Think globally, act locally: approximate controllability through homogenization of an optimal control problem with control on the boundary of certain particles

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Dedicated to the memory of Olga Arsenievna Oleinik (1925-2001) on the occasion of her centenary

Abstract. The phrase "Think globally, act locally" became prominent in the context of sustainable development and environmental activism, encouraging individuals and communities to address global challenges by taking local actions. In this paper, we offer a mathematical framework in which such metaphor can be understood in terms of the global homogenized formulation of some suitable control problem formulated on a domain Ω_ε, a part of a given domain Ω, which is exterior to a periodic distribution of many particles. We assume a linear heat equation on $\Omega_\varepsilon \times (0,T)$ and a Robin-type boundary condition on the boundary of the particles. We prove the "approximate controllability" of the problem, with a final observation, when the control is implemented only on the boundary of certain particles. Firstly, we apply the homogenization process, proving that the solution of the microscopic problem converges, as $\varepsilon \to 0$, to a function $u_0(x,t)$ that is the unique solution to a suitable global state parabolic problem. We consider a microscopic optimal control problem and prove the weak convergence of the state and the optimal control. Finally, we prove the approximate controllability by passing to the limit in a penalty parameter of the cost functional. This conclusion gives a certain mathematical justification for the popular phrase used by ecologists. Moreover, it brings to light some limitations that must be assumed on the local controls to conclude that the result is globally satisfactory.

Keywords: homogenization, perforated domain, critical case, optimal control, "strange" term, boundary control.

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1. Introduction

The phrase "Think globally, act locally" is widely attributed to the Scottish town planner and social activist Patrick Geddes (1854–1932). Geddes was a pioneer in urban planning and environmental thinking, emphasizing the interconnectedness of local and global systems. Geddes did not use the exact phrase but promoted ideas that align with its meaning, particularly in his advocacy for regional planning and considering broader ecological and social impacts while addressing local issues.

The exact wording "Think globally, act locally" is believed to have emerged later and popularized the environmental movement during the late 20th century, particularly in the 1970s and 1980s (see [8]). In this paper, we offer a mathematical justification of this sentence in the framework of Optimal Control Theory applied to heterogeneous media (a spatial domain Ω) including a set of particles with Robin boundary conditions satisfied by the state variable u and when the control operates on the particles' boundary where, as a metaphor, it emphasizes the interaction between global strategies and local actions in a complex system.

The conceptual relationship is understood here on the following basis: "Think globally" can be understood in Optimal Control Theory as a formulation involving a global objective, such as minimizing costs, maximizing efficiency, or achieving a desired state for the system. In a heterogeneous medium, the global objective is obtained through the macroscopic properties of the system, which in mathematics corresponds to the formulation of a homogenized Optimal Control problem. On the other hand, "act locally" can be understood as a formulation in which the controls are implemented on the boundary of some particles in specific subregions. In this way, local decisions (conditions imposed on the boundaries of certain particles) directly affect the global dynamics of the system due to the interdependence of variables.

These conditions exemplify how local effects can be integrated into the global framework of the system. In practice, they represent a balance between external and internal flows, which can be subject to external controls (the "acting locally"). In summary, the phrase connects to the idea that decisions made at the local level (on specific boundaries or particles) have significant implications for the global behavior of the system, and optimal control theory formalizes this interaction through the homogenized partial differential equations on Ω and the "microscopic" action on a certain amount of boundaries of the particles where the control (the action) is implemented.

Our aim is to give a precise mathematical formulation of this metaphor in a simple framework, also showing that some quantitative conditions need to be assumed. For instance, the "intensity" of the local controls should be suitably determined in terms of the scale and the spatial dimension: otherwise, the conclusion does not support the desired global effect.

Many mathematical formulations can be considered. Here, we merely study a simple optimal control problem corresponding to states satisfying a linear parabolic equation. The main formulation, we will consider, concerns an optimal control problem associated to a state function satisfying a linear heat equation in Ω_ε, the exterior of a bounded domain Ω of $\mathbb{R}^n$, $n \geq 3$, containing a set of ε-periodically distributed radially symmetric particles. We assume a homogeneous Dirichlet boundary condition on $\partial\Omega$ and, which is more important, a Robin-type condition on the boundary of each particle. Following an analogy with a global model proposed in climatology (the climate Energy Balance Model

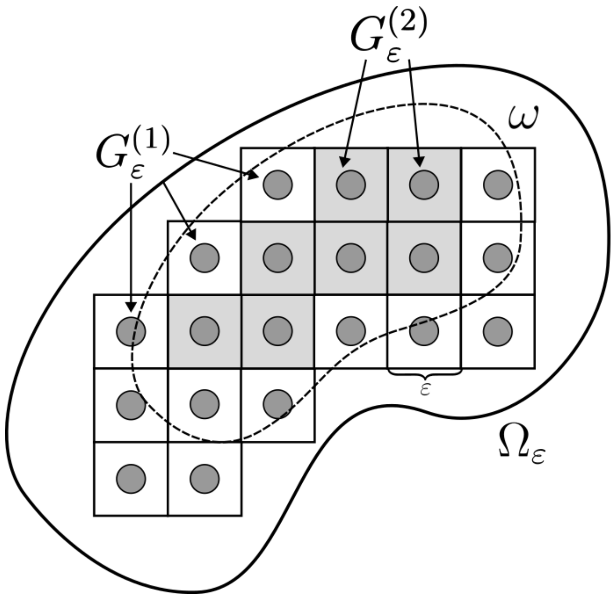

FIGURE 1. The control is implemented only on $S_\varepsilon^{(2)}$, the boundary of some of the internal balls: the ones collected under the notation $G_\varepsilon^{(2)}$.

(EBM), see [17], [3]), the states u_ε represent the averaged Earth surface temperature (here the Earth is represented by the domain Ω) and the EBM equation is represented by a linear heat equation. The "particles" represent the individuals of a population that interfere with their exterior (here represented by a Robin type boundary condition on the boundary of each particle). Since it is impossible to act over the entire spatial domain Ω_ε, it is assumed that only individuals in a small portion of the domain (ω such that $\overline{\omega} \subset \Omega$) can exercise some voluntary control. Thus, the set of boundaries of the internal particles is constituted in the form $S_\varepsilon = S_\varepsilon^{(1)} \bigcup S_\varepsilon^{(2)}$, with $S_\varepsilon^{(2)}$ the set of boundaries of the controlling particles and $S_\varepsilon^{(1)}$ the set of boundaries where no control is implemented. Although, a detailed description will be given in the next section we can anticipate that the state in this control problem will satisfy the following equation and auxiliary conditions

$$\begin{cases} \partial_t u_\varepsilon(v) - \Delta u_\varepsilon(v) = f(x,t), & (x,t) \in \Omega_\varepsilon \times (0,T) = Q_\varepsilon^T, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = 0, & (x,t) \in S_\varepsilon^{(1)} \times (0,T) = S_\varepsilon^{(1),T}, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = \varepsilon^{-\gamma} v(x,t), & (x,t) \in S_\varepsilon^{(2)} \times (0,T) = S_\varepsilon^{(2),T}, \\ u_\varepsilon(v) = 0, & (x,t) \in \partial\Omega \times (0,T) = \Gamma^T, \\ u_\varepsilon(v)(x,0) = 0, & x \in \Omega, \end{cases} \tag{1}$$

where $v \in L^2(S_\varepsilon^{(2),T})$ is the control, $f \in L^2(0,T;L^2(\Omega))$, $a \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 = const > 0$ are known data and ν is the unit outward normal vector to the related surfaces. Notice that by some obvious change of variable that we can also consider the case of a non-zero initial datum. Here we assume a "critical" relation between the problem's parameters (period of the structure ε, order of the particles size, α, and order of the coefficient γ), precisely, $\alpha = \gamma = \frac{n}{n-2}$, since otherwise the global problem we would obtain after the homogenization process (making $\varepsilon \searrow 0$) is less relevant (see, e.g., the exposition made in

[4]). This choice of parameters α and γ is characterized by the emergence of an "strange" term in the effective equations (see [24, 4, 1]).

Our main goal is to prove the "approximate controllability" of the problem with a final observation. This property can be stated in the following terms: given a "target global state", $u_T \in L^2(\Omega)$ (some extra regularity will be assumed in some intermediate steps), and given $\delta > 0$, we want to show the existence of a control $v \in L^2(S_\varepsilon^{(2),T})$ such that we have the estimate

$$\|u_\varepsilon(v)(\cdot,T) - u_T\|^2_{L^2(\Omega_\varepsilon)} \leq \delta. \tag{2}$$

In this way, local actions (the control is placed only on the boundary of some particles) lead to global consequences ($u_\varepsilon(v)(x,T)$ "almost" reach the desired value $u_T(x)$ in the whole domain Ω_ε): something that can be understood as a closed goal to the sentence "think globally, act locally".

Note that while there are several results in the literature on "approximate controllability" in domains with periodic particles (or perforations), our formulation differs from all of them as the controls are applied only on the boundary of certain particles (see, e.g., [2] and its references). Furthermore, the proof techniques employed in some other studies for simpler spatial domains either assume that the control is applied to the entire boundary of the domain (see, e.g., Section 3.1.3 of [12]) or they rely on auxiliary results (the unique continuation property), which is not available for the aforementioned formulation (see Remark 2.14 of [10]).

In order to avoid those difficulties, we will use the homogenization process as a tool. Moreover, to make constructive the proof of the existence of the wanted control, we will follow an idea of Jacques-Louis Lions which consists in the consideration of an auxiliary optimal control problem ([13], [10]). In our case, we first consider a time dependent target function $u_T \in H^1(0,T;H^1_0(\Omega)) \bigcap C(\overline{Q^T})$, and if we denote by $u_\varepsilon(v)$ the solution to the above parabolic problem, the optimal control problem, we will consider, is completed by taking the *cost functional* $J_\varepsilon : L^2(S_\varepsilon^{(2),T}) \to \mathbb{R}$

$$J_\varepsilon(v) = \frac{\theta_1}{2}\|\nabla(u_\varepsilon(v) - u_T)\|^2_{L^2(Q^T_\varepsilon)} + \frac{\theta_2}{2}\|(u(v) - u_T)(\cdot,T)\|^2_{L^2(\Omega_\varepsilon)} + \varepsilon^{-\gamma}\frac{N}{2}\|v\|^2_{L^2(S_\varepsilon^{(2),T})}. \tag{3}$$

Here, we assume the penalty term such that $N \in (0,+\infty)$, and $\theta_1, \theta_2 \geq 0$. Notice that when $\theta_1 = 0$ and $\theta_2 > 0$ the cost functional is well-defined for more general target functions $u_T \in L^2(\Omega)$. The case $\theta_1 > 0$ will considered here in order to extend to this framework some previous results in the literature (see, e.g., [22], [21], [5] and [20]). It is well known that there exists a unique optimal pair $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ (see [12]) with the optimal control $v_\varepsilon \in L^2(S_\varepsilon^{(2),T})$, i.e. satisfying

$$J_\varepsilon(v_\varepsilon) = \min_{v \in L^2(S_\varepsilon^{(2),T})} J_\varepsilon(v).$$

Our strategy will consists in several steps: firstly, we will apply the homogenization process, proving that the extension $\tilde{u}_\varepsilon$ to Q^T, converges, as $\varepsilon \to 0$, to a function $u_0(x,t)$

which is the unique solution of the global state problem

$$\begin{cases} \partial_t u_0 - \Delta u_0 + \mathcal{A}_n \left(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T} \right) u_0 = f + c(x) v_0(x,t)\chi_{\omega^T}, & (x,t) \in Q^T, \\ u_0(x,t) = 0, & (x,t) \in \Gamma^T, \\ u_0(x,0) = 0, & x \in \Omega, \end{cases} \tag{4}$$

where

$$b_1(x) = \frac{a(x)}{a(x) + \mathcal{B}_n}, \quad b_2(x) = \frac{a(x)(a(x) + \mathcal{B}_n)}{(a(x) + \mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n}, \tag{5}$$

and

$$c(x) = \frac{\mathcal{A}_n \mathcal{B}_n}{(a(x) + \mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n}, \tag{6}$$

with $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$, $\mathcal{B}_n = (n-2)C_0^{-1}$, ω_n the surface area of the unit sphere in $\mathbb{R}^n$, $n \geq 3$ and χ_A the characteristic function of the set A. Here, $v_0 \in L^2(\omega^T)$, $\omega^T = \omega \times (0,T)$, is the optimal control associated to a global cost functional $J_0 : L^2(\omega^T) \to \mathbb{R}$ such that $\lim\limits_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0)$, is defined in the following terms:

$$\begin{aligned} J_0(v) \; &= \; \frac{\theta_1}{2} \int\limits_{Q^T} |\nabla(u_0 - u_T)|^2 dxdt + \frac{\theta_2}{2} \int\limits_{\Omega} |(u_0 - u_T)(x,T)|^2 dx \\ &+ \frac{\theta_1 \mathcal{A}_n}{2} \int\limits_0^T \int\limits_{\Omega\setminus\overline{\omega}} b_1^2(x) u_0^2 dxdt + \frac{\theta_1 \mathcal{A}_n}{2} \int\limits_0^T \int\limits_{\omega} b_1(x) b_2(x) u_0^2 dxdt \\ &+ \frac{N}{2} \int\limits_{\omega^T} c(x) v^2 dxdt. \end{aligned} \tag{7}$$

Notice that coefficients $b_2(x)$ and $c(x)$ depend on θ_1 and that no dependence with respect θ_2 arises in the coefficients.

The second step of our strategy is to prove the approximate controllability of the homogenized parabolic problem with final observation: i.e., given the target global state, now $u_T \in L^2(\Omega)$, and given $\delta > 0$, we will show the existence of a control $v \in L^2(\omega\times(0,T))$ such that

$$\|u_0(v)(\cdot,T) - u_T\|^2_{L^2(\Omega)} \leq \delta. \tag{8}$$

We will construct such a control by taking $N \searrow 0$ in the global formulation of the optimal control problem for the case $\theta_1 = 0$ and $\theta_2 = 1$.

Finally, as a third step, we will get the approximate controllability on the starting problem, once we assume ε small enough. In conclusion, we must assume that the number of individual controls must be large enough (such as the ecologist philosophy proclaims). This conclusion gives a certain mathematical justification for the popular phrase used by ecologists. Moreover, it brings to light some limitations that must be assumed on the local controls to conclude that the result is globally satisfactory. For instance, the presence of the terms in $\varepsilon^{-\gamma}$ in the local formulation (in the local state boundary conditions and in the local cost functional) is of capital importance since it is not difficult to show that without them the global optimal control limit problem is entirely different. The critical relation between problems parameters leads to the emergence of some "strange" terms in the limit state problem along with the new term in the limit cost functional and here has an important consequence.

Our main technique of proof consists in characterizing the optimal control v_ε in terms of p_ε, the solution to the related adjoint problem. We will show that this relation is given by the expression $v_\varepsilon = -N^{-1}p_\varepsilon$, where N is the positive constant appearing in the local cost functional J_ε.

The organization of this paper is as follows: Section 2 is devoted to a more precise presentation of the local optimal control problem and to obtain some a priori estimates which will be used later. The detailed statement of the convergence theorem is given in Section 3 and its proof is organized in several subsections in Section 4. Section 5 is devoted to prove the convergence of the cost functionals. Finally, the approximate controllability property is stated and proved in Section 6.

2. Problem statement

Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geq 3)$ with a smooth boundary $\partial\Omega$. For $T > 0$, we use the notation: $Q^T = \Omega \times (0,T)$, $\Gamma^T = \partial\Omega \times (0,T)$. We denote by G_0 the ball of unit radius in $\mathbb{R}^n$ centered at the origin of coordinates. For a domain B and $\delta > 0$, we define set $\delta B = \{\delta^{-1}x \in B\}$. For $\varepsilon > 0$, we consider the domain

$$\widetilde{\Omega}_\varepsilon = \{x \in \Omega : \rho(x, \partial\Omega) > 2\varepsilon\},$$

where $\rho(x,y)$ is the Euclidean distance. We set

$$G_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon}(a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j\in\Upsilon_\varepsilon} G_\varepsilon^j,$$

where $\Upsilon_\varepsilon = \{j \in \mathbb{Z}^n : (a_\varepsilon G_0 + \varepsilon j)\bigcap \widetilde{\Omega_\varepsilon} \neq \emptyset\}$, $|\Upsilon_\varepsilon| \cong d\varepsilon^{-n}$, $d = const > 0$ and where $\mathbb{Z}^n$ is the set of vectors in $\mathbb{R}^n$ with the integer coordinates. Define $Y_\varepsilon^j = \varepsilon Y + \varepsilon j$, $P_\varepsilon^j = \varepsilon j$, where $Y = (-1/2, 1/2)^n$. Note that $\overline{G_\varepsilon^j} \subset Y_\varepsilon^j$ and the center of the ball $G_\varepsilon^j = a_\varepsilon G_0 + \varepsilon j$ coincides with the center of the cube Y_ε^j. We assume that $a_\varepsilon = C_0\varepsilon^\alpha$, $C_0 = const > 0, \alpha > 1$.

We consider a controllable region in the domain Ω given by some domain ω such that $\overline{\omega} \subset \Omega$. We assume the control to be placed only on balls with indexes associated with ω, i.e. we introduce the set $\Upsilon_\varepsilon^{(2)} = \{j \in \Upsilon_\varepsilon : \overline{Y_\varepsilon^j} \subset \omega\}$, $\Upsilon_\varepsilon^{(1)} = \Upsilon_\varepsilon \setminus \Upsilon_\varepsilon^{(2)}$. Based on this indexing, we define sets related to the particles and their boundaries

$$G_\varepsilon^{(1)} = \bigcup_{j\in\Upsilon_\varepsilon^{(1)}} G_\varepsilon^j,\ G_\varepsilon^{(2)} = \bigcup_{j\in\Upsilon_\varepsilon^{(2)}} G_\varepsilon^j,$$
$$S_\varepsilon^{(1)} = \partial G_\varepsilon^{(1)},\ S_\varepsilon^{(2)} = \partial G_\varepsilon^{(2)}.$$

Now, we define the set of particles

$$\Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ Q_\varepsilon^T = \Omega_\varepsilon \times (0,T),\ \omega^T = \omega \times (0,T),$$

with the boundaries

$$\partial\Omega_\varepsilon = \partial\Omega \cup S_\varepsilon,\ S_\varepsilon = S_\varepsilon^{(1)} \cup S_\varepsilon^{(2)},\ \Gamma^T = \partial\Omega \times (0,T),$$

$$S_\varepsilon^T = S_\varepsilon \times (0,T),\ S_\varepsilon^{(i),T} = S_\varepsilon^{(i)} \times (0,T),\ i = 1,2.$$

Let $v \in L^2(0,T;L^2(S_\varepsilon^{(2)}))$. By $u_\varepsilon(v)$ we denote an element of $L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ with $\partial_t u_\varepsilon(v) \in L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))$ that is a solution to the boundary value problem

$$\begin{cases} \partial_t u_\varepsilon(v) - \Delta u_\varepsilon(v) = f(x,t), & (x,t) \in Q_\varepsilon^T, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = 0, & (x,t) \in S_\varepsilon^{(1),T}, \\ \partial_\nu u_\varepsilon(v) + \varepsilon^{-\gamma} a(x) u_\varepsilon(v) = \varepsilon^{-\gamma} v(x,t), & (x,t) \in S_\varepsilon^{(2),T}, \\ u_\varepsilon(v) = 0, & (x,t) \in \Gamma^T, \\ u_\varepsilon(v)(x,0) = 0, & x \in \Omega_\varepsilon, \end{cases} \tag{9}$$

where $f \in L^2(Q^T)$, $a \in C^\infty(\overline{\Omega})$, $a(x) \geq a_0 = \mathit{const} > 0$, ν is the unit outward normal vector to the related surfaces. Here, by $H^1(\Omega_\varepsilon,\partial\Omega)$, we denote the closure with respect to the norm $H^1(\Omega_\varepsilon)$ of the set of infinitely differentiable in $\overline{\Omega}_\varepsilon$ functions vanishing near the boundary $\partial\Omega$. As said before, we assume a "critical" relation between the problem parameters (period of the structure ε, order of the particles size, α, and the order of the coefficient, γ), precisely, $\alpha = \gamma = \frac{n}{n-2}$.

We say that a function $u_\varepsilon(v) \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ with $\partial_t u_\varepsilon(v) \in L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))$ and $u_\varepsilon(x,0) = 0$, is a weak solution to the problem (9) if it satisfies the integral identity

$$\begin{gathered} \int_0^T \langle \partial_t u_\varepsilon(v), \varphi \rangle_{\Omega_\varepsilon} dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon(v) \nabla \varphi dx dt \\ + \varepsilon^{-\gamma} \int_{S_\varepsilon^{(1),T}} a(x) u_\varepsilon(v) \varphi ds dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^{(2),T}} a(x) u_\varepsilon(v) \phi ds dt \\ = \int_{Q_\varepsilon^T} f \varphi dx dt + \varepsilon^{-\gamma} \int_{S_\varepsilon^{(2),T}} v \varphi ds dt, \end{gathered} \tag{10}$$

for an arbitrary function $\varphi \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$. By $\langle\cdot,\cdot\rangle_{\Omega_\varepsilon}$, we denote the duality pairing between $H^1(\Omega_\varepsilon,\partial\Omega)$ and $H^{-1}(\Omega_\varepsilon,\partial\Omega)$.

We consider, at this step, a time dependent target function $u_T \in H^1(0,T;H_0^1(\Omega)) \cap C(\overline{Q^T})$, and if we denote by $u_\varepsilon(v)$ the solution to the above parabolic problem, the optimal control problem we will consider is completed by giving the general cost functional J_ε :

$L^2(0,T;L^2(S_\varepsilon^{(2)})) \to \mathbb{R}$

$$J_\varepsilon(v) = \frac{\theta_1}{2}\|\nabla(u_\varepsilon(v)-u_T)\|^2_{L^2(Q_\varepsilon^T)} + \frac{\theta_2}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon(v)(x,T)-u_T(x,T))^2dx$$
$$+\varepsilon^{-\gamma}\frac{N}{2}\|v\|^2_{L^2(S_\varepsilon^{(2),T})}. \tag{11}$$

Here, we assume the penalty coefficient such that $N \in (0,+\infty)$ and $\theta_1,\, \theta_2 \geq 0$. It is well known that there exists a unique optimal pair $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ (see. [12]), such that

$$J_\varepsilon(v_\varepsilon) = \min_{v\in L^2(S_\varepsilon^{(2),T})} J_\varepsilon(v). \tag{12}$$

One of the goals of this paper is to find the limit as $\varepsilon \to 0$ of the optimal control v_ε and of the cost functional $J_\varepsilon(v_\varepsilon)$.

2.1. **Characterization of the optimal control.** We define the adjoint problem, associated with the state problem (9), in the following terms:

$$\begin{cases} -\partial_t p_\varepsilon - \Delta p_\varepsilon = -\theta_1\Delta(u_\varepsilon - u_T), & (x,t)\in Q_\varepsilon^T,\\ \partial_\nu p_\varepsilon + \varepsilon^{-\gamma}a(x)p_\varepsilon = \theta_1\partial_\nu(u_\varepsilon-u_T), & (x,t)\in S_\varepsilon^{(1),T}\bigcup S_\varepsilon^{(2),T},\\ p_\varepsilon = 0, & (x,t)\in\Gamma^T,\\ p_\varepsilon(x,T) = \theta_2(u_\varepsilon(x,T)-u_T(x,T)), & x\in\Omega_\varepsilon. \end{cases} \tag{13}$$

We say that a function $p_\varepsilon \in L^2(0,T : H^1(\Omega_\varepsilon,\partial\Omega))$ is a weak solution to the problem (13) if $\partial_t p_\varepsilon \in L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))$, $p_\varepsilon(x,T) = \theta_2(u_\varepsilon(x,T)-u_T(x,T))$, and for an arbitrary $\phi \in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$, it satisfies the integral identity

$$-\int\limits_0^T\langle\partial_t p_\varepsilon,\phi\rangle_{\Omega_\varepsilon}dt + \int\limits_{Q_\varepsilon^T}\nabla p_\varepsilon\nabla\phi dxdt + \varepsilon^{-\gamma}\int\limits_{S_\varepsilon^{(1),T}} a(x)p_\varepsilon\phi dsdt$$
$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^{(2),T}} a(x)p_\varepsilon\phi dsdt = \theta_1\int\limits_{Q_\varepsilon^T}\nabla(u_\varepsilon-u_T)\nabla\phi dxdt. \tag{14}$$

Given u_ε and u_T as before, it is well-know the existence and uniqueness of a *weak solution* p_ε of (13).

Theorem 1. *If the pair of functions* $(u_\varepsilon, v_\varepsilon)$ *is optimal for the problem* (9), (11), (12), *then* $v_\varepsilon = -N^{-1}p_\varepsilon\chi_{S_\varepsilon^{(2),T}}$, *where* p_ε *is the weak solution to the problem* (13) *and with* $\chi_{S_\varepsilon^{(2),T}}$ *the characteristic function of the set* $S_\varepsilon^{(2),T}$.

Proof. Let $v \in L^2(0,T;L^2(S_\varepsilon^{(2)}))$ arbitrary. For $\lambda > 0$, we denote $v_\varepsilon^\lambda = v_\varepsilon + \lambda v$. We have

$$J_\varepsilon(v_\varepsilon^\lambda) - J_\varepsilon(v_\varepsilon)$$
$$= \frac{\theta_1}{2}(\|\nabla(u_\varepsilon(v_\varepsilon^\lambda)-u_T)\|^2_{L^2(Q_\varepsilon^T)} - \|\nabla(u_\varepsilon(v_\varepsilon)-u_T)\|^2_{L^2(Q_\varepsilon^T)})$$
$$+\frac{\theta_2}{2}(\|u_\varepsilon(v_\varepsilon^\lambda)(x,T)-u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)} - \|u_\varepsilon(v_\varepsilon)(x,T)-u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)})$$

$$+\varepsilon^{-\gamma}\frac{N}{2}\int\limits_{S_\varepsilon^{(2),T}}((v_\varepsilon^\lambda)^2-v_\varepsilon^2)dsdt$$

$$=\frac{\theta_1}{2}\int\limits_{Q_\varepsilon^T}\nabla(u_\varepsilon(v_\varepsilon^\lambda)-u_\varepsilon(v_\varepsilon))\nabla(u_\varepsilon(v_\varepsilon^\lambda)+u_\varepsilon(v_\varepsilon)-2u_T)$$

$$+\frac{\theta_2}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon(v_\varepsilon^\lambda)-u_\varepsilon(v_\varepsilon))(x,T)(u_\varepsilon(v_\varepsilon^\lambda)+u_\varepsilon(v_\varepsilon)-2u_T)(x,T)dx$$

$$+\varepsilon^{-\gamma}\frac{N}{2}\int\limits_{S_\varepsilon^{(2),T}}(2\lambda v_\varepsilon v+\lambda^2v^2)dsdt.$$

From here, we derive

$$\lim_{\lambda\to 0}\frac{J_\varepsilon(v_\varepsilon+\lambda v)-J_\varepsilon(v_\varepsilon)}{\lambda}=\theta_1\int\limits_{Q_\varepsilon^T}\nabla\theta_\varepsilon\nabla(u_\varepsilon(v_\varepsilon)-u_T)dxdt$$

$$+\theta_2\int\limits_{\Omega_\varepsilon}\theta_\varepsilon(x,T)(u_\varepsilon(v_\varepsilon)(x,T)-u_T(x,T))dx+\varepsilon^{-\gamma}N\int\limits_{S_\varepsilon^{(2),T}}v_\varepsilon v dsdt, \tag{15}$$

where $\theta_\varepsilon=(u_\varepsilon(v_\varepsilon+\lambda v)-u_\varepsilon(v_\varepsilon))\lambda^{-1}$ is the unique solution of the problem

$$\begin{cases}\partial_t\theta_\varepsilon-\Delta\theta_\varepsilon=0, & (x,t)\in Q_\varepsilon^T,\\ \partial_\nu\theta_\varepsilon+\varepsilon^{-\gamma}a(x)\theta_\varepsilon=0, & (x,t)\in S_\varepsilon^{(1),T},\\ \partial_\nu\theta_\varepsilon+\varepsilon^{-\gamma}a(x)\theta_\varepsilon=\varepsilon^{-\gamma}v, & (x,t)\in S_\varepsilon^{(2),T},\\ \theta_\varepsilon=0, & (x,t)\in\Gamma^T,\\ \theta_\varepsilon(x,0)=0, & x\in\Omega_\varepsilon.\end{cases}$$

Note that θ_ε is independent of λ. We say that a function $\theta_\varepsilon\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ with $\partial_t\theta_\varepsilon\in L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))$ and $\theta_\varepsilon(x,0)=0$ is a weak solution of this problem, if the following integral identity holds

$$\int\limits_0^T\langle\partial_t\theta_\varepsilon,\psi\rangle_{\Omega_\varepsilon}dt+\int\limits_{Q_\varepsilon^T}\nabla\theta_\varepsilon\nabla\psi dxdt+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^T}a(x)\theta_\varepsilon\psi dsdt=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^{(2),T}}v\psi dsdt,$$

where $\psi\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ is arbitrary. Taking as a test-function θ_ε in (14), and p_ε as a test-function in the integral identity for θ_ε, we subtract one from the other and get

$$\theta_2\int\limits_{\Omega_\varepsilon}\theta_\varepsilon(x,T)(u_\varepsilon(x,T)-u_T(x,T))dx+\theta_1\int\limits_{Q_\varepsilon^T}\nabla\theta_\varepsilon\nabla(u_\varepsilon-u_T)dx=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^{(2),T}}vp_\varepsilon dsdt.$$

Substituting this expression into (15), we conclude

$$J'_\varepsilon(v_\varepsilon)\cdot v=\varepsilon^{-\gamma}\int\limits_{S_\varepsilon^{(2),T}}vp_\varepsilon dsdt+\varepsilon^{-\gamma}N\int\limits_{S_\varepsilon^{(2),T}}vv_\varepsilon dsdt.$$

As v_ε is the optimal control, then we must have $J'_\varepsilon(v_\varepsilon)\cdot v=0$ for all $v\in L^2(0,T;L^2(S_\varepsilon^{(2)}))$. This implies that $v_\varepsilon=-N^{-1}p_\varepsilon$ for a.e. $(x,t)\in S_\varepsilon^{(2),T}$. ∎

Thus, the optimal control is characterized by the system of equations

$$\begin{cases} \partial_t u_\varepsilon - \Delta u_\varepsilon = f, & (x,t) \in Q_\varepsilon^T, \\ -\partial_t p_\varepsilon - \Delta p_\varepsilon = -\theta_1 \Delta(u_\varepsilon - u_T), & (x,t) \in Q_\varepsilon^T, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} a(x) u_\varepsilon = 0, & (x,t) \in S_\varepsilon^{(1),T}, \\ \partial_\nu u_\varepsilon + \varepsilon^{-\gamma} a(x) u_\varepsilon = -N^{-1}\varepsilon^{-\gamma} p_\varepsilon, & (x,t) \in S_\varepsilon^{(2),T}, \\ \partial_\nu p_\varepsilon + \varepsilon^{-\gamma} a(x) p_\varepsilon = \theta_1 \partial_\nu (u_\varepsilon - u_T), & (x,t) \in S_\varepsilon^{(1),T} \bigcup S_\varepsilon^{(2),T}, \\ u_\varepsilon = p_\varepsilon = 0, & (x,t) \in \Gamma^T, \\ u_\varepsilon(x,0) = 0, & x \in \Omega_\varepsilon, \\ p_\varepsilon(x,T) = \theta_2(u_\varepsilon(x,T) - u_T(x,T)), & x \in \Omega_\varepsilon. \end{cases} \tag{16}$$

Remark 1. *We also have the reverse to Theorem's 1 statement. If the pair* $(u_\varepsilon, p_\varepsilon)$ *is the solution to* (16), *then* $v_\varepsilon = -N^{-1} p_\varepsilon \chi_{S_\varepsilon^{(2),T}}$ *is the optimal control for the problem* (12).

2.2. **Uniform in ε estimates for u_ε and v_ε.** From the integral identities for the problem (16), for functions p_ε and u_ε, we have

$$\int_0^T \partial_t \langle u_\varepsilon, p_\varepsilon \rangle_{\Omega_\varepsilon} dt = \int_{Q_\varepsilon^T} f p_\varepsilon dx dt - N^{-1}\varepsilon^{-\gamma} \int_{S_\varepsilon^{(2),T}} p_\varepsilon^2 ds dt - \theta_1 \int_{Q_\varepsilon^T} \nabla(u_\varepsilon - u_T) \nabla u_\varepsilon dx dt. \tag{17}$$

From here, we deduce

$$\theta_1 \int_{Q_\varepsilon^T} |\nabla(u_\varepsilon - u_T)|^2 dx dt + \varepsilon^{-\gamma} N^{-1} \int_{S_\varepsilon^{(2),T}} p_\varepsilon^2 ds dt + \theta_2 \|u_\varepsilon(x,T) - u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)}$$

$$= \int_{Q_\varepsilon^T} f p_\varepsilon dx dt - \theta_2 \int_{\Omega_\varepsilon} u_T(x,T)(u_\varepsilon(x,T) - u_T(x,T)) dx - \theta_1 \int_{Q_\varepsilon^T} \nabla u_T \nabla(u_\varepsilon - u_T) dx dt. \tag{18}$$

Hence, we conclude

$$\theta_1 \|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q_\varepsilon^T)} + \theta_2 \|u_\varepsilon(x,T) - u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)} + N^{-1}\varepsilon^{-\gamma} \int_{S_\varepsilon^{(2),T}} p_\varepsilon^2 ds dt$$

$$\le \int_{Q_\varepsilon^T} |f||p_\varepsilon| dx dt + C\|\nabla u_T\|^2_{L^2(Q^T)} + C\|u_T(x,T)\|^2_{L^2(\Omega)}, \tag{19}$$

where constant C is independent of ε.

Taking in (14) as a test function p_ε, we get

$$-\frac{\theta_2^2}{2} \|u_\varepsilon(x,T) - u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma} \int_{S_\varepsilon^T} a(x) p_\varepsilon^2 ds dt$$

$$\le \theta_1 \int_{Q_\varepsilon^T} \nabla(u_\varepsilon - u_T) \nabla p_\varepsilon dx dt.$$

From here, we have

$$\|\nabla p_\varepsilon\|^2_{L^2(Q_\varepsilon^T)} + \varepsilon^{-\gamma} \|p_\varepsilon\|^2_{L^2(S_\varepsilon^T)}$$

$$\le C(\theta_1^2 \|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q_\varepsilon^T)} + \theta_2^2 \|u_\varepsilon(x,T) - u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)}). \tag{20}$$

From (19), (20) we derive

$$\theta_1\|\nabla(u_\varepsilon-u_T)\|^2_{L^2(Q^T_\varepsilon)}+\theta_2\|u_\varepsilon(x,T)-u_T(x,T)\|^2_{L^2(\Omega_\varepsilon)}+N^{-1}\varepsilon^{-\gamma}\|p_\varepsilon\|^2_{L^2(S^{(2),T}_\varepsilon)}$$
$$\le C(\|f\|^2_{L^2(Q^T)}+\|\nabla u_T\|^2_{L^2(Q^T)}+\|u_T(x,T)\|^2_{L^2(\Omega)}). \tag{21}$$

Estimates (20), (21) imply

$$\|\nabla p_\varepsilon\|^2_{L^2(Q^T_\varepsilon)}\le C(\|f\|^2_{L^2(Q^T)}+\|\nabla u_T\|^2_{L^2(Q^T)}+\|u_T(x,T)\|^2_{L^2(\Omega)}). \tag{22}$$

Next, we will get some estimates for the time derivatives of u_ε and p_ε

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))}\le C,\quad \|\partial_t p_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))}\le C.$$

Consider the Galerkin's approximations of u_ε and p_ε

$$u^m_\varepsilon=\sum_{k=1}^m a^\varepsilon_{k,m}(t)w^k_\varepsilon(x),\quad p^m_\varepsilon=\sum_{k=1}^m b^\varepsilon_{k,m}(t)w^k_\varepsilon(x),$$

where $\{w^k_\varepsilon(x)\}$ is an orthogonal basis in $H^1(\Omega_\varepsilon,\partial\Omega)$ and an orthonormal basis in $L^2(\Omega_\varepsilon)$. Let $v\in L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))$ arbitrary but such that $\|v\|_{H^1(\Omega_\varepsilon,\partial\Omega)}\le 1$ for a.e. $t\in(0,T)$. Substituting v into the equation for u^m_ε, we have for a.e. $t\in(0,T)$

$$\langle\partial_t u^m_\varepsilon,v\rangle_{\Omega_\varepsilon}=\langle\partial_t u^m_\varepsilon,v^\varepsilon_{1,m}\rangle_{\Omega_\varepsilon}=-\int\limits_{\Omega_\varepsilon}\nabla u^m_\varepsilon\nabla v^\varepsilon_{1,m}dx-\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}a(x)u^m_\varepsilon v^\varepsilon_{1,m}ds$$
$$+\int\limits_{\Omega_\varepsilon}fv^\varepsilon_{1,m}dx-\varepsilon^{-\gamma}N^{-1}\int\limits_{S^{(2)}_\varepsilon}p_\varepsilon v^\varepsilon_{1,m}ds, \tag{23}$$

and substituting it into the equation for p^m_ε, we get

$$\langle\partial_t p^m_\varepsilon,v\rangle_{\Omega_\varepsilon}=\langle\partial_t p^m_\varepsilon,v^\varepsilon_{1,m}\rangle_{\Omega_\varepsilon}=\int\limits_{\Omega_\varepsilon}\nabla p^m_\varepsilon\nabla v^\varepsilon_{1,m}dx$$
$$+\varepsilon^{-\gamma}\int\limits_{S_\varepsilon}a(x)p^m_\varepsilon v^\varepsilon_{1,m}ds-\theta_1\int\limits_{\Omega_\varepsilon}\nabla(u^m_\varepsilon-u_T)\nabla v^\varepsilon_{1,m}dx, \tag{24}$$

where $v=\sum\limits_{k=1}^m g^\varepsilon_{k,m}(t)w^k_\varepsilon(x)+v^\varepsilon_{2,m}\equiv v^\varepsilon_{1,m}+v^\varepsilon_{2,m}$, $(v^\varepsilon_{2,m},w^k_\varepsilon)_{L^2(\Omega_\varepsilon)}=0$, $k=1,\dots,m$. As the functions $\{w^k_\varepsilon\}$ form an orthogonal basis in $H^1(\Omega_\varepsilon,\partial\Omega)$, we have the estimate $\|v^\varepsilon_{1,m}\|_{H^1(\Omega_\varepsilon,\partial\Omega)}\le\|v\|_{H^1(\Omega_\varepsilon,\partial\Omega)}\le 1$.

Note that for u^m_ε and p^m_ε, we have the same estimates as in (21), (22) for u_ε and p_ε. Using these estimations, we derive

$$|\langle\partial_t u^m_\varepsilon,v\rangle_{\Omega_\varepsilon}|\le\|\nabla u^m_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-\gamma}\|u^m_\varepsilon\|_{L^2(S_\varepsilon)}\|v^\varepsilon_{1,m}\|_{L^2(S_\varepsilon)}$$
$$+C\|f\|_{L^2(\Omega_\varepsilon)}+\varepsilon^{-\gamma}N^{-1}\|p_\varepsilon\|_{L^2(S^{(2)}_\varepsilon)}\|v^\varepsilon_{1,m}\|_{L^2(S^{(2)}_\varepsilon)}, \tag{25}$$

and

$$|\langle\partial_t p^m_\varepsilon,v\rangle_{\Omega_\varepsilon}|\le\|\nabla p^m_\varepsilon\|_{L^2(\Omega_\varepsilon)}+C\varepsilon^{-\gamma}\|p^m_\varepsilon\|_{L^2(S_\varepsilon)}\|v^\varepsilon_{1,m}\|_{L^2(S_\varepsilon)}+\theta_1\|\nabla(u^m_\varepsilon-u_T)\|_{L^2(\Omega_\varepsilon)}. \tag{26}$$

From Lemma 2 in [18] and [19] for any function $\phi \in H^1(\Omega_\varepsilon, \partial\Omega)$ and for $\gamma = \frac{n}{n-2}$, we have the estimate

$$\varepsilon^{-\gamma}\|\phi\|^2_{L^2(S_\varepsilon)} \leq K\|\phi\|^2_{H^1(\Omega_\varepsilon,\partial\Omega)}.$$

Applying this estimate in (25), (26), we get

$$\|\partial_t u^m_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))} \leq C, \quad \|\partial_t p^m_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))} \leq C, \tag{27}$$

for a constant C which does not depend on m and ε.

From (27), we have

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))} \leq C, \quad \|\partial_t p_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))} \leq C. \tag{28}$$

Additionally, we conclude

$$\max_{[0,T]}\|u_\varepsilon(x,t)\|_{L^2(\Omega_\varepsilon)} \leq C, \quad \max_{[0,T]}\|p_\varepsilon(x,t)\|_{L^2(\Omega_\varepsilon)} \leq C.$$

Then, if we denote by $\tilde{u}_\varepsilon$, $\tilde{p}_\varepsilon$ the extensions of the functions u_ε, p_ε to Q^T such that $\tilde{u}_\varepsilon \in L^2(0,T;H^1_0(\Omega))$ with $\partial_t\tilde{u}_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ and $\tilde{p}_\varepsilon \in L^2(0,T;H^1_0(\Omega))$ with $\partial_t\tilde{p}_\varepsilon \in L^2(0,T;H^{-1}(\Omega))$ (see [16] for the construction of such extension operator), we get the following estimates

$$\begin{gathered}
\|\tilde{u}_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \leq K\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))}, \quad \|\tilde{p}_\varepsilon\|_{L^2(0,T;H^1_0(\Omega))} \leq K\|p_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))},\\
\|\nabla\tilde{u}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq K\|\nabla u_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}, \quad \|\nabla\tilde{p}_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq K\|\nabla p_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))},\\
\|\partial_t\tilde{u}_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq K\big(\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))} + \|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))}\big).\\
\|\partial_t\tilde{p}_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega))} \leq K\big(\|p_\varepsilon\|_{L^2(0,T;H^1(\Omega_\varepsilon,\partial\Omega))} + \|\partial_t p_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega_\varepsilon,\partial\Omega))}\big).
\end{gathered}$$

where positive constant K doesn't depend on ε. Using these estimates and inequalities (21), (22), (28), we get that there is a subsequence (still denoted by ε) such that as $\varepsilon \to 0$

$$\begin{gathered}
\tilde{u}_\varepsilon \rightharpoonup u_0 \text{ weakly in } L^2(0,T;H^1_0(\Omega)),\\
\tilde{p}_\varepsilon \rightharpoonup p_0 \text{ weakly in } L^2(0,T;H^1_0(\Omega)),\\
\partial_t\tilde{u}_\varepsilon \rightharpoonup \partial_t u_0, \text{ weakly in } L^2(0,T;H^{-1}(\Omega)),\\
\partial_t\tilde{p}_\varepsilon \rightharpoonup \partial_t p_0, \text{ weakly in } L^2(0,T;H^{-1}(\Omega)).
\end{gathered} \tag{29}$$

3. Statement of the main result

The following homogenization theorem holds.

Theorem 2. *Let $n \geq 3$, $\alpha = \gamma = \frac{n}{n-2}$, $f \in L^2(Q^T)$ and let $(u_\varepsilon, p_\varepsilon)$ be the solution to the system (16). Then, their extensions $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ to Q^T converge, as $\varepsilon \to 0$, to the pair of* *functions* (u_0, p_0), *given in* (29), *which is the unique solution to the system*

$$
\begin{cases}
\partial_t u_0 - \Delta u_0 + \mathcal{A}_n\big(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\big)u_0 & \\
= f - N^{-1}c(x)\chi_{\omega^T}p_0, & (x,t)\in Q^T,\\
-\partial_t p_0 - \Delta p_0 + \mathcal{A}_n\big(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\big)p_0 & \\
= -\theta_1\Delta(u_0-u_T) + \mathcal{A}_n\theta_1\big(b_1^2(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_1(x)b_2(x)\chi_{\omega^T}\big)u_0, & (x,t)\in Q^T,\\
u_0(x,t) = p_0(x,t) = 0, & (x,t)\in\Gamma^T,\\
u_0(x,0)=0, \quad p_0(x,T) = \theta_2(u_0(x,T)-u_T(x,T)), & x\in\Omega,
\end{cases}
\tag{30}
$$

where

$$
b_1(x) = \frac{a(x)}{a(x)+\mathcal{B}_n}, \quad b_2(x) = \frac{a(x)(a(x)+\mathcal{B}_n)}{(a(x)+\mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n},
$$
$$
c(x) = \frac{\mathcal{A}_n\mathcal{B}_n}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n},
$$

and $\mathcal{A}_n = (n-2)C_0^{n-2}\omega_n$, $\mathcal{B}_n = \frac{n-2}{C_0}$, ω_n *is the surface area of the unit sphere in* $\mathbb{R}^n$ *and with* χ_A *the characteristic function of the set* A.

As we will show, the pair of functions $(u_0(v), v))$ characterizes the optimal control of the homogenized state problem associated to a suitable cost functional. The state problem, which is related to the system (30), is given by

$$
\begin{cases}
\partial_t u_0(v) - \Delta u_0(v) + \mathcal{A}_n\big(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\big)u_0(v) = & \\
= f + c(x)v\chi_{\omega^T}, & (x,t)\in Q^T,\\
u_0(v)(x,t) = 0, & (x,t)\in\Gamma^T,\\
u_0(v)(x,0) = 0, & x\in\Omega.
\end{cases}
\tag{31}
$$

Now, we introduce the limit cost functional

$$
J_0(v) = \frac{\theta_1}{2}\int\limits_{Q^T} |\nabla(u_0(v)-u_T)|^2 dxdt + \frac{\theta_2}{2}\int\limits_{\Omega}(u_0(v)(x,T)-u_T(x,T))^2dx
$$
$$
+\frac{\mathcal{A}_n\theta_1}{2}\int\limits_0^T\int\limits_{\Omega\setminus\overline{\omega}} b_1^2(x)u_0^2(v)dxdt + \frac{\mathcal{A}_n\theta_1}{2}\int\limits_{\omega^T} b_1(x)b_2(x)u_0^2(v)dxdt + \frac{N}{2}\int\limits_{\omega^T} c(x)v^2dxdt. \tag{32}
$$

and consider the optimal control problem

$$
J_0(v_0) = \min_{v\in L^2(0,T;L^2(\omega))} J_0(v). \tag{33}
$$

Theorem 3. *Under the conditions of Theorem 2, we have*

$$
\lim_{\varepsilon\to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0), \tag{34}
$$

where v_ε *is the optimal control of the problem* (9), (11), (12), *and* v_0 *is the optimal control of the problem* (31)-(33).

Remark 2. *The optimal control* v_0 *is characterized by the system* (30) *and the relation* $v_0 = -N^{-1}\chi_{\omega^T}p_0$.

4. Proof of the homogenization theorem

4.1. **Characterization of the limit of u_ε.** We start by examining the limit of the integral identity for the function u_ε. The most difficult term is the integral over S_ε multiplied by the large growth coefficient $\varepsilon^{-\gamma}$.

For $j \in \mathbb{Z}^n$, we introduce the boundary-value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon}, \\ w_\varepsilon^j = 1, & x \in \partial G^j_\varepsilon, \\ w_\varepsilon^j = 0, & x \in \partial T^j_{\varepsilon/4}, \end{cases} \tag{35}$$

where $T^j_{\varepsilon/4}$ denotes the ball centered in P^j_ε and radius $\varepsilon/4$.

Consider the following function

$$W_\varepsilon^{(2)} = \begin{cases} w_\varepsilon^j(x), & x \in T^j_{\varepsilon/4} \setminus \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon^{(2)}, \\ 1, & x \in \overline{G^j_\varepsilon},\ j \in \Upsilon_\varepsilon^{(2)}, \\ 0, & x \in \Omega \setminus \overline{\bigcup\limits_{j\in\Upsilon_\varepsilon^{(2)}} T^j_{\varepsilon/4}}. \end{cases} \tag{36}$$

It is easy to see that $W_\varepsilon^{(2)} \in H_0^1(\Omega)$ and $W_\varepsilon^{(2)} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ as $\varepsilon \to 0$. Due to the embedding theorem, for some subsequence for which we preserve the notation of the original, we have $W_\varepsilon^{(2)} \to 0$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

In the integral identity (10), we take as a test function $\eta(t) W_\varepsilon^{(2)}(x) \frac{\phi(x)}{a(x)+\mathcal{B}_n}$, where $\eta \in C^1[0,T]$, $\phi \in C_0^\infty(\Omega)$. We get

$$\int_0^T \langle \partial_t u_\varepsilon, \frac{\eta(t) W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} \rangle_{\Omega_\varepsilon} dt + \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla u_\varepsilon \nabla \Big(\frac{\eta(t) W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} \Big) dx dt$$

$$+\varepsilon^{-\gamma} \int_{S_\varepsilon^{(2),T}} a(x) \frac{\eta(t)\phi(x) u_\varepsilon}{a(x)+\mathcal{B}_n} ds dt = \int_{Q_\varepsilon^T} f \frac{\eta(t) W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} dx dt$$

$$-\varepsilon^{-\gamma} N^{-1} \int_{S_\varepsilon^{(2),T}} p_\varepsilon \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} ds dt. \tag{37}$$

Using the properties of the function $W_\varepsilon^{(2)}$, we conclude

$$\lim_{\varepsilon\to0} \int_{Q_\varepsilon^T} f \frac{\eta(t) W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} dx dt = 0. \tag{38}$$

Next, we compute the first integral in (37)

$$\int_0^T \langle \partial_t u_\varepsilon, \frac{\eta(t) W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} \rangle_{\Omega_\varepsilon} dt =$$

$$= -\int_{Q_\varepsilon^T} u_\varepsilon \partial_t \eta(t) \frac{W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} dx dt + \int_{\Omega_\varepsilon} u_\varepsilon(x,T)\eta(T) \frac{W_\varepsilon^{(2)}(x)\phi(x)}{a(x)+\mathcal{B}_n} dx.$$

Using the strong convergence $W_\varepsilon^{(2)} \to 0$ in $L^2(\Omega)$, we conclude

$$\lim_{\varepsilon\to 0} \int_0^T \langle \partial_t u_\varepsilon, \frac{\eta(t) W_\varepsilon^{(2)} \phi(x)}{a(x)+\mathcal{B}_n} \rangle_{\Omega_\varepsilon} dt = 0. \tag{39}$$

Applying that $W_\varepsilon^{(2)} \rightharpoonup 0$ in $H_0^1(\Omega)$ as $\varepsilon \to 0$, we get

$$\sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla u_\varepsilon \nabla\Big(\frac{\eta(t) W_\varepsilon^{(2)} \phi(x)}{a(x)+\mathcal{B}_n}\Big) dxdt$$

$$= \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{T^j_{\varepsilon/4}\setminus\overline{G^j_\varepsilon}} \nabla W_\varepsilon^{(2)} \nabla\Big(u_\varepsilon \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\Big) dxdt + \alpha_\varepsilon,$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Using the definition of $W_\varepsilon^{(2)}$, we have

$$\int_0^T \int_{\Omega_\varepsilon} \nabla W_\varepsilon^{(2)} \nabla\Big(\frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\Big) dxdt$$

$$= \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{\partial T^j_{\varepsilon/4}} \partial_\nu w_\varepsilon^j \frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt + \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{\partial G^j_\varepsilon} \partial_\nu w_\varepsilon^j \frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt$$

$$= -\varepsilon C_0^{n-2}(n-2)4^{n-1} \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{\partial T^j_{\varepsilon/4}} \frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt$$

$$+\mathcal{B}_n \varepsilon^{-\gamma} \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{\partial G^j_\varepsilon} \frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt + \alpha_{1,\varepsilon},$$

where $\alpha_{1,\varepsilon} \to 0$ as $\varepsilon \to 0$.

Applying Lemma 5 from [24], we have

$$\lim_{\varepsilon\to 0} \varepsilon C_0^{n-2}(n-2)4^{n-1} \sum_{j\in\Upsilon_\varepsilon^{(2)}} \int_0^T \int_{\partial T^j_{\varepsilon/4}} \frac{u_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt = \mathcal{A}_n \int_0^T \int_\omega \frac{u_0\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt. \tag{40}$$

Then, from (37)-(40), we have

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon^{(2)}} u_\varepsilon \eta(t)\phi(x) dsdt$$

$$= \int_0^T \int_\omega \frac{\mathcal{A}_n u_0\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt - N^{-1} \lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_0^T \int_{S_\varepsilon^{(2)}} \frac{p_\varepsilon\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt. \tag{41}$$

Now, the problem is with the last term since we do not know yet its limit. To find this limit, we will examine the integral identity for the function p_ε.

4.2. **Characterization of u_0 and p_0.** We take in the integral identity for the function p_ε as a test function $W_\varepsilon^{(2)}(x)\frac{(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}$. We get

$$-\int_0^T \langle \partial_t p_\varepsilon, \frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt$$

$$+\int_0^T\int_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla\Big(\frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\Big) dxdt$$

$$+\varepsilon^{-\gamma}\int_0^T\int_{S_\varepsilon^{(2)}} a(x)\frac{(a(x)+\mathcal{B}_n)p_\varepsilon\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dsdt$$

$$=\theta_1\int_0^T\int_{\Omega_\varepsilon} \nabla(u_\varepsilon-u_T)\nabla\Big(\frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\Big) dxdt. \tag{42}$$

By the same reasoning as above, we conclude

$$\lim_{\varepsilon\to 0}\int_0^T \langle \partial_t p_\varepsilon, \frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt = 0, \tag{43}$$

and for the second integral in the left-hand side, we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega_\varepsilon} \nabla p_\varepsilon \nabla\Big(\frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\Big) dxdt$$

$$=-\mathcal{A}_n\int_0^T\int_{\omega^T} \frac{(a(x)+\mathcal{B}_n)p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dxdt$$

$$+\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\mathcal{B}_n\int_0^T\int_{S_\varepsilon^{(2)}} \frac{(a(x)+\mathcal{B}_n)p_\varepsilon\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dsdt, \tag{44}$$

and for the integral in the right-hand side, we get

$$\lim_{\varepsilon\to 0}\theta_1\int_{Q_\varepsilon^T} \nabla(u_\varepsilon-u_T)\nabla\Big(\frac{W_\varepsilon^{(2)}(x)(a(x)+\mathcal{B}_n)\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\Big) dxdt$$

$$=-\mathcal{A}_n\theta_1\int_{\omega^T} \frac{(a(x)+\mathcal{B}_n)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dxdt$$

$$+\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\theta_1\mathcal{B}_n\int_{S_\varepsilon^{(2),T}} \frac{(a(x)+\mathcal{B}_n)u_\varepsilon\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dsdt. \tag{45}$$

Using the expressions (41)-(45), we get

$$-\mathcal{A}_n\int_{\omega^T} \frac{(a(x)+\mathcal{B}_n)p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dxdt + \varepsilon^{-\gamma}\int_{S_\varepsilon^{(2),T}} \frac{(a(x)+\mathcal{B}_n)^2 p_\varepsilon\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dsdt$$

$$= -\mathcal{A}_n\theta_1 \int\limits_{\omega^T} \frac{(a(x)+\mathcal{B}_n)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt + \theta_1\mathcal{B}_n\mathcal{A}_n \int\limits_{\omega^T} \frac{u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt$$

$$-\varepsilon^{-\gamma}N^{-1}\theta_1\mathcal{B}_n \int\limits_{S_\varepsilon^{(2),T}} \frac{p_\varepsilon\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dsdt + \alpha_\varepsilon,\ \alpha_\varepsilon \to 0, \varepsilon \to 0.$$

From here, we deduce

$$\begin{aligned}
\varepsilon^{-\gamma} \int\limits_{S_\varepsilon^{(2),T}} p_\varepsilon\eta(t)\phi(x)dsdt &= \mathcal{A}_n \int\limits_{\omega^T} \frac{(a(x)+\mathcal{B}_n)p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt \\
&-\mathcal{A}_n\theta_1 \int\limits_{\omega^T} \frac{(a(x)+\mathcal{B}_n)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt \\
&+\mathcal{A}_n\mathcal{B}_n\theta_1 \int\limits_{\omega^T} \frac{u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt + \alpha_\varepsilon \\
&= \mathcal{A}_n \int\limits_{\omega^T} \frac{(a(x)+\mathcal{B}_n)p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt \\
&-\mathcal{A}_n\theta_1 \int\limits_{\omega^T} \frac{a(x)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt + \alpha_\varepsilon.
\end{aligned} \tag{46}$$

where $\alpha_\varepsilon \to 0$, as $\varepsilon \to 0$.

From (41) and (46), we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int\limits_{S_\varepsilon^{(2),T}} u_\varepsilon\eta(t)\phi(x)dsdt &= \mathcal{A}_n \int\limits_{\omega^T} \frac{a(x)+\mathcal{B}_n+\theta_1 N^{-1}}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}u_0\eta(t)\phi(x)dxdt \\
&-\mathcal{A}_n N^{-1} \int\limits_{\omega^T} \frac{p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt,
\end{aligned} \tag{47}$$

and consequently

$$\begin{aligned}
\lim_{\varepsilon\to 0} \int\limits_{S_\varepsilon^{(2),T}} a(x)u_\varepsilon\eta(t)\phi(x)dsdt &= \mathcal{A}_n \int\limits_{\omega^T} \frac{a(x)(a(x)+\mathcal{B}_n+\theta_1 N^{-1})}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}u_0\eta(t)\phi(x)dxdt \\
&-\mathcal{A}_n N^{-1} \int\limits_{\omega^T} \frac{a(x)p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt.
\end{aligned} \tag{48}$$

Now, we introduce the function

$$W_\varepsilon^{(1)} = \begin{cases} w_\varepsilon^j(x), & x \in T_{\varepsilon/4}^j \setminus \overline{G_\varepsilon^j},\ j \in \Upsilon_\varepsilon^{(1)}, \\ 1, & x \in G_\varepsilon^j,\ j \in \Upsilon_\varepsilon^{(1)}, \\ 0, & x \in \Omega \setminus \overline{\bigcup\limits_{j\in\Upsilon_\varepsilon^{(1)}} T_{\varepsilon/4}^j}. \end{cases} \tag{49}$$

We take in the integral identity for u_ε as a test function $W_\varepsilon^{(1)}(x)\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n}$ and get

$$\int_0^T \langle \partial_t u_\varepsilon, W_\varepsilon^{(1)}(x)\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt + \int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla\Big(W_\varepsilon^{(1)}\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n}\Big) dxdt$$

$$+\varepsilon^{-\gamma}\int_{S_\varepsilon^{(1),T}} a(x)u_\varepsilon \frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n} dsdt = \int_{Q_\varepsilon^T} f W_\varepsilon^{(1)}\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n} dxdt.$$

As above, taking into account that $W_\varepsilon^{(1)} \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $W_\varepsilon^{(1)} \to 0$ strongly in $L^2(\Omega)$ as $\varepsilon \to 0$, we conclude

$$\lim_{\varepsilon\to 0}\int_0^T \langle \partial_t u_\varepsilon, W_\varepsilon^{(1)}\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt = 0, \quad \lim_{\varepsilon\to 0}\int_{Q_\varepsilon^T} f W_\varepsilon^{(1)}\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n} dxdt = 0,$$

$$\lim_{\varepsilon\to 0}\int_{Q_\varepsilon^T} \nabla u_\varepsilon \nabla\Big(W_\varepsilon^{(1)}\frac{\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n}\Big) dxdt = -\mathcal{A}_n \int_0^T\int_{\Omega\backslash\overline{\omega}} \frac{u_0\eta(t)\phi(x)a(x)}{a(x)+\mathcal{B}_n} dxdt$$

$$+\lim_{\varepsilon\to 0}\mathcal{B}_n\varepsilon^{-\gamma}\int_{S_\varepsilon^{(1),T}} u_\varepsilon\eta(t)\phi(x)a(x) dsdt.$$

Thus, we have

$$\lim_{\varepsilon\to 0}\varepsilon^{-\gamma}\int_{S_\varepsilon^{(1),T}} a(x)u_\varepsilon\eta(t)\phi(x) dsdt = \mathcal{A}_n\int_0^T\int_{\Omega\backslash\overline{\omega}} \frac{a(x)u_0\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt. \tag{50}$$

Combining (48), (50), we have the following integral identity for u_0

$$\int_0^T \langle \partial_t u_0, \eta(t)\phi(x)\rangle_\Omega dt + \int_{Q^T} \nabla u_0 \nabla(\eta(t)\phi(x)) dxdt$$

$$+\mathcal{A}_n\int_{\omega^T} \frac{a(x)(a(x)+\mathcal{B}_n)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dxdt$$

$$+\mathcal{A}_n\int_0^T\int_{\Omega\backslash\overline{\omega}} \frac{a(x)u_0\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt = \int_{Q^T} f\eta(t)\phi(x) dxdt$$

$$-\mathcal{A}_n\mathcal{B}_n N^{-1}\int_{\omega^T} \frac{p_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} dxdt.$$

By density arguments, this identity is valid for an arbitrary function $\psi \in L^2(0,T;H_0^1(\Omega))$. It means that u_0 is the unique weak solution to the problem

$$\begin{cases} \partial_t u_0 - \Delta u_0 + \mathcal{A}_n \frac{a(x)(a(x)+\mathcal{B}_n)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} \chi_{\omega^T} u_0 \\ +\mathcal{A}_n \frac{a(x)}{a(x)+\mathcal{B}_n} \chi_{(\Omega\setminus\overline{\omega})\times(0,T)} u_0 \\ = f - \mathcal{A}_n\mathcal{B}_n N^{-1} \frac{p_0}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\chi_{\omega^T}, & (x,t)\in Q^T, \\ u_0(x,t)=0, & (x,t)\in\Gamma^T, \\ u_0(x,0)=0, & x\in\Omega. \end{cases} \tag{51}$$

Taking $W_\varepsilon^{(1)} \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}$ as a test function in the integral identity for p_ε, we obtain

$$\int_0^T \langle \partial_t p_\varepsilon, W_\varepsilon^{(1)} \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt + \int_{Q_\varepsilon^T} \nabla p_\varepsilon \nabla\Big(W_\varepsilon^{(1)} \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\Big) dxdt$$
$$+\varepsilon^{-\gamma} \int_{S_\varepsilon^{(1),T}} a(x) p_\varepsilon \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt = \theta_1 \int_{Q_\varepsilon^T} \nabla(u_\varepsilon - u_T)\nabla\Big(W_\varepsilon^{(1)} \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\Big) dxdt. \tag{52}$$

Using the properties of the function $W_\varepsilon^{(1)}$, we conclude

$$\lim_{\varepsilon\to 0} \int_0^T \langle \partial_t p_\varepsilon, W_\varepsilon^{(1)} \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n}\rangle_{\Omega_\varepsilon} dt = 0.$$

As above, using the definition of $W_\varepsilon^{(1)}$, we transform the integrals over Q_ε^T in the left and right parts of the expression (52) and derive

$$\varepsilon^{-\gamma} \int_{S_\varepsilon^{(1),T}} p_\varepsilon \eta(t)\phi(x) dsdt - \mathcal{A}_n \int_0^T \int_{\Omega\setminus\overline{\omega}} p_0 \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt$$
$$= -\mathcal{A}_n\theta_1 \int_0^T \int_{\Omega\setminus\overline{\omega}} u_0 \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt + \mathcal{B}_n\theta_1\varepsilon^{-\gamma} \int_{S_\varepsilon^{(1),T}} u_\varepsilon \frac{\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dsdt + \alpha_\varepsilon, \tag{53}$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.

Due to (50), we derive from (53)

$$\lim_{\varepsilon\to 0} \varepsilon^{-\gamma} \int_{S_\varepsilon^{(1),T}} p_\varepsilon \eta(t)\phi(x) dsdt$$
$$= \mathcal{A}_n \int_0^T \int_{\Omega\setminus\overline{\omega}} \frac{p_0\eta(t)\phi(x)}{a(x)+\mathcal{B}_n} dxdt - \mathcal{A}_n\theta_1 \int_0^T \int_{\Omega\setminus\overline{\omega}} \frac{a(x)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2} dxdt.$$

Now, we are able to pass to the limit as $\varepsilon \to 0$ in the integral identity to p_ε and get the limit relation for p_0

$$-\int_0^T \langle \partial_t p_0, \eta(t)\phi(x)\rangle_\Omega dt + \int_{Q^T} \nabla p_0 \nabla(\eta(t)\phi(x)) dxdt + \mathcal{A}_n \int_0^T \int_{\Omega\setminus\overline{\omega}} \frac{a(x)p_0}{a(x)+\mathcal{B}_n} \eta(t)\phi(x) dxdt$$

$$+\mathcal{A}_n \int\limits_{\omega_T} \frac{a(x)(a(x)+\mathcal{B}_n)p_0}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}\eta(t)\phi(x)dxdt = \theta_1 \int\limits_{Q^T} \nabla(u_0-u_T)\nabla(\eta(t)\phi(x))dxdt$$

$$+\mathcal{A}_n\theta_1 \int\limits_0^T \int\limits_{\Omega\setminus\overline{\omega}} \frac{a^2(x)u_0\eta(t)\phi(x)}{(a(x)+\mathcal{B}_n)^2}dsdt + \mathcal{A}_n\theta_1 \int\limits_{\omega^T} \frac{a^2(x)\eta(t)\phi(x)u_0}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n}dxdt.$$

From this identity, we have that $p_0 \in L^2(0,T;H_0^1(\Omega))$, $\partial_t p_0 \in L^2(0,T;H^{-1}(\Omega))$, $p_0(x,T) = \theta_2(u_0(x,T)-u_T(x,T))$ is the unique weak solution of the problem

$$\begin{cases} -\partial_t p_0 - \Delta p_0 + \mathcal{A}_n \frac{a(x)}{a(x)+\mathcal{B}_n} p_0 \chi_{(\Omega\setminus\overline{\omega})\times(0,T)} & \\ +\mathcal{A}_n \frac{a(x)(a(x)+\mathcal{B}_n)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} p_0 = -\theta_1\Delta(u_0-u_T) & \\ +\mathcal{A}_n\theta_1 \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2} u_0\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + \mathcal{A}_n\theta_1 \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2+\theta_1 N^{-1}\mathcal{B}_n} u_0\chi_{\omega^T}, & (x,T)\in Q^T, \\ p_0(x,t)=0, & (x,t)\in\Gamma^T, \\ p_0(x,T)=\theta_2(u_0(x,T)-u_T(x,T)), & x\in\Omega. \end{cases} \tag{54}$$

This concludes the proof of Theorem 2.

5. Characterization of the cost functional limit

Now, we will show the validity of convergence (34) and prove Theorem 3. For the function $v_\varepsilon = -N^{-1}p_\varepsilon$, we have

$$J_\varepsilon(-N^{-1}p_\varepsilon) = \frac{\theta_1}{2}\int\limits_{Q_\varepsilon^T} |\nabla(u_\varepsilon-u_T)|^2 dxdt + \frac{\theta_2}{2}\int\limits_{\Omega_\varepsilon} (u_\varepsilon(x,T)-u_T(x,T))^2 dx + \frac{\varepsilon^{-\gamma}}{2N}\int\limits_{S_\varepsilon^{(2),T}} p_\varepsilon^2 dsdt.$$

Using u_ε as a test function in the integral identity for p_ε, and taking p_ε as a test function in the integral identity for u_ε, we transform this expression into (see the derivation of (18))

$$J_\varepsilon(-N^{-1}p_\varepsilon) = \frac{1}{2}\int\limits_0^T \Big(\langle\partial_t p_\varepsilon, u_\varepsilon\rangle_{\Omega_\varepsilon} + \langle\partial_t u_\varepsilon, p_\varepsilon\rangle_{\Omega_\varepsilon}\Big)dt + \frac{1}{2}\int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt$$

$$-\frac{\theta_1}{2}\int\limits_{Q_\varepsilon^T} \nabla(u_\varepsilon-u_T)\nabla u_T dxdt - \frac{\theta_2}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon(x,T)-u_T(x,T))u_T(x,T)dx$$

$$= -\frac{\theta_2}{2}\int\limits_{\Omega_\varepsilon}(u_\varepsilon(x,T)-u_T(x,T))u_T(x,T)dxdt + \frac{1}{2}\int\limits_{Q_\varepsilon^T} f p_\varepsilon dxdt - \frac{\theta_1}{2}\int\limits_{Q_\varepsilon^T}\nabla(u_\varepsilon-u_T)\nabla u_T dxdt.$$

Thus, we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(-N^{-1}p_\varepsilon) = -\frac{\theta_2}{2}\int\limits_\Omega (u_0(x,T)-u_T(x,T))u_T(x,T)dx + \frac{1}{2}\int\limits_{Q^T} f p_0 dxdt \tag{55}$$

$$-\frac{\theta_1}{2}\int\limits_{Q^T}\nabla(u_0-u_T)\nabla u_T dxdt = -\frac{\theta_2}{2}\int\limits_\Omega (u_0(x,T)-u_T(x,T))u_T(x,T)dx$$

$$+\frac{1}{2}\int\limits_{Q^T} f p_0 dxdt + \frac{\theta_1}{2}\int\limits_{Q^T}|\nabla(u_0-u_T)|^2 dxdt - \frac{\theta_1}{2}\int\limits_{Q^T}\nabla u_0\nabla(u_0-u_T)dxdt. \tag{56}$$

From the integral identities for the functions u_0 and p_0, we have

$$\int\limits_{Q^T} f p_0 dxdt = \int\limits_0^T \Big(\langle \partial_t u_0, p_0 \rangle_\Omega + \langle \partial_t p_0, u_0 \rangle_\Omega \Big) dt$$

$$+\mathcal{A}_n \mathcal{B}_n N^{-1} \int\limits_{\omega^T} \frac{p_0^2}{(a(x)+\mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n} dxdt$$

$$+\theta_1 \int\limits_{Q^T} \nabla(u_0 - u_T)\nabla u_0 dxdt + \mathcal{A}_n\theta_1 \int\limits_0^T \int\limits_{\Omega\setminus\overline{\omega}} \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2} u_0^2 dxdt$$

$$+\mathcal{A}_n\theta_1 \int\limits_{\omega^T} \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n} u_0^2 dxdt,$$

Substituting this expression into (56), we obtain

$$\lim_{\varepsilon\to 0} J_\varepsilon(-N^{-1}p_\varepsilon) = \frac{\theta_1}{2}\int\limits_{Q^T} |\nabla(u_0 - u_T)|^2 dxdt + \frac{\theta_2}{2}\int\limits_\Omega (u_0(x,T) - u_T(x,T))^2 dx$$

$$+\frac{\mathcal{A}_n\theta_1}{2}\int\limits_0^T\int\limits_{\Omega\setminus\overline{\omega}} \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2} u_0^2 dxdt + \frac{\mathcal{A}_n\theta_1}{2}\int\limits_{\omega^T} \frac{a^2(x)}{(a(x)+\mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n} u_0^2 dxdt$$

$$+\frac{\mathcal{A}_n\mathcal{B}_n N^{-1}}{2}\int\limits_{\omega^T} \frac{p_0^2}{(a(x)+\mathcal{B}_n)^2 + \theta_1 N^{-1}\mathcal{B}_n} dxdt.$$

Hence, we have

$$\lim_{\varepsilon\to 0} J_\varepsilon(-N^{-1}p_\varepsilon) = \frac{\theta_1}{2}\int\limits_{Q^T} |\nabla(u_0 - u_T)|^2 dxdt + \frac{\theta_2}{2}\int\limits_\Omega (u_0(x,T) - u_T(x,T))^2 dx$$

$$+\frac{\mathcal{A}_n\theta_1}{2}\int\limits_0^T\int\limits_{\Omega\setminus\overline{\omega}} b_1^2(x) u_0^2 dxdt + \frac{\mathcal{A}_n\theta_1}{2}\int\limits_{\omega^T} b_1(x)b_2(x) u_0^2 dxdt$$

$$+\frac{1}{2N}\int\limits_{\omega^T} c(x) p_0^2 dxdt \equiv J_0(-N^{-1}p_0\chi_{\omega^T}).$$

This concludes the proof.

6. On the approximate controllability of limit problem and uniform convergence of the controls v_ε. Proof of the main result

As mentioned in the Introduction, we will conclude the approximate controllability of the problem (1) by showing that the limit problem (4) satisfies such a property and by proving that the sequence of optimal controls v_ε is uniformly bounded (with respect to N), when $N \searrow 0$.

Theorem 4. *Let $u_T \in L^2(\Omega)$ and let $u_0(v)$ be the unique solution of the limit problem (4) for a given control $v_0 \in L^2(\omega \times (0,T))$. Then, given $\delta > 0$, there exists a control $v_0 \in L^2(\omega \times (0,T))$ such that*

$$\|u_0(v)(\cdot,T) - u_T\|^2_{L^2(\Omega)} \leq \delta. \tag{57}$$

Moreover, such a control can be obtained as the limit of the optimal controls $v_{0,N}$, associated to the cost functional given by (32) with $\theta_1 = 0$ and $\theta_2 = 1$, i.e.,

$$\begin{aligned} J_{0,N}(v) &= \frac{1}{2}\int_\Omega |u(v)(x,T) - u_T(x)|^2 dx \\ &+\frac{N}{2}\int_{\omega\times(0,T)} c(x)v^2 dxdt, \end{aligned}$$

as $N \searrow 0$.

Proof. First of all we point out that, without any loss of generality, we can assume that $f(t,x) \equiv 0$. Indeed, since the limit problem (4) is linear we can make the change of variable $y_0 = u_0 - Z$, with Z satisfying

$$\begin{cases} \frac{\partial Z}{\partial t} - \Delta Z + \mathcal{A}_n \left(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\right) Z = f, & (x,t) \in \Omega \times (0,T), \\ Z = 0, & (x,t) \in \partial\Omega \times (0,T), \\ Z(x,0) = 0, & x \in \Omega, \end{cases} \tag{58}$$

and then it suffices to substitute the target function u_T by $\widetilde{u}_T = u_T - Z(\cdot,T)$. On the other hand, we can assume $\widetilde{u}_T \neq 0$ a.e. on Ω (since otherwise the conclusion is trivially satisfied by the control $v_0 \equiv 0$). Thus, for a given penalty parameter $N > 0$ we consider the cost functional $J_{0,N}(v)$ where $u(v)$ is the solution of (4) corresponding to $f(t,x) \equiv 0$. Let $v_{0,N}$ the corresponding optimal control (see, e.g., [12]). Notice that since $J_{0,N}(v)$ is weakly continuous, strictly convex and coercive in $L^2(\omega \times (0,T))$ then $v_{0,N}$ exists and it is unique. Arguing as in the proof of Theorem 1 we get that the optimality condition can be written in the following terms:

$$\begin{cases} 0 = J'_{0,N}(v_{0,N})v = N\displaystyle\int_{\omega\times(0,T)} c(x)v_{0,N}v\,dxdt \\ +\displaystyle\int_\Omega (u(v_{0,N})(x,T) - u_T(x))u(v(x,T)dx, \end{cases} \tag{59}$$

for any $v \in L^2(\omega \times (0,T))$. Since $v_{0,N}$ minimizes $J_{0,N}$ on $L^2(\omega \times (0,T))$ we get that, for any $N > 0$

$$J_{0,N}(v_{0,N}) \leq J_{0,N}(0),$$

and, since the solution of $u(v(x,T))$ (4) when $v \equiv 0$ is $u(v(x,T)) \equiv 0$, we get that

$$J_{0,N}(0) = \frac{1}{2}\int_\Omega |u_T(x)|^2 dx.$$

We recall (see (5)) that, since $\theta_1 = 0$,

$$b_1(x) = b_2(x) = \frac{a(x)}{a(x) + \mathcal{B}_n}, \tag{60}$$

and

$$c(x) = \frac{\mathcal{A}_n \mathcal{B}_n}{(a(x) + \mathcal{B}_n)^2}, \tag{61}$$

Then,

$$\|b_2(x)\|_{L^\infty(\omega)} \leq C,$$

for some $C > 0$ and we have also that

$$\frac{N\mathcal{A}_n}{2\mathcal{B}_n} \int\limits_{\omega\times(0,T)} v^2 dxdt \leq \frac{N}{2} \int\limits_{\omega\times(0,T)} c(x) v^2 dxdt.$$

Then, we get that, if $N \in (0,1]$, $\{u(v_{0,N})(.,T) - u_T(.)\}_{N\in(0,1]}$ is a bounded sequence in $L^2(\Omega)$ and

$$\left\{\sqrt{N} v_{0,N})\right\}_{N\in(0,1]} \text{ is a bounded sequence in } L^2(\omega^T). \tag{62}$$

Then, there exists a subsequence, $\xi \in L^2(\Omega)$ and $w \in L^2(\omega^T)$, such that

$$u(v_{0,N})(.,T) - u_T(.) \rightharpoonup \xi \text{ weakly in } L^2(\Omega), \tag{63}$$

and

$$\sqrt{N} v_{0,N} \rightharpoonup w \;\text{ weakly in } L^2(\omega^T).$$

In consequence, from (**??**), we get that

$$\int\limits_\Omega \xi(x) u(v(x,T)) dx = 0,$$

for any $v \in L^2(\omega \times (0,T))$. Let us show that this implies that $\xi \equiv 0$ in Ω. Indeed, we consider the auxiliary problem

$$\begin{cases} -\dfrac{\partial p_N}{\partial t} - \Delta p_N + \mathcal{A}_n \left(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\right) p_N = 0, & (x,t) \in \Omega \times (0,T), \\ p_N = 0, & (x,t) \in \partial\Omega \times (0,T), \\ p_N(x,T) = u(v_{0,N})(.,T) - u_T(.)\ , & x \in \Omega. \end{cases} \tag{64}$$

Then, multiplying (64) by $u(v)$,using the equation satisfied by $u(v)$, and integrating by parts, we get

$$-\int\limits_\Omega (u(v_{0,N})(.,T) - u_T(.)) u((v)(x,T)) dx + N \int\limits_{\omega\times(0,T)} c(x) p_N v dxdt = 0. \tag{65}$$

Then we get that p_N is bounded in $L^2(0,T; H_0^1(\Omega))$ and thus $p_N \rightharpoonup p$ in $L^2(0,T; H_0^1(\Omega))$, as $N \searrow 0$, with p solution of the problem

$$\begin{cases} -\dfrac{\partial p}{\partial t} - \Delta p + \mathcal{A}_n \left(b_1(x)\chi_{(\Omega\setminus\overline{\omega})\times(0,T)} + b_2(x)\chi_{\omega^T}\right) p = 0, & (x,t) \in \Omega \times (0,T), \\ p = 0, & (x,t) \in \partial\Omega \times (0,T), \\ p(x,T) = \xi(x)\ , & x \in \Omega. \end{cases} \tag{66}$$

Then, by (65) we get that $p=0$ on $\omega\times(0,T)$. In consequence, by the Mizohata results (see [15] and its improvement in [9] for the case of bounded coefficients) we deduce that $p\equiv 0$ in $\Omega\times(0,T)$, which implies that $\xi(x)\equiv 0$ on Ω. In addition, we also have the strong convergence in (63) since, from the optimality condition (**??**)

$$N\int\limits_{\omega\times(0,T)} c(x)\left|v_{0,N}\right|^2 dxdt$$
$$+\int\limits_{\Omega}\left|u(v_{0,N})(x,T)-u_T(x))\right|^2 dx\to 0$$

as $N\to 0$.■

Now we are in conditions to prove our main result

Theorem 5. *Let $u_T\in L^2(\Omega)$ and let $u_\varepsilon(v)$ be the unique solution of the problem (1) for a given control $v\in L^2(S_\varepsilon^{(2)}\times(0,T))$. Then, given $\delta>0$, there exists $\varepsilon_0>0$ and there exists $N_0\in(0,\,1)$ (independent of ε_0) such that if $\varepsilon\in(0,\varepsilon_0)$ and $N\in(0,\,N_0)$, the optimal control $v_{\varepsilon,N}\in L^2(S_\varepsilon^{(2)}\times(0,T))$ associated to $J_\varepsilon(v)$, with $\theta_1=0$ and $\theta_2=1$, leads to the approximate controllability property*

$$\|u_\varepsilon(v_{\varepsilon,N})(.,T)-u_T\|^2_{L^2(\Omega_\varepsilon)}\le\delta. \tag{67}$$

Proof. It suffices to apply the above Theorem 4 and the strong convergence $\widetilde{u}_\varepsilon(v)(.,T)\to u_0(v)(.,T)$ in $L^2(\Omega)$ proved in Theorem 2. Indeed, we have

$$\|u_\varepsilon(v)(.,T)-u_T\|^2_{L^2(\Omega_\varepsilon)}\le\|\widetilde{u}_\varepsilon(v)(.,T)-u_0(v)(.,T)\|^2_{L^2(\Omega_\varepsilon)}+\|u_0(v)(.,T)-u_T\|^2_{L^2(\Omega_\varepsilon)}.$$

Moreover, we know (from the optimality condition (**??**)) that the optimal control of the limit problem $v_{0,N}$ satisfies an uniforme estimate (as $N\to 0$, when $N\in(0,\,N_0)$) (see 62). Then, thanks to the characterization of the optimal controls, we can also assume also that this property holds for the microscopic optimal control $v_{\varepsilon,N}$, when $N\in(0,\,N_0)$, for any $\varepsilon\in(0,\varepsilon_0)$.■

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