From a small reactive part of the boundary to the whole interior non-reactive domain: critical scale homogenization, optimal control and controllability

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Abstract. We prove the approximate controllability, for the interior final observation, of the solution of a linear heat equation on $\Omega \times (0, T)$, when the control is placed on a small part of the boundary l_ε which is heterogeneous (with a critical scale) and where we assume a Robin-type boundary condition. One of the motivations is related to some atmosphere – deep ocean models in climatology. We consider the special case of two-dimensional domains Ω which requires suitable coefficients in the Robin-term and in the control. Firstly, we apply the homogenization process proving that the solution of the microscopic problem converges, as $\varepsilon \to 0$, to a function $u_0(x, t)$ that is the unique solution to a suitable global state parabolic problem with a Robin-type boundary condition on a part of the boundary. We consider a microscopic optimal control problem and prove the weak convergence of the state and the optimal control. Finally, we prove the approximate controllability by passing to the limit in a penalty parameter of the cost functional. The conclusion gives a certain mathematical justification to some arguments used by ecologists but it brings to light also some limitations that must be assumed on the local controls to conclude that the result is globally satisfactory.

Keywords: homogenization, small reactive boundary, critical scale, approximate controllability, optimal control, “strange” term.

Subject Classification 35K45, 49K20, 35B27, 92D40.

1. Introduction

This paper deals with the approximate controllability of some heterogeneous problems. We assume that a part of the boundary Γ_2 of the spatial domain Ω is constituted of a set of periodical particles on which the control is placed. One of our motivations is to extend some previous results dealing with approximate controllability of parabolic problems posed on a heterogeneous domain formed by the exterior of a set of periodical particles when the control acts only on the boundary of a small set of those interior particles (see [10]). Our study is also an improvement of our paper [8] where the control is also assumed on the boundary of the domain but in that case the set of periodical particles is placed on a manifold close to the boundary (as in some smart double skin boundaries). The same as in [4], we will consider here the case of two-dimensional set Ω: a case which in many studies is not completely developed since it requires weighted coefficients which are different to the ones arising in n-dimensional domains for $n > 2$. On the other hand, one of the applications which motivate the consideration of this type of problems concerns the study of diffusive climate Energy Balance Models (EBM) (see, e.g. [20] and [3]). The simpler version of those EBMs deals with the mean superficial atmosphere temperature of the Earth, nevertheless, a more realistic version coupling the mean superficial atmospheric temperature with the temperature of a deep ocean has been studied since 1990 (see, e.g. [11], [25] and their many references). In that case, after a suitable identification process, the atmosphere is reduced to a part of the boundary of the domain on which we must assume a Robin-type boundary condition and then the possible control of the global temperature (in both parts, the atmosphere and the ocean) is reduced to a small part of the boundary corresponding to a part of the Earth continents (see Remark 2 below).

After some simple change of variables, we can reformulate the atmosphere boundary as a suitable part of the boundary of the domain. Thus, we will use a simpler formulation which keeps the main difficulties (used in [4] but in the absence of any control) which corresponds to the case of a bounded domain Ω of $\mathbb{R}^2 \cap \{x_2 > 0\}$ for which the boundary consists of two smooth parts $\partial\Omega = \Gamma_1 \cup \Gamma_2$, where $\Gamma_2 = \partial\Omega \cap \{x_2 = 0\} = \{(x_1, 0) : x_1 \in [-l, l]\}$, $l > 0$, $\Gamma_1 = \partial\Omega \cap \{x_2 > 0\}$. To define the active part of the boundary, we imagine a heterogeneos structure (since not all continents can be the base of possible actions). More precisely, we start by introducing the sets

$$Y_1 = \{(y_1, 0) | -\frac{1}{2} < y_1 < \frac{1}{2}\}, \quad \hat{l}_0 = \{(y_1, 0) | -l_0 < y_1 < l_0\} \subset Y_1,$$

where $l_0 \in (0, \frac{1}{2})$. For a small parameter $\varepsilon > 0$, following [4], we consider the "critical size"

$$a_\varepsilon = C_0 \varepsilon \exp(-\frac{\alpha^2}{\varepsilon}), \tag{1}$$

where α, C_0 are positive constants (notice that $0 < a_\varepsilon \ll \varepsilon$). We define

$$\widetilde{G_\varepsilon} = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon \hat{l}_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} l_\varepsilon^j,$$

where $\mathbb{Z}' = \mathbb{Z} \times \{0\}$ is a set of vectors $j = (j_1, 0)$ with an integer j_1. We introduce the set of indices $\Upsilon_\varepsilon = \{j \in \mathbb{Z}' \Big| \overline{l_\varepsilon^j} \subset \{x = (x_1, 0), x_1 \in [-l + 2\varepsilon, l - 2\varepsilon]\}\}$ and

$$l_\varepsilon = \bigcup_{j \in \Upsilon_\varepsilon} l_\varepsilon^j.$$

Let $Y_\varepsilon^j = \varepsilon Y_1 + \varepsilon j$ then it is easy to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$. Next, we define $\gamma_\varepsilon = \Gamma_2 \setminus \overline{l_\varepsilon}$. Note that for any $j \in \mathbb{Z}'$, $|l_\varepsilon^j| = 2a_\varepsilon l_0$, $|l_\varepsilon| \cong d a_\varepsilon \varepsilon^{-1}$, $d = const > 0$.

We introduce the usual notation in evolution problems

$$Q^T = \Omega \times (0, T), \quad \Gamma_1^T = \Gamma_1 \times (0, T), \quad l_\varepsilon^T = l_\varepsilon \times (0, T),$$
$$\gamma_\varepsilon^T = \gamma_\varepsilon \times (0, T), \quad \Gamma_2^T = \Gamma_2 \times (0, T).$$

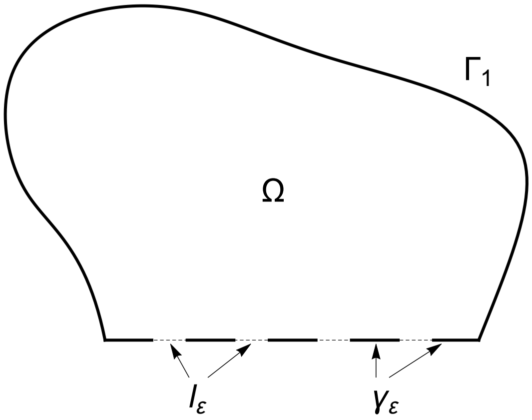

Figure 1. Domain Ω.

For simplicity, we consider only a linear formulation but it is not too hard to extend our results to a nonlinear framework (e.g., see the techniques used in the monograph [5]). As a matter of fact, we are dealing here with the associated "critical scale" since we know that the appearance of some "strange terms" in the homogenization of the problem has the good consequence of regularizing the possible non-Lipschitz non-linear terms (see, e.g, [5]). This fact makes delicate the passing to the limit, as $\varepsilon \to 0$ and the convergence to non-zero limits require the presence of suitable weight coefficients. As shown in [4], for the two-dimensional case with the critical size (1) in order to get a good balance in the heterogeneous boundary condition we need to introduce the weight

$$\beta(\varepsilon) = \exp(\frac{\alpha^2}{\varepsilon}). \tag{2}$$

Then, the control problem we will consider in this paper can be stated in the following terms: given $f \in L^2(Q^T)$ and $a \in C^\infty([-l,l])$, $a(x_1) \geq a_0 = const > 0$, we consider

$$\begin{cases} \partial_t u_\varepsilon(v) - \Delta u_\varepsilon(v) = f(x,t), & (x,t) \in Q^T, \\ \partial_\nu u_\varepsilon(v) + \beta(\varepsilon) a(x_1) u_\varepsilon(v) = \beta(\varepsilon) v, & (x,t) \in l_\varepsilon^T, \\ \partial_\nu u_\varepsilon(v) = 0, & (x,t) \in \gamma_\varepsilon^T, \\ u_\varepsilon(v)(x,0) = 0, & x \in \Omega, \\ u_\varepsilon(v)(x,t) = 0, & (x,t) \in \Gamma_1^T, \end{cases} \tag{3}$$

where $v \in L^2(0,T;L^2(l_\varepsilon))$ is the control and ν is the unit outward normal vector to the related surfaces. Notice that, thanks to the linearity of the model, the fact that we assume zero-initial data (on Ω and on l_ε) is not restrictive since a simple change of variable may reduce other situations to this assumption. We emphasize that the fact that the control is located on the reactive part of the boundary makes the study different from when the control is inside the spatial domain or on the non-reactive part of the boundary.

Our main goal is to prove the "approximate controllability" of the problem with a final observation $u_\varepsilon(v)(\cdot,T)$ on the whole domain Ω. This property can be stated in the following terms: given a "target global state", $u_T \in L^2(\Omega)$ (some extra regularity will be assumed in some intermediate steps), and given $\delta > 0$, we want to show the existence of a control $v \in L^2(0,T;L^2(l_\varepsilon))$ such that we have the estimate

$$\|u_\varepsilon(v)(\cdot,T) - u_T\|^2_{L^2(\Omega_\varepsilon)} \leq \delta. \tag{4}$$

In this way, local actions (the control is placed only on a small part of the boundary on which we assume a reaction term $\beta(\varepsilon)a(x_1)u_\varepsilon(v)$) lead to global consequences ($u_\varepsilon(v)(x,T)$ "almost" reach the desired value $u_T(x)$ in the whole domain Ω_ε). As in [10], we could associate this property as a closed goal to the ecologist sentence "think globally, act locally".

Note, while there are several results in the literature on "approximate controllability" in domains with periodic particles (or perforations), our formulation differs from all of them as in the present paper the controls are applied only on a small part of reactive subsets of the boundary (see, e.g., [1], [2], [18] and its references). Furthermore, the proof techniques employed in some other studies for simpler spatial domains either assume that the control is applied to the entire boundary of the domain (see, e.g., [16]) or they rely on auxiliary results (the unique continuation property), which application may become very subtle when ε is very small (see [14]).

As in [10], we will use the homogenization process as a tool, but with important changes with respect to [10]. To constructively prove the existence of the desired control, we will follow an idea of Jacques-Louis Lions which consists in the consideration of an auxiliary optimal control problem (see [14], [17]). In our case, we first consider a time dependent target function $u_T \in H^1(0,T;H^1_0(\Omega)) \bigcap C(\overline{Q^T})$, and if we denote by $u_\varepsilon(v)$ the solution to the above parabolic problem, the optimal control problem we will consider is completed by introducing the cost functional $J_\varepsilon : L^2(0,T;L^2(l^T_\varepsilon)) \to \mathbb{R}$ defined by

$$\begin{aligned} J_\varepsilon(v) = \frac{\theta_1}{2}\|\nabla(u_\varepsilon(v) - u_T)\|^2_{L^2(Q^T)} + \frac{\theta_2}{2}\int\limits_{\Omega}(u_\varepsilon(v)(x,T) - u_T(x,T))^2 dx \\ + \beta(\varepsilon)\frac{N}{2}\int\limits_{l^T_\varepsilon} v^2 dx_1 dt. \end{aligned} \tag{5}$$

Here, we assume the penalty term such that $N \in (0,+\infty)$, and $\theta_1, \theta_2 \geq 0$. Notice that when $\theta_1 = 0$ and $\theta_2 > 0$ the cost functional is well-defined for more general target functions $u_T \in L^2(\Omega)$. Although our main application concerns the case $\theta_1 = 0$, in some parts of the paper we will assume the general case $\theta_1 \geq 0$ since problems for which $\theta_1 > 0$ have been considered also in the previous literature and we will get here some extension of their results (see, e.g. [7], [23], and [24]).

It is well known that there exists a unique optimal pair $(u_\varepsilon(v_\varepsilon), v_\varepsilon)$ such that

$$J_\varepsilon(v_\varepsilon) = \min_{v \in L^2(l^T_\varepsilon)} J_\varepsilon(v). \tag{6}$$

One of the aims of this paper is to find the limit as $\varepsilon \to 0$ of the optimal control v_ε and of the cost functional $J_\varepsilon(v_\varepsilon)$.

Our strategy consists of several steps: firstly, we will apply the homogenization process, proving that the extension of $\tilde{u}_\varepsilon$ onto Q^T, converges, as $\varepsilon \to 0$, to a function $u_0(x,t)$

which is the unique solution of the global parabolic problem with a Robin-type boundary condition on Γ_2^T

$$\begin{cases} \partial_t u_0(v) - \Delta u_0(v) = f, & (x,t) \in Q^T, \\ \partial_\nu u_0(v) + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} u_0(v) = \frac{\pi \mathcal{C}}{\alpha^2 b_2(x_1)} v, & (x,t) \in \Gamma_2^T, \\ u_0(v) = 0, & (x,t) \in \Gamma_1^T, \\ u_0(v)(x,0) = 0, & x \in \Omega, \end{cases} \tag{7}$$

where

$$\mathcal{C} = \frac{\pi}{2 l_0 C_0 \alpha^2}, \quad b_1(x_1) = a(x_1) + \mathcal{C}, \quad b_2(x_1) = b_1^2(x_1) + \theta_1 N^{-1} \mathcal{C}.$$

We will also prove that the limit cost functional $J_0 : L^2(\Gamma_2^T) \to \mathbb{R}$, i.e. such that $\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0)$, is defined in the following terms

$$J_0(v) = \frac{\theta_1}{2} \int_{Q^T} |\nabla(u_0(v) - u_T)|^2 dx dt + \frac{\theta_2}{2} \int_{\Omega} (u_0(v)(x,T) - u_T(x,T))^2 dx +$$
$$+ \frac{\pi \theta_1}{2\alpha^2} \int_{\Gamma_2^T} \frac{a^2(x_1)}{b_2(x_1)} u_0^2(v) dx_1 dt + \frac{\pi N \mathcal{C}}{2\alpha^2} \int_{\Gamma_2^T} \frac{v^2}{b_2(x_1)} dx_1 dt. \tag{8}$$

We then consider the optimal control problem

$$J_0(v_0) = \min_{v \in L^2(\Gamma_2^T)} J_0(v). \tag{9}$$

Notice that coefficient $b_2(x)$ depends on θ_1 and that no dependence with respect θ_2 arises in the coefficients of the homogenized problem. Obviously, this is due to the fact that the optimal control v_ε depends on the cost J_ε (the coefficient of the homogenized equation is different when the right hand side of the Robin-type boundary condition is a given function g_ε(see, e.g. [5] and [9]).

The second step of our strategy is to prove the approximate controllability of the homogenized parabolic problem with final observation: given the target global state, now $u_T \in L^2(\Omega)$, and given $\delta > 0$, we will show the existence of a control $v \in L^2(\Gamma_2^T)$ such that

$$\|u_0(v)(\cdot, T) - u_T\|^2_{L^2(\Omega)} \leq \delta. \tag{10}$$

We will construct such a control by taking $N \searrow 0$ in the global formulation of the optimal control problem for the case $\theta_1 = 0$ and $\theta_2 = 1$.

Finally, as a third step, we will get the approximate controllability on the original problem, once we assume ε small enough. In conclusion, we must assume that the places where the control v_ε is applied must be large enough (such as the ecologist philosophy proclaims). Notice that our conclusion brings to light some limitations that must be assumed on the local controls to conclude that the result is globally satisfactory. For instance, the presence of the term $\beta(\varepsilon)$ in the local formulation (in the local Robin boundary condition and in the local cost functional $J_\varepsilon(v)$) is of capital importance since it is not difficult to show that without them the global optimal control limit problem is entirely different. As mentioned before, the critical relation between problem's parameters leads to the emergence of some "strange" terms in the limit state problem along with the new term in the limit cost functional and here has an important consequence.

Our main technique of proof consists in characterizing the optimal control v_ε in terms of p_ε, the solution to the related adjoint problem. We will show that this relation is given by the expression $v_\varepsilon = -N^{-1}p_\varepsilon$ a.e. on l_ε^T, where N is the positive constant appearing in the local cost functional J_ε.

The organization of this paper is as follows: Section 2 is devoted to a more precise presentation of the local optimal control problem and to obtain some a priori estimates which will be used later. The proof of the homogenization process is given in Section 3. Section 4 is devoted to prove the convergence of the cost functionals. Finally, the approximate controllability property is stated and proved in Section 5.

2. ON THE OPTIMAL CONTROL PROBLEM

We recall that by $H^1(\Omega, \Gamma_1)$ we denote the closure with respect to the norm $H^1(\Omega)$ of infinitely differentiable in $\overline{\Omega}$ functions vanishing near the boundary Γ_1. We say that a function $u_\varepsilon(v) \in L^2(0,T; H^1(\Omega,\Gamma_1))$ with $\partial_t u_\varepsilon(v) \in L^2(0,T; H^{-1}(\Omega,\Gamma_1))$ and $u_\varepsilon(v)(x,0) = 0$ is a weak solution to the problem (3) if it satisfies the integral identity

$$\begin{aligned}\int\limits_0^T \langle \partial_t u_\varepsilon(v), \phi\rangle dt + \int\limits_{Q^T} \nabla u_\varepsilon(v)\nabla\phi dxdt + \beta(\varepsilon)\int\limits_{l_\varepsilon^T} a(x) u_\varepsilon \phi dx_1 dt = \\ = \int\limits_{Q^T} f\phi dxdt + \beta(\varepsilon)\int\limits_{l_\varepsilon^T} v(x,t)\phi(x,t)dx_1 dt,\end{aligned} \tag{11}$$

where ϕ is an arbitrary function from $L^2(0,T; H^1(\Omega,\Gamma_1))$.

The adjoint problem associated with the state problem (1) takes the form

$$\begin{cases} -\partial_t p_\varepsilon - \Delta p_\varepsilon = -\theta_1 \Delta(u_\varepsilon - u_T), & (x,t)\in Q^T, \\ \partial_\nu p_\varepsilon + \beta(\varepsilon) a(x_1) p_\varepsilon = \theta_1 \partial_\nu (u_\varepsilon - u_T), & (x,t) \in l_\varepsilon^T, \\ \partial_\nu p_\varepsilon = 0, & (x,t)\in \gamma_\varepsilon^T, \\ p_\varepsilon(x,t) = 0, & (x,t)\in \Gamma_1^T, \\ p_\varepsilon(x,T) = \theta_2(u_\varepsilon(x,T) - u_T(x,T)), & x\in\Omega. \end{cases} \tag{12}$$

We say that a function $p_\varepsilon \in L^2(0,T; H^1(\Omega,\Gamma_1))$ with $\partial_t p_\varepsilon \in L^2(0,T; H^{-1}(\Omega,\Gamma_1))$ and $p_\varepsilon(x,T) = \theta_2(u_\varepsilon(x,T) - u_T(x,T))$ is a weak solution of the problem (12) if it satisfies the integral identity

$$\begin{aligned} -\int\limits_0^T \langle \partial_t p_\varepsilon, \phi\rangle dt + \int\limits_{Q^T} \nabla p_\varepsilon \nabla\phi dxdt + \\ +\beta(\varepsilon)\int\limits_{l_\varepsilon^T} a(x) p_\varepsilon \phi dx_1 dt = \theta_1 \int\limits_{Q^T} \nabla(u_\varepsilon - u_T)\nabla\phi dxdt,\end{aligned} \tag{13}$$

for an arbitrary $\phi \in L^2(0,T; H^1(\Omega,\Gamma_1))$.

Theorem 1. *If the pair of functions* $(u_\varepsilon, v_\varepsilon)$ *is optimal for the problem* (3)-(6)*, then* $v_\varepsilon = -N^{-1}p_\varepsilon$*, where* p_ε *is a weak solution to the problem* (12)*.*

Proof. For $\lambda > 0$, we denote by $v_\varepsilon^\lambda = v_\varepsilon + \lambda v$. We have

$$\frac{J_\varepsilon(v_\varepsilon^\lambda) - J_\varepsilon(v_\varepsilon)}{\lambda} = \frac{\theta_1}{2\lambda}(\|\nabla(u_\varepsilon(v_\varepsilon^\lambda) - u_T)\|^2_{L^2(Q^T)} - \|\nabla(u_\varepsilon(v_\varepsilon) - u_T)\|^2_{L^2(Q^T)}) +$$

$$+\frac{\theta_2}{2\lambda}(\|u_\varepsilon(v_\varepsilon^\lambda)(\cdot,T) - u_T(\cdot,T)\|^2_{L^2(\Omega)} - \|u_\varepsilon(v_\varepsilon)(\cdot,T) - u_T(\cdot,T)\|^2_{L^2(\Omega)}) +$$

$$+\beta(\varepsilon)\frac{N}{2\lambda}\int\limits_{l_\varepsilon^T}((v_\varepsilon^\lambda)^2 - v_\varepsilon^2)dx_1dt =$$

$$= \frac{\theta_1}{2}\int\limits_{Q^T}\nabla\frac{(u_\varepsilon(v_\varepsilon^\lambda) - u_\varepsilon(v_\varepsilon))}{\lambda}\nabla(u_\varepsilon(v_\varepsilon^\lambda) + u_\varepsilon(v_\varepsilon) - 2u_T)dxdt +$$

$$+\frac{\theta_2}{2}\int\limits_{\Omega}\frac{(u_\varepsilon(v_\varepsilon^\lambda)(\cdot,T) - u_\varepsilon(v_\varepsilon)(\cdot,T))}{\lambda}(u_\varepsilon(v_\varepsilon^\lambda)(\cdot,T) + u_\varepsilon(v_\varepsilon)(\cdot,T) - 2u_T(\cdot,T))dx +$$

$$+\beta(\varepsilon)\frac{N}{2}\int\limits_{l_\varepsilon^T}(2v_\varepsilon v + \lambda v^2)dx_1dt.$$

From here, we get the optimality condition

$$\lim_{\lambda\to 0}\frac{J_\varepsilon(v_\varepsilon + \lambda v) - J_\varepsilon(v_\varepsilon)}{\lambda} = \theta_1\int\limits_{Q^T}\nabla\theta_\varepsilon\nabla(u_\varepsilon(v_\varepsilon) - u_T)dxdt +$$

$$+\theta_2\int\limits_{\Omega}\theta_\varepsilon(x,T)(u_\varepsilon(v_\varepsilon)(x,T) - u_T(x,T))dx + \beta(\varepsilon)N\int\limits_{l_\varepsilon^T}v_\varepsilon v dx_1dt = 0, \qquad (14)$$

where $\theta_\varepsilon = (u_\varepsilon(v_\varepsilon^\lambda) - u_\varepsilon(v_\varepsilon))/\lambda$ is the unique solution of the problem

$$\begin{cases} \partial_t\theta_\varepsilon - \Delta\theta_\varepsilon = 0, & (x,t) \in Q^T, \\ \partial_\nu\theta_\varepsilon + \beta(\varepsilon)a(x_1)\theta_\varepsilon = \beta(\varepsilon)v, & (x,t) \in l_\varepsilon^T, \\ \partial_\nu\theta_\varepsilon = 0, & (x,t) \in \gamma_\varepsilon^T, \\ \theta_\varepsilon = 0, & (x,t) \in \Gamma_1^T, \\ \theta_\varepsilon(x,0) = 0, & x \in \Omega. \end{cases}$$

Note, θ_ε does not depend on λ. We say that a function $\theta_\varepsilon \in L^2(0,T;H^1(\Omega,\Gamma_1))$ with $\partial_t\theta_\varepsilon \in L^2(0,T;H^{-1}(\Omega,\Gamma_1))$ and $\theta_\varepsilon(x,0) = 0$ is a weak solution of this problem, if it satisfies the following integral identity

$$\int\limits_0^T\langle\partial_t\theta_\varepsilon,\psi\rangle_\Omega dt + \int\limits_{Q^T}\nabla\theta_\varepsilon\nabla\psi dxdt + \beta(\varepsilon)\int\limits_{l_\varepsilon^T}a(x)\theta_\varepsilon\psi dx_1dt = \beta(\varepsilon)\int\limits_{l_\varepsilon^T}v\psi dx_1dt$$

for an arbitrary $\psi \in L^2(0,T;H^1(\Omega,\Gamma_1))$. Taking as a test-function θ_ε in the integral identity (13) and p_ε as a test-function in the integral identity for θ_ε, we subtract one from the other and get

$$\theta_2\int\limits_{\Omega}(u_\varepsilon(x,T)-u_T(x,T))\theta_\varepsilon(x,T)dx=\beta(\varepsilon)\int\limits_{l_\varepsilon^T}vp_\varepsilon dx_1dt-\theta_1\int\limits_{Q^T}\nabla(u_\varepsilon-u_T)\nabla\theta_\varepsilon dxdt.$$

From here and (14), we deduce

$$\beta(\varepsilon)\int\limits_{l_\varepsilon^T}vp_\varepsilon dx_1dt+N\beta(\varepsilon)\int\limits_{l_\varepsilon^T}v_\varepsilon v dx_1dt=0,$$

where $v\in L^2(l_\varepsilon^T)$ is an arbitrary function. Hence $v_\varepsilon=-N^{-1}p_\varepsilon$ a.e. on l_ε^T. ∎

Thus, the optimal control is characterized by the system of equations

$$\begin{cases}\partial_t u_\varepsilon-\Delta u_\varepsilon=f(x,t), & (x,t)\in Q^T,\\ -\partial_t p_\varepsilon-\Delta p_\varepsilon=-\theta_1\Delta(u_\varepsilon-u_T), & (x,t)\in Q^T,\\ \partial_\nu u_\varepsilon+\beta(\varepsilon)a(x)u_\varepsilon=-\beta(\varepsilon)N^{-1}p_\varepsilon, & (x,t)\in l_\varepsilon^T,\\ \partial_\nu p_\varepsilon+\beta(\varepsilon)a(x)p_\varepsilon=\theta_1\partial_\nu(u_\varepsilon-u_T), & (x,t)\in l_\varepsilon^T.\\ \partial_\nu p_\varepsilon=0,\ \partial_\nu u_\varepsilon=0, & (x,t)\in\gamma_\varepsilon^T,\\ p_\varepsilon(x,t)=u_\varepsilon(x,t), & (x,t)\in\Gamma_1^T,\\ u_\varepsilon(x,0)=0,\ p_\varepsilon(x,T)=\theta_2(u_\varepsilon(x,T)-u_T(x,T)), & x\in\Omega,\end{cases}\tag{15}$$

2.1. **Uniform in ε estimates of u_ε and v_ε.** Taking in the integral identity for u_ε as a test function p_ε, and using u_ε as a test-function in the integral identity for p_ε, and, subtracting one identity from the other, we get

$$\int\limits_0^T\Big(\langle\partial_t u_\varepsilon,p_\varepsilon\rangle+\langle\partial_t p_\varepsilon,u_\varepsilon\rangle\Big)dt=$$
$$=\int\limits_{Q^T}fp_\varepsilon dxdt-\beta(\varepsilon)N^{-1}\int\limits_{l_\varepsilon^T}p_\varepsilon^2dx_1dt-\theta_1\int\limits_{Q^T}\nabla(u_\varepsilon-u_T)\nabla u_\varepsilon dxdt.$$

From here, we conclude

$$\theta_2\int\limits_{\Omega}(u_\varepsilon(x,T)-u_T(x,T))^2dx+\beta(\varepsilon)N^{-1}\int\limits_{l_\varepsilon^T}p_\varepsilon^2dx_1dt+\theta_1\int\limits_{Q^T}|\nabla(u_\varepsilon-u_T)|^2dxdt=$$
$$=\int\limits_{Q^T}fp_\varepsilon dxdt-\theta_2\int\limits_{\Omega}u_T(x,T)(u_\varepsilon(x,T)-u_T(x,T))dx-\theta_1\int\limits_{Q^T}\nabla(u_\varepsilon-u_T)\nabla u_T dxdt.\tag{16}$$

Thus, we have

$$\theta_1\|\nabla(u_\varepsilon-u_T)\|^2_{L^2(Q^T)}+\theta_2\|u_\varepsilon(x,T)-u_T(x,T)\|^2_{L^2(\Omega)}+\beta(\varepsilon)N^{-1}\int\limits_{l_\varepsilon^T}p_\varepsilon^2dx_1dt\le$$
$$\le\int\limits_{Q^T}|f||p_\varepsilon|dxdt+C\theta_1\|\nabla u_T\|^2_{L^2(Q^T)}+C\theta_2\|u_T(x,T)\|^2_{L^2(\Omega)}.\tag{17}$$

Taking in (13) as a test function p_ε, we get

$$-\int_0^T \langle \partial_t p_\varepsilon, p_\varepsilon \rangle dt + \|\nabla p_\varepsilon\|^2_{L^2(Q^T)} + \beta(\varepsilon) \int_{l^T_\varepsilon} a(x) p^2_\varepsilon dx_1 dt =$$

$$= \theta_1 \int_{Q^T} \nabla(u_\varepsilon - u_T) \nabla p_\varepsilon dx dt.$$

From here, we derive

$$\begin{gathered} \|\nabla p_\varepsilon\|^2_{L^2(Q^T)} + a_0 \beta(\varepsilon) \|p_\varepsilon\|^2_{L^2(l^T_\varepsilon)} \leq \\ \leq C\theta_1^2 \|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q^T)} + C\theta_2^2 \|u_\varepsilon(x,T) - u_T(x,T)\|^2_{L^2(\Omega)}. \end{gathered} \tag{18}$$

Thus, we have

$$\begin{gathered} \|\nabla p_\varepsilon\|^2_{L^2(Q^T)} + a_0 \beta(\varepsilon) \|p_\varepsilon\|^2_{L^2(l^T_\varepsilon)} \leq \\ \leq C\Big(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \|u_T(x,T)\|^2_{L^2(\Omega)} \Big), \end{gathered} \tag{19}$$

and from (17), we conclude

$$\begin{gathered} \|\nabla(u_\varepsilon - u_T)\|^2_{L^2(Q^T)} + \|u_\varepsilon(x,T) - u_T(x,T)\|_{L^2(\Omega)} + \beta(\varepsilon) N^{-1} \|p_\varepsilon\|^2_{L^2(l^T_\varepsilon)} \leq \\ \leq C\Big(\|f\|^2_{L^2(Q^T)} + \|\nabla u_T\|^2_{L^2(Q^T)} + \|u_T(x,T)\|_{L^2(\Omega)} \Big). \end{gathered} \tag{20}$$

Let us get the estimates

$$\|\partial_t u_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega,\Gamma_1))} \leq K, \;\; \|\partial_t p_\varepsilon\|_{L^2(0,T;H^{-1}(\Omega,\Gamma_1))} \leq K. \tag{21}$$

We need an auxiliary estimate

$$\int_{l^T_\varepsilon} u^2(x_1,0,t) dx_1 dt \leq K\Big(\int_0^T \int_{S^+_{a_\varepsilon l_0}} u^2 ds dt + a_\varepsilon \int_{\Pi^T_\varepsilon} u^2_{x_2} dx_1 dx_2 dt \Big), \tag{22}$$

where

$$S^+_{a_\varepsilon l_0} = \bigcup_{j \in \Upsilon_\varepsilon} (\partial T^j_{a_\varepsilon l_0})^+, \;\; \Pi_\varepsilon = (-l,l) \times (0,\varepsilon), \;\; \Pi^T_\varepsilon = \Pi_\varepsilon \times (0,T),$$

and T^j_r is a ball of radius r centered at the point $(\varepsilon j, 0)$, $(\partial T^j_r)^+ = \partial T^j_r \cap \{x_2 > 0\}$.

Indeed, applying the divergence theorem, we get

$$\int_{\partial (T^j_{a_\varepsilon l_0})^+} f \nu_2 ds = \int_{(T^j_{a_\varepsilon l_0})^+} f_{x_2} dx_1 dx_2.$$

Taking $f = \exp\big(\frac{x_2}{a_\varepsilon l_0}\big) u^2$, we obtain

$$\int_{\partial (T^j_{a_\varepsilon l_0})^+} \exp\Big(\frac{x_2}{a_\varepsilon l_0}\Big) u^2 \nu_2 ds = \int_{(T^j_{a_\varepsilon l_0})^+} \Big(2\exp\Big(\frac{x_2}{a_\varepsilon l_0}\Big) u u_{x_2} + a_\varepsilon^{-1} l_0^{-1} \exp\Big(\frac{x_2}{a_\varepsilon l_0}\Big) u^2 \Big) dx_1 dx_2.$$

Using that $\nu_2 = -1$ if $x_2 = 0$, we get

$$\int\limits_{l_\varepsilon^j} u^2 dx_1 + a_\varepsilon^{-1} l_0^{-1} \int\limits_{(T^j_{a_\varepsilon l_0})^+} \exp(\frac{x_2}{a_\varepsilon l_0}) u^2 dx_1 dx_2 =$$

$$= \int\limits_{(\partial T^j_{a_\varepsilon l_0})^+} \exp(\frac{x_2}{a_\varepsilon l_0}) u^2 \nu_2 ds - \int\limits_{(T^j_{a_\varepsilon l_0})^+} 2\exp(\frac{x_2}{a_\varepsilon l_0}) u u_{x_2} dx_1 dx_2.$$

Note that $\exp(\frac{x_2}{a_\varepsilon l_0}) \leq e$, $|\nu_2| \leq 1$ and $2|u||u_{x_2}| \leq a_\varepsilon^{-1} l_0^{-1} u^2 + a_\varepsilon l_0 |u_{x_2}|^2$. So, we have

$$\int\limits_{l_\varepsilon^j} u^2 dx_1 \leq e \int\limits_{(\partial T^j_{a_\varepsilon l_0})^+} u^2 ds + e a_\varepsilon l_0 \int\limits_{(T^j_{a_\varepsilon l_0})^+} |u_{x_2}|^2 dx_1 dx_2. \tag{23}$$

Summing the inequality (23) over all $j \in \Upsilon_\varepsilon$ and integrating with respect to t from 0 to T, we get (22).

Now, we use the inequality (6) from [22] and obtain

$$\int\limits_{l_\varepsilon^T} u^2(x_1, 0, t) dx_1 dt \leq K\{a_\varepsilon \varepsilon^{-2} \|u\|^2_{L^2(\Pi_\varepsilon^T)} + a_\varepsilon \varepsilon^{-1} \|\nabla u\|^2_{L^2(\Pi_\varepsilon^T)}\}. \tag{24}$$

From (24), we derive

$$\beta(\varepsilon) \int\limits_{l_\varepsilon^T} u^2(x_1, 0, t) dx_1 dt \leq K\{\varepsilon^{-1} \|u\|^2_{L^2(\Pi_\varepsilon^T)} + \|\nabla u\|^2_{L^2(\Pi_\varepsilon^T)}\}. \tag{25}$$

Taking into account the properties of the trace of a function from $H^1(\Omega)$, we have

$$\beta(\varepsilon) \int\limits_{l_\varepsilon^T} u^2(x_1, 0, t) dx_1 dt \leq K \|u\|^2_{L^2(0,T;H^1(\Omega,\Gamma_1))}. \tag{26}$$

Now, we can get the estimate (21). Consider Galerkin's approximations for u_ε and p_ε

$$u_\varepsilon^m = \sum_{k=1}^m a^\varepsilon_{k,m}(t) w^k(x), \;\; p_\varepsilon^m = \sum_{k=1}^m b^\varepsilon_{k,m}(t) w^k(x),$$

where $\{w^k(x)\}$ is orthogonal basis in $H^1(\Omega, \Gamma_1)$ and orthonormal basis in $L^2(\Omega)$. For an arbitrary function $v \in L^2(0,T; H^1(\Omega, \Gamma_1))$ such that $\|v\|_{H^1(\Omega,\Gamma_1)} \leq 1$ for a.e. $t \in (0,T)$. Let us represent $v(\cdot, t) = v_{1,m} + v_{2,m}$, and $v_{1,m} \in \langle w^1, \ldots, w^m \rangle$, $(v_{2,m}, w^k) = 0$, $k = 1, 2, \ldots, m$. Hence, for a.e. $t \in (0,T)$, we have

$$\langle \partial_t u_\varepsilon^m, v \rangle = \langle \partial_t u_\varepsilon^m, v_{1,m} \rangle = -\int\limits_\Omega \nabla u_\varepsilon^m \nabla v_{1,m} dx - \beta(\varepsilon) \int\limits_{l_\varepsilon} a(x) u_\varepsilon^m v_{1,m} dx_1 +$$

$$+ \int\limits_\Omega f v_{1,m} dx - \beta(\varepsilon) N^{-1} \int\limits_{l_\varepsilon} p_\varepsilon^m v_{1,m} dx_1, \tag{27}$$

and

$$\langle \partial_t p_\varepsilon^m, v \rangle = \langle \partial_t p_\varepsilon^m, v_{1,m} \rangle = \int\limits_\Omega \nabla p_\varepsilon^m \nabla v_{1,m} dx +$$

$$+\beta(\varepsilon)\int\limits_{l_\varepsilon} a(x)p_\varepsilon^m v_{1,m}dx_1-\theta_1\int\limits_{\Omega}\nabla(u_\varepsilon^m-u_T)\nabla v_{1,m}dx. \tag{28}$$

Note that for u_ε^m, p_ε^m we have the following estimates

$$\|\nabla u_\varepsilon^m\|^2_{L^2(Q^T)}+\max_{[0,T]}\|u_\varepsilon^m\|^2_{L^2(\Omega)}+\beta(\varepsilon)\|u_\varepsilon^m\|^2_{L^2(l_\varepsilon^T)}\le$$
$$\le K(\|f\|^2_{L^2(Q^T)}+\|\nabla u_T\|^2_{L^2(Q^T)}+\max_{[0,T]}\|u_T\|^2_{L^2(\Omega)}), \tag{29}$$

and

$$\|\nabla p_\varepsilon^m\|^2_{L^2(Q^T)}+\max_{[0,T]}\|p_\varepsilon^m\|^2_{L^2(\Omega)}+\beta(\varepsilon)\|p_\varepsilon^m\|^2_{L^2(l_\varepsilon^T)}\le$$
$$\le K(\|f\|^2_{L^2(Q^T)}+\|\nabla u_T\|^2_{L^2(Q^T)}+\max_{[0,T]}\|u_T\|^2_{L^2(\Omega)}). \tag{30}$$

From (27)-(30), we get

$$|\langle\partial_t u_\varepsilon^m,v\rangle|^2\le K\Big(\|\nabla u_\varepsilon^m\|^2_{L^2(\Omega)}+\beta^2(\varepsilon)\int\limits_{l_\varepsilon}|u_\varepsilon^m|^2dx_1\int\limits_{l_\varepsilon}v_{1,m}^2dx_1+$$
$$+\|f\|^2_{L^2(\Omega)}+\beta^2(\varepsilon)\int\limits_{l^\varepsilon}|p_\varepsilon^m|^2dx_1\int\limits_{l_\varepsilon}v_{1,m}^2dx_1\Big), \tag{31}$$

and

$$|\langle\partial_t p_\varepsilon^m,v\rangle|^2\le K\Big(\|\nabla p_\varepsilon^m\|^2_{L^2(\Omega)}+\beta^2(\varepsilon)\int\limits_{l_\varepsilon}|p_\varepsilon^m|^2dx_1\int\limits_{l_\varepsilon}v_{1,m}^2dx_1+$$
$$+\|\nabla(u_\varepsilon^m-u^T)\|^2_{L^2(\Omega)}\|v_{1,m}\|^2_{H^1(\Omega,\Gamma_1)}\Big). \tag{32}$$

From (26), (29)-(32), we derive the estimates

$$\|\partial_t u_\varepsilon^m\|_{L^2(0,T;H^{-1}(\Omega,\Gamma_1))}\le K,\ \ \|\partial_t p_\varepsilon^m\|_{L^2(0,T;H^{-1}(\Omega,\Gamma_1))}\le K,$$

where K does not depend on m and ε. Taking into account this estimations, we get inequalities (21).

Using estimates (19)-(21), we get that there is a subsequence (still denoted by ε) such that as $\varepsilon\to 0$

$$\begin{gathered}u_\varepsilon\rightharpoonup u_0 \text{ weakly in } L^2(0,T;H^1(\Omega,\Gamma_1)),\\ p_\varepsilon\rightharpoonup p_0 \text{ weakly in } L^2(0,T;H^1(\Omega,\Gamma_1)),\\ \partial_t u_\varepsilon\rightharpoonup \partial_t u_0 \text{ weakly in } L^2(0,T;H^{-1}(\Omega,\Gamma_1)),\\ \partial_t p_\varepsilon\rightharpoonup \partial_t p_0 \text{ weakly in } L^2(0,T;H^{-1}(\Omega,\Gamma_1)).\end{gathered} \tag{33}$$

3. Statement of the main result

Theorem 2. *Let $n=2$, $a_\varepsilon=C_0\varepsilon e^{-\alpha^2/\varepsilon}$, $\beta(\varepsilon)=e^{\alpha^2/\varepsilon}$, $f\in L^2(Q^T)$ and the pair $(u_\varepsilon,p_\varepsilon)$ is a solution to the system* (15). *Then, the pair (u_0,p_0) defined by* (33) *is a solution to* *the system*

$$\begin{cases} \partial_t u_0 - \Delta u_0 = f, & (x,t) \in Q^T, \\ -\partial_t p_0 - \Delta p_0 = -\theta_1 \Delta(u_0 - u_T), & (x,t) \in Q^T, \\ \partial_\nu u_0 + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} u_0 = -\frac{\pi N^{-1}\mathcal{C}}{\alpha^2 b_2(x_1)} p_0, & (x,t) \in \Gamma_2^T, \\ \partial_\nu p_0 + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} p_0 = \theta_1 \partial_\nu (u_0 - u_T) + \frac{\pi \theta_1 a^2(x_1)}{\alpha^2 b_2(x_1)} u_0, & (x,t) \in \Gamma_2^T, \\ u_0(x,t) = p_0(x,t) = 0, & (x,t) \in \Gamma_1^T, \\ u_0(x,0) = 0,\ p_0(x,T) = \theta_2(u_0(x,T) - u_T(x,T)), & x \in \Omega. \end{cases} \tag{34}$$

where

$$\mathcal{C} = \frac{\pi}{2 l_0 C_0 \alpha^2}, \quad b_1(x_1) = a(x_1) + \mathcal{C}, \quad b_2(x_1) = b_1^2(x_1) + \theta_1 N^{-1}\mathcal{C}.$$

As we will show, the pair $(u_0(v), v)$ characterizes the optimal control of the homogenized state problem associated to a suitable cost functional. The state problem is given by

$$\begin{cases} \partial_t u_0(v) - \Delta u_0(v) = f, & (x,t) \in Q^T, \\ \partial_\nu u_0(v) + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} u_0(v) = \frac{\pi \mathcal{C}}{\alpha^2 b_2(x_1)} v, & (x,t) \in \Gamma_2^T, \\ u_0(v) = 0, & (x,t) \in \Gamma_1^T, \\ u_0(v)(x,0) = 0, & x \in \Omega. \end{cases} \tag{35}$$

Now, we introduce the limit cost functional

$$J_0(v) = \frac{\theta_1}{2} \int\limits_{Q^T} |\nabla(u_0(v) - u_T)|^2 dxdt + \frac{\theta_2}{2} \int\limits_{\Omega} (u_0(v)(x,T) - u_T(x,T))^2 dx +$$

$$+ \frac{\pi \theta_1}{2\alpha^2} \int\limits_{\Gamma_2^T} \frac{a^2(x_1)}{b_2(x_1)} u_0^2(v) dx_1 dt + \frac{\pi N \mathcal{C}}{2\alpha^2} \int\limits_{\Gamma_2^T} \frac{v^2}{b_2(x_1)} dx_1 dt, \tag{36}$$

and consider the optimal control problem

$$J_0(v_0) = \min_{v \in L^2(\Gamma_2^T)} J_0(v). \tag{37}$$

Theorem 3. *Under the conditions of Theorem 2, we have*

$$\lim_{\varepsilon \to 0} J_\varepsilon(v_\varepsilon) = J_0(v_0).$$

Remark 1. *Similar to Theorem 1, it is easy to show that* $v_0 = -N^{-1} p_0$.

4. PROOF OF THE HOMOGENIZATION THEOREM

We introduce two auxiliary problems for $j \in \Upsilon_\varepsilon$

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T_{\varepsilon/4}^j \setminus \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 1, & x \in \partial T_{a_\varepsilon}^j, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j, \end{cases} \tag{38}$$

and

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in T_{\varepsilon/4}^j \setminus \overline{l_\varepsilon^j}, \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ q_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j. \end{cases} \tag{39}$$

Note that w_ε^j and q_ε^j are solutions of the problems

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ w_\varepsilon^j = 1, & x \in \partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} w_\varepsilon^j = 0, & x \in \{x_2 = 0\} \cap (T_{\varepsilon/4}^j \setminus \overline{T_{a_\varepsilon}^j}), \end{cases} \tag{40}$$

and

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ q_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j \cap \{x_2 = 0\}) \setminus \overline{l_\varepsilon^j}, \end{cases} \tag{41}$$

where $j \in \Upsilon_\varepsilon$, $l_\varepsilon^j = a_\varepsilon \hat{l}_0 + \varepsilon j$.

Let us introduce two functions

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \\ 1, & x \in (T_{a_\varepsilon}^j)^+, \\ 0, & x \in \Omega \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}, \end{cases} \tag{42}$$

and

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+,\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \setminus \bigcup\limits_{j \in \Upsilon_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases} \tag{43}$$

We have that W_ε, $Q_\varepsilon \in H^1(\Omega, \Gamma_1)$ and Q_ε, $W_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. In addition, the following lemma holds (see Lemma 1 in [4])

Lemma 1. *Let W_ε be a function defined by the formula* (42)*, Q_ε be a function defined by the formula* (43)*. Then*

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \le K\sqrt{\varepsilon}.$$

From this lemma, we conclude that $Q_\varepsilon \rightharpoonup 0$ weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \to 0$. Now, we take $Q_\varepsilon(x)\eta(t)\psi(x)$, where $\eta \in C^1([0,T])$, $\psi \in C^\infty(\overline{\Omega})$ vanishing near the boundary Γ_1 as a test-function in (11). We get

$$\int\limits_0^T \langle \partial_t u_\varepsilon, Q_\varepsilon \eta(t)\psi(x)\rangle dt + \int\limits_{Q^T} \nabla u_\varepsilon \nabla(Q_\varepsilon \eta(t)\psi(x)) dx dt + e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} a(x_1) u_\varepsilon \eta(t)\psi(x) dx_1 dt =$$

$$= \int\limits_{Q^T} f Q_\varepsilon \eta(t)\psi(x) dx dt - N^{-1} e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} p_\varepsilon \eta(t)\psi(x) dx_1 dt. \tag{44}$$

Using the properties of Q_ε and W_ε, we have

$$\int\limits_0^T \langle \partial_t u_\varepsilon, Q_\varepsilon \eta(t)\psi(x)\rangle dt = -\int\limits_0^T \int\limits_\Omega u_\varepsilon Q_\varepsilon \eta'(t)\psi(x) dx dt + \int\limits_\Omega u_\varepsilon(x,T) Q_\varepsilon(x)\eta(T)\psi(x) dx.$$

From here, we derive

$$\lim_{\varepsilon\to 0}\int\limits_0^T \langle \partial_t u_\varepsilon, Q_\varepsilon \eta(t)\psi(x)\rangle dt = 0. \tag{45}$$

Applying Lemma 1, we obtain

$$\begin{aligned} I_{1,\varepsilon} \equiv \int\limits_{Q^T} \nabla u_\varepsilon \nabla (Q_\varepsilon \eta(t)\psi(x)) dxdt = \int\limits_{Q^T} \nabla u_\varepsilon \nabla ((Q_\varepsilon - W_\varepsilon)\eta(t)\psi(x)) dxdt + \\ + \int\limits_{Q^T} \nabla u_\varepsilon \nabla (W_\varepsilon(x)\eta(t)\psi(x)) dxdt = \int\limits_{Q^T} \nabla W_\varepsilon \nabla (u_\varepsilon \eta(t)\psi(x)) dxdt + \alpha_\varepsilon, \end{aligned} \tag{46}$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.

Note that $w_\varepsilon^j(x) = \frac{\ln(4r/\varepsilon)}{\ln(4a_\varepsilon/\varepsilon)}$, $r = |x - P_\varepsilon^j|$, $\partial_\nu w_\varepsilon^j\Big|_{x\in \partial T^j{}_{\varepsilon/4}} = \frac{4}{\varepsilon \ln(\frac{4a_\varepsilon}{\varepsilon})} = -\frac{4}{\alpha^2} + \alpha_\varepsilon$, $\partial_\nu w_\varepsilon^j\Big|_{\partial T_{a_\varepsilon}^j} = \frac{e^{\alpha^2/\varepsilon}}{C_0\alpha^2} + e^{\alpha^2/\varepsilon}\alpha_\varepsilon$, where $\alpha_\varepsilon \to 0$, as $\varepsilon \to 0$. Thus from (46), we have

$$I_{1,\varepsilon} = \frac{e^{\frac{\alpha^2}{\varepsilon}}}{C_0\alpha^2} \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^T \int\limits_{(\partial T_{a_\varepsilon}^j)^+} u_\varepsilon \eta(t)\psi(x) dsdt - \frac{4}{\alpha^2} \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^T \int\limits_{(\partial T_{\varepsilon/4}^j)^+} u_\varepsilon \eta(t)\psi(x) dsdt + \alpha_\varepsilon, \tag{47}$$

where $\alpha_\varepsilon \to 0$, as $\varepsilon \to 0$.

Now, we apply the inequality (34) from [4]

$$\Big| e^{\alpha^2/\varepsilon} \frac{\pi}{2l_0} \int\limits_{l_\varepsilon^T} u_\varepsilon \eta(t)\psi(x) dx_1 dt - e^{\alpha^2/\varepsilon} \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^T \int\limits_{(\partial T_{a_\varepsilon}^j)^+} u_\varepsilon \eta(t)\psi(x) dsdt \Big| \leq K\sqrt{\varepsilon}. \tag{48}$$

From (48), we have

$$I_{1,\varepsilon} = \frac{\pi e^{\alpha^2/\varepsilon}}{2l_0 C_0 \alpha^2} \int\limits_{l_\varepsilon^T} u_\varepsilon \eta(t)\psi(x) dx_1 dt - \frac{4}{\alpha^2} \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^T \int\limits_{(\partial T_{\varepsilon/4}^j)^+} u_\varepsilon \eta(t)\psi(x) dsdt + \alpha_\varepsilon, \tag{49}$$

where $\alpha_\varepsilon \to 0$, $\varepsilon \to 0$

Using that

$$\int\limits_{Q^T} f Q_\varepsilon \eta(t)\psi(x) dxdt \to 0, \text{ as}\, \varepsilon \to 0,$$

from (44), (45), (49), we deduce

$$\begin{aligned} e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} (a(x) + \frac{\pi}{2l_0 C_0 \alpha^2}) u_\varepsilon \eta(t)\psi(x) dx_1 dt = \\ = \frac{4}{\alpha^2} \sum_{j\in\Upsilon_\varepsilon} \int\limits_0^T \int\limits_{(\partial T_{\varepsilon/4}^j)^+} u_\varepsilon \eta(t)\psi(x) dsdt - N^{-1} e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} p_\varepsilon \eta(t)\psi(x) dx_1 dt + \alpha_\varepsilon, \end{aligned}$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$.

Taking into account that (see [4], [5])

$$\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+} u_\varepsilon\psi(x)\eta(t)dsdt=\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}u_0\psi(x)\eta(t)dx_1dt+\alpha_\varepsilon,$$

we have (after the change $\psi\mapsto\psi/(a(x_1)+\frac{\pi}{2l_0C_0\alpha^2}))$

$$e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}u_\varepsilon\eta(t)\psi(x)dx_1dt=\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}u_0\eta(t)\frac{\psi(x)}{(a(x_1)+\frac{\pi}{2l_0C_0\alpha^2})}dx_1dt-$$
$$-N^{-1}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}p_\varepsilon\eta(t)\frac{\psi(x)}{(a(x_1)+\frac{\pi}{2l_0C_0\alpha^2})}dx_1dt+\alpha_\varepsilon, \tag{50}$$

where $\alpha_\varepsilon\to0$ as $\varepsilon\to0$.

Let us take $Q_\varepsilon\eta(t)\psi(x)$ as a test-function in the integral identity (13). We get

$$-\int\limits_0^T\langle\partial_tp_\varepsilon,Q_\varepsilon\eta(t)\psi(x)\rangle dt+\int\limits_{Q^T}\nabla p_\varepsilon\nabla(Q_\varepsilon\eta(t)\psi(x))dxdt+$$
$$+e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}a(x_1)p_\varepsilon\eta(t)\psi(x)dx_1dt=\theta_1\int\limits_{Q^T}\nabla(u_\varepsilon-u_T)\nabla(Q_\varepsilon\eta(t)\psi(x))dxdt. \tag{51}$$

Using the properties of Q_ε and applying the estimates of p_ε, we have

$$\lim_{\varepsilon\to0}\int\limits_0^T\langle\partial_tp_\varepsilon,Q_\varepsilon\eta(t)\psi(x)\rangle dt=0. \tag{52}$$

In addition to this, we have

$$I_{2,\varepsilon}\equiv\int\limits_{Q^T}\nabla p_\varepsilon\nabla(Q_\varepsilon\eta(t)\psi(x))dxdt=$$
$$=\int\limits_{Q^T}\nabla p_\varepsilon\nabla((Q_\varepsilon-W_\varepsilon)\eta(t)\psi(x))dxdt+\int\limits_{Q^T}\nabla p_\varepsilon\nabla(W_\varepsilon\eta(t)\psi(x))dxdt=$$
$$=\int\limits_{Q^T}\nabla W_\varepsilon\nabla(p_\varepsilon\eta(t)\psi(x))dxdt+\alpha_\varepsilon.$$

It follows that

$$I_{2,\varepsilon}=\frac{e^{\alpha^2/\varepsilon}}{C_0\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{a_\varepsilon})^+}p_\varepsilon\eta(t)\psi(x)dsdt-\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}p_\varepsilon\eta(t)\psi(x)dsdt+\alpha_\varepsilon=$$
$$=\frac{\pi}{2l_0C_0\alpha^2}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}p_\varepsilon\eta(t)\psi(x)dx_1dt-\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}p_0\eta(t)\psi(x)dx_1dt+\alpha_\varepsilon.$$

So, the left hand side of (51) has the form

$$e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}(a(x)+\frac{\pi}{2l_0C_0\alpha^2})p_\varepsilon\eta(t)\psi(x)dx_1dt-\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}p_0\eta(t)\psi(x)dx_1dt+\alpha_\varepsilon. \tag{53}$$

Now, we transform the right-hand side of (51). We have

$$\theta_1\int\limits_{Q^T}\nabla(u_\varepsilon-u_T)\nabla(Q_\varepsilon\eta(t)\psi(x))dxdt=$$

$$=\theta_1\int\limits_{Q^T}\nabla u_\varepsilon\nabla((Q_\varepsilon-W_\varepsilon)\eta(t)\psi(x))dxdt+\theta_1\int\limits_{Q^T}\nabla u_\varepsilon\nabla(W_\varepsilon\eta(t)\psi(x))dxdt=$$

$$=\frac{\theta_1}{C_0\alpha^2}e^{\alpha^2/\varepsilon}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{a_\varepsilon})^+}u_\varepsilon\eta(t)\psi(x)dsdt-\frac{4\theta_1}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}u_\varepsilon\eta(t)\psi(x)dsdt+\alpha_\varepsilon=$$

$$=\frac{\pi\theta_1}{2l_0C_0\alpha^2}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}u_\varepsilon\eta(t)\psi(x)dx_1dt-\frac{\theta_1\pi}{\alpha^2}\int\limits_{\Gamma_2^T}u_0\eta(t)\psi(x)dx_1dt+\alpha_\varepsilon. \tag{54}$$

Thus, we have

$$e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}(a(x_1)+\frac{\pi}{2l_0C_0\alpha^2})p_\varepsilon\eta(t)\psi(x)dx_1dt-\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}p_0\eta(t)\psi(x)dx_1dt=$$

$$=\frac{\pi\theta_1}{2l_0C_0\alpha^2}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}u_\varepsilon\eta(t)\psi(x)dx_1dt-\frac{\theta_1\pi}{\alpha^2}\int\limits_{\Gamma_2^T}u_0\eta(t)\psi(x)dx_1dt+\alpha_\varepsilon. \tag{55}$$

From (50) and (55), we derive

$$e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}p_\varepsilon\eta(t)\psi(x)dx_1dt=\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{b_1(x_1)}{b_2(x_1)}\eta(t)\psi(x)p_0dx_1dt-$$

$$-\frac{\pi\theta_1}{\alpha^2}\int\limits_{\Gamma_2^T}u_0\eta(t)\psi(x)\frac{a(x_1)}{b_2(x_1)}dx_1dt+\alpha_\varepsilon, \tag{56}$$

where

$$\mathcal{C}=\frac{\pi}{2l_0C_0\alpha^2},\quad b_1(x_1)=a(x_1)+\mathcal{C},\quad b_2(x_1)=b_1^2(x_1)+\theta_1N^{-1}\mathcal{C}.$$

From (56), we have

$$\lim_{\varepsilon\to0}e^{\alpha^2/\varepsilon}\int\limits_{l_\varepsilon^T}a(x_1)p_\varepsilon\eta(t)\psi(x)dx_1dt=$$

$$=\frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{a(x_1)b_1(x_1)}{b_2(x_1)}p_0\eta(t)\psi(x)dx_1dt-\frac{\pi\theta_1}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{a^2(x_1)}{b_2(x_1)}u_0\eta(t)\psi(x)dx_1dt.$$

Thus, we have that p_0 is a weak solution of the problem

$$\begin{cases} -\partial_t p_0 - \Delta p_0 = -\theta_1 \Delta(u_0 - u_T), & (x,t) \in Q^T, \\ \partial_\nu p_0 + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} p_0 = \theta_1 \partial_\nu (u_0 - u_T) + \frac{\pi \theta_1 a^2(x_1)}{\alpha^2 b_2(x_1)} u_0, & (x,t) \in \Gamma_2^T, \\ p_0 = 0, & (x,t) \in \Gamma_1^T, \\ p_0(x,T) = \theta_2(u_0(x,T) - u_T(x,T)), & x \in \Omega. \end{cases} \tag{57}$$

From (50) and (56), we get

$$e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} a(x) u_\varepsilon \eta(t) \psi(x) dx_1 dt + N^{-1} e^{\alpha^2/\varepsilon} \int\limits_{l_\varepsilon^T} p_\varepsilon \eta(t) \psi(x) dx_1 dt =$$

$$= \frac{\pi}{\alpha^2} \int\limits_{\Gamma_2^T} u_0 \eta(t) \psi(x) \frac{a(x_1) b_1(x_1)}{b_2(x_1)} dx_1 dt + \frac{\pi N^{-1} \mathcal{C}}{\alpha^2} \int\limits_{\Gamma_2^T} p_0 \eta(t) \psi(x) \frac{1}{b_2(x_1)} dx_1 dt + \alpha_\varepsilon. \tag{58}$$

It follows that u_0 is a weak solution of the problem

$$\begin{cases} \partial_t u_0 - \Delta u_0 = f, & (x,t) \in Q^T, \\ \partial_\nu u_0 + \frac{\pi a(x_1) b_1(x_1)}{\alpha^2 b_2(x_1)} u_0 = -\frac{\pi N^{-1} \mathcal{C}}{\alpha^2 b_2(x_1)} p_0, & (x,t) \in \Gamma_2^T, \\ u_0(x,t) = 0, & (x,t) \in \Gamma_1^T, \\ u_0(x,0) = 0, & x \in \Omega. \end{cases} \tag{59}$$

5. COST FUNCTIONAL LIMIT

For the function $v_\varepsilon = -N^{-1} p_\varepsilon$, we have

$$J_\varepsilon(-N^{-1} p_\varepsilon) = \frac{\theta_1}{2} \int\limits_{Q^T} |\nabla(u_\varepsilon - u_T)|^2 dx dt + \frac{\theta_2}{2} \int\limits_{\Omega} (u_\varepsilon(x,T) - u_T(x,T))^2 dx + \frac{\beta(\varepsilon)}{2N} \int\limits_{l_\varepsilon^T} p_\varepsilon^2 dx_1 dt.$$

From the integral identities for u_ε and p_ε, we derive

$$\int\limits_0^T (\langle \partial_t u_\varepsilon, p_\varepsilon \rangle + \langle \partial_t p_\varepsilon, u_\varepsilon \rangle) dt + \theta_1 \int\limits_{Q^T} |\nabla(u_\varepsilon - u_T)|^2 dx dt + N^{-1} \beta(\varepsilon) \int\limits_{l_\varepsilon^T} p_\varepsilon^2 dx_1 dt =$$

$$= -\theta_1 \int\limits_{Q^T} \nabla(u_\varepsilon - u_T) \nabla u_T dx dt + \int\limits_{Q^T} f p_\varepsilon dx dt. \tag{60}$$

From (60), we conclude

$$J_\varepsilon(-N^{-1} p_\varepsilon) = -\frac{\theta_1}{2} \int\limits_{Q^T} \nabla(u_\varepsilon - u_T) \nabla u_T dx dt -$$

$$-\frac{\theta_2}{2} \int\limits_{\Omega} (u_\varepsilon(x,T) - u_T(x,T)) u_T(x,T) dx + \frac{1}{2} \int\limits_{Q^T} f p_\varepsilon dx dt. \tag{61}$$

Thus, we have

$$\lim_{\varepsilon \to 0} J_\varepsilon(-N^{-1} p_\varepsilon) = -\frac{\theta_1}{2} \int\limits_{Q^T} \nabla(u_0 - u_T) \nabla u_T dx dt - \frac{\theta_2}{2} \int\limits_{\Omega} (u_0(x,T) - u_T(x,T)) u_T(x,T) dx +$$

$$+\frac{1}{2}\int\limits_{Q^T} fp_0dxdt=\frac{\theta_1}{2}\int\limits_{Q^T}|\nabla(u_0-u_T)|^2dxdt+\frac{\theta_2}{2}\int\limits_{\Omega}(u_0(x,T)-u_T(x,T))^2dx-$$

$$-\frac{\theta_1}{2}\int\limits_{Q^T}\nabla u_0\nabla(u_0-u_T)dxdt-\frac{\theta_2}{2}\int\limits_{\Omega}u_0(x,T)(u_0(x,T)-u_T(x,T))dx+\frac{1}{2}\int\limits_{Q^T}fp_0dxdt.$$

Using that

$$\int\limits_0^T(\langle\partial_t u_0,p_0\rangle+\langle\partial_t p_0,u_0\rangle)dt+\frac{\pi\theta_1}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{a^2(x_1)}{b_2(x_1)}u_0^2dx_1dt+\frac{\pi N^{-1}\mathcal{C}}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{p_0^2}{b_2(x_1)}dx_1dt=$$

$$=\int\limits_{Q^T}fp_0dxdt-\theta_1\int\limits_{Q^T}\nabla(u_0-u_T)\nabla u_0dxdt,$$

we get

$$\int\limits_{Q^T}fp_0dxdt=\theta_1\int\limits_{Q^T}\nabla(u_0-u_T)\nabla u_0dxdt+\theta_2\int\limits_{\Omega}u_0(x,T)(u_0(x,T)-u_T(x,T))dx+$$

$$+\frac{\pi\theta_1}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{a^2(x_1)}{b_2(x_1)}u_0^2dx_1dt+\frac{\pi N^{-1}\mathcal{C}}{\alpha^2}\int\limits_{\Gamma_2^T}\frac{p_0^2}{b_2(x_1)}dx_1dt. \tag{62}$$

Thus, we have

$$\lim_{\varepsilon\to0}J_\varepsilon(-N^{-1}p_\varepsilon)=\frac{\theta_1}{2}\int\limits_{Q^T}|\nabla(u_0-u_T)|^2dxdt+\frac{\theta_2}{2}\int\limits_{\Omega}(u_0(x,T)-u_T(x,T))^2dx+$$

$$+\frac{\pi\theta_1}{2\alpha^2}\int\limits_{\Gamma_2^T}\frac{a^2(x_1)}{b_2(x_1)}u_0^2dx_1dt+\frac{\pi N^{-1}\mathcal{C}}{2\alpha^2}\int\limits_{\Gamma_2^T}\frac{p_0^2}{b_2(x_1)}dx_1dt.$$

6. ON THE APPROXIMATE CONTROLLABILITY OF LIMIT PROBLEM AND UNIFORM CONVERGENCE OF THE CONTROLS v_ε

We will conclude the approximate controllability of the problem (3) by showing that the limit problem for $\theta_1=0$ and $\theta_2=1$

$$\begin{cases}\partial_t u_0(v)-\Delta u_0(v)=f, & (x,t)\in Q^T,\\ \partial_\nu u_0(v)+\frac{\pi a(x_1)b_1(x_1)}{\alpha^2 b_2(x_1)}u_0(v)=\frac{\pi\mathcal{C}}{\alpha^2 b_2(x_1)}v, & (x,t)\in\Gamma_2^T,\\ u_0(v)=0, & (x,t)\in\Gamma_1^T,\\ u_0(v)(x,0)=0, & x\in\Omega,\end{cases} \tag{63}$$

satisfies such a property and by proving that the sequence of optimal controls v_ε is uniformly bounded (with respect to N), when $N\searrow0$.

Theorem 4. *Let $u_T\in L^2(\Omega)$ and let $u_0(v)$ be the unique solution of the limit problem (63) for a given $v\in L^2(\Gamma_2^T)$. Then, for given $\delta>0$, there exists a control $v_0\in L^2(\Gamma_2^T)$ such that*

$$\|u_0(v_0)(\cdot,T)-u_T\|_{L^2(\Omega)}\le\delta.$$

Moreover, such control can be obtained as a limit of the optimal controls $v_{0,N}$ *associated to the cost functional given by* (3) *with* $\theta_1 = 0$ *and* $\theta_2 = 1$, *i.e.,*

$$J_{0,N}(v) = \frac{1}{2}\int\limits_{\Omega} |u_0(v)(x,T) - u_T(x)|^2 dx + \frac{N}{2}\int\limits_{\Gamma_2^T} h(x_1)v^2 dx_1 dt,$$

as $N \searrow 0$, *where* $h(x_1) = \frac{\pi\mathcal{C}}{\alpha^2 b_2(x_1)}$.

Proof. Without any loss of generality, we can suppose that $f \equiv 0$. Indeed, since the limit problem (63) is linear, we can make the change of variable $u(v) = u_0(v) - z$, where z is a solution of the problem

$$\begin{cases} \partial_t z - \Delta z = f, & (x,t) \in Q^T, \\ \partial_\nu z + \frac{\pi a(x_1)}{\alpha^2 b_1(x_1)} z = 0, & (x,t) \in \Gamma_2^T, \\ z(x,t) = 0, & (x,t) \in \Gamma_1^T, \\ z(x,0) = 0, & x \in \Omega. \end{cases} \tag{64}$$

Thus, for a given penalty parameter $N > 0$, we consider cost functional $J_{0,N}(v)$ where $u(v)$ is the solution of the problem

$$\begin{cases} \partial_t u(v) - \Delta u(v) = 0, & (x,t) \in Q^T, \\ \partial_\nu u(v) + \frac{\pi a(x_1)}{\alpha^2 b_1(x_1)} u(v) = h(x_1)v, & (x,t) \in \Gamma_2^T, \\ u(v) = 0, & (x,t) \in \Gamma_1^T, \\ u(v)(x,0) = 0, & x \in \Omega. \end{cases} \tag{65}$$

Let $v_{0,N}$ be the corresponding optimal control. Notice, since $J_{0,N}(v)$ is weakly continuous, strictly convex and coercive in $L^2(\Gamma_2^T)$, then $v_{0,N}$ exists and it is unique. In addition, we have that the optimality condition can be written in the following form

$$0 = J_{0,N}^{'}(v_{0,N})v = \frac{N}{2}\int\limits_{\Gamma_2^T} h(x_1)v_{0,N} v dx_1 dt + \frac{1}{2}\int\limits_{\Omega} (u(v_{0,N})(x,T) - u_T(x))u(v)(x,T)dx, \tag{66}$$

where v is an arbitrary function from $L^2(\Gamma_2^T)$. Since $v_{0,N}$ minimizes $J_{0,N}(v)$ on $L^2(\Gamma_2^T)$, we get that for any $N > 0$,

$$J_{0,N}(v_{0,N}) \leq J_{0,N}(0).$$

Taking into account that if $v \equiv 0$, then the solution of (65) $u(0) \equiv 0$, we get

$$J_{0,N}(0) = \frac{1}{2}\int\limits_{\Omega} u_T^2(x)dx.$$

Using that $h(x_1) \geq a_0 = \mathit{const} > 0$, we obtain

$$\|\sqrt{N} v_{0,N}\|_{L^2(\Gamma_2^T)} \leq K, \ \|u(v_{0,N})(\cdot,T) - u_T(\cdot)\|_{L^2(\Omega)} \leq K, \tag{67}$$

where K does not depend on N.

Thus, there exists a subsequence, functions $\xi \in L^2(\Omega)$ and $w \in L^2(\Gamma_2^T)$, such that as $N \to 0$, we have convergences

$$u(v_{0,N})(\cdot,T) - u_T(\cdot) \rightharpoonup \xi \text{ weakly in } L^2(\Omega), \tag{68}$$

and

$$\sqrt{N} v_{0,N} \rightharpoonup w \text{ weakly in } L^2(\Gamma_2^T). \tag{69}$$

From here and from (66), we have

$$\int\limits_{\Omega} \xi(x) u(v)(x,T) dx = 0, \tag{70}$$

for any $v \in L^2(\Gamma_2^T)$. Let us show that $\xi \equiv 0$ in Ω. Indeed, we consider the auxiliary adjoint problem

$$\begin{cases} -\partial_t p - \Delta p = 0, & (x,t) \in Q^T, \\ \partial_\nu p + \frac{\pi a(x_1)}{\alpha^2 b_1(x_1)} p = 0, & (x,t) \in \Gamma_2^T, \\ p(x,t) = 0, & (x,t) \in \Gamma_1^T, \\ p(x,T) = \xi(x), & x \in \Omega. \end{cases} \tag{71}$$

Multiplying the integral identity corresponding to (71) by $u(v)$ and taking p as a test function in the integral identity to (65), we get

$$-\int\limits_{Q^T} \partial_t p u(v) dx dt + \int\limits_{Q^T} \nabla p \nabla u(v) dx dt + \frac{\pi}{\alpha^2} \int\limits_{\Gamma_2^T} \frac{a(x_1)}{b_1(x_1)} p u(v) dx_1 dt = 0, \tag{72}$$

and

$$\int\limits_{Q^T} \partial_t u(v) p dx dt + \int\limits_{Q^T} \nabla u(v) \nabla p dx dt + \frac{\pi}{\alpha^2} \int\limits_{\Gamma_2^T} \frac{a(x_1)}{b_1(x_1)} u(v) p dx_1 dt = \int\limits_{\Gamma_2^T} h(x_1) p v dx_1 dt. \tag{73}$$

Subtracting from the equality (73) the equality (72) we obtain

$$\int\limits_{Q^T} (\partial_t u(v) p + u(v) \partial_t p) dx dt = \int\limits_{\Gamma_2^T} h(x_1) p v dx_1 dt.$$

From here, we conclude

$$\int\limits_{\Omega} \xi(x) u(v)(x,T) dx = \int\limits_{\Gamma_2^T} h(x_1) p v dx_1 dt, \tag{74}$$

where v is an arbitrary function from $L^2(\Gamma_2^T)$. From (70) and (74), we derive that $p \equiv 0$ on Γ_2^T. Hence, we have for p

$$\begin{cases} -\partial_t p - \Delta p = 0, & (x,t) \in Q^T, \\ \partial_\nu p = 0, & (x,t) \in \Gamma_2^T, \\ p(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \end{cases} \tag{75}$$

thus, we deduce that $p \equiv 0$ in Q^T. Indeed, we can apply an argument already used in [13] (Theorem 2.3) which reduces the question to the case of the unique continuation property when the function vanishes in an interior subdomain of the cylinder of definition, since we can consider a connected domain $\widehat{\Omega}$ given as

$$\widehat{\Omega} = \Omega \cup \omega,\ \partial\widehat{\Omega} \in C^2,\ \overline{\omega} \cap (\partial\Omega \setminus \Gamma_1) = \phi \text{ (the empty set)},$$

where ω is a connected domain in $\mathbb{R}^2$. We extend function p by zero on $\omega^T = \omega \times (0,T)$ producing the function $\widehat{p}$. Then, since $\partial_\nu p = 0$ on Γ_2^T, we get that $\widehat{p}$ satisfies (weakly)

$$\begin{cases} -\partial_t \widehat{p} - \Delta \widehat{p} = 0, & (x,t) \in \widehat{Q}^T, \\ \widehat{p}(x,t) = 0, & (x,t) \in \partial\widehat{\Omega} \times (0,T), \end{cases} \tag{76}$$

where $\widehat{Q}^T = \widehat{\Omega} \times (0,T)$ and $\widehat{p} \equiv 0$ on ω^T, and by applying well-known unique continuation results (see, e.g. [19], [12]) we conclude that $\widehat{p} \equiv 0$ on $\widehat{Q}^T$, and thus $\xi(x) \equiv 0$ on Ω. Setting in the optimality condition (66) $v = v_{0,N}$, we get the strong convergence in (68)

$$\frac{N}{2} \int\limits_{\Gamma_2^T} h(x_1) v_{0,N}^2 dx dt + \frac{1}{2} \int\limits_{\Omega} (u(v_{0,N})(x,T) - u_T(x))^2 dx =$$
$$= \frac{1}{2} \int\limits_{\Omega} u_T(x)(u_T(x) - u(v_{0,N})(x,T)) dx \to 0 \text{ as } N \to 0.$$

▮

Now, we can prove the following result

Theorem 5. *Let $u_T \in L^2(\Omega)$ and let $u_\varepsilon(v)$ be the unique solution of the problem* (3) *for a given control $v \in L^2(l_\varepsilon^T)$. Then for given $\delta > 0$, there exists $\varepsilon_0 > 0$ and there exists $N_0 \in (0,1)$ (independent of ε_0) such that if $\varepsilon \in (0,\varepsilon_0)$ and $N \in (0,N_0)$, the optimal control $v_{\varepsilon,N} \in L^2(l_\varepsilon^T)$, associated to $J_{\varepsilon,N}$ with $\theta_1 = 0$, $\theta_2 = 1$, leads to the approximate controllability property*

$$\|u_\varepsilon(v_{\varepsilon,N})(\cdot,T) - u_T\|_{L^2(\Omega_\varepsilon)} \le \delta. \tag{77}$$

Proof. We take an arbitrary $\delta > 0$. We have

$$\|u_\varepsilon(v_{\varepsilon,N})(\cdot,T) - u_T\|_{L^2(\Omega)} \le$$
$$\le \|u_\varepsilon(v_{\varepsilon,N})(\cdot,T) - u_0(v_{0,N})(\cdot,T)\|_{L^2(\Omega)} + \|u_0(v_{0,N})(\cdot,T) - u_T\|_{L^2(\Omega)}. \tag{78}$$

Now from the Theorem 5, for an arbitrary $\delta > 0$ we find such $N_0 > 0$, that for any $N \in (0,N_0)$ we have that

$$\|u_0(v_{0,N})(\cdot,T) - u_T\|_{L^2(\Omega)} \le \delta/2. \tag{79}$$

Using the optimality condition to $v_{\varepsilon,N}$ we get

$$0 = J_\varepsilon^{'}(v_{\varepsilon,N})v = \int\limits_{\Omega} u_\varepsilon(v)(x,T)(u_\varepsilon(v_{\varepsilon,N})(x,T) - u_T(x)) dx + N\beta(\varepsilon) \int\limits_{l_\varepsilon^T} v_{\varepsilon,N} v dx_1 dt,$$

where v is an arbitrary function from $L^2(l_\varepsilon^T)$. Taking here $v = v_{\varepsilon,N}$, we obtain

$$0 = \int\limits_{\Omega} u_\varepsilon(v_{\varepsilon,N})(x,T)(u_\varepsilon(v_{\varepsilon,N})(x,T) - u_0(v_{0,N})(x,T)) dx +$$
$$+ \int\limits_{\Omega} u_\varepsilon(v_{\varepsilon,N})(x,T)(u_0(v_{0,N})(x,T) - u_T(x)) dx + N\beta(\varepsilon) \int\limits_{l_\varepsilon^T} v_{\varepsilon,N}^2 dx_1 dt =$$

$$= \int\limits_{\Omega} (u_\varepsilon(v_{\varepsilon,N})(x,T) - u_0(v_{0,N})(x,T))^2 dx +$$

$$+ \int\limits_{\Omega} u_0(v_{0,N})(x,T)(u_\varepsilon(v_{\varepsilon,N})(x,T) - u_0(v_{0,N})(x,T))dx +$$

$$+ \int\limits_{\Omega} u_\varepsilon(v_{\varepsilon,N})(x,T)(u_0(v_{0,N})(x,T) - u_T(x))dx + N\beta(\varepsilon) \int\limits_{l_\varepsilon^T} v_{\varepsilon,N}^2 dx_1 dt. \tag{80}$$

From (80), we derive

$$N\beta(\varepsilon) \int\limits_{l_\varepsilon^T} v_{\varepsilon,N}^2 dx_1 dt + \int\limits_{\Omega} (u_\varepsilon(v_{\varepsilon,N})(x,T) - u_0(v_{0,N})(x,T))^2 dx =$$

$$= \int\limits_{\Omega} u_0(v_{0,N})(x,T)(u_0(v_{0,N})(x,T) - u_\varepsilon(v_{\varepsilon,N})(x,T))dx +$$

$$+ \int\limits_{\Omega} u_\varepsilon(v_{\varepsilon,N})(x,T)(u_0(v_{0,N})(x,T) - u_T(x))dx \equiv I_{\varepsilon,N}^1 + I_{\varepsilon,N}^2.$$

Using that $\|u_\varepsilon(v_{\varepsilon,N})(\cdot,T)\|_{L^2(\Omega)} \le K$, where $K = const > 0$, which does not depend of ε and N, we have for $I_{\varepsilon,N}^2$

$$|I_{\varepsilon,N}^2| \le K\|u_0(v_{0,N})(\cdot,T) - u_T\|_{L^2(\Omega)}. \tag{81}$$

Using the approximate controllability property of $v_{0,N}$ there exists such N_2, that for any $N \in (0, N_2)$

$$K\|u_0(v_{0,N})(\cdot,T) - u_T\|_{L^2(\Omega)} \le \delta^2/8.$$

From here and (81), we deduce

$$|I_{\varepsilon,N}^2| \le \delta^2/8. \tag{82}$$

Now, we fix $N \in (0, \min(N_1, N_2))$ and using the weak convergence $u_\varepsilon(v_{\varepsilon,N})(\cdot,N) \rightharpoonup u_0(v_{0,N})(\cdot,N)$ in $L^2(\Omega)$ as $\varepsilon \to 0$, we can choose $\varepsilon_0 = \varepsilon_0(N)$ such that if $\varepsilon \in (0, \varepsilon_0)$ then

$$|I_{\varepsilon,N}^1| \le \delta^2/8. \tag{83}$$

From (82), (83), we conclude that for any fixed $N \in (0, N_0)$ we can find the interval $(0, \varepsilon_0)$, such that the right hand side of (78) is less than δ. ∎

Remark 2. *We would like to emphasize the relevance of the obtained results when applied to Energy Balance Models coupled with the averaged temperature of a deep ocean, as in* [11] *(and its many references). For instance, the finite volumes method, used in this paper, and the numerical algorithms introduced in* [14] *for the optimal control and the approximate controlability of parabolic systems could be applied to get interesting numerical experiences on the solution of the homogenized control problem instead of the very complex initial small parameter formulation.*

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