Electron beams: partially flat solutions of a nonlinear elliptic equation with a singular absorption term

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Dedicated to my Master, Haïm Brezis, who trusted me and gave me wings to fly higher.

Abstract. In the so-called Child-Langmuir, established since 1911, an electron beam is formed linking two electrodes, which are assumed to be two parallel plates of area A, separated to a finite distance D. When $D \ll \sqrt{A}$, "edge effects" are negligible and the modelling is reduced to a nonlinear boundary problem for a singular ordinary differential equation in which a constant coefficient (the generated electric current j) must be found in order to get simultaneously Dirichlet and Neumann homogeneous boundary conditions in one of the extremes. If $D > \sqrt{A}$, then the problem becomes much more difficult since the "edge effects" arise in the plane (x, y) and the electric current (now $j(x)$ due to the presence of a very large perpendicular magnetic field) must be determined in order to get solutions $u(x, y)$ of a singular semilinear equation which are partially flat ($u = \frac{\partial u}{\partial n} = 0$ on a part of the boundary). In this paper, we offer a rigorous mathematical treatment of some former studies (Joel Lebowitz and Alexander Rokhenko (2003) and Alexander Rokhenko (2006)), where several open questions were left open: for instance, the need for a singularity of $j(x)$ near the cathode edge to get such partially flat solutions.

Key words: Electron beams, space charge, Child-Langmuir law, singular semilinear equation, partially flat solution, super and subsolution method, H^1-matching and its generalizations.
Subject Classification: 35Q99, 78A20, 35J75, 35J60.

1 Introduction

In 1911, the American physicist Clement D. Child (1868 – 1933) proposed in [11] an experimental law ("Child's law") that he deduced after modelling the electric current that flows between the plates of a vacuum tube: the main components in electronics from about 1905 to 1960, when transistors and integrated circuits mostly supplanted them. For some time (1907-1908), Child was visiting J.J. Thompson (1856-1940), 1906 Nobel Prize in Physics, who built the cathode ray tubes, proving in 1897 the existence and charge of the electron. Child's law is still a staple of textbooks treating charged particle motion in vacuum and in solids. After normalizing the variable and the unknown, and by considering the case in which the anode and cathode are simulated by planes at a finite distance, Child proposed a singular nonlinear ordinary differential equation which perhaps is one of the older singular equations considered in the literature. Its formulation was the following: find $j > 0$ (modelling the maximum generated electric current) to get a positive solution of the nonlinear boundary problem

$$\begin{cases} -u''(y) + \dfrac{j}{\sqrt{u(y)}} = 0 & y \in (0,1), \\ u(0) = 0 \quad u(1) = 1, \end{cases} \tag{1}$$

such that

$$u'(0) = 0 \text{ and } u > 0 \text{ on } (0,1).$$

Notice that since $u(0) = 0$ the singularity of the equation is present on one of the boundaries ($y = 0$), representing the cathode. Condition $u'(0) = 0$ says that the electron flux is assumed to start from rest, since the electric field is zero there. This law was later considered in a series of papers (see, e.g., [38]), and improved by Irving Langmuir (1881-1957), 1932 Chemistry Nobel Prize. Since then, the above law is called the "Child-Langmuir law". In physical terms, it is known as the "three-halves-power law" (see Remark 1), since it says that the electric current J is proportional to the 3/2-power of the voltage, $V^{3/2}$ (in contrast to Ohm's law on electric circuits for which $J = kV$ for some $k > 0$). We will see, in Section 3, that in the renormalized formulation

$$j \equiv \frac{4}{9} \text{ and } u(y) = y^{4/3} \text{ for } x \in (0,1),$$

in contrast with the 1823 Ohm's law [$u(y) = ky$, for some $k > 0$].

In the Child-Langmuir modelling, the electrodes are assumed to be two parallel plates of area A, separated by a finite distance D ($D = 1$ after some change of variables). They assumed that $D \lll \sqrt{A}$, so that "edge effects "are negligible.

The study of the problem in which edge effects are taken into account is much more difficult and was a subject of research until our days (see, e.g., the survey [51]). A more realistic modelling, already considered by many authors (see, e.g., [45], [46]), takes place in the plane (X, y), in which the cathode is only an interval $(-a, a) \times \{0\}$ of the device (the set of points $(X, y) \in (-a-b, a+b) \times (0,1)$, for a given $b > 0$), and it is assumed the presence of a strong magnetic field. This makes the electron beam well confined in such a way that the electric current is only X-dependent, $J(X, y) = j(X)$. The primitive question regarded by the Child-Langmuir law can now be stated in the following terms as an overdetermined problem: given $a, b > 0$, find sufficient conditions on a x-dependent function $j : (-a-b, a+b) \to [0, +\infty)$, with

$$\begin{cases} j(X) > 0 & \text{if } X \in (-a, a),\ j \in L^1_{loc}(-a, a), \\ j(X) = 0 & \text{if } X \in (-a-b, -a) \cup (a, a+b), \end{cases} \tag{2}$$

to get the solvability of the singular nonlinear boundary value problem

$$\widehat{P}_{a,b,j} = \begin{cases} -\Delta u + \frac{j(X)}{\sqrt{u}} = 0 & X \in (-a-b, a+b),\ y \in (0,1), \\ u(X, 0) = 0 & X \in (-a-b, a+b), \\ u(X, 1) = 1 & X \in (-a-b, a+b), \\ u(\pm(a+b), y) = y & y \in (0,1), \end{cases} \tag{3}$$

with the additional conditions

$$\widehat{AC}_{a,b} = \begin{cases} \frac{\partial u}{\partial y}(X, 0) = 0 & X \in (-a, a), \\ u(X, y) > 0 & X \in (-a-b, a+b),\ y \in (0,1). \end{cases} \tag{4}$$

The more important fact considered in the literature is the study of the adaptation of profiles $u(\cdot, y)$ from the external profile (which is assumed to be given by $u(b, y) = y$, as in Ohm's law) to the profile in the center of the cathode $u(0, y)$, where the authors expect to have the profile correspondent to the Child-Langmuir law, $u(0, y) = y^{4/3}$ for $y \in (0,1)$.

A very delicate question is the behaviour of $j(X)$ near $X = \pm a$ since it must generate complicated edge effects (for instance, by the *strong maximum principle*, we know that $\frac{\partial u}{\partial y}(X, 0) > 0$ if $(-a-b, -a) \cup (a, a+b)$, since u is harmonic on $((-a-b, -a) \cup (a, a+b)) \times (0, 1)$).

To simplify the formulation, due to the symmetry of the problem, it is usual to consider only (see, e.g., [45], [46], [44]) the transmission of the profiles $u(X_0, y)$, $y \in (0, 1)$, when $X_0 \in [0, a+b]$ in one of the two symmetrical parts of the device (for example, the one on the right). To give the edge behaviour (on the boundary of the cathode) a central presence in the formulation, it is traditional to introduce the change of variable $x = X - a$, $X \in (0, a+b)$ and then $x \in (-a, b)$ for a given $b > 0$. In this way, $(-a, 0)$ becomes the center of the cathode and $(0, 0)$ the edge of the right side of the cathode. We arrive then to the formulation which will be considered in this paper: given $a, b > 0$, we consider the problem of finding the coefficient $j(x)$ such that there exists a solution of the singular boundary value problem on $\Omega = (-a, b) \times (0, 1)$

$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & \text{in } \Omega, \\ u(x, 0) = 0 & x \in (-a, b), \\ u(x, 1) = 1 & x \in (-a, b), \\ u(-a, y) = y^{4/3} & y \in (0, 1), \\ u(b, y) = y & y \in (0, 1), \end{cases} \tag{5}$$

satisfying the additional conditions

$$AC_{a,b} = \begin{cases} \frac{\partial u}{\partial y}(x, 0) = 0 & x \in (-a, 0), \\ u > 0 & \text{in } \Omega. \end{cases} \tag{6}$$

Notice that the additional conditions imply a failure of the "unique continuation property" and the Hopf-Oleinik lemma related to the strong maximum principle (see, e.g., the exposition made in [25]): indeed, $u > 0$ in $(-a, b) \times (0, 1)$ although $u = \frac{\partial u}{\partial y} = 0$ on $(-a, 0) \times \{0\}$. The additional condition, $\frac{\partial u}{\partial y}(x, 0) = 0$ for $x \in (-a, 0)$, represents the vanishing of the electric field on the half of the cathode under consideration. This type of behaviour arises in many *free boundary problems* ([18]) for semilinear equations with a non-Lipschitz absorption. It is quite similar to the requirement of the so-called *flat solutions* arising, for instance, in the study of linear ([20], [25]) and nonlinear ([26]) Schrödinger equations. Here, the constraint on the function j and the singularity of the nonlinear absorption term $(1/\sqrt{u})$ generate some important additional difficulties.

Despite the relevance of the space-charge-limited flows in many applications (vacuum and solid electronic devices, electron guns, particle accelerators, high current diodes, tubes, thyristors, etc., see, e.g., [37], [35], [34]) the study of 2-d simple geometries as here considered presents important difficulties, and it was the benchmark for almost a century, in particular for the study of the generated current $j(x)$ in the vicinity of the cathode edges (see Figure 1). Many previous studies, based on numerical and asymptotic methods (see, e.g., [50], [45], [46], [44] and their many references) claim that due to the important constraint on $j(x)$ (independence with respect to y) it is needed to assume that

$$\lim_{x \nearrow 0} j(x) = +\infty,$$

(see, in Figure 1, the singularities of the function $j(x)$ at the extremes $\pm a$, for different widths a of the cathode). A rigorous mathematical proof of it was left as an open problem in the cited references.

One of the main goals of this paper is to prove, and make it precise, a conjecture proposed in many numerical analysis experiences (see, e.g., [44] and its references) concerning the formation of "wings" in the electric current (i.e., the formation of a singularity of $j(x)$ near the cathode edge). In addition, we will prove a stronger version of the positivity of the solution: in fact, the solution is non-degenerate (see subsection 4.4).

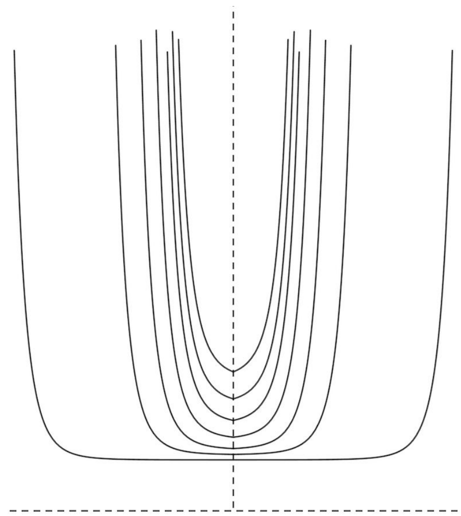

Figure 1: Wings of $j(x)$ in some numerical experiments for different values of the cathode width a: adapted from [48].

Theorem 1 *There exists* $A_0, b_0 > 0$ *and* $\beta_0 \in (0, \frac{1}{2})$*, such that, if* $b \geq b_0 > 0$*, and we assume*

$$\begin{cases} j(x) = \frac{A}{(-x)^{\beta}}, & \text{for } x \in (-a, 0), \\ j(x) = 0 & \text{for } x \in (0, b), \end{cases} \tag{7}$$

with

$$0 \leq \beta < \beta_0 \text{ and } A \in (0, A_0), \tag{8}$$

then, there exists a solution $u \in L^2(\Omega; \delta)$*, with* $\delta = d((x,y), \partial\Omega)$*, of problem* $P_{a,b,j}$ *and the additional conditions* $AC_{a,b}$*. Moreover*

$$0 < \underline{C}\delta(x,y)^{\overline{\alpha}} \leq u(x,y) \leq \overline{C}\delta(x,y)^{\underline{\alpha}} \text{ a.e. } (x,y) \in (-a,0) \times (0,1),$$

for some $\underline{C}$*,*$\overline{C} > 0$*, with* $\alpha_0 \leq \overline{\alpha} \leq \underline{\alpha} \leq \frac{4}{3}$*, given by*

$$\overline{\alpha} = \frac{2}{3}(2 - \overline{\beta}),\ \underline{\alpha} = \frac{2}{3}(2 - \underline{\beta}),$$

where $\alpha_0 = \frac{2}{3}(2 - \beta_0)$*,* $\overline{\beta}$*,* $\underline{\beta} \in (0, \beta_0]$*. Finally, u is unique in the class of non-degenerate solutions.* □

The proof will be made by constructing suitable super and subsolutions for several auxiliary problems that are obtained by matching some strategic functions on different subdomains. In particular, we will need to consider the nonlinear singular eigenvalue type problem

$$\begin{cases} -U''(s) + \frac{C}{\sqrt{U(s)}} = \lambda U(s) & s \in (-R, R), \\ U(\pm R) = 0, \end{cases} \tag{9}$$

for which the global bifurcation diagram (in terms of the parameter λ) will be completely characterized (see Figure 2). Here the positive constants C and R are given. We will show: i) there is a bifurcation from the infinity for λ near $\lambda_1(R)$ (the first eigenvalue of the linear problem with $C = 0$), ii) the bifurcation curve is strictly decreasing (which implies the uniqueness of the nonnegative solution U) and iii) the curve

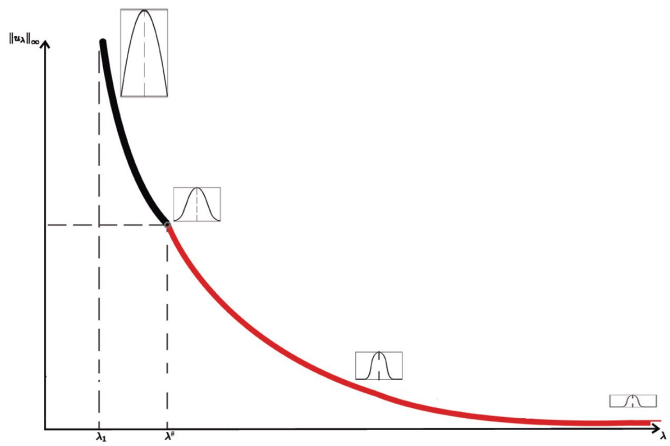

Figure 2: Bifurcation from infinity and critical values of λ for a flat solution.

is not C^1 for a suitable value $\lambda = \lambda^* > \lambda_1(R)$ corresponding to a "flat solution" (i.e. the solution U is such that $U'(\pm R) = 0$ and $U(s) > 0$ for any $s \in (-R, R)$). This extends several results in the previous literature (see, e.g., [27], [24], and their references).

Theorem 1 confirms many numerical experiences that lacked a rigorous proof. The progressive changes in the profiles can be illustrated by Figure 3.

We point out that the nature of the problem changes radically when the singular nonlinearity arises on the other side of the equality (as a forcing term): see, e.g. [13] and [28]. Most of the motivations to consider singular absorption terms have their foundation in fields such as chemical reactions, non-Newtonian flows and other applied subjects very different from the one of space-charge-limited flows (see, e.g., [31], [29], [12] and [18]). A survey on singular elliptic equations can be found in [33].

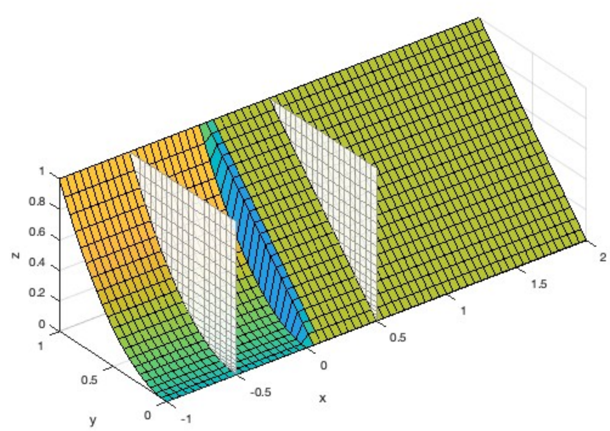

Figure 3: Progressive changes of profiles $u(\cdot, y)$.

We will need to construct several super and subsolutions satisfying suitable qualitative properties. This will require matching different functions. We point out that two remarkable matching studies in the literature were those of G. Gamov in [32] (when proving the tunnelling effect) and the one by H. Brezis in [4] (proving the compact support of solutions of suitable variational inequalities: see the exposition made in [19]). Here we will need to use a matching method that is not the usual H^1-matching criterion but a looser one that allows the formation of singularities as long as they have a certain sign (see Remark 14 below).

The organization of this paper is the following: Section 2 is devoted to the modelling of the problem. In particular, we will make mention to some optimization criterion on $j(x)$ (see Remark 5). The one-dimensional case, including the Child-Langmuir law, is revisited in Section 3. The elegant study of the one-dimensional case is due to H. Brezis (personal communication in 2004). We also consider the non-autonomous case $j = j(x)$ since it will be applied later. The main goal of this work is the study of the two-dimensional formulation that will be developed in the last Section 4 divided into different subsections. We will give details on the construction of a positive subsolution and a supersolution, both satisfying the additional condition, and prove the uniqueness of solutions in the class of non-degenerate solutions.

2 On the modelling of the 2-d space-charge-limited flow

Following [45] (see also [43]), a general formulation of the problem under consideration could start with the consideration of the stationary Maxwell system of equations for the electric and magnetic fields $(\mathbf{E}, \mathbf{B})$ defined on a set $\widetilde{\Omega} \subset \mathbb{R}^3$, $\widetilde{\Omega} = \mathbb{R}\times(0, D)\times \mathbb{R}$, with $D > 0$ given, separating two conducting electrodes placed on the planes $Y = 0$ (cathode) and $Y = D$ (anode) [with $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_1$]:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0} \\ \nabla \cdot \mathbf{B} = 0 \\ \nabla \times \mathbf{E} = \mathbf{0} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \end{cases} \tag{10}$$

Here $\rho(X, Y, Z)$ is the electron-charge density, ϵ_0 the free space permittivity and $\mathbf{J}(X, Y, Z)$ denotes the current density (here we are using a variable notation (X, Y, Z) different from the one of the Introduction). Since we are assuming that ρ is stationary we get that

$$div\ \mathbf{J} = 0 \text{ in } \widetilde{\Omega}.$$

Now we assume that the *cathode* is in the (X, Z) plane, $Y = 0$, and it has a width $2A$. We also assume that there is a *very strong magnetic field* $\mathbf{B}$, which is perpendicular to the electrodes ($\mathbf{B}$ is in the Y-direction), inhibiting the transversal components of the electron velocities $v(X, Y)$, and then $\rho(X, Y, Z) = \rho(X, Y)\chi_{\{|X|\leq A\}}(X, Y)$, where $\chi_{\{|X|\leq A\}}(X, Y)$ is the characteristic function

$$\chi_{\{|X|\leq A\}}(X, Y) = \begin{cases} 1 & \text{if } |X| \leq A \\ 0 & \text{otherwise.} \end{cases}$$

The fact that ρ is not constant is the reason to call this type of problems as *space charge*. Due to the assumption on $\mathbf{B}$, we know that *the potential* U of the *electric field* $(\mathbf{E} = -\nabla U)$ is Z-independent, i.e., $U = U(X, Y)$.

We assume that *the emitted electrons leave the cathode with zero velocity* and thus, if we take $U = U(X, Y) = 0$ in the cathode, the total *mechanical energy* is $E_0 = 0$. If e and m represent the *charge* and the *mass* of the electron, the *conservation of the mechanical energy* leads to the equation

$$\frac{m}{2} v^2(X, Y) = eU(X, Y). \tag{11}$$

i.e.,

$$v(X,Y)=\sqrt{\frac{2eU(X,Y)}{m}}.$$

Remember that the mechanical force (by a negative charge) is given by $\mathbf{F}=(-e)\mathbf{E}=-e(-\nabla U)=e\nabla U$ and thus the potential energy is $-eU$. From this, we deduce that the current density is only dependent on X, $\mathbf{J}(X,Y,Z)=J(X)\chi_{\{|X|\leq A\}}(X,Y)\mathbf{e}_2$ and determines the velocity of electrons. We also recall that

$$\mathbf{J}=-\rho v\mathbf{e}_2:=J(X)\chi_{\{|X|\leq A\}}(X,Y)\mathbf{e}_2.$$

Then

$$\rho(X,Y)=-\frac{J(X)\chi_{\{|X|\leq A\}}(X,Y)}{\sqrt{\frac{2eU(X,Y)}{m}}}.$$

We introduce now the *dimensionless variables* (once again, the notation is different from the one used in the Introduction)

$$x=\frac{X}{D},y=\frac{Y}{D}\text{ and }a=\frac{A}{D},$$

and the *dimensionless functions*

$$u(x,y)=\frac{U(X,Y)}{V}\text{ and }j(x)=\frac{9}{4}\sqrt{\frac{m}{2e}}\frac{J(X)}{\epsilon_0 V^{3/2}}.$$

Then, the first equation of the Maxwell system and the conservation of the mechanical energy lead to the singular nonlinear Poisson equation

$$\Delta u(x,y)=-4\pi\rho(x,y)=\frac{j(x)}{\sqrt{u(x,y)}},$$

with $\rho(x,y)=0$ (i.e. $j(x)=0$) if $|x|>a$.

Remark 1 In the dimensionless process, it is possible to choose a different new expression for $J(X)$. So, for instance, in [45] it was used the function $\widetilde{j}(x)=\frac{4}{9}j(x)$, so that the one-dimensional case corresponds to $\widetilde{j}(x)\equiv 1$. In any case, the "three-halves-power law" Child-Langmuir law says that in the one-dimensional case

$$J=\frac{4}{9}\epsilon_0\sqrt{\frac{2e}{m}}\frac{V^{3/2}}{D^2}. \tag{12}$$

Remark 2 Most of the results of the following sections of this paper can be applied in the framework of two concentric cylindrical diodes (see the formulation considered in [10]). A different modelling was considered in [9] where they study a section through a thin wire at high potential contained within an earthed conductor at zero potential.

Remark 3 The study of the mathematical treatment of this problem was suggested to me by Haïm Brezis, during my visit to his Department of the University of Paris VI, in March 2003. Brezis attended a seminar by Joel Lebowitz (on his article [45]), at his workplace, Rutgers University (where Brezis would later have a contract as Distinguished Visiting Professor from 2004 until the date of his death on 2024). I received a more concrete formulation in the fax of Brezis on May, 21, 2004 (containing the Theorem 2 below) and I sent him a first set of my results on the two-dimensional problem on June and September of 2004. We make several working sessions during my stay in Paris, as Invited Professor of Paris VI, from February 15, to March 14 of 2005, but without finding a positive subsolution (see subsection 4.2 below). In this occasion, we received a visit from Joel Lebowitz to Paris VI University, and we exchanged opinions in a joint session the three of us together (see Remark 4 below and the picture at the end of this paper). I produced a new draft, sent to both of them, on June 27, 2006. I made a public presentation of the results in the Opening Workshop of the European FIRST Project on June 30, 2009, in Orsay (University of Paris Sud). Later, I presented a first version of the results of this article at the conference, "Recent and new perspectives in Nonlinear Analysis", held in Urbino (Italy), November 3-4, 2022 and at the IMEIO PhD Course, of the Universidad Politécnica de Madrid, on January 16, 2023.

Remark 4 (Suggestion by Joel Lebowitz, Paris, 2005). For other different geometrical electrode shapes, we can assume that the open (possibly unbounded) set $\widetilde{\Omega} \subset \mathbb{R}^3$ has two components defining its finite boundary $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_1$ and the problem is to find a pair $(\mathbf{J}(x,y,z), u(x,y,z))$ satisfying (in some weak sense) the following set of conditions:

$$\begin{gathered} div\mathbf{J} = 0 \text{ in } \widetilde{\Omega}, \\ u(x,y,z) \geq 0 \text{ in } \widetilde{\Omega}, \\ \sqrt{u}\Delta u = j(x,y,z) \text{ in } \widetilde{\Omega}, \\ j(x,y,z) = \left|\mathbf{J}(x,y,z)\right|, \\ \frac{\mathbf{J}(x,y,z)}{|\mathbf{J}(x,y,z)|} = \frac{\nabla u(x,y,z)}{|\nabla u(x,y,z)|} \\ u\left|_{\Gamma_0}\right. = 0 \text{ and } u\left|_{\Gamma 1}\right. = 1. \end{gathered}$$

Remark 5 As a matter of fact, already in the pioneering works by Child and Langmuir it was mentioned that the interesting case corresponds to a current density J^* defined as the "largest current" that can be emitted without time-dependent behavior and thus independent of any thermodynamic effect on the cathode. This will be illustrated later for the one-dimensional case (when justifying why $j = 4/9$). For a general formulation, as the one presented in Remark 4, this kind of optimization criterion can be stated as follows (suggestion by Joel Lebowitz, Paris, 2005): from the first of the above conditions we deduce that there exists a constant $\alpha = \alpha(\mathbf{J})$ such that for any simple curve Γ (when $\widetilde{\Omega}$ is unbounded Γ can be unbounded without multiple intersection points) we have that

$$\int_\Gamma \mathbf{J}\cdot\mathbf{n}d\sigma = \alpha.$$

We define the set C of "admissible solutions" $(\mathbf{J}(x,y,z), u(x,y,z))$ by means of the set of conditions given in Remark 4 and then the problem is now to find $(\mathbf{J}^*, u^*) \in \mathcal{C}$ such that

$$\alpha(\mathbf{J}^*) = \max_{(\mathbf{J},u)\in\mathcal{C}} \alpha(\mathbf{J}).$$

Remark 6 As already mentioned in the Introduction, when $a = +\infty$ the problem can be associated with a one-dimensional formulation for which we get $\mathbf{J}^*(x,y,z) = \frac{4}{9}\mathbf{e}_2$ and, as we will prove in the next Section, $u^*(y) = y^{4/3}$. An interesting open question can be stated in this framework**:** is it true that if $(\mathbf{J}, u) \in \mathcal{C}$ then necessarily $\mathbf{J}(x,y,z) = j\mathbf{e}_2$ with j constant and $u = u(y)$?

Remark 7 In the case of space-charge-limited flows ($0 < a + b < +\infty$) the boundary condition $u(\pm(a + b), y) = y$ for $y \in (0,1)$ was first proposed in [45], since the external electric field $\mathbf{E}$ must behave, at least at very long distances, in a similar way to the case without any cathode ($a = 0$).

3 The one-dimensional Child-Langmuir law revisited

3.1 The autonomous case

The problem associated with the one-dimensional Child-Langmuir law can be formulated in the following terms: find $j > 0$ in order to get a function $u \in W^{2,1}(0,1)$ solving the boundary problem

$$\begin{cases} -u''(y) + \dfrac{j}{\sqrt{u(y)}} = 0 & y \in (0,1), \\ u(0) = 0 \quad u(1) = 1, \end{cases} \tag{13}$$

and such that

$$u'(0) = 0 \text{ and } u > 0. \tag{14}$$

We recall that, in this physical case, the only interest is in positive solutions $u(y) > 0$ for any $y \in (0,1)$ but the problem also arises in other fields (for instance, in Chemical Engineering) where the possibility of having *a dead core* (a subinterval $(0,\xi) \subset (0,1)$ on which $u = 0$) makes perfect sense for suitable values of j.
Here $W^{2,1}(0,1)$ denotes the usual Sobolev space requiring that $u'' \in L^1(0,1)$ (see, e.g., [5]). Since $W^{2,1}(0,1) \subset C^1([0,1])$ condition (14) is well justified

We point out that since the singular nonlinear term $\frac{1}{\sqrt{u}}$ is neither monotonically non-decreasing, nor Lipschitz continuous, even in the case in which $j > 0$ is given, we cannot apply general results in the literature on the uniqueness of solutions.

As a matter of fact, given $j > 0$, some different notions of solution of (13) can be introduced (being compatible with the supplementary condition (14)). A natural notion of solution arises when problem (13) is understood in the framework of the Calculus of Variations. We define

$$K = \left\{ u \in H^1(0,1) \text{ such that } u(0) = 0, u(1) = 1 \text{ and } u \geq 0 \text{ on } (0,1) \right\},$$

and

$$J(u) = \int_0^1 \left(\frac{1}{2}\, u'(y)^2 + 2j\sqrt{u(y)} \right) dy,$$

and then we call *variational solution of* (13) to a function $u \in K$ such that

$$J(u) = \min_{v \in K} \left\{ \int_0^1 \left(\frac{1}{2} v'(y)^2 + 2j\sqrt{v(y)} \right) dy \right\}. \tag{15}$$

As said before, one of the reasons why the singular problem is quite relevant in the applications is for the possible occurrence of a *free boundary* (according to the values of $j > 0$) associated to a solution u of (13). In the one-dimensional setting, the *free boundary* is given simply by a point $\xi \in [0,1)$ such that

$$\begin{cases} u(y) = 0 & \text{if } y \in [0,\xi], \\ u(y) > 0 & \text{if } y \in (\xi,1], \text{ and } u'(\xi) = 0. \end{cases} \tag{16}$$

Notice that the supplementary condition (14) corresponds to the case $\xi = 0$. Many authors used to say that, in that case, u is a "*flat solution*" (on $y = 0$). This is the more interesting case for us, since the electric fields should vanish on the cathode.

The treatment of free boundary solutions (not necessarily being a variational solution) leads to the notion of strong solutions with a free boundary:

Definition 1 *We say that $u \in C^1([0,1)) \cap K$ is a strong solution with a free boundary if there exists $\xi \in [0,1)$ such that (14) holds and, in addition, $\frac{1}{\sqrt{u}} \in L^1(\xi,1)$, and $u''(y) = \dfrac{j}{\sqrt{u(y)}}$ a.e. $y \in (\xi,1)$.*

Notice that, in this case, we must understand the differential equation as

$$-u''(y) + \frac{j}{\sqrt{u(y)}} \chi_{\{u>0\}} = 0 \quad y \in (0,1),$$

where $\chi_{\{u>0\}}$ denotes the characteristic function of the set $\{y \in (0,1) : u(y) > 0\}$, since otherwise the singular term is not well defined on $[0,\xi]$.

The following result was communicated to me by Haïm Brezis on 2004. Although there are different similar treatments in the literature (see, e.g., [17]) the uniqueness part and the argument used in the proof is completely new.

Theorem 2 *(H. Brezis) There exists a unique variational solution. Moreover, if we define* $j^* = \frac{4}{9}$*, then:*
a) if $j = j^*$ *the variational solution is a flat solution and it is given by*

$$u(y) = y^{4/3}. \tag{17}$$

b) if $j > j^*$*the variational solution is a free boundary solution with*

$$\xi = 1 - \frac{1}{\sqrt{\frac{9}{4}j}}, \tag{18}$$

and it is given by $u(y) = A(y-\xi)_+^{4/3}$ *with* $A = (\frac{9}{4}j)^{2/3}$.
c) if $0 < j < j^*$ *then the variational solution is such that* $u > 0$ *on* $(0,1]$ *and* $u'(0) = K_0 > 0$, *for some* $K_0 = K_0(j)$.

Remark 8 In fact, from the proof we will see that if $j \to +\infty$ then $\xi \to 1$ (a sort of limit of "boundary layer" type). Moreover, if $j \to +0$ then $u(y) \to y$ in $C^0([0,1])$.

Proof. Let us start by proving the uniqueness of variational free boundary type solutions. By well-known results, (15) admits always a minimizer which satisfies $u'' = \frac{j}{\sqrt{u}}$ on the set $[\xi < y < 1]$. Then, by multiplying by u' we get

$$\begin{cases} \dfrac{du}{dy} = \sqrt{4j}\,(u(y))^{1/4} \\ u(1) = 1, \end{cases} \tag{19}$$

a problem with the uniqueness of solutions since $u^{1/4}$ is monotone increasing. By Leibniz's formula for separable ordinary differential equations, we get

$$u(y) = \left(\frac{3\sqrt{j}}{2}\right)^{4/3} (y-\xi)_+^{4/3},$$

which proves a) and b). Let us consider now the case of $j \in (0, \frac{4}{9})$. Note that if we arrive to prove that $\frac{du}{dy}(0) = K_0 > 0$ then after multiplying by u' we will get that

$$u'(y) = \sqrt{4j\sqrt{u(y)} + K_0^2},$$

and the Leibniz formula would lead to the implicit expression of the solution

$$\int_0^{u(y)} \frac{du}{\sqrt{4j\sqrt{u} + k_0^2}} = y. \tag{20}$$

A more direct way to apply the above argument is the following: let

$$H(\sigma) = \int_0^\sigma \frac{ds}{\sqrt{s^{1/2}+1}}, \quad \text{for } \sigma > 0. \tag{21}$$

Then $H(0) = 0$ and

$$H'(\sigma) = \frac{1}{\sqrt{\sqrt{\sigma}+1}} > 0. \tag{22}$$

Thus H can be inverted, $\sigma = H^{-1}(\tau)$, and we get that $H^{-1}(0) = 0$ and

$$\frac{d}{d\tau}H^{-1}(\tau) = \sqrt{\sqrt{H^{-1}(\tau)} + 1} > 0. \tag{23}$$

Let us prove that any variational solution can be written in the form $u(y) = \frac{1}{A}H^{-1}(By)$, for suitable constants A and B. Note that

$$u'(y) = \frac{B}{A}\sqrt{\sqrt{Au(y)} + 1} \quad \text{and thus } u'(0) = \frac{B}{A}. \tag{24}$$

Indeed, since $H(Au(y)) = By$ we get

$$H'(Au)A\frac{du}{dy} = B,$$

i.e.,

$$\frac{1}{\sqrt{\sqrt{Au(y)} + 1}}A\frac{du}{dy} = B,$$

which implies (24). Moreover

$$\begin{aligned}
\frac{d^2u}{dy^2}(y) &= \tfrac{B}{A}\tfrac{1}{2}\left(\sqrt{Au(y)} + 1\right)^{-1/2}\tfrac{1}{2}\left(Au(y)\right)^{-1/2}A\frac{du}{dy}(y) \\
&= \frac{B}{A}\frac{1}{2}\left(\sqrt{Au(y)} + 1\right)^{-1/2}\frac{1}{2}\left(Au(y)\right)^{-1/2}B\sqrt{\sqrt{Au(y)} + 1} \\
&= \frac{B^2}{4A}\frac{1}{\sqrt{Au(y)}} \\
&= \frac{B^2}{4A^{3/2}}\frac{1}{\sqrt{u(y)}}.
\end{aligned} \tag{25}$$

Thus, we must determine A and B by the conditions $\frac{B^2}{4A^{3/2}} = j$ and $u(1) = 1$. Then we get the condition

$$H(A) = B = 2\sqrt{j}A^{3/4}, \tag{26}$$

which has a unique solution A (and then a unique value of B) since H is strictly increasing, $H(0) = 0$ and $\lim_{\sigma\to+\infty} H(\sigma) = +\infty$. Then

$$u'(0) = \frac{B_0}{A_0} = \frac{H(A_0)}{A_0},$$

with $A_0 > 0$ solution of

$$\frac{H(A_0)}{A_0^{3/4}} = 2\sqrt{j}.$$

The uniqueness of the constants A and B also proves the uniqueness of the variational solution, and part c) follows. □

Remark 9 In cases a) and b) we can improve the regularity: we have that $u \in W^{2,p}(\xi, 1) \quad \forall p \in [1, \frac{3}{2})$, and in case c) $u \in W^{2,p}(0,1) \quad \forall p \in [1, 2)$. In fact, we have the *sharp gradient estimate*

$$|u'(y)| \le Cu^{1/4}(y), \text{ for some } C > 0, \text{ for any } y \in (0,1). \tag{27}$$

Estimates of this nature play an important role in the study of the existence of solutions of the associated parabolic problem (the so-called "quenching problem"): see, e.g., [42] and [14], among others.

Remark 10 Note also that $j^* = \sup\{j > 0 : u'(0) > 0 \text{ with } u \text{ variational solution of (15)}\}$. Something similar was already mentioned in the space charge literature (see, e.g., [37] p. 381 and [39] p. 2372).

3.2 The distributed one-dimensional current density

For different purposes, it is useful to consider a similar problem to the above formulation but now for some $j = j(y)$ with $j \in L^1_{loc}(0,1)$, $j \geq 0$. The problem under consideration is

$$\begin{cases} -u''(y) + \dfrac{j(y)}{\sqrt{u(y)}}\chi_{\{u>0\}} = 0 \quad y \in (0,1), u \geq 0 \\ u \geq 0 \quad y \in (0,1), \\ u(0) = 0 \quad u(1) = 1, \end{cases} \tag{28}$$

where $\chi_{\{u>0\}}$ denotes again the characteristic function of the set $\{y \in (0,1) : u(y) > 0\}$. Since the non-linear term is decreasing, we will apply the method of super and subsolutions.

Definition 2 *Given $p \geq 1$, a function $\overline{u} \in H^1(0,1)$ is a $p-$strong supersolution of problem (28), if*

i) $\overline{u} \geq 0$ and $\dfrac{j}{\sqrt{\overline{u}}} \in L^p(0,1)$,

ii) $\overline{u}'' \in L^1(0,1)$,

iii) $-\overline{u}'' + \dfrac{j}{\sqrt{\overline{u}}}\chi_{\{\overline{u}>0\}} \geq 0$ a.e. in $(0,1)$,

iv) $\overline{u}(0) = 0$ and $\overline{u}(1) \geq 1$.

The notion of $p-$strong subsolution $\underline{u} \in H^1(0,1)$ is similar, by replacing the inequalities $\geq$ in iii) and iv) by the inequality $\leq$, but always with $\underline{u}(0) = 0$. We recall here a variation of the iterative method of super and subsolutions (see, e.g., the exposition and general ideas in the monograph [41]) which applies to our framework with a singular absorption. The proof is a special case of a more general statement, which will be given later (see Theorem 6 and Remark 13).

Theorem 3 *Let $j \in L^1_{loc}(0,1)$, $j \geq 0$. Assume that there exists $p > 1$, a $p-$strong supersolution u^0 and a $p-$strong subsolution u_0, such that*

$$0 < u_0(y) \leq u^0(y) \text{ a.e. } y \in (0,1). \tag{29}$$

Then, problem (28) has a maximal solution u^* and a minimal solution u_* on the "interval" $[u_0, u^0]$ of $H^1(0,1)$, i.e., such that

$$u_0 \leq u_* \leq u^* \leq u^0 \text{ a.e. } y \in (0,1). \tag{30}$$

Moreover u''_*, $u''^* \in L^p(0,1)$. □

We can consider some special cases of $j(y)$ that allow to get some results in the line of Theorem 2:

$$j(y) = \lambda y^q \text{ with } q \in (-\frac{1}{2}, 1), \ \lambda > 0. \tag{31}$$

Note that, obviously, $q = 0$ corresponds to the case treated before and that our study will consider cases in which $j(0) = 0$ ($q \in (0,1)$) as well as cases in which $j(0) = +\infty$ ($q \in (-\frac{1}{2}, 0)$).

Theorem 4 *Assume (31). Then if we define $\lambda^*_q = \frac{2(1+2q)(2+q)}{9}$ we have:*
a) if $\lambda = \lambda^*_q$ the function $u(y) = y^{(4+2q)/3}$ is a flat solution of the problem.
b) if $\lambda > \lambda^*_q$ the function $u(y) = A(y-\xi)_+^{(4+2q)/3}$ is a free boundary solution with $\xi = 1 - \dfrac{1}{\left[\frac{\lambda}{\lambda^*_q}\right]^{1/(2+q)}}$,

and $A = (\frac{\lambda}{\lambda^*_q})^{2/3}$. Moreover,

$$\frac{j(y)}{\sqrt{u(y)}}\chi_{\{u>0\}} = \frac{\lambda}{\sqrt{A}}(y-\xi)^{-\frac{(2-2q)}{3}}\chi_{\{y>\xi\}} \in L^1(0,1).$$

c) if $0<\lambda<\lambda_q^*$*, then the unique solution* u *is such that* $u>0$ *on* $(0,1]$ *and* $u'(0)=K_0>0,$ *for some* $K_0=K_0(\lambda)$.

Remark 11 If we use a different notation, $\beta=-q$ and $\alpha=(4-2\beta)/3$, then we get the existence of solutions of the form $u(y)=y^{\alpha}$, with $\alpha\in(1,4/3)$, once we assume $\beta\in(0,1/2)$. Curiously enough, the constraint $3\alpha/2+\beta=2$ arises also in Theorem 1 dealing with the two-dimensional problem (see Section 4).

The key idea of the proof is to build a family of explicit solutions of the differential equation inspired by Chapter 2 of [18].

Lemma 1 *Let* $u_m(y)=Cy^{\frac{2}{1+m}}$*, with* $C>0$ *and* $m\in(0,1)$*. Then*

$$-u_m''(y)+\frac{j_m(y)}{\sqrt{u_m(y)}}=0, \tag{32}$$

where

$$j_m(y)=\frac{2C\sqrt{C}(1-m)}{(1+m)^2}y^{\frac{(1-2m)}{(1+m)}} \quad y\in(0,1). \tag{33}$$

Proof. It suffices to check that

$$(u_m)'(y)=C\frac{2}{1+m}y^{\left(\frac{2}{1+m}-1\right)}=\frac{2C}{1+m}y^{\frac{1-m}{1+m}},$$

$$(u_m)''(y)=\frac{2C(1-m)}{(1+m)^2}y^{\left(\frac{1-m}{1+m}-1\right)}=\frac{2C(1-m)}{(1+m)^2}y^{\frac{-2m}{1+m}}. \tag{34}$$

Since $\sqrt{u_m(y)}=\sqrt{C}y^{\frac{1}{1+m}}$, we get

$$-(u_m)''(y)=\frac{-j_m(y)}{\sqrt{u_m(y)}}=\frac{-1}{\sqrt{C}}y^{\frac{-1}{1+m}}j_m(y).$$

Applying (34) we arrive at (32) with $j_m(y)$ given by (33). □

Corollary 1 *Given* $j(y)=\lambda y^q$ *with* $q\in(-\frac{1}{2},1)$ *and* $\lambda>0$*, the function* $u_q(y)=C_q(\lambda)y^{\frac{4+2q}{3}}$*,with*

$$C_q(\lambda)=\left[\frac{9\lambda}{2(1+2q)(2+q)}\right]^{2/3}, \tag{35}$$

is a solution of

$$-u_q''+\frac{\lambda y^q}{\sqrt{u_q}}=0, \text{ in } (0,1).$$

Proof. We impose $\frac{(1-2m)}{(1+m)}=q$ and

$$\frac{2C\sqrt{C}(1-m)}{(1+m)^2}=\lambda.$$

Then we get $m=\frac{1-q}{2+q}$ and $C(\lambda)$ given by (35). Then the conclusion is a direct consequence of Corollary 1. □

Proof of Theorem 4. Conclusions a) and b) are a direct consequence of Corollary 1. To prove part c) it suffices to build a subsolution $\underline{u}(y)\le y$ satisfying the searched properties, since $\overline{u}(y)=y$ is a supersolution satisfying that $\underline{u}(y) \leq \overline{u}(y)$ a.e. $y \in (0,1)$, and then, by Theorem 3, there exists a minimal solution u_* of (28) satisfying

$$\underline{u}(y) \leq u_*(y) \leq \overline{u}(y) \text{ a.e. } y \in (0,1).$$

Then if $\underline{u}'(0) > 0$ we conclude, as desired, that $u_*'(0) > 0$.
A construction of a suitable subsolution is the following: let $\varepsilon > 0$ be small enough, to be determined later, let $u_\#(y) = y^{(4+2q)/3}$ and define

$$\underline{u}(y) = \frac{u_\#(y) + \varepsilon y}{1+\varepsilon}.$$

Then it is clear that $\underline{u}(0) = 0$, $\underline{u}(1) = 1$ and since $\underline{u}'(y) = \frac{u_\#'(y)+\varepsilon}{1+\varepsilon}$ we get $\underline{u}'(0) = \frac{\varepsilon}{1+\varepsilon} > 0$. Moreover,

$$\underline{u}''(y) = \frac{u_\#''(y)}{1+\varepsilon} = \frac{\lambda_q^* y^q}{(1+\varepsilon)\sqrt{u_\#(y)}}.$$

Then, concerning the differential equation, we can have the condition for subsolution

$$-\underline{u}''(y) + \frac{\lambda y^q}{\sqrt{\underline{u}(y)}} = -\frac{\lambda_q^* y^q}{(1+\varepsilon)\sqrt{u_\#(y)}} + \frac{\sqrt{1+\varepsilon}\lambda y^q}{\sqrt{u_\#(y)+\varepsilon y}} \leq 0$$

if we have

$$\frac{\sqrt{1+\varepsilon}\lambda}{\sqrt{u_\#(y)+\varepsilon y}} \leq \frac{\lambda_q^*}{(1+\varepsilon)\sqrt{u_\#(y)}}. \tag{36}$$

But, obviously

$$\frac{\lambda_q^*}{(1+\varepsilon)\sqrt{u_\#(y)+\varepsilon y}} \leq \frac{\lambda_q^*}{(1+\varepsilon)\sqrt{u_\#(y)}}.$$

Then we arrive at the searched inequality (36) if we have

$$\lambda \leq \frac{\lambda_q^*}{(1+\varepsilon)^{3/2}}.$$

This is clearly true since $\lambda < \lambda_q^*$: it suffices to take $\varepsilon > 0$ such that

$$\varepsilon < \left(\frac{\lambda_q^*}{\lambda}\right)^{2/3} - 1.$$

□

The functions $u_q(y)$ given in Corollary 1 can be used as super and subsolutions to get the existence of a flat solution for a more general function $j(y)$. The result that follows is just a sample of many other possible results.

Theorem 5 *Let $-\frac{1}{2} < r < q < 0$. Assume*

$$\lambda_r y^r \geq j(y) \geq \lambda_q y^q \quad \text{a.e. } y \in (0,1),$$

with

$$\lambda_q = \lambda_q^* \text{ and } \lambda_r > \lambda_r^* \text{ with } \lambda_r \text{ large enough,} \tag{37}$$

or

$$\lambda_q < \lambda_q^* \text{ and } \lambda_r = \lambda_r^* \text{ with } \lambda_q \geq 0 \text{ small enough.} \tag{38}$$

Then there exists a maximal solution u^* and a minimal solution u_* of problem (28). Moreover, in case of (37)

$$A_r(y-\xi_r)_+^{\frac{4+2r}{3}} \leq u_*(y) \leq u^*(y) \leq y^{\frac{4+2q}{3}} \text{ a.e. } y \in (0,1),$$

for some $A_r > 0$ and $\xi_r \in (0,1)$, and in case (38)

$$y^{\frac{4+2r}{3}} \leq u_*(y) \leq u^*(y) \leq u_q(y) \text{ a.e. } y \in (0,1),$$

with $u_q(y)$ the solution of problem (28) corresponding to $j(y) = \lambda_q y^q$ (when $\lambda_q \in (0, \lambda_q^*)$) mentioned in part c) of Theorem 4. Moreover if $\lambda_q = 0$ we can take as u_q the function $u_q(y) = y$.

Proof. We recall that for $a > b \geq 0$ we have $0 < y^a < y^b < 1$ for any $y \in (0,1)$. Thus, equivalently, if $-a < -b \leq 0$ then $y^{-a} > y^{-b} > 1$ for any $y \in (0,1)$.
We take now $-a = r$ and $-b = q$ so $y^r > y^q$. The conclusion will be obtained through the application of Theorem 4 to different choices of the super and subsolution. In case (37) we take as supersolution $u^0(y) = y^{\frac{4+2q}{3}}$ since

$$-u^{0\prime\prime}(y) + \frac{j(y)}{\sqrt{u^0(y)}} \geq -u^{0\prime\prime}(y) + \frac{\lambda_q^* y^q}{\sqrt{u^0(y)}} = 0.$$

On the other hand, as a subsolution we can take $u_0(y) = A_r(y - \xi_r)_+^{\frac{4+2r}{3}}$ for some $A_r > 0$ and $\xi_r \in (0,1)$ as indicated in part b) of Theorem 4 (remember that $\lambda_r > \lambda_q^*$) since

$$-u_0''(y) + \frac{j(y)}{\sqrt{u_0(y)}} \leq -u_0''(y) + \frac{\lambda_r y^r}{\sqrt{u_0(y)}} = 0.$$

We have $u_0(0) = u^0(0) = 0$, $u_0(1) = u^0(1) = 1$. Moreover, if λ_r is large enough we have

$$u_0(y) = A_r(y - \xi_r)_+^{\frac{4+2r}{3}} < y^{\frac{4+2q}{3}} = u^0(y) \text{ for any } y \in (0,1),$$

(in spite that $0 < \frac{4+2r}{3} < \frac{4+2q}{3}$ and thus $y^{\frac{4+2q}{3}} < y^{\frac{4+2r}{3}}$ for $y \in (0,1)$). Indeed, we can take $A_r > 0$ and $\xi_r \in (0,1)$ so that $u_0'(1) > u^{0\prime}(1) = \frac{4+2q}{3}$ and thus both functions are well-ordered.
In case (38) we take as subsolution the function $u_0(y) = y^{\frac{4+2r}{3}}$ and as supersolution the function $u^0(y) = u_q(y)$ the solution of problem (28) corresponding to $j(y) = \lambda_q y^q$ when $\lambda_q \in (0, \lambda_q^*)$ mentioned in part c) of Theorem 4. The conditions in terms of the differential equation are satisfied, as before. Moreover, again, $u_0(0) = u^0(0) = 0$, $u_0(1) = u^0(1) = 1$, and the inequality $u_0(y) = y^{\frac{4+2r}{3}} < u_q(y) = u^0(y)$ for any $y \in (0,1)$, holds once that $\lambda_q > 0$ is small enough (since we know that $u_q(y) \to y$ when $\lambda_q \to 0$). If $\lambda_q = 0$ the function $u_q(y) = y$ is a supersolution and the proof ends. □

Remark 12 The gradient estimate mentioned when j is a constant is no longer valid when $j(y) = \lambda y^{-\beta}$ with $\beta \in (0, \frac{1}{2})$. If $\lambda \geq \lambda_\beta$, for some $\lambda_\beta > 0$

$$|u'(y)| \leq C u^{\frac{1+2\beta}{4+2\beta}}(y), \quad \text{for some } C > 0, \text{ for any } y \in (0,1). \tag{39}$$

4 The 2-d parallel-plate geometry

As mentioned in the Introduction, Theorem 1 makes precise, a conjecture by A. Rokhlenko [44]: if $j(x)$ behaves as $A/|x|^\beta$, for some $\beta \in (0, 1/2)$, for $x \in (-a, 0)$ then there exists a weak solution $u(x,y)$ of (3), and u "behaves" (near the cathode $[-a,a] \times \{0\}$) as y^α for some $\alpha \in (1, 4/3)$, with $\alpha = 4/3 - 2\beta$. We will obtain other auxiliary results for the more general case of $j : (-a,b) \to [0, +\infty)$ such that

$$\begin{cases} j(x) > 0 & \text{if } x \in (-a,0),\ j \in L^1_{loc}(-a,0), \\ j(x) = 0 & \text{if } x \in (0,b), \end{cases} \tag{40}$$

with a possible singularity of the current density function $j(x)$ only expected at the origin $x = 0$. Our main goal is, therefore, to get the transition, near $x = 0$, between flat and linear profiles $u(., y)$.

As mentioned before, since in other frameworks the solutions may vanish in some parts of the spatial domain (*dead cores*), we can reformulate the PDE as

$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}}\chi_{\{u>0\}} = 0 & x \in (-a,b),\ y \in (0,1), \\ u(x,0) = 0 & x \in (-a,b), \\ u(x,1) = 1 & x \in (-a,b), \\ u(-a,y) = y^{4/3} & y \in (0,1), \\ u(b,y) = y & y \in (0,1). \end{cases} \tag{41}$$

This Section will be structured in several subsections. In subsection 4.1 we present the adaptation of the super and subsolutions method to the problem $P_{a,b,j}$. The construction of *positive subsolutions* satisfying the additional conditions $AC_{a,b}$ will be presented in subsection 4.2. The proof is made by constructing suitable super and subsolutions for several auxiliary problems and matching suitably the corresponding solutions. In particular, we will use the global bifurcation diagram (in terms of the parameter λ) associated to the auxiliary problem (9). In subsection 4.3 we will end the proof of Theorem 1 by constructing a suitable supersolution which is flat on $(-a,0)\times\{0\}$. Finally, in subsection 4.4 we will prove the uniqueness of non-degenerate solutions.

4.1 On the existence of solutions of $P_{a,b,j}$ via super and subsolutions method

Due to the singularity of the nonlinear term, the super and subsolutions method needs to be applied under some adequate conditions. Let $\Omega = (-a,b)\times(0,1)$ and j as in (40).

Definition 3 *A function $u^0 \in W^{1,1}(\Omega)$, is said a $p-$positive supersolution of $P_{a,b,j}$ if $u^0 \geq 0$, $\frac{j}{\sqrt{u^0}} \in L^p(\Omega)$, for some $p \geq 1$, and it verifies in a very weak sense that*

$$\begin{cases} -\Delta u^0 \geq -\frac{j(x)}{\sqrt{u^0}} & in\ \Omega, \\ u^0(x,0) \geq 0 & x \in (-a,b), \\ u^0(x,1) \geq 1 & x \in (-a,b), \\ u^0(-a,y) \geq y^{4/3} & y \in (0,1), \\ u^0(b,y) \geq y & y \in (0,1). \end{cases}$$

The notion of $p-$positive subsolution u_0 is introduced similarly, i.e., $u_0 \in W^{1,1}(\Omega)$ with $u_0 > 0$ and $\frac{j}{\sqrt{u_0}} \in L^p(\Omega)$, satisfies that

$$\begin{cases} -\Delta \mathrm{u}_0 \leq -\frac{j(x)}{\sqrt{\mathrm{u}_0}} & in\ \Omega, \\ \mathrm{u}_0(x,0) = 0 & x \in (-a,b), \\ \mathrm{u}_0(x,1) \leq 1 & x \in (-a,b), \\ \mathrm{u}_0(-a,y) \leq y^{4/3} & y \in (0,1), \\ u^0(b,y) \leq y & y \in (0,1). \end{cases}$$

Theorem 6 *Assume that*

$$\left\{ \begin{array}{c} \textit{there exists } p > 1, \textit{ a } p-\textit{positive supersolution } u^0 \textit{ and a } p-\textit{positive subsolution } u_0 \\ \textit{of } P_{a,b,j} \textit{ such that } u_0 \leq u^0 \textit{ a.e. in } \Omega. \end{array} \right. \tag{42}$$

Then problem $P_{a,b,j}$ possesses a minimal and a maximal solutions u_* and u^* in the interval $[u_0, u^0]$, i.e.,

$$u_0 \leq u_* \leq u^* \leq u^0 \ \ a.e. in\ \Omega.$$

Proof. We define the iterative schemes (starting with u^0 and u_0)

$$\begin{cases} -\Delta u^n = -\frac{j(x)}{\sqrt{u^{n-1}}} & \text{in } \Omega, \\ u^n(x,0) = 0 & x \in (-a,b), \\ u^n(x,1) = 1 & x \in (-a,b), \\ u^n(-a,y) = y^{4/3} & y \in (0,1), \\ u^n(b,y) = y & y \in (0,1), \end{cases}$$

and

$$\begin{cases} -\Delta u_n = -\frac{j(x)}{\sqrt{u_{n-1}}} & \text{in } \Omega, \\ u_n(x,0) = 0 & x \in (-a,b), \\ u_n(x,1) = 1 & x \in (-a,b), \\ u_n(-a,y) = y^{4/3} & y \in (0,1), \\ u_n(b,y) = y & y \in (0,1). \end{cases}$$

By using the comparison principle for the Laplace operator, we get that

$$0 < u_0 \leq u_1 \leq ... \leq u_n \leq ... \leq u^n \leq ... \leq u^1 \leq u^0 \quad \text{a.e. in } \Omega,$$

and so the sequences $\{u^n\}, \{u_n\}$ converge (monotonically) in $L^p(\Omega)$ to some functions u_* *and* u^* and the sequences $\{\frac{j}{\sqrt{u^{n-1}}}\}$, $\{\frac{j}{\sqrt{u_{n-1}}}\}$ are bounded in $L^p(\Omega)$ and converge also (monotonically) in $L^p(\Omega)$. ∎

Remark 13 With some minor modifications, the above result holds when the super and subsolutions are in the weighted space $\frac{j}{\sqrt{u^0}} \in L^p(\Omega, \delta)$, for some $p \geq 1$, with $\delta = d((x,y), \partial\Omega)$. See, e.g., [7], [47], [30] and [40] among many other papers (see also some applications in [8]). This allows a greater generality to treat the singular term: for instance, the function $u^0(x,y) = y$ is such that $\frac{1}{\sqrt{u^0}} \in L^p(\Omega)$ for $p \in [1,2)$, $\frac{1}{\sqrt{u^0}} \notin L^2(\Omega)$ but $\frac{1}{\sqrt{u^0}} \in L^2(\Omega, \delta)$.

Remark 14 By some standard approximating arguments, it is well-known that the above notion of $p-$super and subsolutions of $P_{a,b,j}$ can be extended to the case in which the diffusion term generates an additional distribution over a simple curve Γ separating Ω in two different parts (matching without a $W^{1,p}$-contact: see [36] and [3]). Then, for instance, the subsolution u_0 of problem $P_{a,b,j}$ is allowed to satisfy

$$-\Delta u_0 + \frac{j(x)}{\sqrt{u_0}} \leq 0 \text{ in } \mathcal{D}'(\Omega).$$

The existence of a (not-flat) supersolution u^0 can be easily proved.

Lemma 2 *Let $j(x)$ satisfying (40). Then the function $u^0(x,y) = y$ is a positive $p-$supersolution of $P_{a,b,j}$ for any $p \in [1,2)$.*

Proof. It is a trivial fact since

$$-\Delta u^0 = 0 \geq -\frac{j(x)}{\sqrt{u^0}} \quad x \in (-a,b), \ y \in (0,1),$$

and

$$\begin{cases} u^0(x,0) = 0 & x \in (-a,b), \\ u^0(x,1) = 1 & x \in (-a,b), \\ u^0(-a,y) \geq y^{4/3} & y \in (0,1), \\ u^0(b,y) = y & y \in (0,1). \end{cases}$$

Moreover, $\frac{j(x)}{\sqrt{u^0}} \in L^p(\Omega)$ for any $p \in [1, 2)$. □

The construction of a positive subsolution is a very delicate task, which will be presented in the following subsection. Before proving Theorem 1 we can get an existence and uniqueness result of a positive solution under a more general assumption on $j(x)$ than in Theorem 1 but without ensuring that it is flat on $(-a, 0) \times \{0\}$. We define

$$\delta(x, y) := \mathrm{dist}((x, y), \partial\Omega),$$

which sometimes we shall denote simply as δ.

Theorem 7 *Assume that $j(x)$ satisfies*

$$0 \leq j(x) \leq \frac{A}{(-x)^\beta}, \text{ for } x \in (-a, 0) \text{ and } j(x) = 0 \text{ if } x \in (0, b),$$

with

$$0 \leq \beta < 1/2 \text{ and } A > 0 \text{ small enough.} \tag{43}$$

Then there exists a weak solution $u \in L^2(\Omega; \delta)$ of

$$P_{a,b,j} = \begin{cases} -\Delta u + \frac{j(x)}{\sqrt{u}} = 0 & x \in (-a, b),\ y \in (0, 1), \\ u(x, 0) = 0 & x \in (-a, b), \\ u(x, 1) = 1 & x \in (-a, b), \\ u(-a, y) = y^{4/3} & y \in (0, 1), \\ u(b, y) = y & y \in (0, 1), \end{cases} \tag{44}$$

such that

$$C\delta(x, y)^\alpha \leq u(x, y) \leq y \text{ a.e. } (x, y) \in (-a, b) \times (0, 1), \tag{45}$$

for some $C > 0$, with $1 < \alpha < \frac{4}{3}$, given by

$$\alpha = \frac{2}{3}(2 - \beta).$$

□

The solutions satisfying inequalities of the type (45) are called as *nondegenerate* solutions. The uniqueness of solutions given in Theorem 7 is a consequence of the techniques introduced in the paper [21]. Indeed, let $\nu \in (0, 4/3]$, and define the class of functions

$$\mathcal{M}(\nu) := \Big\{ u \in L^2(\Omega; \delta) \;\Big|\; \text{such that } u(x, y) \geq C\delta(x, y)^\nu \quad \text{in } \Omega, \text{ for some } C > 0 \Big\}. \tag{46}$$

We have

Theorem 8 *Assume $j(x)$ as in Theorem 7. Then, there exists at most a solution $u \in \mathcal{M}(\nu)$ of $P_{a,b,j}$.* □

The proof will be obtained through some smoothing estimates for some suitable parabolic problem (see subsection 4.4). Finally, the proof of the main result of this paper (Theorem 1) will be a direct consequence of the method of super and subsolutions, and the following result on partially flat supersolutions:

Theorem 9 *There exist $A_0, b_0 > 0$ and $\beta_0 \in (0, \frac{1}{2})$ such that, if $b \geq b_0 > 0$, and if we assume (7) and (43) then there exists a partially flat supersolution $\overline{u}(x, y)$ of problem $P_{a,b,j}$, i.e., such that $\overline{u} \in L^2(\Omega; \delta)$,*

$$\begin{cases} -\Delta\overline{u} + \frac{j(x)}{\sqrt{\overline{u}}} \geq 0 & x \in (-a, b),\ y \in (0,1), \\ \overline{u}(x, 0) \geq 0 & x \in (-a, b), \\ \overline{u}(x, 1) \geq 1 & x \in (-a, b), \\ \overline{u}(-a, y) \geq y^{4/3} & y \in (0,1), \\ \overline{u}(b, y) \geq y & y \in (0,1), \end{cases} \tag{47}$$

and

$$0 < \overline{u}(x, y) \leq C\delta(x, y)^{\alpha} \ \text{a.e.}\ (x, y) \in (-a, 0) \times (0, 1), \tag{48}$$

for some $C > 0$, with $\alpha \in (\alpha_0, \frac{4}{3}]$ given by

$$\alpha = \frac{2}{3}(2 - \beta),\ \text{where}\ \alpha_0 = \frac{2}{3}(2 - \beta_0).$$

□

Remark 15 In order to have the zero flux condition on $(-a, 0)$ we need some special unbounded behaviour on $j(x)$ near $x = 0$. Given $x_0 \in (-a, 0)$ we can construct (as in [2], [22], [18]) a "barrier function" of the form

$$\overline{u}(x, y : x_0) = K \left\|(x, y) - (x_0, 0)\right\|^{4/3} = K[(x - x_0)^2 + y^2]^{2/3}, \tag{49}$$

for a suitable $K > 0$ in such a way that $\overline{u}$ let a local supersolution once we assume

$$j(x) \geq C(-x)_+^{-\delta},\ x \in (-a, a)$$

for some suitable $\delta > 0$.More precisely we can assume

$$j(x) \geq 4\max(1, \frac{\sqrt{2}}{\sqrt{-x}}),\ x \in (-a, 0)\ \text{ and } j(x) = 0 \text{ for } x \in (0, a), \tag{50}$$

and that

$$j(x)[(x - x_0)^2 + y^2]^{-1/3} \in L^p(\Omega_{x_0}),\ \text{for some } p > 1,$$

where $\Omega_{x_0} = \{(x, y) \in (-a, 0) \times (0, 1) : (x - x_0)^2 + y^2 < (-x_0)^2\}$, for any $x_0 \in (-a, 0)$. Assume also that there exists and a p-nonnegative subsolution u_0 such that $u_0(x, y) \leq y$ a.e. in Ω. Then, if u satisfies the problem in the interval $[u_0, y]$, i.e., such that

$$u_0 \leq u \leq y \quad a.e.in\ \Omega$$

we have that

$$u(x, y) \leq \max(1, \frac{2^{1/3}}{(-x_0)^{1/3}})[(x - x_0)^2 + y^2]^{2/3}, \tag{51}$$

for any $(x, y) \in (-a, 0) \times (0, 1)$ and $x_0 \in (-a, 0)$ such that $(x - x_0)^2 + y^2 < (-x_0)^2$. In particular $\frac{\partial u}{\partial y}(x, 0) = 0$ for $x \in (-a, 0)$. The difficulty in this approach is to prove the positivity of the solution.

4.2 On the construction of some strict positive subsolutions with zero flux on $(-a,0)\times\{0\}$

The key point in the proof of Theorem 7 is the construction of a flat positive subsolution. We start by reducing the difficulty of this task by splitting the domain into two different subdomains. We define the subsets

$$\Omega_- := \{(x,y)\in\Omega : x\in(-a,0)\}, \text{ and } \Omega_+ := \{(x,y)\in\Omega : x\in(0,b)\}.$$

We will need to introduce an *artificial boundary condition* on the points $(0,y)$, $y\in(0,1)$, corresponding to the external boundary of this half of the cathode. We will prove a rigorous result that, in some sense, is connected to the study made in [44] by using asymptotic techniques and numerical analysis. Obtaining the correct profile function in the external boundary condition, at $x=0$, is already an important conclusion, since several choices were already considered in many papers in the literature on space charge problems. It was expected that the correct behaviour at the external border of the cathode is greater than the profile $y^{4/3}$ in the middle of the cathode, corresponding to the Child-Langmuir law, but no concrete function has been proposed for this external profile. The precise artificial boundary condition on the exterior of the cathode, which we will introduce (as a Dirichlet boundary condition), is the following:

$$u(0,y) = hy^{\alpha},\ y\in(0,1), \text{for some } h\in(0,1] \text{ and for some } \alpha\in(1,4/3). \tag{52}$$

The following result will simplify the task of finding a global positive subsolution since it will allow us to pass from the study of a discontinuous absorption coefficient $j(x)$ to a problem with a strictly positive one.

Proposition 1 *Let $j(x)$ satisfying (40). Let $\alpha\in(1,4/3)$. Given $h\in(0,1]$, consider the problem on Ω_-*

$$P_{a,0,j} = \begin{cases} -\triangle u_- + \dfrac{j(x)}{\sqrt{u_-}} = 0 & \text{in } \Omega_-, \\ u_-(-a,y) = y^{4/3} & y\in(0,1), \\ u_-(0,y) = hy^{\alpha} & y\in(0,1), \\ u_-(x,0) = 0 & x\in(-a,0), \\ u_-(x,1) = 1 & x\in(-a,0). \end{cases} \tag{53}$$

Assume that there exists $u_{0,-}(x,y)$, a p_0-subsolution of problem $P_{a,0,j}$, for some $p_0\geq 1$, such that

$$\frac{\partial u_{0,-}}{\partial x}(0,y) \leq 0 \text{ for } y\in(0,1), \tag{54}$$

satisfying the additional conditions

$$AC_{a,0} = \begin{cases} \frac{\partial u_{0,-}}{\partial y}(x,0) = 0 & x\in(-a,0), \\ u_{0,-}(x,y) > 0 & x\in(-a,0),\ y\in(0,1). \end{cases} \tag{55}$$

Then problem $P_{a,b,j}$ has a p_0-subsolution $\underline{u}$ satisfying the additional conditions $AC_{a,b}$.

Proof. Let $u_+(x,y)$ be the unique classical solution $u_+\in C^2(\Omega_+)\cap C^0(\overline{\Omega_+})$ of the linear problem

$$P_{0,b,0} = \begin{cases} -\triangle u_+ = 0 & \text{in } \Omega_+, \\ u_+(0,y) = hy^{\alpha} & y\in(0,1), \\ u_+(b,y) = y & y\in(0,1), \\ u_+(x,0) = 0 & x\in(0,b), \\ u_+(x,1) = 1 & x\in(0,b). \end{cases} \tag{56}$$

Define the function

$$\underline{u}(x,y)=\begin{cases} u_-(x,y) & \text{if } (x,y)\in\Omega_-,\\ u_+(x,y) & \text{if } (x,y)\in\Omega_+.\end{cases}$$

It is clear that $\underline{u}$ is a continuous function $\underline{u}\in C^0(\Omega)$ but its gradient has a discontinuity in the segment $x=0, y\in(0,1)$, since $j(x)$ is discontinuous in that segment. All the boundary conditions of $P_{a,b,j}$ are fulfilled and also the additional conditions $AC_{a,b}$. In order to check that $\underline{u}(x,y)$ is a p_0-subsolution of problem $P_{a,0,j}$ we will apply Corollary I.1 of [3]. To do this, if we define the segment $\Gamma=\{(0,y),\ y\in(0,1)\}$ then it suffices to check that

$$\frac{\partial u_-}{\partial \mathbf{n}}\le\frac{\partial u_+}{\partial \mathbf{n}}\quad \text{on } \Gamma, \tag{57}$$

where $\mathbf{n}$ is the unit exterior normal vector to Ω_-. In our case, $\mathbf{n}=\mathbf{e_1}$ and then condition (57) is expressed as

$$\frac{\partial u_-}{\partial x}(0,y)\le\frac{\partial u_+}{\partial x}(0,y)\qquad y\in(0,1). \tag{58}$$

By the assumption (54), it suffices to check that

$$\frac{\partial u_+}{\partial x}(0,y)\ge 0 \text{ for } y\in(0,1).$$

To do that, let us consider the function $U_+(x,y)=u_+(x,y)-y$. Then

$$\begin{cases} -\triangle U_+=0 & \text{in } \Omega_+,\\ U_+(0,y)=y^\alpha-y & y\in(0,1),\\ U_+(b,y)=0 & y\in(0,1),\\ U_+(x,0)=0 & x\in(0,b),\\ U_+(x,1)=0 & x\in(0,b).\end{cases} \tag{59}$$

We can define, now, the auxiliary function,

$$\underline{U}(x,y)=(b-x)(hy^\alpha-y),\ (x,y)\in\Omega_+.$$

Then we get that

$$\begin{cases} -\triangle \underline{U}\le 0 & \text{in } \Omega_+,\\ \underline{U}(0,y)=\omega(y)=hy^\alpha-y & y\in(0,1),\\ \underline{U}(b,y)=0 & y\in(0,1),\\ \underline{U}(x,0)=0 & x\in(0,b),\\ \underline{U}(x,1)=0 & x\in(0,b).\end{cases}$$

Thus, by the maximum principle we get that $\underline{U}(x,y)\le U_+(x,y)$ on Ω_+. But since $\underline{U}(0,y)=U_+(0,y)$ and we have that $\frac{\partial \underline{U}}{\partial x}(0,y)\ge 0$ for $y\in(0,1)$, we deduce that necessarily, $\frac{\partial U_+}{\partial x}(0,y)\ge0$ for $y\in(0,1)$, which leads to the required inequality. □

According Proposition 1, to finish with the construction of the global subsolution u_0, we must justify the existence of a function $u_{0,-}(x,y)$ solution of the nonlinear problem $P_{a,0,j}$, raised on Ω_-, and to check that $u_{0,-}(x,y)$ satisfies the additional conditions (55) and (54).

In order to construct the subsolution for the case of a possible unbounded $j(x)$, satisfying that $j(x)\le\frac{A}{(-x)^\beta}$ on $(-a,0)$, we will use some ideas coming from the study of Fluid Mechanics in the consideration of spatial domains with corners (see, e.g., [1]). We will try to find the subsolution $u_{0,-}(x,y)$ in the form

$$u_{0,-}(x,y)=\phi(r,\theta)=kr^\alpha U(\theta), \text{ for some } k>0,\ \alpha>1, \tag{60}$$

where $r=\sqrt{x^2+y^2}$ and $\theta=\arctan(y/x)$. Then, the partial differential inequality becomes

$$-\triangle\phi+\frac{j(r\cos\theta)}{\sqrt{\phi}}=-\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial\phi}{\partial r})-\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{j(r\cos\theta)}{\sqrt{\phi}}\leq 0. \tag{61}$$

We will assume $r\in[0,R)$, for a suitable $R>0$, and $\theta\in(\frac{\pi}{2},\pi)$. The additional conditions (55) will require $\phi(r,\theta)>0$ if $r>0$ and

$$\phi(r,\pi)=\frac{\partial\phi}{\partial\theta}(r,\pi)=0 \text{ for } r\in[0,R).$$

We make now the *structural condition* (40) on $j(x)$ and thus

$$j(r\cos\theta)\leq\frac{A}{(-r\cos\theta)^\beta},\text{ if }\theta\in(\frac{\pi}{2},\pi),\text{ for some }A>0\text{ and }\beta\in(0,1). \tag{62}$$

The partial differential inequality (61) leads to the study of the ordinary differential equation

$$-U''(\theta)+\frac{V(\theta)}{\sqrt{U(\theta)}}=\lambda U(\theta)\quad\theta\in(\tfrac{\pi}{2},\pi), \tag{63}$$

once we assume the constraint

$$\alpha+2\beta=4/3, \tag{64}$$

and then with

$$V(\theta)=\frac{A}{(-\cos\theta)^\beta}\text{ and }\lambda=\alpha^2. \tag{65}$$

The complemmentary conditions $AC_{a,0}$ become now

$$\begin{cases} U(\theta)>0 & \theta\in(\frac{\pi}{2},\pi),\\ U(\pi)=U'(\pi)=0. \end{cases}$$

Notice that the potential $V(\theta)$ is singular only for $\theta=\frac{\pi}{2}$. On the other hand, we will need to match this subsolution with another function which is positive for $\theta=\frac{\pi}{2}$. Then, we will construct the subsolution in two different pieces (see Figures 4 and 8)

$$U(\theta)=\begin{cases} v_1(\theta) & \text{if }\theta\in(\frac{\pi}{2},\pi-R_0),\\ v_2(\theta) & \text{if }\theta\in(\pi-R_0,\pi), \end{cases}$$

for some $R_0\in(0,\frac{\pi}{2})$, and with $v_2(\theta)\in[0,1]$ such that

$$\begin{cases} -v_2''(\theta)+\frac{V_0}{\sqrt{v_2(\theta)}}=\lambda v_2(\theta) & \theta\in(\pi-R_0,\pi),\\ v_2(\pi)=v_2'(\pi)=0, \end{cases} \tag{66}$$

where

$$V(\theta)\geq V_0\text{ if }\theta\in(\frac{\pi}{2},\pi-R_0).$$

On the other hand, $v_1(\theta)$ must take into account the singularity of the potential $V(\theta)$ on the interval $(\frac{\pi}{2},\pi-R_0)$ and this will require a careful matching (see Figure 4). In addition, we must guarantee a good match with the function $u_+(x,y)$ defined on Ω_+, as indicated in Proposition 1. This means that we want to have

$$v_1'(\frac{\pi}{2})\geq 0,$$

since we require $\frac{\partial u_{0,-}}{\partial x}(0,y)\leq 0$ for $y\in(0,1)$, and from $u_{0,-}(0,y)=kr^\alpha U(\theta)|_{\theta=\frac{\pi}{2}}$, we have

$$\frac{\partial u_{0,-}}{\partial x}(0,y)=\cdots=-Ar^{\alpha-1}U'(\theta)\big|_{\theta=\frac{\pi}{2}}\leq 0.$$

Before presenting the details on the construction of $v_1(\theta)$ and $v_2(\theta)$, it is very useful to consider the auxiliary nonlinear ODE of eigenvalue type (9) presented in the Introduction.

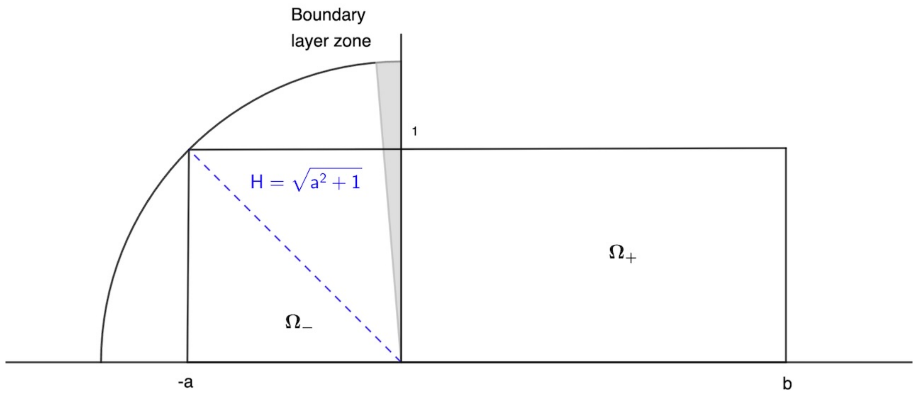

Figure 4: Boundary layer

4.2.1 Bifurcation curve and flat solution for an auxiliary related nonlinear eigenvalue ordinary differential problem

Since the equation (66) can be understood as a nonlinear eigenvalue problem, it is useful to start by considering the following auxiliary related problem:

$$\begin{cases} -U''(s) + \frac{V_0}{\sqrt{U(s)}} = \lambda U(s) & s \in (-R, R), \\ U(\pm R) = 0, \end{cases} \tag{67}$$

where the positive constants V_0 and R are given. The following result extends several results in the previous literature (see, e.g., [23], [27], [24] and some of the expositions made in [8]).

Theorem 10 *Given the positive constants V_0 and R then:*
i) there is a bifurcation from infinity for λ near $\lambda_1(R) = (\frac{\pi}{2R})^2$ (the first eigenvalue of the linear problem with $V_0 = 0$),
ii) the bifurcation curve is strictly decreasing (which implies the uniqueness of nonnegative solutions),
iii) the curve is not C^1 for a suitable value $\lambda = \lambda^* > \lambda_1(R)$ corresponding to a "flat solution" (i.e. the solution U is such that $U'(\pm R) = 0$ and $U(s) > 0$).

To show the qualitative behaviour of solutions of problem (67) we make the change of variables

$$u_{\lambda,V_0}(x) = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} u(\sqrt{\lambda}x),$$

where u is now the solution of the renormalized problem

$$P(L) \begin{cases} -u'' = f(u) & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases} \tag{68}$$

with

$$f(u) = u - \frac{1}{\sqrt{u}}$$

and $L=\sqrt{\lambda}R$. By multiplying by u and integrating by parts, we get that nontrivial solutions may exist only if $\lambda>\lambda_1=\frac{\pi^2}{4R^2}$, the first eigenvalue to the linear problem

$$\begin{cases} -u''=\lambda u & \text{in } (-R,R), \\ u(\pm R)=0. \end{cases} \tag{69}$$

Notice that now the role of the "eigenvalue" λ is transferred to the length of the new interval $L=\sqrt{\lambda}R$. We introduce

$$F(r)=\int_0^r f(s)ds=\frac{r^2}{2}-2\sqrt{r},$$

and note that $f(s)<0$ if $0<s<1:=r_f$ and $f(s)>0$ if $1<s$. On the other hand $F(s)<0$ if $0<s<r_F=2\sqrt[3]{2}$ ($\approx 2,51$) and $F(s)>0$ for $s>r_F$ (see Figure 5).

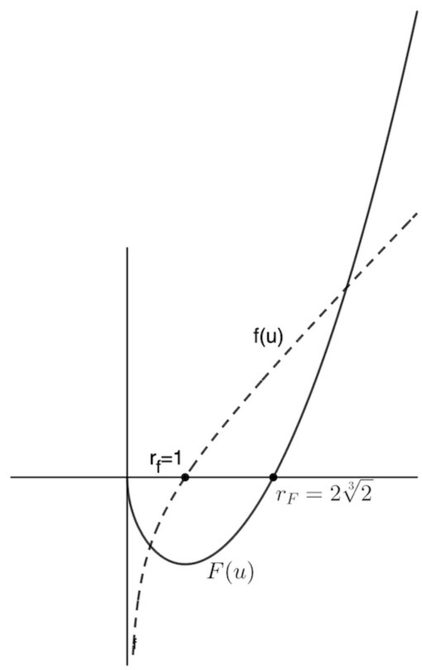

Figure 5: Functions f and F.

By multiplying by u', integrating by parts and denoting $\mu:=\|u\|_{L^\infty}$ for $\mu\in[r_F,\infty)$ we get that a function u is a positive solution of problem $P(L)$ if and only if

$$\frac{1}{\sqrt{2}}\int_{u(x)}^{\mu}\frac{dr}{(F(\mu)-F(r))^{1/2}}=|x|\,,\ \text{for}\ |x|\leq L,$$

and μ and $L>0$ are related by the equation

$$\gamma(\mu)=L,$$

where $\gamma:[r_F,+\infty)\to\mathbb{R}$ is given by

$$\gamma(\mu):=\frac{1}{\sqrt{2}}\int_0^{\mu}\frac{dr}{(F(\mu)-F(r))^{1/2}}. \tag{70}$$

Now we use the following fact, whose proof is exactly as in [23] and [27]: a function u is a positive solution of problem $P(L)$ if and only if

$$\frac{1}{\sqrt{2}}\int_{u(x)}^{\mu}\frac{dr}{(F(\mu)-F(r))^{1/2}}=|x|\,,\ \text{for}\ |x|\le L.$$

Moreover

$$u'(\pm L)=\mp\sqrt{2}\sqrt{F(\mu)}. \tag{71}$$

Thus, $u'(\pm R)=0$ corresponds to the case in which the maximum of the solution is r_F. It can be computed (using the *Gauss-Lobatto rules:* a remark made by G. Díaz to the author) that $\gamma(r_F)\approx 2,09$.

We will start by proving the existence of a branch of positive solutions for a bounded interval of the parameter, $\lambda\in((\frac{\pi}{2R})^2,\lambda_1^*)$.

Theorem 11 *We define with* $F(r)=\frac{r^2}{2}-2\sqrt{r}$. *Let* $r_F=2\sqrt[3]{2}$. *Then the mapping* $\gamma:[r_F,+\infty)\to\mathbb{R}$ *has the following properties*
(i) $\gamma\in C[r_F,\infty)\cap C^1(r_F,\infty)$;
(ii) For any $\mu>r_F$ $\gamma'(\mu)<0$,
(iii) $\gamma'(\mu)\to-\xi$ as $\mu\downarrow r_F$, for some $\xi>0$,
(iv) $\lim_{\mu\to+\infty}\gamma(\mu)=\frac{\pi}{2}$.

Concerning the flat solution and the possible solutions with compact support we have:

Theorem 12 *Let*

$$\lambda_1^*=\frac{1}{2R^2}\left(\int_0^{r_F}\frac{dr}{(F(\mu)-F(r))^{1/2}}\right)^2 \tag{72}$$

then we have:

a) if $\lambda\in(0,(\frac{\pi}{2R})^2)$ *there is no positive solution,*

b) if $\lambda\in((\frac{\pi}{2R})^2,\lambda_1^*)$ *there is a unique positive solution* u_{λ,V_0}. *Moreover* $\partial u_{\lambda,V_0}/\partial n(\pm R)<0$ *and* $\|u_{\lambda,V_0}\|_{L^\infty(-R.R)}=\left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}}\gamma^{-1}(\sqrt{\lambda}R)$,

c) if $\lambda=\lambda_1^*$ *there is only one positive solution* $u_{\lambda_1^*,V_0}$. *Moreover* $u'_{\lambda_1^*,V_0}(\pm R)=0$

$$\left\|u_{\lambda_1^*,V_0}\right\|_{L^\infty(-R,R)}=\left(\frac{4V_0}{\lambda_1^*}\right)^{\frac{2}{3}}.$$

d) if $\lambda>\lambda_1^*$, *there is a family of nonnegative solutions that are generated by extending by zero the function* $u_{\lambda_1^*,V_0}$ *outside* $(-R,R)$ *(and which we label again as* $u_{\lambda_1^*,V_0}$*). In particular, if* $\lambda=\lambda_1^*\omega$ *with* $\omega>1$ *we have a family* $S_1(\lambda)$ *of compact support nonnegative solutions with connected support defined by*

$$u_{\lambda,V_0}(x)=\frac{1}{\omega^{\frac{2}{3}}}u_{\lambda_1^*,V_0}(\sqrt{\omega}x-z),$$

where the shifting argument z *is arbitrary among the points* $z\in(-R,R)$ *such that support of* $u_{\lambda,V_0}(.)\subset(-R,R)$. *Moreover, for* $\lambda>\lambda_1^*$ *large enough we can build, similarly, a subset of* $S_j(\lambda)$ *of compact support nonnegative solutions with the support formed by* j *components, with* $j\in\{1,2,...,N\}$, *for some suitable* $N=N(\lambda)$ *and then the set of nontrivial and nonnegative solutions of* $P(\lambda)$ *is formed by* $S(\lambda)=\cup_{j=1}^N S_j(\lambda)$. *In any case those solutions satisfy that*

$$\|u_{\lambda,V_0}\|_{L^\infty(-R,R)}=\frac{1}{\omega^{\frac{2}{3}}}\left\|u_{\lambda_1^*,V_0}\right\|_{L^\infty(-R,R)}=\frac{1}{\omega^{\frac{2}{3}}}\left(\frac{4V_0}{\lambda_1^*}\right)^{\frac{2}{3}},\ \textit{for any}\ \omega=\lambda/\lambda_1^*>1.$$

Proofs of Theorems 11 and 12. The proof of property i) is exactly the same as the one presented in [27]. For the proof of (ii) and (iii), we have

$$\gamma'(\mu) = \int_0^{\mu} \frac{\theta(\mu) - \theta(r)}{(F(\mu) - F(r))^{1/2}} dr$$

where $\theta(t) = 2F(t) - tf(t) = -3\sqrt{t}$, and differentiating we get for, any $t > 0$

$$\theta'(t) = -\frac{3}{2\sqrt{t}} < 0.$$

Hence $\gamma'(\mu) < 0$ for any $\mu > r_F$ (which proves (ii)).
For the proof of (iii) it suffices to see that as $\mu \downarrow r_F$, the integrand of $\gamma'(\mu)$ converges pointwise to $(-F(r))^{-3/2}$ near $r = 0$ and in our case $(-F(r))^{-3/2}$ behaves as $r^{-3/4}$ near $r = 0$ and thus $\gamma'(\mu)$ converges to a number $-\xi$ as $\mu \downarrow r_F$, for some $\xi > 0$.
Finally, to prove (iv), we note that

$$\gamma(\mu) \leq \frac{\mu}{2} \int_0^1 \frac{dt}{(\frac{\mu^2}{2}(1 - t^2))^{1/2}} = \int_0^1 \frac{dt}{\sqrt{1 - t^2}} = \frac{\pi}{2}. \tag{73}$$

Moreover, we have

$$\gamma(\mu) = \frac{\mu}{\sqrt{2}} \int_0^1 \frac{dt}{\left(\frac{\mu}{\sqrt{2}}((1 - t^2) - \frac{1}{\mu^{3/2}}(1 - \sqrt{t})\right)}, \tag{74}$$

and if $\mu \to +\infty$, by using Lebesgue's Theorem, we get

$$\lim_{\mu \to +\infty} \gamma(\mu) = \frac{\pi}{2}.$$

Now we define $L_0 = \frac{\pi}{2}$ and L^* given by

$$L^* = \gamma(r_F) = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/2}} = \frac{1}{\sqrt{2}} \int_0^{r_F} \frac{dr}{(-2\sqrt{r} + \frac{r^2}{2})^{1/2}}.$$

We know that $u'(\pm R) = 0$ corresponds to the value L^* and that the maximum of the solution is r_F.
So, qualitatively, function γ is described by the Figure 6.
If we now go back with our change of variables, we get

$$\|u_{\lambda,V_0}\|_{L^\infty(-R,R)} = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} r_F,$$

and we obtain, finally, the bifurcation diagram given by the first branch of Figure 2, where solutions for $\lambda > \lambda^*$ are compact supported solutions originated as in [27] from the extension by zero of the free boundary solution u_{λ^*} satisfying

$$\begin{cases} -u''_{\lambda^*} + \frac{V_0}{\sqrt{u_{\lambda^*}}} = \lambda^* u_{\lambda^*} & \text{in } (-R, R), \\ u_{\lambda^*}(\pm R) = u'_{\lambda^*}(\pm R) = 0. \end{cases} \tag{75}$$

The rest of the details are completely analogous to the similar parts of the paper [27]. □

Remark 16 *Once that we know that for $\lambda > \lambda_1^*$ we have that*

$$u_{\lambda,V_0}(x) = \frac{1}{\omega^{\frac{2}{3}}} u_{\lambda_1^*,V_0}(\sqrt{\omega}x - z),$$

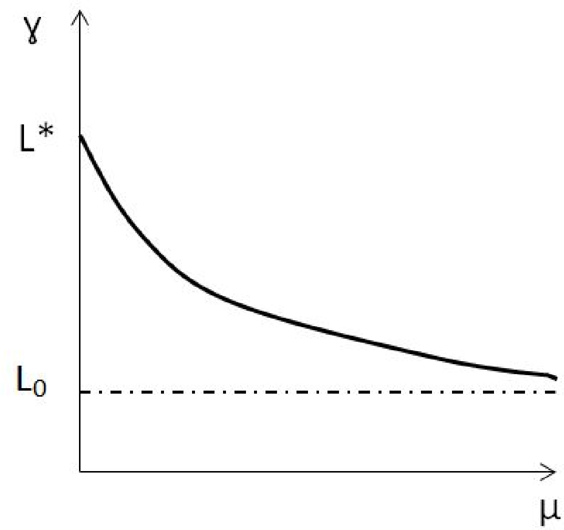

Figure 6: Critical value L^*.

then we get that the bifurcating curve Λ *is not* C^1 *at* $\lambda = \lambda_1^*$ *since* $\Lambda'(\lambda_1^*-) = \xi < 0$ *and* $\Lambda'(\lambda_1^*+) = -\frac{2C}{3(\lambda_1^*)^{5/3}}$*, with* $C = 4^{2/3}(V_0)^{2/3}$*. In addition, for* $\lambda > \lambda_1^*$ *we can express other norms (different from the* L^∞*-norm) in terms of* λ*. For instance, we have that*

$$\left\| u'_{\lambda,V_0} \right\|_{L^\infty(-R,R)} = C\lambda^{-\frac{1}{6}}$$

for a suitable constant $C > 0$ *independent of* λ*. This proves that* $\left\| u'_{\lambda,V_0} \right\|_{L^\infty(-R,R)} \to 0$ *as* $\lambda \to +\infty$*.*

4.2.2 On the construction of the subsolution over Ω_-: continuation

Now, let us come back to the construction of the positive subsolution over Ω_- mentioned in Proposition 1. We recall that, for functions of the form (60) the partial differential equation (61) leads to the ordinary differential equation

$$-U''(\theta) + \tfrac{V(\theta)}{\sqrt{U(\theta)}} = \lambda U(\theta) \quad \theta \in (\tfrac{\pi}{2}, \pi), \tag{76}$$

once we assume the constraint

$$\alpha + 2\beta = 4/3, \tag{77}$$

with

$$V(\theta) = \frac{A}{(-\cos\theta)^\beta} \text{ and } \lambda = \alpha^2. \tag{78}$$

Notice that the potential $V(\theta)$ is singular only for $\theta = \frac{\pi}{2}$. On the other hand, we need to match this subsolution with another function which is positive for $\theta = \frac{\pi}{2}$. Then, given $\varepsilon > 0$ small enough, we will construct the subsolution in two different pieces

$$U(\theta) = \begin{cases} v_1(\theta) & \text{if } \theta \in (\frac{\pi}{2}, \pi - R_0), \\ v_2(\theta) & \text{if } \theta \in (\pi - R_0, \pi), \end{cases}$$

for some R_0 with $\pi - R_0 \in (\frac{\pi}{2} + \varepsilon, \frac{\pi}{2} + 2\varepsilon)$, and with $v_2(\theta)$ such that

$$\begin{cases} -v_2''(\theta) + \frac{V_\varepsilon}{\sqrt{v_2(\theta)}} = \lambda v_2(\theta) \quad \theta \in (\pi - R_0, \pi), \\ v_2(\pi) = v_2'(\pi) = 0, \end{cases} \tag{79}$$

where

$$V(\theta) \le V_\varepsilon \text{ if } \theta \in (\frac{\pi}{2} + \varepsilon, \pi).$$

See Figure 7.

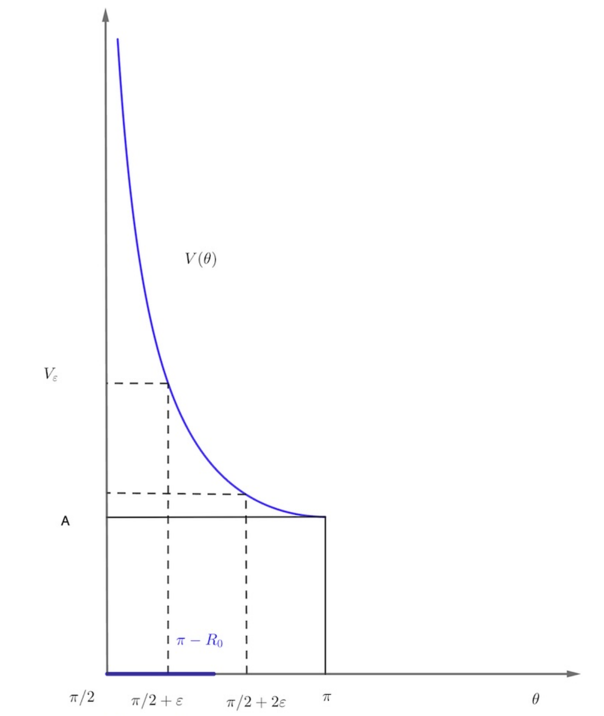

Figure 7: Singular potential function $V(\theta)$.

Proof of Theorem 7. We recall the change of variable

$$u_{\lambda,V_0}(x) = \left(\frac{V_0}{\lambda}\right)^{\frac{2}{3}} u(\sqrt{\lambda}x),$$

which links the question of the distinguished eigenvalue $\lambda^* = (\alpha^*)^2 = \frac{1}{2R^2}\gamma(r_F)^2$ with a special length L^* for which the solution is flat on the boundary.
The computations $r_F = 4^{2/3}$ ($\approx 2,51$) and $\gamma(r_F) \approx 2,09$ allow to see that the corresponding L^* leads to $\lambda^* = \frac{16}{9}$ and thus $\alpha^* = \frac{4}{3}$ and $R = \frac{\pi}{2}$. This corresponds to the case $\beta = 0$ thanks to the reciprocal relation

$$\beta = 2 - \frac{3\alpha}{2}.$$

Moreover, we must take A small enough (in fact $A = \frac{4}{9}$) and there is a kind of *fortunate coincidence.*

Notice that although $\|u_{\lambda,V_0}\|_{L^\infty} > 1$ there is no difficulty with this since we can take $u_{0,-}(x,y) = \phi(r,\theta) = kr^\alpha U(\theta)$, for some $k > 0$ small enough.

If we consider the case of $\beta \in (0, \frac{1}{2})$ (i.e. $\alpha^* \in (1, \frac{4}{3})$) then the problem is not autonomous (it appears $V(\theta) = \frac{A}{(-\cos\theta)^\beta}$), the corresponding $R > 0$, given by the equation $\lambda^* = (\alpha^*)^2 = \frac{1}{2R^2}\gamma(r_F)^2$, is such that $R > \frac{\pi}{2}$ and we must truncate $V(\theta)$ on an interval $(\pi - R_0, \pi)$, for instance by taking $R_0 < R$ such that $u_{\lambda, V_0}(\pi - R_0) = 1$ (see Figure 7).
We see then that the matching between $v_1(\theta)$ and $v_2(\theta)$ must take place at $\pi - R_0$ (see Figure 8).

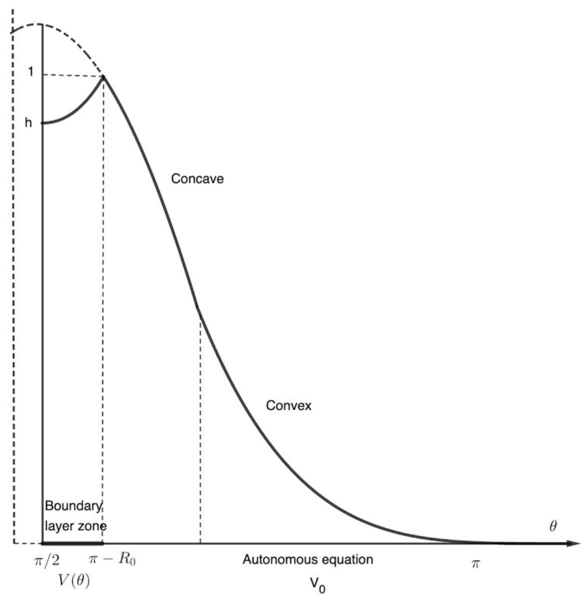

Figure 8: Matching functions $v_1(\theta)$ and $v_2(\theta)$ to get a subsolution.

The construction of $v_1(\theta)$, in the so-called "boundary layer zone", can be carried out as in the one-dimensional problem with a distributed potential $V(\theta)$. Indeed, given $h \in (0, 1)$ we take $v_1(\theta)$ as the solution of

$$\begin{cases} -v_1''(\theta) + \frac{V(\theta)}{\sqrt{v_1(\theta)}} \leq 0 & \theta \in (\frac{\pi}{2}, \pi - R_0), \\ v_1(\frac{\pi}{2}) = h, \, v_1(\pi - R_0) = 1. \end{cases} \tag{80}$$

Notice that using that

$$\cos\theta \leq \frac{2}{\pi}(\frac{\pi}{2} - \theta), \text{ if } \theta \in (\frac{\pi}{2}, \pi),$$

then the structural assumption on $V(\theta)$ implies that

$$V(\theta) \leq \frac{A\pi^\beta}{2^\beta(\theta - \frac{\pi}{2})^\beta} := \frac{C}{(\theta - \frac{\pi}{2})^\beta}.$$

and we can take

$$-v_1''(\theta) + \frac{\frac{C}{(\theta - \frac{\pi}{2})^\beta}}{\sqrt{v_1(\theta)}} = 0$$

with the above boundary conditions. Thus, we can prove that

$$v_1'(\frac{\pi}{2}) \geq 0$$

and that

$$v_1'(\pi - R_0) \geq 0 \geq v_2'(\pi - R_0).$$

We get that the graph of the matched function presents a cusp at $\theta = \pi - R_0$, which is far from being *natural*, but it is sufficient for our purposes. Finally, the boundary inequality

$$u_{0,-}(x,y) = kr^{\alpha}U(\theta) \leq 1 \text{ if } y = 1$$

holds for k small enough such that

$$kH^{\alpha}U(\theta) \leq 1, \text{ where } H = \sqrt{a^2+1}. \tag{81}$$

and the proof of the existence of $u_{0,-}(x,y)$, *a* p_0-*subsolution* of problem $P_{a,0,j}$, assumed in Proposition 1 is now completed, which proves the nondegenerate estimate (45) of Theorem 7 (notice that the estimate from above on Ω_+ is trivially satisfied since $u_{0,-}(x,y)$ is a harmonic function).
The proof of *ii*) is a direct consequence of the Lemma 2 showing that $u^0(x,y) = y$ is a universal supersolution. Finally, the assumptions of the uniqueness result Theorem 8 are satisfied, and thus the proof of Theorem 7 is complete. □

4.3 Construction of a partially flat supersolution

We make now the structural assumptions, (7) and (8), stated in Theorem 1. Thus, in particular, we already know, by Theorems 7 and 8 that the mere existence of a flat nondegenerate subsolution implies the uniqueness of the solution. We concentrate our attention now in proving that the above conditions on $j(x)$ are sufficient conditions for the existence of a partially flat supersolution, i.e., being flat only on the half of the cathode region under study $[-a,0] \times \{0\}$.

Since we only need to work with the inequality $\geq$ in the equation, it is enough to use (as before) that

$$j(r,\theta) = \frac{A}{(-r\cos\theta)^{\beta}} \geq V_0 \text{ for } x \in (-a,0).$$

We will search for the supersolution by some matching arguments, now in three different regions, as indicated in the Figure 9 :

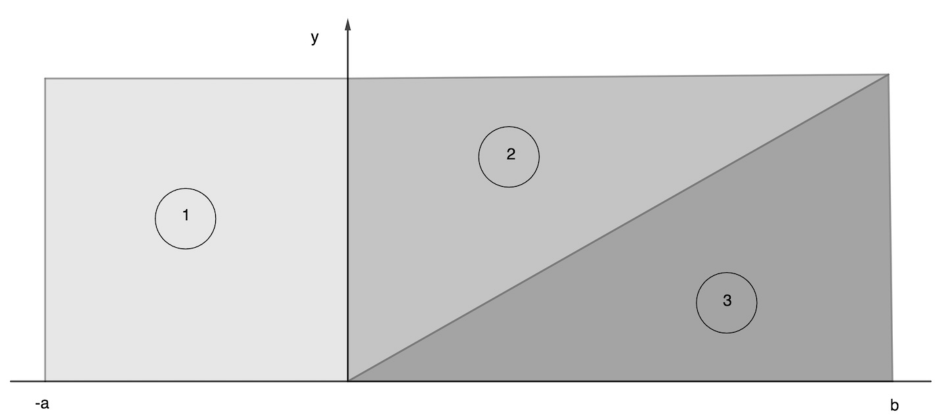

Figure 9: Decomposition in subdomains.

In the first region Ω_- we will construct the subsolution $u_-^0(x,y)$ in the same form than the subsolution, i.e.,

$$u_-^0(x,y)=\phi(r,\theta)=kr^\alpha U(\theta), \text{ for some } k>0,\ \alpha>1,$$

and thus with $U(\theta)$ solution of

$$-U''(\theta)+\frac{V_0}{\sqrt{U(\theta)}}=\lambda U(\theta)\quad \theta\in(\tfrac{\pi}{2},\pi). \tag{82}$$

So that no boundary layer needs to be taken into account. This means that, in terms of the proof of Theorem 7, $U(\theta)=v_2(\theta)$.

As before, if we consider the case of $\beta\in(0,\frac{1}{2}]$ (i.e. $\alpha^*\in(1,\frac{4}{3}]$) then the corresponding $R>0$, given by the equation $\lambda^*=(\alpha^*)^2=\frac{1}{2R^2}\gamma(r_F)^2$, is such that $R\geq\frac{\pi}{2}$ ($R=\frac{\pi}{2}$ if $\alpha^*=\frac{4}{3}$) and thus

$$v_2'(\frac{\pi}{2})\leq 0.$$

The matching with the second region will follow a different argument. In this part, the supersolution must be superharmonic, and it is searched in the form

$$u_{+,2}^0(x,y)=Kr^\alpha\sin(\alpha\theta),\ \ \theta\in(\theta_b,\frac{\pi}{2}),$$

with $\tan\theta_b=b$. It is not difficult to check that

$$-\Delta u_{+,2}^0=K\alpha(\alpha-1)r^{\alpha-2}\sin(\alpha\theta)\geq 0.$$

The matching is now more delicate. For a H^1-matching we must have:

$$\begin{cases} u_-^0(0,y)=u_{+,2}^0(0,y) & y\in(0,1),\\ \nabla u_-^0(0,y)=\nabla u_{+,2}^0(0,y) & y\in(0,1).\end{cases}$$

The first condition holds if

$$\frac{k}{K}=\frac{\sin(\frac{\alpha\pi}{2})}{U(\frac{\pi}{2})}.$$

The second condition (once we allow the formation of singularities with a good sign, i.e., with a non-negative generated implicit distribution) leads to

$$U'(\frac{\pi}{2})\geq\frac{K}{k}\alpha\cos(\frac{\alpha\pi}{2})=\alpha U(\frac{\pi}{2})\frac{\cos(\frac{\alpha\pi}{2})}{\sin(\frac{\alpha\pi}{2})}=\alpha U(\frac{\pi}{2})\cot(\frac{\alpha\pi}{2}).$$

But, $\frac{\alpha\pi}{2}\in(\frac{\pi}{2},\frac{2\pi}{3})\subset(0,\pi)$ so that $\cot(\frac{\alpha\pi}{2})<0$ (see the Figure 10).
Thus we must investigate for which $\alpha\in(1,\frac{4}{3}]$ we have some kind of Robin type boundary inequality on $U(\theta)$ for $\theta=\frac{\pi}{2}$:

$$\frac{-U'(\frac{\pi}{2})}{U(\frac{\pi}{2})}\leq\alpha(-\cot(\frac{\alpha\pi}{2})). \tag{83}$$

This condition trivially holds if $\alpha=\frac{4}{3}$ (since, in that case, $U'(\frac{\pi}{2})=0$). Then by continuity in α, there exists $\alpha_0\in(1,\frac{4}{3})$ such that (83) holds for any $\alpha\in(1,\frac{4}{3})$. This justifies the assumption (7) in Theorem 9 (and in Theorem 1).

Finally, in the third region, we extend the above function $u_{+,2}^0(x,y)$, by a kind of interpolation argument (remember that we must have the boundary condition $u_{+,3}^0(b,y)\geq y$, for any $y\in(0,1)$). Then we define

$$u_{+,3}^0(x,y)=C_3r\sin(\alpha\theta)+\widehat{C}_3r^\alpha\sin(\alpha\theta_b)\quad\text{if } (x,y)\in\Omega_{+,3},$$

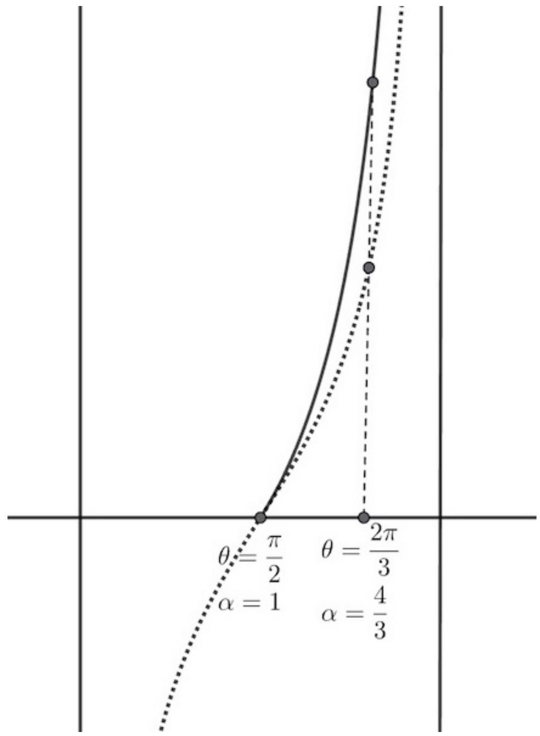

Figure 10: Representation of the functions -$\alpha\cot(\frac{\alpha\pi}{2})$

with $\tan\theta_b = b$. It can be proved that $u^0_{+,3}(x,y)$ is a super-harmonic function ($-\Delta u^0_{+,3} \geq 0$) and that (if b is large enough) the positive constants C_3 and $\widehat{C}_3$ can be chosen such that $u^0_{+,3}(x,y)$ satisfies the correct inequalities on the boundaries

$$\begin{cases} u^0_{+,3}(b,y) \geq y, & \text{for any } y \in (0,1), \\ u^0_{+,3}(x,0) \geq 0 & \text{for any } x \in (0,b), \end{cases}$$

and it correctly matches with $u^0_{+,2}(x,y)$ (generalizing, at most, a "good signed" measure) on the matching boundary $\theta = \theta_b$. This proves Theorem 9 (and also Theorem 1, except the proof of the uniqueness of solution which will be presented in the next subsection). $\square$

4.4 Proof of the uniqueness of nondegenerate solutions

The proof of Theorem 8 will be a consequence of some previous results concerning the parabolic problem

$$PP_{a,b,j,u_0} = \begin{cases} u_t - \Delta u + \frac{j(x)}{\sqrt{u}}\chi_{\{u>0\}} = 0 & t>0,\ x \in (-a,b),\ y \in (0,1), \\ u(t,x,0) = 0 & t>0,\ x \in (-a,b), \\ u(t,x,1) = 1 & t>0,\ x \in (-a,b), \\ u(t,-a,y) = y^{4/3} & t>0,\ y \in (0,1), \\ u(t,b,y) = y & t>0,\ y \in (0,1), \\ u(0,x,y) = u_0(x,y) & x \in (-a,b),\ y \in (0,1). \end{cases} \tag{84}$$

Notice that the above parabolic problem is not related to the evolution problem associated with the electron beam formulation. It would be related to some associated evolution problem if the modelling under study were concerned with the Chemical Engineering problem dealing with a chemical kinetics with a reaction of order -1/2. Here, we are only interested in the fact that this artificial parabolic problem leads us (under suitable assumptions) to the uniqueness of solutions of the mathematical model for a stationary electron beam.

The existence of solutions to problem PP_{a,b,j,u_0} is an easy variation of results dealing with parabolic quenching type problems (see, e.g., [16], [15], [21] and [14]). We point out that no flat condition is required in this formulation. As in the elliptic problem, we can prove that for certain initial data, the solutions are non-degenerate in the sense that for any $t > 0$ the solution u is such that $u(t)$ belongs to the following set

$$\mathcal{M}(\nu) := \left\{ u \in L^2(\Omega;\delta) \;\middle|\; \text{such that } u(x,y) \geq C\delta(x,y)^\nu \quad \text{in } \Omega, \text{ for some } C > 0 \right\},\ \nu \in \left(0, \frac{4}{3}\right].$$

The following result is an adaptation to this framework of the main theorem of [21]. It gives the continuous dependence of solutions for the initial data (implying, obviously, the uniqueness of solutions), as well as a smoothing effect concerning the initial datum. In the proof, we will use some Hilbertian techniques. We consider two initial data, and we will prove that if the respective solutions are such that $u(t), v(t) \in \mathcal{M}(\nu)$ (i.e. with $\delta^{-\nu}u, \delta^{-\nu}v \geq C$) then we can estimate the $L^2(\Omega)$-norm of $\delta^{-\gamma}[u(t) - v(t)]_+$ for some suitable $\gamma \in (0, 1]$ in terms of the $L^2(\Omega;\delta)$-norm of $[u_0 - v_0]_+$. Notice that this automatically implies an estimate on the $L^2(\Omega)$-norm of $[u(t) - v(t)]_+$.

Theorem 13 *Let $u_0, v_0 \in L^1(\Omega) \cap L^2(\Omega;\delta)$. Let u, v be weak solutions of PP_{a,b,j,u_0} and PP_{a,b,j,v_0}, respectively such that $u(t), v(t) \in \mathcal{M}(\nu)$ for some $\nu \in \left(0, \frac{4}{3}\right]$. Then, for any $t \in (0, \infty)$, we have*

$$\left\| \delta^{-\gamma}[u(t) - v(t)]_+ \right\|_{L^2(\Omega)} \leq C t^{-\frac{2\gamma+1}{4}} \left\| [u_0 - v_0]_+ \right\|_{L^2(\Omega;\delta)}, \tag{85}$$

with

$$\gamma := \min\left\{ \frac{3\nu}{2}, 1 \right\} \tag{86}$$

and for some constant $C > 0$ independent of t. In particular, $u_0 \leq v_0$ implies that for any $t \in [0, +\infty)$,

$$u(t,\cdot) \leq v(t,\cdot) \quad \text{a.e. in } \Omega$$

and

$$\left\| \delta^{-\gamma} \left(u(t) - v(t) \right) \right\|_{L^2(\Omega)} \leq C t^{-\frac{2\gamma+1}{4}} \left\| u_0 - v_0 \right\|_{L^2(\Omega;\delta)}. \tag{87}$$

Before referring to the proof of Theorem 13, we will prove that this implies the uniqueness of the positive solution for the stationary problem $P_{a,b,j}$ presented in Theorem 8.

Proof of Theorem 8. Let us call u_∞ and v_∞ two possible solutions of $P_{a,b,j}$ in the class $\mathcal{M}(\nu)$. By taking $u_0 = u_\infty$ and $v_0 = v_\infty$ as initial data in PP_{a,b,j,u_0} and PP_{a,b,j,v_0}, since u_∞ and v_∞ are, obviously, solutions of the respective parabolic problems, we get that $u_\infty - v_\infty$ satisfies

$$\left\| \delta^{-1} \left(u_\infty - v_\infty \right)_+ \right\|_{L^2(\Omega)} \leq C t^{-\frac{2\gamma+1}{4}} \left\| (u_\infty - v_\infty)_+ \right\|_{L^2(\Omega,\delta)}.$$

Making $t \nearrow +\infty$ and reversing the role of u_∞ and v_∞, we get that $u_\infty = v_\infty$. □

The proof of Theorem 13 follows some slight modifications of the paper [21]. Nonetheless, for the sake of completeness, we present here the main lines of the proof.

Without loss of generality, we can use the notion of *mild solution* on $L^1(\Omega)$, i.e. $u \in \mathcal{C}([0,T); L^1(\Omega))$, for any $T > 0$: $j(x)u^{-1/2} \in L^1(\Omega \times (0,T))$ and u fulfills the identity

$$u(\cdot,t) = S(t)u_0(\cdot) - \int_0^t S(t-s)(\chi_{\{u>0\}}u^{-1/2}(\cdot,s))ds, \quad \text{in } L^1(\Omega), \tag{88}$$

where $S(t)$ is the $L^1(\Omega)$-semigroup corresponding to the Laplace operator with the corresponding Dirichlet (stationary) boundary conditions (see, e.g., [14]). We shall need some well-known auxiliary results. The first one is a singular version of the Gronwall's inequality (due to Brezis and Cazenave [6]), which is especially useful in the study of non-global Lipschitz perturbations of the heat equation.

Lemma 3 *Let $T > 0$, $A \geq 0$, $0 \leq a, b \leq 1$ and let f be a non-negative function with $f \in L^p(0,T)$ for some $p > 1$ such that $\max\{a,b\} < 1/p'$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Consider a non-negative function $\varphi \in L^\infty(0,T)$ such that, for almost every $t \in (0,T)$,*

$$\varphi(t) \leq At^{-a} + \int_0^t (t-s)^{-b} f(s)\varphi(s)\, ds. \tag{89}$$

Then, there exists $C > 0$ only depending on T, a, b, p and $\|f\|_{L^p(0,T)}$ such that, for almost every $t \in (0,T)$,

$$\varphi(t) \leq ACt^{-a}. \tag{90}$$

We shall also use some regularizing effects properties satisfied by the semigroup $S(t)$ of the heat equation with zero Dirichlet boundary conditions (since we will apply it to the difference of two solutions of the parabolic problem PP_{a,b,j,u_0}) due to different authors, among them [49], [16] and [47]: see details, in [21].

Lemma 4

1. *There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^2(\Omega)$,*

$$\|\nabla S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{1}{2}}\|u_0\|_{L^2(\Omega)}. \tag{91}$$

2. *There exists $C > 0$ such that, for any $t > 0$ and any $u_0 \in L^1(\Omega)$,*

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{N}{4}}\|u_0\|_{L^1(\Omega)}. \tag{92}$$

3. *There exists $C > 0$ such that, for any $t > 0$, any $m \in (0,1]$ and any $u_0 \in L^2(\Omega; \delta^{2m})$,*

$$\|S(t)u_0\|_{L^2(\Omega)} \leq Ct^{-\frac{m}{2}}\|u_0\|_{L^2(\Omega,\delta^{2m})}. \tag{93}$$

4. *There exists $C > 0$ such that, for any $t > 0$, any $p \in [1, +\infty)$ and any $u_0 \in L^p(\Omega, \delta)$,*

$$\|S(t)u_0\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2p}}\|u_0\|_{L^p(\Omega,\delta)}. \tag{94}$$

Finally, we can give the main arguments of the proof.

Proof of Theorem 13. By the constant variations formula, we know that for any $t \in [0,T]$,

$$u(t) - v(t) = S(t)(u_0 - v_0) + \int_0^t S(t-s)\,(h(u(s)) - h(v(s)))\; ds \quad \text{in } \Omega, \tag{95}$$

where $h(x,u) := j(x)u^{-1/2}$. By the convexity of the function $u \mapsto u^{-1/2}$ and the assumption that $u(t), v(t) \in \mathcal{M}(\nu)$, we deduce that

$$h(x,u) - h(x,v) \leq Cj(x)\delta^{-3\nu/2}(u-v)_+ \quad \text{in } \Omega. \tag{96}$$

Thus, if we denote $w := u - v$, we get for any $\tau, t \in [0,T]$ with $\tau \leq t$

$$w_+(t) \leq S(t-\tau)w_+(\tau) + C\int_\tau^t S(t-s)(j(x)\delta^{-3\nu/2}w_+(s))\,ds. \tag{97}$$

We multiply (97) by the weight $\delta^{-\gamma}$, with $\gamma \in [0,1]$ to be chosen later, and take the L^2-norms. Then,

$$\|\delta^{-\gamma}w_+(t)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}S(t-\tau)w_+(\tau)\|_{L^2(\Omega)} + C\int_\tau^t \|S(t-s)j(x)\delta^{-[(\beta+1)\nu+\gamma]}w_+(s)\|_{L^2(\Omega)}\,ds.$$

Let us fix $s,t>0$ and let us call $\psi := S(t-s)j(x)\delta^{-(\beta+1)\nu}w_+(s)$. Then, by Hölder inequality,

$$\|\delta^{-\gamma}\psi\|^2_{L^2(\Omega)} = \int_\Omega \frac{\psi^2}{\delta^{2\gamma}}\,dx \leq \left(\int_\Omega \frac{\psi^2}{\delta^2}\,dx\right)^\gamma \left(\int_\Omega \psi^2\,dx\right)^{1-\gamma}$$

(note that the limit cases $\gamma \equiv 0$ and $\gamma \equiv 1$ are allowed). Then, applying the Hardy inequality,

$$\|\delta^{-\gamma}\psi\|_{L^2(\Omega)} \leq C\|\nabla\psi\|^\gamma_{L^2(\Omega)}\|\psi\|^{1-\gamma}_{L^2(\Omega)}.$$

By property 1 of Lemma 4, for $\frac{t-s}{2}$, we get

$$\|\delta^{-[(\beta+1)\nu+\gamma]}S(t-s)w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{\gamma}{2}}\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)}. \tag{98}$$

Analogously, using property 4 of Lemma 4, and that $j \in L^2(\Omega)$ (remember the assumptions made in Theorem 7)

$$\|\delta^{-\gamma}S(t)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\frac{\gamma}{2}}\|S\left(\frac{t}{2}\right)j(x)w_+(0)\|_{L^2(\Omega)} \leq Ct^{-\left(\frac{\gamma}{2}+\frac{1}{4}\right)}\|w_+(0)\|_{L^2(\Omega;\delta)}. \tag{99}$$

In order to apply the singular Gronwall's inequality, we must relate the weights $\delta^{-\gamma}$ and $\delta^{-3\nu/2}$ keeping in mind that $\gamma \in [0,1]$. To do that, we apply property 3 of Lemma 4 for some $m \in [0,1]$. We shall take

$$3\nu/2 = \gamma + m. \tag{100}$$

Indeed, if $(\beta+1)\nu \in (1,2]$, then we take $\gamma = 1$, $m = 3\nu/2 - 1$ and we apply point 3 of Lemma 4 to the initial datum:

$$\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(m+1)}w_+(s)\|_{L^2(\Omega)} \leq C(t-s)^{-\frac{m}{2}}\|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}. \tag{101}$$

On the other hand, if $3\nu/2 \in [0,1]$, we can take $\gamma = 3\nu/2$ and thus, since $S(t-s)$ is a contraction in $L^2(\Omega)$, we get

$$\|S\left(\frac{t-s}{2}\right)j(x)\delta^{-(\beta+1)\nu}w_+(s)\|_{L^2(\Omega)} = \|S\left(\frac{t-s}{2}\right)j(x)\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)} \leq \|\delta^{-\gamma}w_+(s)\|_{L^2(\Omega)}, \tag{102}$$

which corresponds to (100) with $m=0$. In other words,

$$\gamma = \min\{1, 3\nu/2\}$$

and

$$m = \max\{3\nu/2 - 1, 0\}.$$

Collecting the previous inequalities, we arrive to

$$\|\delta^{-\gamma} w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|w_+(0)\|_{L^2(\Omega;\delta)} + C \int_0^t (t-s)^{-\frac{m}{2}} \|\delta^{-\gamma} w_+(s)\|_{L^2(\Omega)}.$$

Thus, we can apply Lemma 3 with $a = \frac{2\gamma+1}{4} \in \left[\frac{1}{4}, \frac{3}{4}\right]$, $b = \frac{m}{2}$ and $A = C\|w_+(0)\|_{L^2(\Omega;\delta)}$ to deduce that

$$\|\delta^{-\gamma} w_+(t)\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \|w_+(0)\|_{L^2(\Omega;\delta)}.$$

□

Remark 17 *As in the proof of Theorem 8, if u_∞ is the degenerate solution of problem $P_{a,b,j}$, for any $u_0 \in L^2(\Omega;\delta)$ leading to a non-degenerate solution $u(t)$ of the parabolic problem PP_{a,b,j,u_0}, we get a quantitative estimate on the asymptotic stability of u_∞ in the class of non-degenerate stationary solutions, since, for any $t \in (0,\infty)$, we have*

$$\left\|\delta^{-\gamma}[u(t) - u_\infty]\right\|_{L^2(\Omega)} \leq Ct^{-\frac{2\gamma+1}{4}} \left\|u_0 - u_\infty\right\|_{L^2(\Omega;\delta)},$$

with $\gamma := \min\left\{\frac{3\nu}{2}, 1\right\}$, for some constant $C > 0$ independent of t.

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Figure 11: Haïm Brezis, Joel Lebowitz and the author, in Paris, 2005.

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