.* Let u_j, $j = 1, 2$ be the two solutions of Eq (1.1), and let $\Phi_t^j(x)$ be the two flows corresponding to u_j. It suffices to show that the two flows coincide.
Fix $T > 0$. For any $0 < t < T$, we set

$$\delta(t) = \frac{1}{|\Omega_T|} \int_{\Omega_T} \left|\Phi_t^1(x) - \Phi_t^2(x)\right| dx\,,$$

where $\Omega_T := B(0, r(T))$, and $r(T)$ is defined as in (4.20).
We remind that $\mathrm{supp}(u_j(., t)) \subset \Omega_T$, $j = 1, 2$ for all $t \in [0, T]$. Now, we want to show that $\delta(t) = 0$ for all $t \in [0, T]$.
Remind that

$$\Phi_t^j(x) = x + \int_0^t \vec{v}_j(\Phi_\tau^j(x), \tau)\, d\tau, \quad j = 1, 2\,.$$

Then,

$$\begin{aligned} \delta(t) \leq &\frac{1}{|\Omega_T|} \left| \int_{\Omega_T} \int_0^t \left[\vec{v}_1(\Phi_\tau^1(x), \tau) - \vec{v}_1(\Phi_\tau^2(x), \tau)\right] d\tau dx \right| \\ &+ \frac{1}{|\Omega_T|} \left| \int_{\Omega_T} \int_0^t \left[\vec{v}_1(\Phi_\tau^2(x), \tau) - \vec{v}_2(\Phi_\tau^2(x), \tau)\right] d\tau dx \right| := \mathbf{T}_1 + \mathbf{T}_2\,. \end{aligned} \tag{4.24}$$

We first study $\mathbf{T}_1$. It is known that vector velocity $\nabla(-\Delta)^{-1}[w](x)$ satisfies the log-Lipschitz property (see, e.g., [3, 27, 28]). Precisely, we have

$$\left|\nabla(-\Delta)^{-1}[w](x) - \nabla(-\Delta)^{-1}[w](y)\right| \leq C_N \|w\|_X h(|x - y|), \quad \forall x, y \in \mathbb{R}^N\,, \tag{4.25}$$

where

$$h(s) = \begin{cases} s(1 - \ln s) & \text{if } s < 1, \\ 1 & \text{if } s \geq 1. \end{cases}$$

Note that h is a concave function. Applying (4.25) to $w = u_1^m$ yields

$$\begin{aligned} \left|\vec{v}_1(\Phi_\tau^1(x), \tau) - \vec{v}_1(\Phi_\tau^2(x), \tau)\right| &\leq C\|u^m(\tau)\|_X\, h\left(|\Phi_\tau^1(x) - \Phi_\tau^2(x)|\right) \\ &\leq C\|u_0\|_X^m\, h\left(|\Phi_\tau^1(x) - \Phi_\tau^2(x)|\right)\,. \end{aligned} \tag{4.26}$$

With the last inequality noted, we deduce from the Fubini theorem, and the Jensen inequality that

$$\mathbf{T}_1 \leq C\|u_0\|_X^m \int_0^t \frac{1}{|\Omega_T|} \int_{\Omega_T} h\left(\Phi_\tau^1(x) - \Phi_\tau^2(x)\right) dx d\tau \leq C\|u_0\|_X^m \int_0^t h\left(\delta(\tau)\right) d\tau\,. \tag{4.27}$$

Concerning $\mathbf{T}_2$, observe that

$$\begin{aligned} \vec{v}_2(\Phi_t^2(x), t) &= -\frac{1}{\omega_N} \int_{\mathbb{R}^N} \frac{\Phi_t^2(x) - y}{|\Phi_t^2(x) - y|^N} u_2^m(y, t)\, dy \\ &= -\frac{1}{\omega_N} \int_{\mathbb{R}^N} \frac{\Phi_t^2(x) - \Phi_t^2(z)}{|\Phi_t^2(x) - \Phi_t^2(z)|^N} \frac{u_0^m(z)}{1 + m u_0^m(z) t} \det\left(\frac{\partial \Phi_t^2(z)}{\partial z}\right) dz \\ &= -\frac{1}{\omega_N} \int_{\mathbb{R}^N} \frac{\Phi_t^2(x) - \Phi_t^2(z)}{|\Phi_t^2(x) - \Phi_t^2(z)|^N} \frac{u_0^m(z)}{(1 + m u_0^m(z) t)^{1 - \frac{1}{m}}}\, dz\,, \end{aligned}$$

and

$$\begin{aligned} \vec{v}_1(\Phi_t^2(x), t) &= -\frac{1}{\omega_N} \int_{\mathbb{R}^N} \frac{\Phi_t^2(x) - y}{|\Phi_t^2(x) - y|^N} u_1^m(y, t)\, dy \\ &= -\frac{1}{\omega_N} \int_{\mathbb{R}^N} \frac{\Phi_t^2(x) - \Phi_t^1(z)}{|\Phi_t^2(x) - \Phi_t^1(z)|^N} \frac{u_0^m(z)}{1 + m u_0^m(z) t} \det\left(\frac{\partial \Phi_t^1(z)}{\partial z}\right) dz \end{aligned}$$

$$= -\frac{1}{\omega_N}\int_{\mathbb{R}^N}\frac{\Phi^2_t(x)-\Phi^1_t(z)}{|\Phi^2_t(x)-\Phi^1_t(z)|^N}\frac{u_0^m(z)}{(1+mu_0^m(z)t)^{1-\frac{1}{m}}}\,dz\,.$$

Thanks to the Fubini theorem, we get

$$\begin{aligned}
&\left|\frac{1}{|\Omega_T|}\int_{\Omega_T}\left[\vec{v}_1(\Phi^2_t(x),t)-\vec{v}_2(\Phi^2_t(x),t)\right]dx\right| \\
&\le \frac{1}{|\Omega_T|\omega_N}\int_{\Omega_T}\int_{\Omega_T}\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right|\frac{u_0^m(z)\,dzdx}{(1+mu_0^m(z)t)^{1-\frac{1}{m}}} \\
&\le \frac{\|u_0\|^m_{L^\infty(\mathbb{R}^N)}}{|\Omega_T|\omega_N}\int_{\Omega_T}\int_{\Omega_T}\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right|dxdz\,. \qquad (4.28)
\end{aligned}$$

Next, let us set

$$d = \left|\Phi^1_t(z)-\Phi^2_t(z)\right|.$$

Then, we rewrite the integral in (4.28) into the sum of $\mathbf{J}_1+\mathbf{J}_2+\mathbf{J}_3$, with

$$\begin{cases}
\mathbf{J}_1 = \dfrac{1}{|\Omega_T|}\displaystyle\int_{\Omega_T}\int_{|\Phi^2_t(x)-\Phi^1_t(z)|<2d}\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right|dxdz, \\
\mathbf{J}_2 = \dfrac{1}{|\Omega_T|}\displaystyle\int_{\Omega_T}\int_{|\Phi^2_t(x)-\Phi^2_t(z)|<2d}\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right|dxdz, \\
\mathbf{J}_3 = \dfrac{1}{|\Omega_T|}\displaystyle\int_{\Omega_T}\int_{|\Phi^2_t(x)-\Phi^2_t(z)|\ge 2d,|\Phi^2_t(x)-\Phi^1_t(z)|\ge 2d}\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right|dxdz.
\end{cases}$$

We first study $\mathbf{J}_1$. Since $|\Phi^2_t(x)-\Phi^1_t(z)|<2d$, then it is obvious that

$$d \le |\Phi^2_t(x)-\Phi^2_t(z)| \le 3d\,.$$

Therefore,

$$\begin{aligned}
&\left|K(\Phi^2_t(x)-\Phi^1_t(z))-K(\Phi^2_t(x)-\Phi^2_t(z))\right| \\
&\le C_N\left(\frac{1}{\left|\Phi^2_t(x)-\Phi^1_t(z)\right|^{N-1}}+\frac{1}{\left|\Phi^2_t(x)-\Phi^2_t(z)\right|^{N-1}}\right).
\end{aligned}$$

This implies that

$$\begin{aligned}
\mathbf{J}_1 &\le \frac{C_N}{|\Omega_T|}\int_{\Omega_T}\int_{|\Phi^2_t(x)-\Phi^1_t(z)|<2d}\frac{1}{\left|\Phi^2_t(x)-\Phi^1_t(z)\right|^{N-1}}dxdz \\
&+ \frac{C_N}{|\Omega_T|}\int_{\Omega_T}\int_{|\Phi^2_t(x)-\Phi^2_t(z)|<3d}\frac{1}{\left|\Phi^2_t(x)-\Phi^2_t(z)\right|^{N-1}}dxdz \\
&\le \frac{C_N}{|\Omega_T|}\int_{\Omega_T}\left(\int_0^{2d}\frac{1}{r^{N-1}}\omega_N r^{N-1}dr+\int_0^{3d}\frac{1}{r^{N-1}}\omega_N r^{N-1}dr\right)dz \\
&\le C_N\frac{1}{|\Omega_T|}\int_{\Omega_T}\left|\Phi^1_t(z)-\Phi^2_t(z)\right|dz = C_N\delta(t)\,.
\end{aligned}$$

Similarly as in the proof of $\mathbf{J}_1$, we also obtain

$$\mathbf{J}_2 \le C_N\delta(t)\,.$$

Finally, we treat $\mathbf{J}_3$. Since $\mathrm{supp}(u_j(.,t))\subset\Omega_T$, $j=1,2$ for all $t\in[0,T]$, then there exists a radius $R_0(T)>0$ such that

$$\left|\Phi^2_t(x)-\Phi^1_t(z)\right|+\left|\Phi^2_t(x)-\Phi^2_t(z)\right|<R_0(T)$$

for all $x, z \in \Omega_T$.
On the other hand, since $\left|\Phi_t^2(x) - \Phi_t^j(z)\right| \geq 2d$ for $j = 1, 2$, then it follows from the mean value theorem that

$$\left|K(\Phi_t^2(x) - \Phi_t^1(z)) - K(\Phi_t^2(x) - \Phi_t^2(z))\right| \leq C_N \max\left\{\frac{\left|\Phi_t^1(z) - \Phi_t^2(z)\right|}{\left|\Phi_t^2(x) - \Phi_t^1(z)\right|^N}, \frac{\left|\Phi_t^1(z) - \Phi_t^2(z)\right|}{\left|\Phi_t^2(x) - \Phi_t^2(z)\right|^N}\right\}.$$

Thus, we find that

$$\begin{aligned}
\mathbf{J}_3 &\leq \frac{C}{|\Omega_T|}\int_{\Omega_T}\int_{2d \leq \left|\Phi_t^2(x) - \Phi_t^1(z)\right| < R_0(T)} \frac{d}{\left|\Phi_t^2(x) - \Phi_t^1(z)\right|^N} dxdz \\
&+ \frac{C}{|\Omega_T|}\int_{\Omega_T}\int_{2d \leq \left|\Phi_t^2(x) - \Phi_t^2(z)\right| < R_0(T)} \frac{d}{\left|\Phi_t^2(x) - \Phi_t^2(z)\right|^N} dxdz \\
&\lesssim \frac{1}{|\Omega_T|}\int_{\Omega_T} d\left(\int_{2d}^{R_0(T)} \frac{1}{r^N}\omega_N r^{N-1} dr\right) dz \\
&\lesssim \frac{1}{|\Omega_T|}\int_{\Omega_T} d\left(\ln R_0(T) - \ln 2d\right) dz \\
&\lesssim \frac{1}{|\Omega_T|}\int_{\Omega_T} h\left(\left|\Phi_t^1(z) - \Phi_t^2(z)\right|\right) dz \leq h\left(\delta(t)\right) .
\end{aligned}$$

Note that the last inequality was obtained by the concavity of h.
By inserting the estimates of $\mathbf{J}_j$, $j = 1, 2, 3$ into (4.28), we obtain

$$\left|\frac{1}{|\Omega_T|}\int_{\Omega_T}\left[\vec{v}_1(\Phi_t^2(x), t) - \vec{v}_2(\Phi_t^2(x), t)\right] dx\right| \leq C\|u_0\|^m_{L^\infty(\mathbb{R}^N)} h(\delta(t)) . \tag{4.29}$$

Combining (4.29), (4.27), and (4.24) yields

$$\delta(t) \leq C\int_0^t h(\delta(\tau)), \quad \delta(0) = 0 , \tag{4.30}$$

where $C = C(T, \|u_0\|_X, m, N)$.
Thanks to the fact

$$\int_0^s \frac{d\tau}{h(\tau)} = +\infty, \quad \text{for } s > 0 ,$$

we deduce from (4.30) that $\delta(t) = 0$ for all $t \in [0, T]$. So, we obtain the uniqueness result. This puts an end to the proof of Theorem 1.1.

4.2. **Proof of Theorem 1.2.** To obtain the result, we just repeat the proof of Theorem 1.1 for the relative estimates in terms of $C^\gamma(\mathbb{R}^N)$-norm.

• **$C^\gamma(\mathbb{R}^N)$-estimate.** To establish the $C^\gamma(\mathbb{R}^N)$-estimate of u_k, it suffices to control $\|u_k(t)\|_{C^\gamma(\mathbb{R}^N)}$ by means of $\|u_0\|_{C^\gamma(\mathbb{R}^N)}$ for all $k \geq 1$. For brief, we drop the dependence on k of u_k. Acting ∇ to both sides of (4.17) yields

$$\frac{d}{dt}\nabla\Phi_t(x) = \nabla\vec{v}(\Phi_t(x), t)\nabla\Phi_t(x) .$$

By integrating both sides of the above equation on $(0, t)$, we get

$$\nabla\Phi_t(x) = \mathbb{I}_N - \int_0^t \nabla\vec{v}(\Phi_\tau(x), \tau)\nabla\Phi_\tau(x)\, d\tau , \tag{4.31}$$

where $\mathbb{I}_N$ is the identity matrix of order N.
Therefore, we deduce that

$$\left\|\nabla\Phi_t\right\|_{L^\infty(\mathbb{R}^N)} \leq C(N) - \int_0^t \left\|\nabla\vec{v}(\Phi_\tau(.),\tau)\right\|_{L^\infty(\mathbb{R}^N)} \|\nabla\Phi_\tau\|_{L^\infty(\mathbb{R}^N)}\, d\tau\,.$$

Applying Grönwall's inequality (see Lemma 2.3) yields

$$\left\|\nabla\Phi_t\right\|_{L^\infty(\mathbb{R}^N)} \leq C(N)\exp\left\{\int_0^t \left\|\nabla\vec{v}(\Phi_\tau(.),\tau)\right\|_{L^\infty(\mathbb{R}^N)}\, d\tau\right\}\,. \tag{4.32}$$

Note that $\nabla\vec{v}(x,t) = -\nabla^2(-\Delta)^{-1}u^m(x,t) = -\mathcal{R}\mathcal{R}[u^m](x,t)$.
Since $\mathcal{R}_j$ map $\dot{C}^\gamma(\mathbb{R}^N) \to \dot{C}^\gamma(\mathbb{R}^N)$ for all $j = 1,\ldots,N$ (see Section 2), then we obtain

$$\left|\nabla\vec{v}(\Phi_t(.),t)\right|_{C^\gamma(\mathbb{R}^N)} \lesssim \left|\mathcal{R}[u^m(\Phi_t(.),t)]\right|_{C^\gamma(\mathbb{R}^N)} \lesssim \left|u^m(\Phi_t(.),t)\right|_{C^\gamma(\mathbb{R}^N)}\,. \tag{4.33}$$

On the other hand, it follows from the mean value theorem, (1.12), and (4.19) that

$$\begin{aligned}
\left|u^m(\Phi_t(.),t)\right|_{C^\gamma(\mathbb{R}^N)} &= \sup_{x\neq y}\frac{\left|u^m(\Phi_k(x,t),t) - u^m(\Phi_k(y,t),t)\right|}{|x-y|^\gamma}\\
&= \sup_{x\neq y}\frac{\left|\frac{u_0^m(x)}{1+mtu_0^m(x)} - \frac{u_0^m(y)}{1+mtu_0^m(y)}\right|}{|x-y|^\gamma}\\
&\leq C(T,m)\|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}|u_0|_{C^\gamma(\mathbb{R}^N)}, \quad \forall t\in(0,T)\,.
\end{aligned} \tag{4.34}$$

Combining (4.34) and (4.33) yields

$$\left|\nabla\vec{v}(\Phi_t(.),t)\right|_{C^\gamma(\mathbb{R}^N)} \leq C(T,m,N,\gamma)\|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}|u_0|_{C^\gamma(\mathbb{R}^N)}, \quad \forall t\in(0,T)\,. \tag{4.35}$$

By the interpolation inequality in Lemma 2.2, the L^p-boundedness of $\mathcal{R}$, $p>1$, and (4.35), we obtain

$$\begin{aligned}
\left\|\nabla\vec{v}(\Phi_t(.),t)\right\|_{L^\infty(\mathbb{R}^N)} &\lesssim \left\|\nabla\vec{v}(\Phi_t(.),t)\right\|^{\frac{\gamma}{\gamma+N/q}}_{L^q(\mathbb{R}^N)}\left|\nabla\vec{v}(\Phi_t(.),t)\right|^{\frac{N/q}{\gamma+N/q}}_{C^\gamma(\mathbb{R}^N)}\\
&\lesssim \left(\left\|-\mathcal{R}\mathcal{R}[u^m(\Phi_t(.),t)]\right\|_{L^q(\mathbb{R}^N)}\right)^{\frac{\gamma}{\gamma+N/q}}\left(\|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}|u_0|_{C^\gamma(\mathbb{R}^N)}\right)^{\frac{N/q}{\gamma+N/q}}\\
&\lesssim \left(\left\|u^m(\Phi_t(.),t)\right\|_{L^q(\mathbb{R}^N)}\right)^{\frac{\gamma}{\gamma+N/q}}\left(\|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}|u_0|_{C^\gamma(\mathbb{R}^N)}\right)^{\frac{N/q}{\gamma+N/q}}\\
&\lesssim \left(\|u_0\|^m_{L^{qm}(\mathbb{R}^N)}\right)^{\frac{\gamma}{\gamma+N/q}}\left(\|u_0\|^{m-1}_{L^\infty(\mathbb{R}^N)}|u_0|_{C^\gamma(\mathbb{R}^N)}\right)^{\frac{N/q}{\gamma+N/q}}
\end{aligned} \tag{4.36}$$

for all $t\in(0,T)$, and for $q>1$.
A combination of (4.32) and (4.36) implies that

$$\left\|\nabla\Phi_t(.)\right\|_{L^\infty(\mathbb{R}^N)} \leq C, \quad \forall t\in(0,T)\,, \tag{4.37}$$

where $C = C(u_0,T,N,m,\gamma,q)$.
Next, we try to obtain a priori estimate of $\left|\nabla\Phi_t(.)\right|_{C^\gamma(\mathbb{R}^N)}$ in terms of $|u_0|_{C^\gamma(\mathbb{R}^N)}$.
Taking the semi-norm $|.|_{C^\gamma(\mathbb{R}^N)}$ to both sides of (4.31), we obtain

$$\begin{aligned}
\left|\nabla\Phi_t(.)\right|_{C^\gamma} &\leq \int_0^t \left|\nabla\vec{v}(\Phi_\tau(.),\tau)\nabla\Phi_\tau(.)\right|_{C^\gamma}\, d\tau\\
&\leq \int_0^t \left(\left|\nabla\vec{v}(\Phi_\tau(.),\tau)\right|_{C^\gamma}\left\|\nabla\Phi_\tau(.)\right\|_{L^\infty} + \left\|\nabla\vec{v}(\Phi_\tau(.),\tau)\right\|_{L^\infty}\left|\nabla\Phi_\tau(.)\right|_{C^\gamma}\right) d\tau\,.
\end{aligned}$$

Thanks to (4.36) and (4.37), it follows from the last inequality that

$$\left|\nabla\Phi_t(.)\right|_{C^\gamma} \leq C_1 + \int_0^t \left\|\nabla\vec{v}(\Phi_\tau(.),\tau)\right\|_{L^\infty}\left|\nabla\Phi_\tau(.)\right|_{C^\gamma} d\tau, \quad \forall t\in(0,T),$$

where constant $C_1 > 0$ depends on the parameters, and u_0 as in (4.37).
Again, by applying the Grönwall inequality, and by (4.35), we arrive

$$\left|\nabla\Phi_t(.)\right|_{C^\gamma} \leq C_1 \exp\left\{\int_0^t \left|\nabla\vec{v}(\Phi_\tau(.),\tau)\right|_{C^\gamma} d\tau\right\} \leq C_2\,. \tag{4.38}$$

In summary, $\left\|\nabla\Phi_t(.)\right\|_{C^\gamma}$ and $\left\|\nabla\vec{v}(\Phi_t(.),t)\right\|_{C^\gamma}$ are controlled by $\|u_0\|_{L^1} + |u_0|_{C^\gamma}$.
Thanks to Lemma 4.1, we find that

$$u(x,t) = \frac{u_0(\Phi_t^{-1}(x))}{\left[1 + mtu_0^m(\Phi_t^{-1}(x))\right]^{\frac{1}{m}}}\,.$$

This implies that

$$\|u(t)\|_{C^\gamma(\mathbb{R}^N)} \leq C, \quad \forall t \in (0,T)\,, \tag{4.39}$$

where constant C merely depends on u_0, T, and the parameters involved.
Since u_k verifies (4.39) for all $k \geq 1$, then $u_k(t) \longrightarrow u(t)$ in $L^\infty(0,T;C^\gamma(\mathbb{R}^N))$ as $k \to \infty$ according to the Arzelà–Ascoli theorem (up to a subsequence). By repeating the proof of passing to the limit as $k \to \infty$ as in Theorem 1.1, one obtains $u \in L^\infty(0,T;C^\gamma(\mathbb{R}^N))$ is a unique weak solution to Eq (1.1). Hence, we complete the proof of Theorem 1.2.

Remark 4.2. *We emphasize that* (4.39) *is used to obtain the compactness result for solutions* u_k *in the proof of Theorem 1.1.*

5. Asymptotic behavior of solutions

In this section, we investigate the asymptotic behavior of solutions to Eq (1.1) via the vortex patch of solution when $t \to \infty$, and we give the proof of Theorem 1.3. Our proof is similar to the one in [3, Theorem 3.1], but with a slight difference concerning the nonlinearity of $u^m(x,t)$, $m > 1$. Note that the nonlinear term does not conserve the mass for $t > 0$. This fact plays a crucial role in "frozen in time estimate of the velocity at the boundary", see the proof of [3, Theorem 3.1].

Proof of Theorem 1.3. As mentioned above, we just present some points in the proof having slight differences according to the appearance of nonlinear term u^m. We first prove Theorem 1.3 for densities $u_0 \in C_c^1(\mathbb{R}^N)$ to be sure that vector field $\Phi_t(x) = \Phi(x,t)$ is differentiable in time. After that, the case $u_0 \in H^s(\mathbb{R}^N)$ (resp. $u_0 \in C^\alpha(\mathbb{R}^N)$) with compact support can be obtained by using the smoothing effect to u_0 as in [3, Theorem 3.1].
Now, let us set

$$W(x,t) = \frac{h_0 \mathbf{1}_{B(0,R(t))}}{(1 + mh_0^m t)^{\frac{1}{m}}},\ \ h_0 = \|u_0\|_{L^\infty(\mathbb{R}^N)},$$

and

$$\begin{cases} R(t) = \left(1 + mh_0^m t\right)^{\frac{1}{Nm}},\ \ r(t) = R(t)(1 + E(t))^{\frac{1}{N}}, \\ E(t) = E_0\left(1 + mh_0^m t\right)^{\frac{-N2^{1-N}}{m}},\ \ E_0 = r_0^N - 1. \end{cases} \tag{5.1}$$

Since we assume that $\|u_0\|_{L^\infty(\mathbb{R}^N)} = 1$ in Theorem 1.3, then we necessarily have

$$h_0 = 1, \quad R(t) = (1 + mt)^{\frac{1}{Nm}}$$

in the following.
• **Change of variables.** We define the solution $U(y,\tau)$ of the renormalized flow associated to a solution $u(x,t)$ as follows:

$$y = \frac{x}{R(t)}, \quad \tau = \ln\left[(1 + mt)^{\frac{1}{m}}\right], \quad U(y,\tau) = (1 + mt)^{\frac{1}{m}} u(x,t)\,, \tag{5.2}$$

as well as the corresponding change for pressure and velocity:

$$P(y,\tau) = (-\Delta)^{-1}U^m(y,\tau) = [R(t)]^{mN-2}(-\Delta)^{-1}u^m(x,t) = [R(t)]^{mN-2}p(x,t)\,,$$

and

$$V(y,\tau) = -\nabla P(y,\tau) = [R(t)]^{mN-1}v(x,t)\,.$$

By a straightforward computation, one can verify that $U(y,\tau)$ satisfies

$$U_\tau = \mathrm{div}_y\left(U(\nabla P + \frac{y}{N})\right), \quad \Delta_y P = -U^m(y,\tau)\,. \tag{5.3}$$

See the proof of (5.3) in the Appendix section.
Next, for every $t > 0$, it is clear that function $g(l) = \frac{l}{\left(1+ml^m t\right)^{\frac{1}{m}}}$ is increasing in $(0,\infty)$.
As a result, one has

$$u(x,t) \le \frac{1}{(1+mt)^{\frac{1}{m}}}, \quad \forall x \in \mathbb{R}^N$$

since $\sup_{t>0} \|u(t)\|_{L^\infty(\mathbb{R}^N)} \le \|u_0\|_{L^\infty(\mathbb{R}^N)} = 1$.
Therefore, we deduce from (5.2) that

$$U(y,\tau) \le 1, \quad \forall (y,\tau) \in \mathbb{R}^N \times (0,\infty)\,. \tag{5.4}$$

In addition, we observe that

$$\int_{\mathbb{R}^N} U(y,\tau)\,dy = R^N(t)\int_{\mathbb{R}^N} u(R(t)y,t)\,dy = \int_{\mathbb{R}^N} u(x,t)\,dx = \omega_N, \quad \forall \tau > 0. \tag{5.5}$$

• **Estimate of the size of the support.**

Lemma 5.1 (Frozen in time estimate of the velocity at the boundary)**.** *Let $\mu \in L^\infty_c(\mathbb{R}^N)$ be such that*

$$\mathrm{supp}(\mu) \subset \overline{B(0,r)}, \ \|\mu(t)\|_{L^1(\mathbb{R}^N)} \le \omega_N, \ \|\mu\|_{L^\infty(\mathbb{R}^N)} \le 1$$

for some $r \ge 1$. Note that we necessarily have $r \ge 1$ since $\|\mu\|_{L^1(\mathbb{R}^N)} \le \omega_N$. Then, the velocity field

$$\vec{v}(y) = -\nabla(-\Delta)^{-1}\mu(y) - \frac{y}{N}$$

satisfies

$$\vec{v}(y)\cdot y \le -2^{1-N}r^{2-N}\left(r^N - 1\right), \quad \forall |y| = r, \tag{5.6}$$

for some constant $C = C(N)$.

Remark 5.1. *Since we shall apply Lemma 5.1 to $\mu = U^m(y,\tau)$, which does not conserve the mass whenever $m > 1$, then we must extend* [3, Lemma 3.1] *to the case $\|\mu\|_{L^1(\mathbb{R}^N)} \le \mu_0$ for some $\mu_0 > 0$. Fortunately, the proof of* [3, Lemma 3.1] *still works very well on this case. Compare to the linear case $\mu = U(x,t)$, which conserves the mass $\|u_0\|_{L^1(\mathbb{R}^N)}$ for all $t > 0$. That allows us to "frozen in time estimate of the velocity at the boundary".*

Proof of Lemma 5.1. The proof is similar to the one of [3, Lemma 3.1] with a different scale. For convenience, we give the proof here.
Let $y \in \mathbb{R}^N$ be such that $|y| = r$. Thanks to the fact

$$-\nabla(-\Delta)^{-1}\mathbf{1}_{B(0,r)}(y) = \frac{y}{N}$$

we can rewrite

$$\vec{v}(y) = \nabla(-\Delta)^{-1}\left[\mathbf{1}_{B(0,r)} - \mu\right](y).$$

Since $\mathrm{supp}(\mu) \subset \overline{B(0,r)}$ and $\|\mu\|_{L^\infty(\mathbb{R}^N)} \le 1$, then it is clear that

$$\mathbf{1}_{B(0,r)} - \mu \ge 0, \text{ and } \left\|\mathbf{1}_{B(0,r)} - \mu\right\|_{L^1(\mathbb{R}^N)} = \left|B(0,r)\right| - \omega_N\,.$$

Next, for every $z \in B(0,r)$, we find easily that

$$|y-z|^2 = |y|^2 - 2y\cdot z + |z|^2 \le 2|y|^2 - 2y\cdot z = 2(y-z)\cdot y\,.$$

Then,

$$\begin{aligned}
-\vec{v}(y)\cdot y &= \frac{1}{\omega_N}\int_{B(0,r)} \frac{(y-z)\cdot y}{|y-z|^N}\,(\mathbf{1}_{B(0,r)}-\mu)(z)\,dz \\
&\ge \frac{1}{2\omega_N}\int_{B(0,r)} |y-z|^{2-N}\,(\mathbf{1}_{B(0,r)}-\mu)(z)\,dz\,. \\
&\ge \frac{1}{2\omega_N}\int_{B(0,r)} (2r)^{2-N}\,(\mathbf{1}_{B(0,r)}-\mu)(z)\,dz \\
&= \frac{2^{1-N}r^{2-N}}{\omega_N}\left(\big|B(0,r)\big| - \|\mu\|_{L^1(B(0,r))}\right) \\
&\ge \frac{2^{1-N}r^{2-N}}{\omega_N}\left(\big|B(0,r)\big| - \omega_N\right) = 2^{1-N}r^{2-N}\left(r^N-1\right).
\end{aligned} \tag{5.7}$$

This yields the proof of Lemma 5.1. □

Now, we are ready to apply Lemma 5.1 to $\mu = U^m(y,\tau)$. Thus, it suffices to verify that $U^m(y,\tau)$ satisfies the conditions in this lemma. In fact, it follows from (5.4) and (5.5) that

$$\|U^m(\tau)\|_{L^\infty(\mathbb{R}^N)} \le \|U(\tau)\|^m_{L^\infty(\mathbb{R}^N)} \le 1, \tag{5.8}$$

and

$$\|U^m(\tau)\|_{L^1(\mathbb{R}^N)} = \|U(\tau)\|^m_{L^m(\mathbb{R}^N)} \le \|U(\tau)\|_{L^1(\mathbb{R}^N)}\|U(\tau)\|^{m-1}_{L^\infty(\mathbb{R}^N)} \le \omega_N \tag{5.9}$$

for all $\tau > 0$.
Next, we define $\Omega_\tau = \text{supp}(U(.,\tau))$ and $L(\tau) = \sup_{x\in\Omega_\tau}|x|$, and put $\vec{V}_1(y,\tau) = \nabla P(y,\tau) + \frac{y}{N}$.
From Eq. (5.3), we observe that $\vec{V}_1$ satisfies the characteristic equation:

$$\frac{d}{dt}\Phi_\tau(x) = \vec{V}_1(\Phi_\tau(x),\tau), \quad \Phi_\tau(x)_{|_{\tau=0}} = x\,.$$

Fix a time $\tau \ge 0$ and choose x_0 so that $\Phi_\tau(x_0) \in \Omega_\tau$, and $|\Phi_\tau(x_0)| = L(\tau)$.
Applying Lemma 5.1 to $\vec{V}_1$ yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}L^2(\tau) &= \Phi_\tau(x_0)\cdot\frac{d}{d\tau}\Phi_\tau(x_0) = \Phi_\tau(x_0)\cdot\vec{V}_1(\Phi_\tau(x_0),\tau) \\
&\le -2^{1-N}|\Phi_\tau(x_0)|^{2-N}\left(|\Phi_\tau(x_0)|^N - 1\right) = -2^{1-N}L^{2-N}(\tau)\left(L^N(\tau)-1\right).
\end{aligned}$$

Thus, we obtain

$$\frac{d}{dt}L^N(\tau) \le -N2^{1-N}\left(L^N(\tau)-1\right). \tag{5.10}$$

Thanks to (5.10) and Lemma 5.1, we can repeat the proof of [3, Lemma 3.2] for U, but with a different scale according to (5.8)-(5.9). Then, we obtain

$$\text{supp}(U(\tau)) \subset B(0,\tilde{r}(\tau)), \quad \forall\tau > 0,$$

where $\tilde{r}$ satisfies the ODE:

$$\frac{d}{d\tau}\tilde{r}^N(\tau) = -N2^{1-N}(\tilde{r}^N(\tau)-1), \quad \tilde{r}(0) = r_0\,. \tag{5.11}$$

Note that $\text{supp}(U(0)) = \text{supp}(u_0) \subset B(0,r_0)$.
By (5.10), going back to the original variable this translates into:

$$\frac{r^N(t)}{R^N(t)} - 1 \le \left(r_0^N - 1\right)(1+mt)^{\frac{-N2^{1-N}}{m}} = E(t)\,. \tag{5.12}$$

Now, it suffices to prove (1.14). Observe that $u(x,t) \le W(x,t)$ in $B(0,R(t))$, and $u(x,t) \ge W(x,t)$ outside $B(0,R(t))$. Using moreover the fact that both $W(x,t)$ and $u(x,t)$ have mass ω_N we easily find that

$$\begin{aligned}\left\|u(t) - W(t)\right\|_{L^1(\mathbb{R}^N)} &= \int_{|x|\le R(t)} [W(x,t) - u(x,t)]\,dx + \int_{|x|>R(t)} u(x,t)\,dx \\ &= 2\int_{|x|>R(t)} u(x,t)\,dx = 2\int_{R(t)<|x|<r(t)} u(x,t)\,dx\,. \end{aligned} \tag{5.13}$$

Note that

$$\int_{R(t)<|x|<r(t)} u(x,t)\,dx \le \int_{R(t)<|x|<r(t)} \frac{dx}{(1+mt)^{1/m}} = \omega_N\left(\frac{r^N(t)}{R^N(t)} - 1\right) = \omega_N E(t)\,. \tag{5.14}$$

The last equality follows from (5.12).
Combining (5.14) and (5.13) yields

$$\left\|u(t) - W(t)\right\|_{L^1(\mathbb{R}^N)} \le 2\omega_N E(t), \quad \text{for } t > 0\,.$$

Hence, we obtain (1.14) for $q = 1$.
For $q \in (1,\infty)$, it follows from the interpolation inequality and universal bound (1.13) that

$$\begin{aligned}\left\|u(t) - W(t)\right\|_{L^q(\mathbb{R}^N)} &\le \left\|u(t) - W(t)\right\|_{L^\infty(\mathbb{R}^N)}^{\frac{q-1}{q}} \left\|u(t) - W(t)\right\|_{L^1(\mathbb{R}^N)}^{\frac{1}{q}} \\ &\le C(q)(mt)^{-\frac{q-1}{qm}}(2\omega_N E(t))^{\frac{1}{q}} \\ &\le C(q,m,N)(r_0^N - 1)^{\frac{1}{q}} t^{-\frac{q-1+N2^{1-N}}{qm}}, \quad \text{for } t > 0. \end{aligned}$$

This completes the proof of Theorem 1.3. □

As a consequence, we have the following asymptotic of $U(y,\tau)$.

Corollary 5.1. *Let $U(y,\tau)$ be defined as in (5.2). Then, $U(y,\tau)$ converges to $U^*(y) := \mathbf{1}_{\{|y|\le 1\}}$ in L^q-norm, $1 \le q < \infty$ when $\tau \to \infty$. Specifically, we have*

$$\left\|U(\tau) - U_*\right\|_{L^q(\mathbb{R}^N)} \le C(N,q)\left(\frac{r_0^N - 1}{e^{N2^{1-N}\tau}}\right)^{1/q}, \quad \forall \tau > 0. \tag{5.15}$$

Proof of Corollary 5.1. Observe that $U(y,\tau) \le U^*(y)$ in $B(0,1)$ from (5.4), and $U(y,\tau) \ge U^*(y)$ outside $B(0,1)$. Moreover, since U^* and $U(.,\tau)$ have the same mass ω_N for all $\tau > 0$, then

$$\begin{aligned}\left\|U(\tau) - U_*\right\|_{L^1(\mathbb{R}^N)} &= \int_{|y|<1} [U_*(y) - U(y,\tau)]\,dy + \int_{|y|\ge 1} U(y,\tau)\,dy \\ &= 2\int_{|y|\ge 1} U(y,\tau)\,dy\,. \end{aligned}$$

With the last equation noted, and by the definition of U in (5.2), we observe that

$$\begin{aligned}\left\|U(\tau) - U_*\right\|_{L^1(\mathbb{R}^N)} &= 2\int_{|y|\ge 1} U(y,\tau)\,dy = 2R^N(t)\int_{|y|\ge 1} u(yR(t),t)\,dy \\ &= 2\int_{|x|\ge R(t)} u(x,t)\,dx = \int_{R(t)<|x|<r(t)} u(x,t)\,dx \\ &\le 2\omega_N E(t)\,. \end{aligned}$$

The last inequality follows from (5.14). Remind that $\tau = \ln(1+mt)^{1/m}$.
Then, we deduce that

$$\left\|U(\tau) - U_*\right\|_{L^1(\mathbb{R}^N)} \le 2\omega_N \frac{r_0^N - 1}{e^{n2^{1-N}\tau}}, \quad \forall \tau > 0\,.$$

Finally, the estimate for $\left\|U(\tau) - U_*\right\|_{L^q(\mathbb{R}^N)}$ can be done similarly as in the proof of $\left\|u(t) - W(t)\right\|_{L^q(\mathbb{R}^N)}$ above. Then, we leave its details to the reader. □

6. Estimates via symmetrization

In this part, we study the symmetrization of solutions to Eq (1.1). Here, we will prove that, even if the problem concerns an hyperbolic/elliptic system it is possible to compare, in a suitable sense, the solutions of the problem corresponding to an initial datum u_0 with supersolutions corresponding to a radially symmetric initial datum U_0 with equimesurable level sets with respect to u_0. This kind of symmetrization process is well-known in the context of parabolic and elliptic problems but, as far as we know, this one has not been developed for hyperbolic/elliptic systems. Our depart point is the paper [20] (see also [19]) concerning a related system arising in chemotaxis.

Now, let $T > 0$, and let $u : Q_T \to [0, \infty)$ be a measurable function. For $t \in [0, T]$, we set $u(t) : \mathbb{R}^N \to [0, \infty)$, $u(t)(x) = u(t, x)$. Then, we will write $u_*(t, s) = u(t)_*(s)$ for $t \in [0, T]$ and $s \in [0, \infty)$.
If $u, U \in L^\infty(0, T; L^1(\mathbb{R}^N))$ are nonnegative, then the *concentration mass comparison* $u(t) \prec U(t)$ can be equivalently formulated as

$$k(t, s) \leq K(t, s), \quad \text{for any } t \in [0, T],\, s \geq 0,$$

where

$$k(t, s) = \int_0^s u_*(t, \sigma)\, d\sigma, \qquad K(t, s) = \int_0^s U_*(t, \sigma)\, d\sigma\,.$$

For the proof of Theorem 1.4 we will start by proving some similar results for the case of Eq (3.1), but with U_ε solution of the problem

$$\begin{cases} \partial_t U = \varepsilon \Delta U + \operatorname{div}(U \nabla P) & \text{in } \mathbb{R}^N \times (0, T)\,, \\ -\Delta P = M_0^{m-1} U\,, & \\ U(x, 0) = U_0(x) & \text{in } \mathbb{R}^N\,. \end{cases} \tag{6.1}$$

Here, we remind that $M_0 = \|U_0\|_{L^\infty(\mathbb{R}^N)}$.

Theorem 6.1. *Let $m \geq 1$, $N \geq 2$. Let $u_0, U_0 \in L_c^\infty(\mathbb{R}^N)$ be nonnegative such that*

$$\|u_0\|_{L^\infty(\mathbb{R}^N)} \leq \|U_0\|_{L^\infty(\mathbb{R}^N)}\,, \qquad \|u_0\|_{L^1(\mathbb{R}^N)} \leq \|U_0\|_{L^1(\mathbb{R}^N)}\,. \tag{6.2}$$

Let u_ε (resp. U_ε) be the unique nonnegative bounded weak solution of Eq (3.1) (resp. Eq (6.1)). Then, for any $t \in [0, T]$, there holds

$$\int_0^\infty s^{\frac{2(N-1)}{N}} \left[k(t, s) - K(t, s)\right]_+^2 ds \leq e^{tC(\varepsilon)} \int_0^\infty s^{\frac{2(N-1)}{N}} \left[k_0(s) - K_0(s)\right]_+^2 ds, \tag{6.3}$$

with

$$C(\varepsilon) = M_0^m + \frac{\overline{C}}{\varepsilon}, \qquad \overline{C} = \frac{M_0^{2(m-1)}}{2N^2 \omega_N^{\frac{2}{N}}}\,.$$

Furthermore, if $u_0 \prec U_0$, then for any $t > 0$, we have $u_\varepsilon(t) \prec U_\varepsilon(t)$, and $p_\varepsilon(t) \prec P_\varepsilon(t)$ for all $\varepsilon > 0$.

Proof of Theorem 6.1. Let us put

$$\mu(t, \theta) = \operatorname{meas}\{x \in \Omega, u(t, x) > \theta\}\,.$$

The proof is just an adaptation and generalization of the proof of [20, Proposition 3.2] (see also [19, Theorem 2 and Proposition A.1]). At the beginning, we assume that initial data u_0 and U_0 are regular enough. Specifically, we can pick $u_0, U_0 \in C_c^1(\mathbb{R}^N)$. Then, we have

$$\int_0^{\mu(t,\theta)} \frac{\partial u(t,\sigma)_*}{\partial t} d\sigma = \int_{\{u(t)>\theta\}} \frac{\partial u(t,x)}{\partial t} dx, \text{ for a.e. } t \in (0,T).$$

Thus, it is possible to extend the computations in [19, Lemma 4] to the case of the spatial domain $\Omega = \mathbb{R}^N$. These computations lead, in case of u solution of problem (3.1), to the following auxiliary problem

$$\begin{cases} \frac{\partial k}{\partial t} - \varepsilon d(s)\frac{\partial^2 k}{\partial s^2} - \left(\int_0^{\mu(t,\theta)} (u(t,\sigma)_*)^m \, d\sigma\right) \frac{\partial k}{\partial s} \le 0 & a.e. \text{ in } (0,T)\times(0,+\infty), \\ k(t,0) = 0,\, k(t,+\infty) = \int_{\mathbb{R}^N} u_0 dx & t \in (0,T), \\ k(0,s) = k_0(s) = \int_0^s u_0(\sigma)_* d\sigma & s \ge 0, \end{cases}$$

with $k \in L^\infty((0,T)\times(0,+\infty)) \cap H^1(0,T : W^{1,q}_{loc}(0,+\infty)) \cap L^2(0,T : W^{2,q}_{loc}(0,+\infty))$, for some $q > N$, and $d(s) = N^2\omega_N^{\frac{2}{N}} s^{\frac{2(N-1)}{N}}$.
Since

$$0 \le [u(t,\sigma)_*]^m \le M_0^{m-1} u(t,\sigma)_* \quad \text{for a.e. } (t,\sigma) \in [0,T]\times(0,+\infty),$$

then we get as in [20, Proposition 3.1] that

$$\frac{\partial k}{\partial t} - \varepsilon d(s)\frac{\partial^2 k}{\partial s^2} - M_0^{m-1} k \frac{\partial k}{\partial s} \le 0 \quad \text{for a.e. } (t,\sigma) \in [0,T]\times(0,+\infty).$$

Next, let us set $z(t,s) = \int_0^s p(t,\sigma)_* \, d\sigma$. Then, it satisfies

$$\begin{cases} -d(s)\frac{\partial^2 z}{\partial s^2} \le k & \text{a.e. in } (0,T)\times(0,+\infty), \\ z(t,0) = 0,\ \lim_{s\to\infty} \frac{\partial z}{\partial s}(t,s) = 0, & t \in (0,T). \end{cases} \tag{6.4}$$

Repeating the proof as in [19, Lemma 6], we get

$$\begin{cases} \frac{\partial K}{\partial t} - \varepsilon d(s)\frac{\partial^2 K}{\partial s^2} - M_0^{m-1} K \frac{\partial K}{\partial s} = 0 & a.e. \text{ in } (0,T)\times(0,+\infty), \\ K(t,0) = 0,\, K(t,+\infty) = \displaystyle\int_{\mathbb{R}^N} U_0(x)\, dx & t \in (0,T), \\ K(0,s) = K_0(s) = \displaystyle\int_0^s U_0(\sigma)_* \, d\sigma & s \ge 0. \end{cases}$$

On the other hand, if we set $Z(t,s) = \int_0^s P(t,\sigma)_* \, d\sigma$, then we see that

$$\begin{cases} -d(s)\frac{\partial^2 Z}{\partial s^2} = K & a.e. \text{ in } (0,T)\times(0,+\infty), \\ Z(t,0) = 0,\ \lim_{s\to\infty} \frac{\partial Z}{\partial s}(t,s) = 0 & t \in (0,T). \end{cases} \tag{6.5}$$

Now, we modify the comparison result in [20, Proposition 3.2] (see also [19, Proposition A1]). By defining $w = k - K$, and by (6.2), we get

$$\begin{cases} \frac{\partial w}{\partial t} - \varepsilon d(s)\frac{\partial^2 w}{\partial s^2} - M_0^{m-1}(k\frac{\partial k}{\partial s} - K\frac{\partial K}{\partial s}) \le 0 & a.e. \text{ in } (0,T)\times(0,+\infty), \\ w(t,0) = 0,\, w(t,+\infty) \le 0 & t \in (0,T), \\ w(0,s) = k_0(s) - K_0(s) & s \ge 0. \end{cases}$$

Multiplying the differential inequality by $s^{2(N-1)/N}w_+$, integrating by parts over (δ, L), with $0 < \delta < 1 < L$, and using the fact from Theorem 3.1 that

$$\frac{\partial k}{\partial s} = u_* \leq \|u_0\|_{L^\infty(\mathbb{R}^N)} \leq M_0,$$

we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\delta^L s^{\frac{2(N-1)}{N}}|w_+|^2\,ds + \varepsilon N^2\omega_N^{\frac{2}{N}}\int_\delta^L \left|\frac{\partial w_+}{\partial s}\right|^2 ds \\
&\leq M_0^{m-1}\int_\delta^L s^{\frac{2(N-1)}{N}}\left\{|w_+|^2\frac{\partial k}{\partial s} + w_+\frac{\partial w_+}{\partial s}K\right\}ds + \varepsilon R(t,\delta,L) \\
&\leq M_0^m\int_\delta^L s^{\frac{2(N-1)}{N}}|w_+|^2\,ds + M_0^{m-1}\int_\delta^L s^{\frac{2(N-1)}{N}}w_+\frac{\partial w_+}{\partial s}K\,ds + \varepsilon R(t,\delta,L),
\end{aligned}$$

where

$$R(t,\delta,L) = C\left\{\left|\frac{\partial w}{\partial s}(t,\delta)\right|w_+(t,\delta) + \left|\frac{\partial w}{\partial s}(t,L)\right|w_+(t,L)\right\}$$

satisfies $R(t,\delta,L) \leq C'$ for some constant $C' > 0$, and

$$R(t,\delta,L) \to 0 \text{ as } \delta \to 0 \text{ and } L \to \infty\,.$$

Thanks to the Cauchy inequality, we have

$$\begin{aligned}
M_0^{m-1}\int_\delta^L s^{2(N-1)/N}w_+\frac{\partial w_+}{\partial s}K\,ds &\leq \frac{1}{2}\varepsilon N^2(\omega_N)^{2/N}\int_\delta^L(\frac{\partial w_+}{\partial s})^2\,ds \\
&\quad + \frac{\overline{C}}{\varepsilon}\int_\delta^L s^{2(N-1)/N}(w_+)^2\,ds,
\end{aligned}$$

with $\overline{C} = \frac{M_0^{2(m-1)}}{2N^2(\omega_N)^{2/N}}$. Then, for $t \in (0,T)$ one has

$$\frac{d}{dt}\int_\delta^L s^{\frac{2(N-1)}{N}}|w_+|^2\,ds \leq \left(M_0^m + \frac{\overline{C}}{\varepsilon}\right)\int_\delta^L s^{\frac{2(N-1)}{N}}|w_+|^2\,ds + \varepsilon R(t,\delta,L)\,.$$

Then, letting $\delta \to 0$ and $L \to \infty$, by Grönwall's inequality we conclude that

$$\int_0^\infty s^{\frac{2(N-1)}{N}}|w_+(t,s)|^2\,ds \leq e^{t\left(M_0^m + \frac{\overline{C}}{\varepsilon}\right)}\int_0^\infty s^{\frac{2(N-1)}{N}}|w_+(0,s)|^2\,ds\,,$$

which proves estimate (6.3).
Thus, if u_0 is less concentrated than U_0, then we find that $w_+(0,s) = 0$ on $(0,+\infty)$. It follows from the last inequality that for any $t > 0$ there holds

$$w_+(t,s) = 0 \quad \text{for } s \in (0,+\infty)\,.$$

Hence, we obtain

$$u_\varepsilon(t) \prec U_\varepsilon(t), \quad \text{for } t > 0\,. \tag{6.6}$$

Using (6.4) and (6.5), by the theory of rearrangement (see [40]), we deduce that

$$p_\varepsilon(t) \prec P_\varepsilon(t), \quad \text{for } t > 0\,. \tag{6.7}$$

Thus, we get the proof of Theorem 6.1 for regular initial data $u_0, U_0 \in C_c^1(\mathbb{R}^N)$.
Finally, if $u_0, U_0 \in L_c^\infty(\mathbb{R}^N)$, then by using the smoothing effect to initial data u_0, U_0, one can obtain (6.6) and (6.7).
It is well-known that due to a result by Hardy–Littlewood–Pólya (see, e.g., Talenti [40]) the above comparison in mass concentration implies L^q-estimate (1.19) for $q \in (1,\infty)$.
This puts an end to the proof of Theorem 6.1. □

Now, it suffices to give the proof of Theorem 1.4.

Proof of Theorem 1.4. At the beginning, we show that (6.6) and (6.7) hold for (u, U) and (p, P) respectively. Remind that u, U are the unique solutions of the two equations (1.1), (6.1) with densities u_0, U_0 respectively.
It suffices to show that (6.6) holds true for (u, U); and (6.7) is true for (p, P).
Thanks to the uniqueness result, and by the same argument as in the proof of Theorem 1.1, we observe that solution $u_\varepsilon(t)$ (resp. $U_\varepsilon(t)$) of Eq (3.1) (resp. Eq (6.1)) converges weakly to $u(t)$ (resp. $U(t)$) in $L^q(\mathbb{R}^N)$, $1 < q < \infty$ as $\varepsilon \to 0$ (up to a subsequence if necessary).
Next, $u_\varepsilon(t)$ (resp. $U_\varepsilon(t)$) is uniformly bounded in X with respect to ε for all $t > 0$. As a consequence, the restriction of $\{u_\varepsilon(t)\}_{\varepsilon>0}$ (resp. $\{U_\varepsilon(t)\}_{\varepsilon>0}$) to Ω is equi-integrable for any bounded set Ω in $\mathbb{R}^N$. Thanks to the Dunford–Pettis theorem, $\{u_\varepsilon(t)\}_{\varepsilon>0}$ (resp. $\{U_\varepsilon(t)\}_{\varepsilon>0}$) is a relatively compact subset in $L^1(\Omega)$ with the weak topology. Thus, for any ball $B(0, r) \subset \mathbb{R}^N$, we have

$$\begin{cases} \displaystyle\int_{B(0,r)} u_\varepsilon(x,t)\psi(x)\,dx \to \int_{B(0,r)} u(x,t)\psi(x)\,dx, \quad \forall\psi \in L^\infty(B(0,r))\,, \\ \displaystyle\int_{B(0,r)} U_\varepsilon(x,t)\psi(x)\,dx \to \int_{B(0,r)} U(x,t)\psi(x)\,dx, \quad \forall\psi \in L^\infty(B(0,r))\,, \end{cases}$$

as $\varepsilon \to 0$.
Therefore, it follows from (6.6) that $u(t) \prec U(t)$ for all $t > 0$.
By the same argument, we also obtain from (6.7) that $p(t) \prec P(t)$ for all $t > 0$. As before, the above comparison in mass concentration implies (1.19).
To finish the proof, it remains to prove (1.21). By (1.20), and by the conservation of mass, for any $t \in [0, T]$ we have

$$\int_0^{+\infty} u(t,\sigma)_*\,d\sigma = \int_0^{+\infty} u_0(\sigma)_*\,d\sigma = \int_0^{+\infty} U_0(\sigma)_*\,d\sigma = \int_0^{+\infty} U(t,\sigma)_*\,d\sigma\,.$$

Since $u(t) \prec U(t)$ for $t \in [0, T]$, then for any $s \in [0, \infty)$ we find that

$$\begin{aligned} \int_s^{+\infty} u(t,\sigma)_*\,d\sigma &= \int_0^{+\infty} u(t,\sigma)_*\,d\sigma - \int_0^{s} u(t,\sigma)_*\,d\sigma \\ &\geq \int_0^{+\infty} U(t,\sigma)_*\,d\sigma - \int_0^{s} U(t,\sigma)_*\,d\sigma = \int_s^{+\infty} U(t,\sigma)_*\,d\sigma\,. \end{aligned} \tag{6.8}$$

For any $t \in [0, T]$, let $[0, R_u(t)]$ and $[0, R_U(t)]$ be the supports of $u(t)_*$ and $U(t)_*$ respectively. Since $\displaystyle\int_{R_u(t)}^{+\infty} u(t,\sigma)_*\,d\sigma = 0$, then it follows from (6.8) that

$$\int_{R_u(t)}^{+\infty} U(t,\sigma)_*\,d\sigma = 0\,.$$

In addition, since $U(t,\sigma)_*$ is non-increasing and nonnegative, then we deduce that $R_u(t) \geq R_U(t)$. This yields the desired result.
Thus, we complete the proof of Theorem 1.4. □

Remark 6.1. *Arguing as in* [18]*, it seems possible to get alternative estimates to the one given in* (6.3) *[in L^2 with a weight] but now in $L^\infty(0, +\infty)$.*

Remark 6.2. *An iterative implicit Euler discretization method was used to get a result similar to Theorem 6.1 in references* [25, 15] *when $m = 1$. Note that this method does not give estimate* (6.3)*. Unfortunately, the proof of Lemma A.6 given in* [25] *is not complete (the proof of that for any regular vectorial function the operator is accretive requires some nontrivial additional arguments).*

7. Appendix

Proposition 7.1. *Let $s > 0$ be noninteger, and let Γ be a Lipschitz function on $\mathbb{R}$ such that $\Gamma(0) = 0$. Then, there exists a constant $C = C(\Gamma) > 0$ such that*

$$\|\Gamma(u)\|_{H^s(\mathbb{R}^N)} \leq C\left\|\Gamma'\right\|_{L^\infty(\mathbb{R}^N)}\|u\|_{H^s(\mathbb{R}^N)}$$

Proof of Proposition 7.1. From the condition $\Gamma(0) = 0$, we have that

$$\|\Gamma(u)\|_{L^2(\mathbb{R}^N)} \leq \|\Gamma'\|_{L^\infty(\mathbb{R})}\|u\|_{L^2(\mathbb{R}^N)}\,.$$

Moreover, it is known that $H^s(\mathbb{R}^N)$, $s > 0$ coincides with the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^N)$. Thus,

$$\begin{aligned}
\|\Gamma(u)\|^2_{H^s(\mathbb{R}^N)} &\approx \|\Gamma(u)\|^2_{W^{s,2}(\mathbb{R}^N)} = \|\Gamma(u)\|^2_{L^2(\mathbb{R}^N)} + \int\int \frac{|\Gamma(u)(x) - \Gamma(u)(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&\leq \|\Gamma'\|^2_{L^\infty(\mathbb{R})}\|u\|^2_{L^2(\mathbb{R}^N)} + \|\Gamma'\|^2_{L^\infty(\mathbb{R})}\int\int \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dxdy \\
&= \|\Gamma'\|^2_{L^\infty(\mathbb{R})}\|u\|^2_{W^{s,2}(\mathbb{R}^N)} \approx \|\Gamma'\|^2_{L^\infty(\mathbb{R})}\|u\|^2_{H^s(\mathbb{R}^N)}\,.
\end{aligned}$$

Combining the two indicated inequalities yields the desired result. □

Proposition 7.2 (Proof of (5.3))**.**

We have

$$\begin{aligned}
U_\tau(y,\tau) &= \frac{d}{d\tau}\left[e^\tau u\left(e^{\frac{\tau}{N}}y, \frac{e^{m\tau}-1}{m}\right)\right] = U(y,\tau) + e^\tau\left[\nabla_x u(x,t)\cdot\frac{y}{N}e^{\tau/N} + \partial_t u(x,t)e^{m\tau}\right] \\
&= U(y,\tau) + R^{N+1}(t)\nabla_x u(x,t)\cdot\frac{y}{N} + R^{(m+1)N}(t)\,\mathrm{div}_x(u\nabla p) \\
&= U + R^{N+1}(t)\nabla_x u(x,t)\cdot\frac{y}{N} + R^{(m+1)N}(t)\nabla_x u\cdot\nabla p(x,t) - [R^N(t)u(x,t)]^{m+1}.
\end{aligned}$$

And

$$\begin{aligned}
\mathrm{div}_y\left(U\left(\nabla P + \frac{y}{N}\right)\right) &= \mathrm{div}_y\,(U\nabla P) + \mathrm{div}_y\left(U\frac{y}{N}\right) \\
&= \mathrm{div}_y\,(U\nabla P) + \nabla_y U\cdot\frac{y}{N} + U \\
&= \nabla_y U\cdot\left(\nabla P + \frac{y}{N}\right) + U - U^{m+1} \\
&= R^{N+1}(t)\nabla_x u\cdot\left(-[R(t)]^{mN-1}\nabla p(x,t) + \frac{y}{N}\right) + U - U^{m+1} \\
&= -R^{(m+1)N}(t)\nabla_x u\cdot\nabla p(x,t) + R^{N+1}(t)\nabla_x u\cdot\frac{y}{N} + U - U^{m+1}.
\end{aligned}$$

Then, it is easy to see that

$$U_\tau(y,\tau) = \mathrm{div}_y\left(U\left(\nabla P + \frac{y}{N}\right)\right).$$

Acknowledgements. The research of N. A. Dao is supported by Vietnam Ministry of Education and Training under grant number B2024–KSA–02. The research of JID was partially supported by the project PID2023-146754NB-I00 funded by MCIU/AEI/10.13039/501100011033 and FEDER, EU. MCIU/AEI/10.13039/-501100011033/FEDER, EU and the Research Group MOMAT (Ref. 910480) of the UCM.
There is no conflict of interest, and the data availability is within the manuscript.

References

[1] L. Ambrosio, S. Serfaty, A gradient flow approach to an evolution problem arising in superconductivity. Comm. Pure Appl. Math. **61** (2008), no. 11, 1495-1539.

[2] L. Ambrosio, E. Mainini, S. Serfaty, Gradient flow of the Chapman–Rubinstein–Schatzman model for signed vortices. Annales IHP, Analyse non linéaire **28** (2011), no. 2, 217-246.

[3] A. L. Bertozzi, T. Laurent, F. Léger, Aggregation and spreading via the Newtonian potential the dynamics of patch solutions. Mathematical Models and Methods in Applied Sciences **22** (2012) 1140005.

[4] P. Bénilan, H. Brézis and M. Crandall, A semilinear equation in $L^1(\mathbb{R}^N)$. Ann. Scuola Norm. Sup. Pisa 4 (1975), 523–555.

[5] P. Biler, C. Imbert, G. Karch, The nonlocal porous medium equation: Barenblatt profiles and other weak solutions. Arch. Ration. Mech. Anal. **215** (2015), 497-529.

[6] M. Bonforte, A. Figalli, X. Ros-Otón, Infinite speed of propagation and regularity of solutions to the fractional porous medium equation in general domains, Comm. Pure Appl. Math. **70** (2017), 1472-1508.

[7] M. Bonforte, A. Figalli, J. L. Vázquez, Sharp global estimates for local and non-local porous medium-type equations in bounded domains. Anal. PDE **11** (2018), 945-982.

[8] M. Bonforte, Y. Sire, Y., J. L. Vázquez, Existence, uniqueness and asymptotic behaviour for fractional porous medium equations on bounded domains, Discrete Contin. Dyn. Syst. **35**(2015), 5725-5767.

[9] M. Bonforte, J. L. Vázquez, Quantitative local and global a priori estimates for fractional nonlinear diffusion equations. Adv. Math. **250** (2014), 242-284.

[10] L. A. Caffarelli, J. L. Vázquez, Nonlinear porous medium flow with fractional potential pressure. Arch. Rational Mech. Anal. **202** (2011), 537-565.

[11] L. A. Caffarelli, F. Soria, J. L. Vázquez, Regularity of solutions of the fractional porous medium flow. J. Eur. Math. Soc. **15** (2013), 1701-1746.

[12] L. A. Caffarelli, J. L. Vázquez, Regularity of solutions of the fractional porous medium flow With exponent $1/2$, St. Petersburg Mathematical Journal. **27** (2016), 437-460.

[13] J. A. Carrillo, D. Gómez-Castro, J. L. Vázquez, A fast regularisation of a Newtonian vortex equation, Ann. Inst. H. Poincaré Anal. Non Linéaire **39** (2022), 705-747.

[14] J. A. Carrillo, D. Gómez-Castro, J. L. Vázquez, Vortex formation for a non-local interaction model with Newtonian repulsion and superlinear mobility, Adv. Nonlinear. Anal. **11** (2022), 937-967.

[15] J. A. Carrillo, S. Hittmeir, B. Volzone, Y. Yao, Nonlinear Aggregation-Diffusion Equations: Radial Symmetry and Long Time Asymptotics. Inventiones Mathematicae **218** 3 (2019), 889-977.

[16] S. J. Chapman, J. Rubinstein, M. Schatzman, A mean-field model for superconducting vortices. Eur. J. Appl. Math. **7** (1996), 97-111.

[17] N. A. Dao, J. I. Díaz, Energy and large time estimates for nonlinear porous medium flow with nonlocal pressure in $\mathbb{R}^N$. Arch. Rat. Mech. Anal. **238** (2020), 299-345.

[18] J. I. Díaz, Symmetrization of nonlinear elliptic and parabolic problems and applications: a particular overview. In *Progress in partial differential equations.elliptic and parabolic problems (C.* Bandle, ed.), Pitman Research Notes in Mathematics No 266, Longman, Harlow, Essex, 1992, 1-16.

[19] J. I. Díaz, T. Nagai, Symmetrization in a parabolic-elliptic system related to chemotaxis. Advances in Mathematical Sciences and Applications, **5** 2 (1995), 659-680.

[20] J. I. Díaz, T. Nagai, J. M. Rakotoson, Symmetrization techniques on unbounded domains: application to a Chemotaxis system on $\mathbb{R}^N$. *Journal of Differential Equations*, **145** 1 (1998), 156-183.

[21] R. J. DiPerna, P.-L. Lions, Ordinary differential equations, transport theory and Sobolev spaces, Invent. Math. **98** (3) (1989), 511-547.

[22] J. Dolbeault, A. Zhang, Flows and functional inequalities for fractional operators. Applicable Analysis **96** (2017), 1547-1560.

[23] Q. Du, P. Zhang, Existence of weak solutions to some vortex density models. SIAM J. Math. Anal. **34** (2003), 1279-1299.

[24] T. Kato, G. Ponce, Commutator estimates and the Euler and Navier–Stokes equations. Comm. Pure Appl. Math. **41** (1988), 891-907.

[25] I. Kim, Y. Yao, The Patlak–Keller–Segel Model and Its Variations: Properties of Solutions via Maximum Principle. Siam Journal On Mathematical Analysis 44.2 (2012), pp. 568–602.

[26] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Ural'tseva, Linear and Quasilinear Equations of Parabolic Type. Transl. Math. Monographs **23**, Amer. Math. Soc., 1968.

[27] F. H. Lin, P. Zhang. On the hydrodynamic limit of Ginzburg–Landau vortices. Discrete Cont. Dyn. Syst. **6** (2000), 121-142.

[28] G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density. J. Math. Pures Appl. (9) 86 (2006), no. 1, 68-79.
[29] A. J. Majda, A. L. Bertozzi, Vorticity and Incompressible Flow. Cambridge texts in applied mathematics, 2002.
[30] N. Masmoudi, P. Zhang, Global solutions to vortex density equations arising from sup-conductivity, Ann. I. H. Poincaré- AN 22 (2005) 441-458.
[31] Q.-H. Nguyen, J. L. Vázquez, Porous medium equation with nonlocal pressure in a bounded domain. Communications in Partial Differential Equations 43 10 (2018), 1502-1539.
[32] J. Nieto, F. Poupaud, J. Soler, High-field limit for the Vlasov–Poisson–Fokker–Planck system, Arch. Rational Mech. Anal. **158** (2001), 29-59.
[33] A. Porretta. Existence Results for Nonlinear Parabolic Equations via Strong Convergence of Truncations. Annali di Matematica pura ed applicata. (IV), **CLXXVII** (1999), 143-172.
[34] Jean-Michel Rakotoson, *Réarrangement relatif. Un instrument d'estimations dans les problèmes aux limites,* SMAI-Springer, Paris, 2008.
[35] R. Robert, Unicité de la solution faible à support compact de l'equation de Vlasov Poisson, C. R. Acad. Sci. Sér. I **324** (1997), 873-877.
[36] S. Serfaty, J. L. Vázquez. A Mean Field Equation as Limit of Nonlinear Diffusion with Fractional Laplacian Operators. Calc. Var. PDEs. **49** (2014), 1091-1120.
[37] D. Stan, F. del Teso, J. L. Vázquez, Transformations of self-similar solutions for porous medium equations of fractional type, Nonlinear Analysis: Theory, Methods & Applications 119 (2015), 62-73.
[38] D. Stan, F. del Teso, J. L. Vázquez. Finite and infinite speed of propagation for porous medium equations with fractional pressure. Jour. Diff. Eqns. **260** (2016), 1154-1199.
[39] E. Stein. Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton, 1970.
[40] G. Talenti. The Art of Rearranging. Milan Journal of Mathematics **84** (2016), 105-157.
[41] M. E. Taylor. Partial Differential Equations. III: Nonlinear Equations, 2nd edn, xxii, p. 715. Applied Mathematical Sciences 117. Springer, New York, 2011.
[42] V. I. Yudovich. Nonstationary flow of an ideal incompressible liquid ZH. Vych. Math. **3** (1963), 1032-1066. (in Russian): USSR Comput. Math. Math. Phys. **3** (1963), 1407-1456.
[43] E. Weinan. Dynamics of vortex liquids in Ginzburg–Landau theories with applications to superconductivity. Phys. Rev. B **50** (1994), 1126-1135.
[44] P. Zhang, Y. Zheng. On the Existence and Uniqueness of Solutions to an Asymptotic Equation of a Variational Wave Equation. Acta Mathematica Sinica **15** (1999), 115-130.

N. A. Dao: School of Economic Mathematics and Statistics, University of Economics Ho Chi Minh City (UEH), Ho Chi Minh City, Viet Nam
Email address: anhdn@ueh.edu.vn

Díaz: Instituto de Matemática Interdisciplinar, Universidad Complutense de Madrid, 28040 Madrid Spain
Email address: jidiaz@ucm.es