Stochastic diffusive energy balance climate model with a multiplicative noise modeling the Solar variability

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Dedicated to Roger Temam, always admired, on the occasion of his 85th birthday

Abstract

We prove existence, uniqueness, and comparison of solutions for a nonlinear stochastic parabolic partial differential equation that includes the Solar variability in terms of a multiplicative Wiener cylindrical noise in the term of the absorbed radiative energy in a simplified diffusive one-dimensional Energy Balance Model. We introduce a hybrid co-albedo nonlinear term, which has the advantages of both the Sellers model, as it is a continuous function, and the Budyko model, as it has an infinite derivative at $u = -10°C$ (the temperature at which ice is white), allowing the location of the polar ice caps to be easily detected. We show that, despite the lack of differentiability of this function, the method of successive approximations can be satisfactorily applied.

1 Introduction

The radiative energy balance climate models (EBMs) are a class of tools for representing the evolution of the global climate (spatial distribution of temperature at the Earth's surface) over large time scales. Despite their simplicity, the EBMs give a useful, representation of Earth's climate by capturing the fundamental mechanisms governing its behaviour. They were proposed in 1969, independently (during the "Cold War") by an American, Sellers [52], and a Russian, Budyko [9], maintaining an enormous resemblance that confirms their great robustness. The EBMs assume that the averaged atmospheric temperature evolves (on a large scale of time) according to the radiation balance of the budget, i.e. the difference in the radiations absorbed and emitted by the planet. The Solar radiation (is the primary input) though must be corrected by the Earth's co-albedo (which depends of the temperature as a feedback) and the balance is taken with the infrared radiation emitted by the Earth and also in the presence of a surface diffusion. When describing the constitutive laws of the absorbed and emitted radiation it is needed to take into account important elements such as the Solar constant, the Solar insolation (depending of the spatial distribution), the atmospheric composition (which appears as coefficient when applying the Stefan-Boltzmann for the outgoing longwave radiation) etc. Since the pioneering work of 1969, mentioned above, EBMs have been the subject of numerous research studies and monographs ([48], [32], [35], [23], [44], [33], [1], [26], [7], [25], [24], [37], [5], [10], etc.). Some version of the EBMs can be also obtained by averaging in the *primitive equations* (see, e.g., [39], [40], [41]), as it was presented in [38]. A different averaging approach was developed by Hasselmann [34] (see also [2] and [15]). EBMs are also very useful in the study of past climates ([12]).

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Deterministic EBMs do not include unpredictable external forcings, as, for instance, volcanic emissions (there are currently more than 500 active volcanoes), etc. From the mathematical point of view, this has been treated by means of an additive white noise ([47], [19], [28], [36], [20], [43], [14], [15]). The mathematical models can be also coupled with some simple modelling of the deep ocean temperature ([58], [29], [30], [27], [16]) but, for simplicity in the formulation, we will not follow this coupling in this paper.

EBMs with stochastic noise allow us to justify, through scientific arguments and the available data, the possible increase in extreme events due to Climate Change, that is not possible under purely deterministic approaches (see, e.g. [15]).

The main goal of this paper is to study a mathematical model taking into account the influence on the climate of the abrupt changes in the Solar radiation (the Solar storms). The assumption that Solar emission is constant must be replaced by a more realistic study that takes Solar variability into account ([51], [57]). In fact, there is a whole family of space satellites whose primary mission is to analyze and measure this Solar variability. The Earth Radiation Budget Satellite (ERBS) was a NASA scientific research satellite. The satellite was one of three satellites in NASA's research program, named Earth Radiation Budget Experiment (ERBE), to investigate the Earth's radiation budget. NASA's CERES instruments have continued the ERB data record after 1997. We recall that the Solar energy that falls annually on Earth's surface is about ten thousand times the energy demand of the world's population (7.7 billion people). Phenomena such as Solar flares or coronal mass ejections (extreme Solar events) can cause brief increases in radiation. Occasionally, flares heat the Sun's surface, reaching temperatures of about 45 million degrees Celsius, much higher than those in the core. Long-term variations are minimal (the magnitude of the short-term fluctuations are small, typically less than 0.1% of the average value) but relevant in the climate and paleoclimate studies ([12]).

The main factor that generates variations is the Solar activity. The so-called S. Schabe (1789-1875) cycles vary over 11 years ([45]). This is too short a period to have any impact on the climate, though it is very relevant in other aspects. This was previously measured by counting sunspots, and is now measured using satellite radiance measurements. Other types of cycles (the so-called W. Gleissberg (1903-1986) cycle)) has an oscillation amplitude similar to the Schwabe cycles but its duration is approximately 87 years (70-100 years) and has a greater impact on climate due to its duration ([50]). It is related to the well-identified past periods (Maunder Minimum (1645-1715) and the Dalton Minimum (1800-1830)) of an extraordinarily low Solar activity. There is also the well-known Milankowitz cycles, based on Celestial Mechanics, which are justified in another way (on scales of thousands of years) and can be considered as periodic versions of the Solar constant (see a mathematical study on a pure time periodical Solar datum $Q(t)$ in [3]).

The mathematical model that we will consider in this paper includes the Solar variability in terms of a multiplicative Wiener noise in the term of the absorbed radiative energy in a simplified diffusive one-dimensional Energy Balance Model:

$$(\mathrm{E}_{\beta,\varepsilon})\begin{cases} du_t - \dfrac{\partial}{\partial x}\left(\left(1-x^2\right)\dfrac{\partial u_t}{\partial x}\right) + g(u_t) = \mathrm{QS}(x)\beta(u_t)\left(1+\varepsilon d\mathrm{W}_t\right), \\ u(x,0) = u_0(x). \end{cases}$$

where $x \in \mathrm{I} \doteq (-1,1)$, $x = \cos\phi$, with ϕ the spherical latitude, $t > 0$ and $\varepsilon \geq 0$. Notice that we are following the usual dynamical system notation (see, e.g., [55]), $u = u(x,t;\omega) = u_t(x;\omega)$ where $x \in \mathrm{I}, \doteq (-1,1)$, $x = \cos\phi$, with ϕ the spherical latitude, $t \geq 0$ and ω is in the probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$. The *cylindrical Wiener processes* $\mathrm{W}_t(x;\omega)$ is not time differentiable and thus the notation used for deterministic models $\frac{\partial u}{\partial t}$ is not well justified. In what follows, we will use the notation $\rho(x) = 1 - x^2,\ x \in \mathrm{I}$ for the degenerate diffusion operator coefficient. Note that in this formulation, it is not necessary to specify any boundary condition on $\partial\mathrm{I}$, since the physical problem is posed on the sphere as a Riemannian manifold without boundary. This explains the degeneracy of the boundary operator on $\partial\mathrm{I}$.

Although we can assume greater generality, here we mainly assume that $\mathrm{Q} > 0$, and that the Earth emitted radiation is given by the term $g(u)$ such that

($\mathbf{H}_g$) $g : \mathbb{R} \to \mathbb{R}$ is a continuous increasing function.
The co-albedo term is assumed to be such that
($\mathbf{H}_\beta$) β is a *bounded* maximal monotone graph in $\mathbb{R}^2$, i.e. $\beta(s) \in [m, \mathrm{M}]$, $\forall s \in \mathbb{R}$.

We recall that one of the important differences in the 1969 modelling of these terms was concerning their regularity: In [52] it was assumed that $\beta(s)$ is a Lipschitz continuous non-decreasing function with a huge slope near $u = -10°$ (the Celsius temperature at which the ice goes from transparent to white), in contrast with [9], who assumed that the co-albedo is a discontinuous function at $u = -10$, to better parameterize the regions occupied by the polar caps. Here we are identifying the possible discontinuous non-decreasing function $\beta(s)$ with the associated maximal monotone graph ([8]) by including the entire jump interval at the discontinuity points. Although the result leads to a possible multivalued expression, we will simplify the writing by keeping the symbol $=$ in the equation.

On the *insolation function* $\mathrm{S}(x)$ we assume
($\mathbf{H}_s$) $\mathrm{S} : \mathrm{I} \to \mathbb{R},\ \mathrm{S} \in \mathrm{L}^\infty(\mathrm{I}),\ \mathrm{S}_1 \geq \mathrm{S}(x) \geq \mathrm{S}_0 > 0$ a.e. $x \in \mathrm{I}$.
Despite the presence of multiplicative noise $d\mathrm{W}_t$, there are many abstract results which can be applied when β is, as in the Sellers' option, a globally Lipschitz continuous function of the unknown (see, e.g., [13]). On the other hand, the study of the multiplicative noise in the presence of a discontinuous co-albedo function (as in the Budyko case) looks very complex to be considered in a first approach. In that paper, we will follow an intermediate option by considering a class of co-albedo functions β that, being continuous, present a singularity in their derivative at the critical value $u = -10$, which, following Budyko's motivation, allows us to easily recognize the regions occupied by the polar caps. More specifically, let us define the co-albedo function given by

$$\beta_{-10}(u) = \begin{cases} \beta_i, & \text{if } u < -10, \\ (\beta_w - \beta_i)\theta_{\delta+10}(u+10) + \beta_i, & \text{if } -10 \leq u \leq -10+\delta, \\ \beta_w, & \text{if } u > -10+\delta, \end{cases} \tag{1}$$

with $0 < \delta < 1$, where the function $\theta_\delta(u)$ is given by

$$\theta_\delta(u) = (\beta_w - \beta_i)\frac{u \ln u}{\delta \ln \delta}, \quad u \geq 0. \tag{2}$$

We note that $\beta \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1\big(\mathbb{R} \setminus \{-10, -10+\delta\}\big)$, with

$$\begin{cases} \beta'_{-10}(-10^-) = 0 \quad \text{and} \quad \beta'_{-10}(-10^+) = +\infty, \\ \beta'_{-10}\big((-10+\delta)^-\big) > 0 \quad \text{and} \quad \beta'_{-10}\big((-10+\delta)^+\big) = 0. \end{cases}$$

Thus, this choice of the co-albedo function (that we will call the *hybrid co-albedo function* in what follows) presents a sudden change at the critical temperature for which ice becomes white while offering a seamless transition from ice to water (see Figure 1 below where the profile of the co-albedo function was transferred to the origin).

We recall that the uniqueness of solutions when the right-hand side term is not Lipschitz continuous is a very delicate question. For deterministic parabolic PDEs, an important contribution was offered in the paper [31]. They prove that the usual uniqueness criterion for non-monotone elliptic equations is not enough for the parabolic equation, and they prove the uniqueness of solutions by asking an Osgood-type condition on the non-linear term. We will extend this type of results, in the stochastic framework, in two different directions which, in the best of our knowledge, were not presented in [31], nor on its generalizations): our proof will be constructive (since we will prove that the successive approximations method can be applied even in the absence of differentiability on β). In addition, we will get some comparison results (in terms of two different initial data).

Faced with a wide range of possible choices, in this article, we will model the *cylindrical Wiener noise* $\mathrm{W}_t(x;\omega)$ produced by the erratic Solar storms using a series expansion of the eigenfunctions

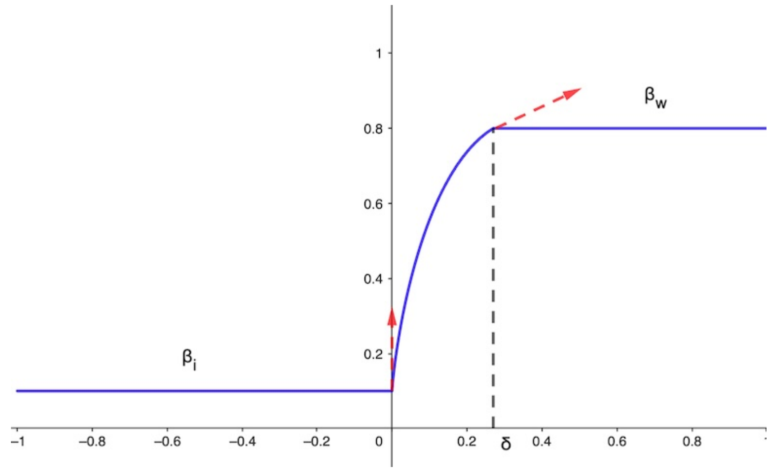

Figure 1: Hybrid co-albedo profile

generated by the Legendre diffusion mentioned above. We recall that the sequence of eigenvalues are $\mu_n = n(n+1)$ and that after a normalization the eigenfunctions are given by

$$\mathrm{e}_n(x) = \sqrt{\frac{2}{2n+1}}\mathrm{P}_n(x), \quad -1 \leq x \leq 1,$$

with (the Rodrigues formula)

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x+1)^{n-k}(x-1).$$

Since $\mu_0 = 0$, to avoid strong difficulties, we artificially introduce a given (but arbitrary) $\mu > 0$ and then we will work with the perturbed diffusion $-\frac{\partial}{\partial x}\left((1-x^2)\frac{\partial u}{\partial x}\right) + \mu u$ (the artificial term μu will also be added to the corresponding right-hand side). So that the same family of eigenfunctions corresponds now to the new sequence of eigenvalues $\widehat{\mu}_n = \mu_n + \mu > 0$. Thus, the noise we consider will be given by

$$\mathrm{W}_t = \sum_{n\geq 0}\frac{1}{\sqrt{\widehat{\mu}_n}}\mathbf{B}_t^n \mathrm{e}_n, \quad t \geq 0, \tag{3}$$

where the processes $\{\mathbf{B}_t^n\}_{t\geq 0}$ is a family of Brownian motions mutually independent (see [11, 13, 42], and Section 3, for a definition). We note that finally the parameter μ only appear in the noise (see (3)).

It is important to highlight that the main difficulty in the study of the multiplicative noise comes from the presence of the non-linear term $\beta(u)$ since if this term were a linear function in $\beta(u)$ the problem would be reduced to applying a clever change of variable that leads to a deterministic problem dependent on a parameter (see Remark 11).

The main contribution in this paper is

Theorem 1 *Let us assume* $\mathrm{u}_0 \in \mathrm{L}^\infty(\mathrm{I})$*, let* $\mu > 0$ *arbitrarily given. Assume* $(\mathbf{H}_g)$, $(\mathbf{H}_s)$ *and* β *given by (1). Then there exists a unique mild solution* $\mathrm{u}^{\mathrm{u}_0}$*, in the sense of* (48)*, of the climate diffusive energy balance model* $(\mathrm{E}_{\beta,\varepsilon})$*. Furthermore, if* $\mathrm{u}^{\mathrm{u}_0}$ *and* $\mathrm{u}^{\widehat{\mathrm{u}}_0}$ *denote the solutions of the problems relative to these data, one has the continuous dependence inequality*

$$\left\|\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\right\|_{\mathbb{B}_t}^2 \leq 4\mathrm{M}^2\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathrm{H}}^2 + \widehat{\mathrm{C}}_{\mathrm{T}}\int_0^{\mathrm{T}} \theta_{\mathrm{F,B}}\left(\left\|\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\right\|_{\mathbb{B}_s}^2\right)ds \tag{4}$$

where the constants $\mathrm{M}, \widehat{\mathrm{C}}_{\mathrm{T}}$ *and the function* $\theta_{\mathrm{F,B}}$ *are given in Proposition 2 and the Banach space* $\mathbb{B}_t$ *in* (52). *In addition, introducing a suitable function* (15), *we have the growth estimate*

$$\Psi_{4\mathrm{M}^2\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|^2_{\mathrm{H}}}\left(\|\mathrm{u}^{\mathrm{u}_0}-\mathrm{u}^{\widehat{\mathrm{u}}_0}\|^2_{\mathbb{B}_t}\right)\leq \widehat{\mathrm{C}}_{\mathrm{T}}t, \quad t\in[0,\mathrm{T}], \tag{5}$$

provided $\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|_{\mathrm{H}}>0$. *Finally, we have the quantitative comparison estimate*

$$\left\|\left(\mathrm{u}^{\mathrm{u}_0}-\mathrm{u}^{\widehat{\mathrm{u}}_0}\right)_+\right\|^2_{\mathbb{B}_t}\leq 4\mathrm{M}^2\left\|\left(\mathrm{u}_0-\widehat{\mathrm{u}}_0\right)_+\right\|^2_{\mathrm{H}}+\widehat{\mathrm{C}}_{\mathrm{T}}\int_0^{\mathrm{T}}\theta_{\mathrm{F,B}}\left(\left\|\left(\mathrm{u}^{\mathrm{u}_0}-\mathrm{u}^{\widehat{\mathrm{u}}_0}\right)_+\right\|^2_{\mathbb{B}_s}\right)ds \tag{6}$$

for $t\in[0,\mathrm{T}]$ *holds, where* $r_+=\max\{r,0\}$. *In consequence, we have the comparison of solutions:*

$$\mathrm{u}_0\leq\widehat{\mathrm{u}}_0 \quad \text{in } \mathrm{L}^2(\mathrm{I}) \quad \Rightarrow \quad \mathrm{u}^{\mathrm{u}_0}_t(x;\omega)\leq \mathrm{u}^{\widehat{\mathrm{u}}_0}_t(x;\omega) \quad \text{for all } (x,t)\in\mathrm{I}\times[0,\mathrm{T}],\ \text{a.e. } \omega\in\Omega.$$

We point out that in a separate study ([21]) we will offer some extension and related results to those presented in this article, but in some different contexts: we will prove a strictly deterministic version of the aforementioned improvements of the important article [31], and we will also explain how the stochastic problem can be addressed by applying the results in an abstract framework that offers possible applications to problems not necessarily originating from climate models ([22]).

The organization of this paper is the following. Some useful properties of the above-presented hybrid co-albedo function will be given in Section 2. In Section 3 we collect some auxiliary and technical results which will be useful in this stochastic framework. Finally, the proof of the main Theorem will be given in Section 4.

2 On the hybrid co-albedo function

In this Section, we collect some results on some properties of the concave and increasing continuous function $\theta_\delta(u)$ used in the definition of the hybrid co-albedo function. For simplicity in the notation, we drop the dependence with respect to the constant δ. So, let

$$\theta(u)=(\beta_w-\beta_i)\frac{u\ln u}{\delta\ln\delta}, \quad u\geq 0. \tag{7}$$

For Section 4, it will be very useful the study of the nonlinear integral equation

$$\mathrm{v}(t)=\mathrm{v}_0+\alpha\int_0^t\theta\big(\mathrm{v}(s)\big)ds, \quad 0\leq s\leq\mathrm{T}. \tag{8}$$

It will have a key role in proving the existence part of the main theorem.

Theorem 2 *Let* $\mathrm{v}_0,\alpha>0$ *and let* $\mathrm{v}(t)$ *be any nonnegative integrable function satisfying*

$$\mathrm{v}(t)\leq\mathrm{v}_0+\alpha\int_0^t\theta\big(\mathrm{v}(s)\big)ds, \quad 0\leq s\leq\mathrm{T}\leq+\infty. \tag{9}$$

Then we have the implicit estimate,

$$\int_{\mathrm{v}_0}^{\mathrm{v}(t)}\frac{ds}{\theta(s)}\leq\alpha t, \quad 0\leq t\leq\mathrm{T}. \tag{10}$$

In fact, the unique nonnegative function $\mathrm{v}(t)$ *such that*

$$\mathrm{v}(t)\leq\alpha\int_0^t\theta\big(\mathrm{v}(s)\big)ds, \quad 0\leq s\leq\mathrm{T} \tag{11}$$

is the null function. Moreover, if $\mathrm{v}_0\geq 0$ *and* $\alpha>0$, *under the Osgood condition*

$$\int_{0^+}\frac{du}{\theta(u)}=\infty, \tag{12}$$

the nonlinear integral equation

$$\mathrm{v}(t)=\mathrm{v}_0+\alpha\int_0^t\theta\big(\mathrm{v}(s)\big)ds,\quad 0\le t\le \mathrm{T}, \tag{13}$$

admits a unique global nonnegative solution, on $[0,\mathrm{T}]$.

PROOF As it is well known, if $\mathrm{v}_0>0$, the integral equation (8) is equivalent to the Cauchy problem

$$\begin{cases}\mathrm{v}'(t)=\alpha\theta\big(\mathrm{v}(t)\big),\\ \mathrm{v}(0)=\mathrm{v}_0,\end{cases} \tag{14}$$

whose positive and continuous global solution is represented, thanks to the Leibnitz's formula, by

$$\int_{\mathrm{v}_0}^{\mathrm{v}(t)}\frac{ds}{\theta(s)}=\alpha t,\quad 0\le t\le \mathrm{T}.$$

When $\mathrm{v}(t)$ satisfies the inequality (9), we require a sharper refinement because an equivalence as (8) and (14) does not hold in general. So, we introduce the positive and non decreasing function $\mathrm{V}(t)=\max\limits_{0\le\tau\le t}\mathrm{v}(\tau)=\mathrm{v}(\tau_t)$, for some $\tau_t\in[0,t]$. Next, we define

$$\widehat{\mathrm{V}}(t)=\mathrm{v}_0+\alpha\int_0^t\theta\big(\mathrm{V}(s)\big)ds>0,\quad t>0,$$

that satisfies $\widehat{\mathrm{V}}(0)=\mathrm{v}_0$, as well as

$$\mathrm{V}(t)=\mathrm{v}(\tau_t)\le\mathrm{v}_0+\alpha\int_0^{\tau_t}\theta\big(\mathrm{v}(s)\big)\le\mathrm{v}_0+\alpha\int_0^t\theta\big(\mathrm{V}(s)\big)=\widehat{\mathrm{V}}(t)$$

and

$$\begin{cases}\widehat{\mathrm{V}}'(t)=\alpha\theta\big(\mathrm{V}(t)\big)\le\alpha\theta\big(\widehat{\mathrm{V}}(t)\big),\\ \widehat{\mathrm{V}}(0)=\mathrm{v}_0>0\end{cases}$$

quite similar to (14). Hence a kind of Leibnitz inequality

$$\int_{\mathrm{v}_0}^{\mathrm{v}(t)}\frac{dr}{\theta(r)}\le\int_{\mathrm{v}_0}^{\widehat{\mathrm{V}}(t)}\frac{dr}{\vartheta(r)}=\int_0^t\frac{\widehat{\mathrm{V}}'(t)dt}{\theta\big(\widehat{\mathrm{V}}(t)\big)}\le\alpha t<+\infty$$

holds, and then (10) follows.
On the other hand, when $\mathrm{v}_0=0$, if we suppose $\mathrm{v}(t)>0$ in some interval $t\in]0,t_1]\subset[0,\mathrm{T}]$ the above reasoning shows that $\widehat{\mathrm{V}}(0)=0$, $0<\mathrm{V}(t)\le\widehat{\mathrm{V}}(t)$ and $\widehat{\mathrm{V}}'(t)\le\alpha\theta\big(\widehat{\mathrm{V}}(t)\big)$, from which we deduce that $\widehat{\mathrm{V}}(t)>0$ in $t\in]0,t_1]$ and

$$\int_0^{\widehat{\mathrm{V}}(t_1)}\frac{dr}{\theta(r)}=\int_0^{t_1}\frac{\widehat{\mathrm{V}}'(t)dt}{\theta\big(\widehat{\mathrm{V}}(t)\big)}\le\alpha t<+\infty,$$

contrary to the condition (12). □

Since θ is continuous and increasing we may introduce the increasing function

$$\Psi_{\mathrm{v}_0}(\mathrm{v})\doteq\int_{\mathrm{v}_0}^{\mathrm{v}}\frac{ds}{\theta(s)},\quad \mathrm{v}\ge\mathrm{v}_0, \tag{15}$$

provided $\mathrm{v}_0>0$. Then the inequality (10) can be rewritten as

$$\mathrm{v}(t)\le\Psi_{\mathrm{v}_0}^{-1}(\alpha t),\quad 0\le t\le\mathrm{T}, \tag{16}$$

provided $\mathrm{v}(0)=\Psi_{\mathrm{v}_0}^{-1}(0)>0$. We emphasize that if $\mathrm{v}_0=0$, the inequality (16) has not sense because the unique non negative function solving (11) is the constant function $\mathrm{v}(t)\equiv0$. It is easy to see that the function given by (7) verifies the Osgood assumption (12)

$$\int_{0^+}\frac{ds}{\theta(s)}=+\infty.$$

Remark 1 Several generalizations of the above result will be presented in [[22, 21]]. We note that in general no convex function $\theta(u)$ satisfies (12). Certainly, if (12) holds the function θ is not integrable near 0. For instance the functions satisfying

$$\frac{\theta(u)}{u} \leq \ln \frac{1}{u}, \quad \ln \frac{1}{u} \ln \overset{(n}{\cdots} \ln \frac{1}{u}, \quad n \geq 0, \quad \text{near } u = 0, \tag{17}$$

provide other examples for which (12) holds near the origin. The conditions (12) and (17) coincide with the classical Osgood's criterion (see [49]). On the other hand, we also note that the examples given by (17) verify

$$\theta(u)|\ln u| \leq u|\ln u| \ln \frac{1}{u} \ln \overset{(n}{\cdots} \ln \frac{1}{u} = u\big(\ln u\big)^{n+2}, \quad n \geq 0, \quad \text{near v} = 0,$$

and thus they satisfy the so-called Dini condition

$$\lim_{u \searrow 0} \theta(u)|\ln u = 0. \tag{18}$$

□

We also emphasize that the inequality

$$|\theta(u) - \theta(v)| \leq \theta(|u - v|), \quad u, v \geq 0. \tag{19}$$

holds (see Remark 3 below).

Remark 2 In order to provide inequalities as (19) we may consider real functions $\Phi : [0, \infty[\to [0, \infty[$ satisfying $\Phi(0) \geq 0$ with the subadditive property

$$\Phi(u) + \Phi(w) \geq \Phi(u + w) \quad \Leftrightarrow \quad \Phi(u + w) - \Phi(u) \leq \Phi(w), \quad u, w \geq 0. \tag{20}$$

holds. So that, let $u, v \geq 0$ such that $v \geq u$. We define $w = v - u \geq 0$ for which

$$0 \leq \Phi(v) - \Phi(u) = \Phi(w + u) - \Phi(u) \leq \Phi(w) = \Phi(v - u),$$

whenever Φ is nondecreasing. By means of a similar reasoning, we conclude that

$$|\Phi(u) - \Phi(v)| \leq \Phi(|u - v|), \quad u, v \geq 0, \tag{21}$$

whenever Φ is nondecreasing. □

Remark 3 The sub-additive property (20) is satisfied by real concave functions $\Phi : [0, \infty[\to [0, \infty[$ satisfying $\Phi(0) \geq 0$. Indeed, it follows

$$\Phi(\lambda z) = \Phi\big(\lambda z + (1 - \lambda)0\big) \geq \lambda\Phi(z) + (1 - \lambda)\Phi(0) \geq \lambda\Phi(z), \quad z \geq 0, \ 0 < \lambda < 1.$$

In particular, given $u, w > 0$ by choosing $\lambda_{u,w} = \dfrac{u}{u + w} \in]0, 1[$ it follows

$$\begin{cases} \Phi(u) = \Phi\big(\lambda_{u,w}(u + w)\big) \geq \lambda_{u,w}\Phi(u + w), \\ \Phi(w) = \Phi\big((1 - \lambda_{u,w})(u + w)\big) \geq (1 - \lambda_{u,w})\Phi(u + w), \end{cases}$$

whence one concludes the subadditive property

$$\Phi(u) + \Phi(w) \geq \Phi(u + w).$$

Arguing as in [18, Lemma 4.3], we also may obtain the subadditive property (20) by transfer without concavity settings. Indeed, assume

$$q(u) + q(v) \geq q(u + v), \quad u, v \geq 0$$

and

$$\frac{\Phi(u)}{q(u)} \quad \text{is non increasing.} \tag{22}$$

Then

$$\Phi(u)+\Phi(v)=q(u)\frac{\Phi(u)}{q(u)}+q(v)\frac{\Phi(v)}{q(v)}\geq (q(u)+q(v))\frac{\Phi(u+v)}{q(u+v)}\geq \Phi(u+v),$$

thus, the sub-additivity of the function $q(u)$ is transferred to the function $\Phi(u)$ provided (22). In particular, any function Φ such that

$$\frac{\Phi(u)}{u^m} \quad \text{is non increasing}$$

for some $0<m\leq 1$ is sub-additive and the inequality (21) holds whenever Φ is nondecreasing. $\square$

Finally, we come back to the hybrid co-albedo profile (centered at the origin) of the Introduction

$$\beta(u)=\begin{cases}\beta_i, & \text{if } u<0,\\ \left(\beta_w-\beta_i\right)\dfrac{u\ln u}{\delta\ln\delta}+\beta_i, & \text{if } 0\leq u\leq\delta,\\ \beta_w, & \text{if } u>\delta,\end{cases} \tag{23}$$

with $0<\delta<1$ that governs the co-albedo function (see Figure 1). We note that the profile verifies $\beta\in\mathcal{C}(\mathbb{R})\cap\mathcal{C}^1\left(\mathbb{R}\setminus\{0,\delta\}\right)$, with

$$\begin{cases}\beta'(0^-)=0 \quad \text{and} \quad \beta'(0^+)=+\infty,\\ \beta'\left(\delta^-\right)>0 \quad \text{and} \quad \beta'(\delta^+)=0.\end{cases}$$

We claim that the profile β satisfies

$$|\beta(u)-\beta(v)|\leq\theta(|u-v|), \quad u,v\in\mathbb{R}. \tag{24}$$

(see (19)). Indeed, when $v\leq 0$ one has

$$\beta(u)-\beta(v)=\theta(u)-\theta(0)=\theta(u)\leq\theta(u-v), \quad v\leq 0\leq u\leq\delta.$$

Analogously, if $u\geq\delta$ one has

$$\beta(u)-\beta(v)=\theta(\delta)-\theta(v)\leq\theta(\delta-v)\leq\theta(u-v), \quad u\geq\delta\geq v\geq 0.$$

Finally, the claim follows from

$$\begin{cases}\beta(u)-\beta(v)=\theta(u)-\theta(v)\leq\theta(u-v), & \delta\geq u\geq v\geq 0,\\ \beta(u)-\beta(v)=\theta(\delta)-\theta(0)=\theta(\delta)\leq\theta(u-v), & u\geq\delta\geq 0\geq v.\end{cases}$$

Moreover, from (18) the function β satisfies the Osgood's criterion (12) (see also Theorem 2).

3 On the cylindrical well-adapted to the Legendre diffusion Wiener noise

By introducing the change of variable $\mathrm{X}_t=\mathrm{u}_t-10$, $\xi=\mathrm{u}_0-10$, the problem $(\mathrm{E}_{\beta,\varepsilon})$ corresponds to a choice of the general semilinear equation

$$\begin{cases}d\mathrm{X}_t+\mathcal{A}\mathrm{X}_t dt=\mathrm{F}_t\left(\mathrm{X}_t\right)dt+\mathrm{B}_t(\mathrm{X}_t)d\mathrm{W}_t,\\ \mathrm{X}_0=\xi,\end{cases} \tag{25}$$

which will be treated on a separable Hilbert space H where we are considering the measurable processes $X : \Omega_T \to H$, $\Omega_T = [0,T] \times \Omega$, posed in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with a complete right continuous filtration $\{\mathcal{F}_t\}_{t\in[0,T]} \subset \mathcal{F}$. Moreover, as usual, we denote by $\mathcal{P}_T = \mathcal{B}([0,T]) \otimes \mathcal{F}$ the predictable σ- fields on defined by

$$\mathcal{P}_T \doteq \sigma\left(\{]s,t] \times A_s : \ 0 \le s < t \le T, \ A_s \in \mathcal{F}_s\} \cup \{\{0\} \times A_0 : A_0 \in \mathcal{F}_0\}\right).$$

They may also be introduced by

$$\mathcal{P}_T \doteq \sigma\left(Y : \ \Omega_T \to \mathbb{R} : \ Y \text{ is left continuous and adapted to } \mathcal{F}_t, \ t \in [0,T]\right),$$

more according to our purposes (see below).

Next, we make precise the framework where (25) will be formulated. Motivated by the Stochastic Partial Differential Equation of $(E_{\beta,\varepsilon})$, we consider the differential operator $A : \ D(A) \to H$ with

$$\begin{cases} D(A) = \{v \in H : \ Av \in H\}, \\ Av(x) = -\dfrac{d}{dx}\left((1-x^2)\dfrac{d}{dx}v(x)\right), \quad x \in I =]-1,+1[, \quad \text{if } v \in D(A), \end{cases} \tag{26}$$

defined in $H = L^2(I)$, equipped with its usual norm

$$\|u\| = \left(\int_I |u(x)|^2 dx\right)^{\frac{1}{2}}, \quad u \in H$$

and the scalar product

$$\langle u, v\rangle = \int_I u(x)v(x)dx, \quad u, v \in H.$$

In [23] (see also [35]) it was proved that A is a maximal monotone operator densely defined, thus $\overline{D(A)} = H$. More precisely, according to [23] or [20], the domain of the operator is associated with a suitable energy space related to the Legendre diffusion operator. It is the weighted space

$$V = \{w \in L^2(I) : \ w' \in L^2(I; \rho)\}$$

where

$$L^2(I:\rho) = \left\{w : \ \int_I \rho|w|^2 dx < +\infty\right\},$$

equipped with its norm

$$\|w\|^2_{L^2(I:\rho)} = \int_I \rho|w|^2 dx$$

(we recall that $\rho(x) = 1 - x^2$). Here, $H = L^2(I)$ is the so-called Hilbert pivot space. Notice that V is a separable Hilbert space related to the norm

$$\|w\|_V = \|w\|_{L^2(I)} + \|w'\|_{L^2(I:\rho)}.$$

Next, we introduce the abstract version of the diffusion operator by means of the functional operator $\mathcal{A} : \ V \to V'$ given by

$$\mathcal{A}u \doteq -\frac{\partial}{\partial x}\left(\rho\frac{\partial}{\partial x}u\right), \quad u \in V. \tag{27}$$

Working with semigroup theory ([8]), it is useful to define the above operator A as the realization

$$\begin{cases} D(A) = \{v \in H : \ \mathcal{A}v \in H\}, \\ Av(x) = \mathcal{A}v \quad \text{if } v \in D(A). \end{cases}$$

Then, it was shown in [23] that the the operator A can be written as the subdifferential $\mathrm{Av} = \partial\varphi(\mathrm{v})$ of the convex and lower semicontinuous functional, $\varphi : \mathrm{H} \to \mathbb{R} \cup \{+\infty\}$, given by

$$\varphi(\mathrm{v}) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\mathrm{I}} \rho(x) \left|\dfrac{\partial \mathrm{v}}{\partial x}\right|^2 dx & \text{if } \mathbf{v} \in \mathrm{V}, \\ +\infty & \text{otherwise,} \end{cases} \tag{28}$$

that A is densely defined (see [23, Proposition 1]) and that $\partial\varphi(\mathbf{v})$ generates a compact semigroup of contractions on H (see [23, Lemma 1]). Then, for any $\mathrm{u}_0 \in \mathrm{H}$ there exists a unique function $\mathrm{u} \in \mathcal{C}([0,\mathrm{T}] : \mathrm{H})$, with the smoothing effect that $\mathrm{u}(t) \in \mathrm{D(A)}$ for *a.e.* $t \in (0, \mathrm{T}]$, such that u is the mild solution of the abstract problem

$$\begin{cases} \dfrac{d\mathrm{u}}{dt}(t) + \mathrm{Au}(t) = 0, & t > 0, \\ \mathrm{u}(0) = \mathrm{u}_0 \in \mathrm{H}. \end{cases} \tag{29}$$

Remark 4 In fact, in [23] it was obtained the complementary regularity

$$t^{\frac{1}{2}} \frac{d\mathrm{u}(t)}{dt} \in \mathrm{L}^2\big(0, \mathrm{T} : \mathbb{H}\big)$$

of the above mild solution. Moreover, if $\mathrm{u}_0 \in \mathrm{L}^p(\mathrm{I}),\ 1 \le p \le \infty$, then $\mathrm{u}(t) \in \mathrm{L}^p(\mathrm{I})$ for *a.e.* $t \in (0, \mathrm{T}]$. In fact, if $\mathrm{u}_0 \in \mathrm{V} \subset \mathrm{H}$ one has $\dfrac{d\mathrm{u}}{dt} \in \mathrm{L}^2\big(0, \mathrm{T} : \mathrm{H}\big)$. □

On the other hand, we also know that H admits a Hilbertian basis given by the eigenvectors $\{\mathrm{e}_n\}_{n\ge 0} \subset \mathrm{D(A)}$ of the operator A, defined through the orthonormal Legendre polynomials of degree n, defined by the property

$$\mathrm{AP}_n = n(n-1)\mathrm{P}_n \quad \text{on H.}$$

So that, the constants $\mu_n = n(n+1)$ are the corresponding eigenvalues. Since we have

$$\langle \mathrm{P}_n, \mathrm{P}_m \rangle_{\mathrm{H}} = \frac{2}{2n+1}\delta_{n,m},$$

the normalized eigenvectors of A are the functions

$$\mathrm{e}_n(x) = \sqrt{\frac{2}{2n+1}}\mathrm{P}_n(x), \quad -1 \le x \le 1.$$

It also follows the Rodrigues formula that

$$\mathrm{P}_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\big(x^2-1\big)^n = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}^2 (x+1)^{n-k}(x-1),$$

as well as the recurrence identity

$$(n+1)\mathrm{P}_{n+1}(x) + n\mathrm{P}_{n-1}(x) = (2n+1)x\mathrm{P}_n(x),$$

(see, e.g., [48]). Then, from the direct computation $\mathrm{P}_0(x) \equiv 1$ and $\mathrm{P}_1(x) = x$, we deduce that $\mathrm{P}_2(x) = \dfrac{3x^2-1}{2}$, $\mathrm{P}_3(x) = \dfrac{x(5x^2-3)}{2}, \ldots$ and so on (see, e.g., [46]).

As mentioned in the Introduction, we emphasize that the first eigenvalue is the null value $\mu_0 = 0$. This implies that A is not invertible. In order to avoid loss of invertibility, given $\mu > 0$, we replace the differential operator A by

$$\mathrm{A}_\mu \mathrm{u} = \mathrm{Au} + \mu\mathrm{u}, \quad \mathrm{u} \in \mathrm{D(A)}.$$

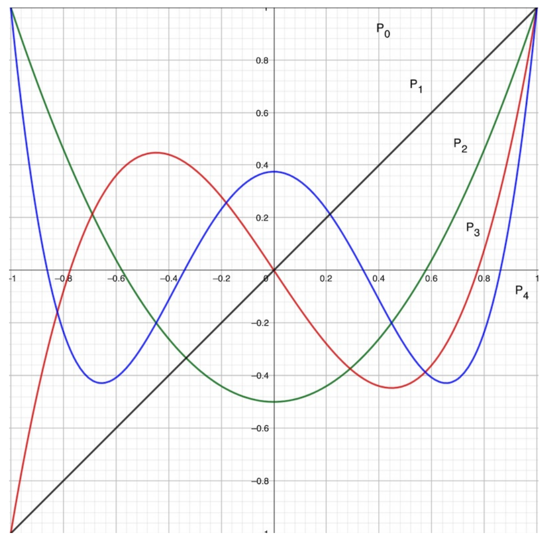

Figure 2: Legendre polynomials

Then, we get the same Hilbertian basis of H, given by the same eigenvectors $\{\mathrm{e}_n\}_{n\geq 0} \subset \mathrm{D(A)}$ of the operator A_μ

$$\mathrm{e}_n(x) = \sqrt{\frac{2}{2n+1}}\mathrm{P}_n(x), \quad -1 \leq x \leq 1,$$

but now, the corresponding eigenvalues are $\widehat{\mu}_n = \mu_n + \mu > 0$. So that, for any $\mathrm{f} \in \mathrm{H}$ there exists a unique solution of $\mathrm{A}_\mu \mathrm{v} = \mathrm{f}$ given by the representation

$$\mathrm{v} = \sum_{n\geq 0} \frac{\langle \mathrm{f}, \mathrm{e}_n\rangle}{\widehat{\mu}_n}\mathrm{e}_n.$$

Denoting by $\{\mathrm{S}_t^\mu\}_{t\geq 0}$ the semigroup generated by A_μ, for every $\mathrm{u}_0 \in \mathrm{H}$, the mild solution of

$$\begin{cases} \dfrac{d\mathrm{u}}{dt}(t) + \mathrm{A}_\mu \mathrm{u}(t) = 0, & t > 0, \\ \mathrm{u}(0) = \mathrm{u}_0, \end{cases}$$

admits the representation

$$\mathrm{u}(t) = \mathrm{S}_t^\mu \mathrm{u}_0 = e^{-\mu t}\sum_{n\geq 0}\langle \mathrm{u}_0, \mathrm{e}_n\rangle_{\mathrm{H}} \mathrm{e}^{-\mu_n t}\mathrm{e}_n, \quad t \geq 0. \tag{30}$$

If we consider now the non-autonomous problem

$$\begin{cases} \dfrac{d\mathrm{u}}{dt}(t) + \mathrm{A}_\mu \mathrm{u}(t) = \mathrm{f}(t), & t > 0, \\ \mathrm{u}(0) = \mathrm{u}_0 \in \mathrm{H}, \end{cases} \tag{31}$$

when $\mathrm{f} \in \mathrm{L}^1\big(0, \mathrm{T}; \mathrm{D(A)}\big)$, we may solve (31) via the generalized Duhamel formula (or constants variations formula) (see [8])

$$\mathrm{u}(t) = \mathrm{S}_t^{\widehat{\mu}}\mathrm{u}_0 + \int_0^t \mathrm{S}_{t-s}^{\widehat{\mu}}\mathrm{f}(s)ds, \quad 0 \leq s < \mathrm{T},$$

thus

$$\mathrm{u}(t)=\sum_{n\geq 0}\left(\langle \mathrm{u}_0,e_n\rangle_{\mathrm{H}}e^{-\widehat{\mu}_n t}+e^{-\widehat{\mu}_n t}\int_0^t\langle \mathrm{f}(s),\mathrm{e}_n\rangle_{\mathrm{H}}e^{\widehat{\mu}_n s}ds\right)\mathrm{e}_n,\quad t\geq 0, \tag{32}$$

and u is called the mild solution of (31).

In order to make precise the stochastic framework, we introduce the operator $\mathcal{Q}_\mu : \mathrm{H} \overset{\mathrm{A}_\mu^{-1}}{\to} \mathrm{D(A)} \hookrightarrow \mathrm{H}$, thus $\mathcal{Q}_\mu \mathrm{u}\in \mathrm{H}$ is the solution of $\mathrm{A}_\mu \mathcal{Q}_\mu \mathrm{u}=\mathrm{u}$ with

$$\mathcal{Q}_\mu \mathrm{u}=\sum_{n\geq 0}\frac{\langle \mathrm{u},\mathrm{e}_n\rangle}{\widehat{\mu}_n}\mathrm{u}_n,\quad \mathrm{u}\in\mathrm{H}$$

for which

$$\|\mathcal{Q}_\mu \mathrm{u}\|^2=\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n^2}\langle \mathrm{u},\mathrm{e}_n\rangle^2,\quad \mathrm{u}\in\mathrm{H}.$$

Since

$$\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n^2}<\sum_{n\geq 1}\frac{1}{\mu_n^2}<\sum_{n\geq 1}\frac{1}{\mu_n}=\sum_{n\geq 1}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\lim_{n\to\infty}\frac{1}{n+1}=1,$$

the operator $\mathcal{Q}_\mu$ is a bounded, positive and symmetric operator with a finite trace

$$\text{Trace }\mathcal{Q}_\mu=\sum_{n\geq 1}\langle \mathcal{Q}_\mu \mathrm{e}_n,\mathrm{e}_n\rangle=\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n}<\frac{1}{\mu}+\sum_{n\geq 1}\frac{1}{\mu_n}=\frac{1}{\mu}+1.$$

Remark 5 Since $\mathcal{Q}_\mu \mathrm{e}_n=\dfrac{1}{\widehat{\mu}_n}\mathrm{e}_n,\ n\geq 0$ the eigenvalues $\widehat{\lambda}_n$ of $\mathcal{Q}_\mu$ coincide with the inverse values $\dfrac{1}{\widehat{\mu}_n}$ of those of the differential operator A_μ. □

Remark 6 If we consider the nonnegative square root operator of $\mathcal{Q}_\mu$, the inequality

$$\|\mathcal{Q}_\mu^{\frac12}\mathrm{u}-\sum_{k=1}^{j}\langle \mathcal{Q}_\mu^{\frac12}\mathrm{u},\mathrm{e}_k\rangle \mathrm{e}_k\|^2\leq\sum_{k=j+1}^{\infty}|\langle \mathcal{Q}_\mu^{\frac12}\mathrm{u},\mathrm{e}_k\rangle|^2\leq\|\mathrm{u}\|^2\sum_{k=j+1}^{\infty}\|\mathcal{Q}_\mu^{\frac12}\mathrm{e}_k\|^2\leq\|\mathrm{u}\|^2\sum_{k=j+1}^{\infty}\langle \mathcal{Q}_\mu \mathrm{e}_k,\mathrm{e}_k\rangle,$$

shows that $\mathcal{Q}_\mu^{\frac12}$ is a compact operator because is a limit of finite rank operators, whence $\mathcal{Q}_\mu=\mathcal{Q}_\mu^{\frac12}\mathcal{Q}_\mu^{\frac12}$ is also a compact operator. Moreover,

$$\left\{\begin{array}{l}\mathcal{Q}_\mu \mathrm{u}=\displaystyle\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n}\langle \mathrm{u},\mathrm{e}_n\rangle \mathrm{e}_n,\\ \mathcal{Q}_\mu^{\frac12}\mathrm{u}=\displaystyle\sum_{n\geq 0}\frac{1}{\sqrt{\widehat{\mu}_n}}\langle \mathrm{u},\mathrm{e}_n\rangle e_n,\end{array}\right.\quad \mathrm{u}\in\mathrm{H}.$$

Finally,

$$\|\mathcal{Q}_\mu^{\frac12}\|^2_{\mathcal{L}_2(\mathrm{H})}=\sum_{n\geq 0}\|\mathcal{Q}_\mu^{\frac12}\mathrm{e}_n\|^2=\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n}=\|\mathcal{Q}_\mu\|_{\mathcal{L}_1^+(\mathrm{H})}=\text{Trace }\mathcal{Q}_\mu<+\infty,$$

we have proved that $\mathcal{Q}_\mu^{\frac12}$ is a Hilbert-Schmidt operator on H.

Remark 7 We note the well-known representation of those Hilbert-Schmidt operators

$$\mathcal{Q}_\mu^{\frac12}\mathrm{u}(x)=\int_{\mathrm{I}}\left(\sum_{n\geq 0}\frac{1}{\sqrt{\widehat{\mu}_n}}\mathrm{e}_n(x)\mathrm{e}_n(y)\right)\mathrm{u}(y)dy,\quad x\in\mathrm{I},\quad \mathrm{u}\in\mathrm{H},$$

whose kernel is

$$k(x,y)\doteq\sum_{n\geq 0}\frac{1}{\sqrt{\widehat{\mu}_n}}\mathrm{e}_m(x)\mathrm{e}_n(y)\quad\text{with}\quad\int_{\mathrm{I}\times\mathrm{I}}|k(x,y)|^2dxdy\leq\sum_{n\geq 0}\frac{1}{\widehat{\mu}_n}=\text{Trace }\mathcal{Q}_\mu.$$

□

The operator $\mathcal{Q}_\mu$ determines the so-called Cameron-Martin space $\mathrm{H}_{\mathcal{Q}_\mu} = \mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{H}$, proper subspace dense in H, that enables us the relation

$$\mathrm{u} = \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{v} \quad \Leftrightarrow \quad \mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{u} = \mathrm{v},$$

thus the operator $\mathcal{Q}_\mu^{-\frac{1}{2}}$ is defined on the Hilbert subspace $\mathrm{H}_{\mathcal{Q}_\mu}$ endowed with

$$\|\mathrm{u}\|_{\mathcal{Q}_\mu} \doteq \|\mathcal{Q}_{\widehat{\mu}}^{-\frac{1}{2}}\mathrm{u}\|_{\mathrm{H}},$$

coming from the identity

$$\langle \mathrm{u}, \mathrm{v}\rangle_{\mathcal{Q}_\mu} \doteq \langle \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{u}, \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{v}\rangle, \quad \mathrm{u}, \mathrm{v} \in \mathrm{H}_{\mathcal{Q}_\mu}.$$

Remark 8 If there exists $\mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{u} = \mathrm{v} \in \mathrm{H}$ the property

$$\mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{v} = \mathrm{u} \quad \Leftrightarrow \quad \frac{1}{\sqrt{\widehat{\mu}_n}}\langle \mathrm{v}, \mathrm{e}_n\rangle = \langle \mathrm{u}, \mathrm{e}_n\rangle$$

requires

$$\|\mathrm{v}\|^2 = \sum_{n\geq 0} \widehat{\mu}_n \big|\langle \mathrm{u}, \mathrm{e}_n\rangle\big|^2 < +\infty.$$

Since $\mathrm{u} \in \mathrm{H}_{\mathcal{Q}_\mu}$ implies $\mathrm{u} = \mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{w}$ for some $\mathrm{w} \in \mathrm{H}$ one has

$$\langle \mathrm{u}, \mathrm{e}_n\rangle = \frac{1}{\sqrt{\widehat{\mu}}_n}\langle \mathrm{w}, \mathrm{e}_n\rangle.$$

Then

$$\sum_{n\geq 0} \sqrt{\widehat{\mu}}_n \big|\langle \mathrm{u}, \mathrm{e}_n\rangle\big|^2 = \sum_{n\geq 0} \big|\langle \mathrm{w}, \mathrm{e}_n\rangle\big|^2 = \|\mathrm{w}\|^2.$$

Hence, we have the representation

$$\begin{cases} \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{u} = \displaystyle\sum_{n\geq 0} \sqrt{\widehat{\mu}_n}\langle \mathrm{u}, \mathrm{e}_n\rangle \mathrm{e}_m, \\ \langle \mathrm{u}, \mathrm{v}\rangle_{\mathcal{Q}_\mu} = \displaystyle\sum_{n\geq 0} \widehat{\mu}_n \langle \mathrm{u}, \mathrm{e}_m\rangle\langle \mathrm{v}, \mathrm{e}_n\rangle = \langle \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{u}, \mathcal{Q}_\mu^{-\frac{1}{2}}\mathrm{v}\rangle, \end{cases}, \quad \mathrm{u} \in \mathrm{H}_{\mathcal{Q}_\mu}.$$

In particular, $\sqrt{\widehat{\mu}_n}\mathrm{e}_n \in \mathrm{H}_{\mathcal{Q}_\mu}$ and $\big\|\sqrt{\widehat{\mu}_n}\mathrm{e}_n\big\|_{\mathcal{Q}_\mu} = \|\mathrm{e}_n\|_{\mathrm{H}} = 1.$ □

After the above notations and commentaries we consider a $\mathcal{Q}_\mu$-cylindrical Wiener process on H denoted by $\{\mathrm{W}_t\}_{t\geq 0}$ satisfying

i) $\{\mathrm{W}_t\}_{t\geq 0}$ has continuous trajectories and $\mathrm{W}_0 = 0$,

ii) $\{\mathrm{W}_t\}_{t\geq 0}$ has independent increments and

$$\mathcal{L}\big(\mathrm{W}_t - \mathrm{W}_s\big) = \mathcal{N}\big(0, (t-s)\mathcal{Q}_\mu\big) = \mathcal{L}\big(\mathrm{W}_{t-s}\big), \quad t \geq s \geq 0$$

ii) $\mathcal{L}\big(\mathrm{W}_t\big) = \mathcal{L}\big(-\mathrm{W}_t\big),\ t \geq 0,$

(see [13]). From the above definition, $\{\mathrm{W}_t\}_{t\geq 0}$ is a Gaussian process on H with

$$\mathbb{E}[\mathrm{W}_t] = 0 \quad \text{and} \quad \mathrm{Cov}\ \mathrm{W}_t = t\mathcal{Q}_\mu, \quad t \geq 0.$$

Then, we may introduce the Brownian motions

$$\mathbf{B}_t^n \doteq \sqrt{\widehat{\mu}}_n\langle \mathrm{W}_t, \mathrm{e}_n\rangle, \quad t \geq 0. \tag{33}$$

Since

$$\begin{aligned}\mathbb{E}\big[\mathbf{B}_t^n\mathbf{B}_s^{n'}\big] &= \sqrt{\widehat{\mu}_n\widehat{\mu_{n'}}}\,\mathbb{E}\big[\langle \mathrm{W}_t,\mathrm{e}_m\rangle\langle \mathrm{W}_s,\mathrm{e}_{m'}\rangle\big]\\ &= \sqrt{\widehat{\mu}_n\widehat{\mu_{n'}}}\,\big(\mathbb{E}\big[\langle \mathrm{W}_t-\mathrm{W}_s,\mathrm{e}_n\rangle\langle \mathrm{W}_s,\mathrm{e}_{n'}\rangle\big]+\mathbb{E}\big[\langle \mathrm{W}_s,\mathrm{e}_n\rangle\langle \mathrm{W}_s,\mathrm{e}_{n'}\rangle\big]\big)\\ &= \sqrt{\widehat{\mu}_n\widehat{\mu_{n'}}}\,s\langle \mathcal{Q}\mathrm{e}_n,\mathrm{e}_{n'}\rangle = s\delta_{n,n'},\quad 0\le s\le t,\end{aligned}$$

the processes $\big\{\mathbf{B}_t^m\big\}_{t\ge 0}$ are mutually independent. Moreover $\mathbb{E}\big[|\mathbf{B}_t^k|^2\big]=t$, uniformly on k, implies

$$\mathbb{E}\left[\left\|\sum_{k=j}^n \frac{1}{\sqrt{\widehat{\mu}_n}}\mathbf{B}_t^k\mathrm{e}_k\right\|^2\right] = t\sum_{k=j}^n\langle \mathcal{Q}_\mu\mathrm{e}_k,\mathrm{e}_k\rangle = t\sum_{k=j}^n\frac{1}{\sqrt{\mu_k}}.$$

Thus, the property Trace $\mathcal{Q}_\mu<\infty$ enables us to admit the representation on H

$$\mathrm{W}_t=\sum_{n\ge 0}\frac{1}{\sqrt{\widehat{\mu}}_n}\mathbf{B}_t^n\mathrm{e}_n,\quad t\ge 0, \tag{34}$$

for which the Wiener isometry

$$\mathbb{E}\big[\|\mathrm{W}_t\|^2\big]=\mathbb{E}\left[\sum_{n\ge 0}\frac{1}{\widehat{\mu}_n}\|\mathbf{B}_t^n\mathrm{e}_n\|^2\right]=t\sum_{n\ge 1}\frac{1}{\widehat{\mu}_n}=t\mathrm{Trace}\ \mathcal{Q}_\mu \tag{35}$$

holds.

Remark 9 In fact, from the maximality martingale inequality (see [13])

$$\mathbb{P}\left(\sup_{t\in[0,\mathrm{T}]}\sum_{k=j}^n\frac{1}{\sqrt{\widehat{\mu}_k}}\|\mathbf{B}_t^k\mathrm{e}_k\|>r\right)\le\frac{4}{r}\mathbb{E}\left[\|\sum_{k=j}^n\frac{1}{\sqrt{\widehat{\mu}_k}}\mathbf{B}_\mathrm{T}^j\mathrm{e}_k\|^2\right]\le\frac{4\mathrm{T}}{r}\sum_{k=j}^n\frac{1}{\sqrt{\widehat{\mu}_k}},$$

one proves that the serie (34) is uniformly convergent on $[0,\mathrm{T}]$ $\mathbb{P}$-$a.s$ (see [13, Theorem 4.3]). □

Remark 10 The isometry (35), as well as

$$\mathbb{E}\big[\langle \mathrm{W}_t,\mathrm{u}\rangle\langle \mathrm{W}_s,\mathrm{v}\rangle\big]=\mathbb{E}\left[\sum_{n\ge 0}\frac{1}{\widehat{\mu}_n}\mathbf{B}_t^n\mathbf{B}_s^n\langle \mathrm{e}_n,\mathrm{u}\rangle\langle \mathrm{e}_n,\mathrm{v}\rangle\right]=t\wedge s\langle \mathcal{Q}_\mu\mathrm{u},\mathrm{v}\rangle,$$

show the covariance property of the operator $\mathcal{Q}_\mu$. □

On the other hand, we denote by $\mathcal{L}_{\mathcal{Q}_\mu}=\mathcal{L}_2(\mathrm{H}_{\mathcal{Q}_\mu},\mathrm{H})$ the space of the Hilbert-Schmidt operators $\mathcal{D}:\ \mathrm{H}_{\mathcal{Q}_\mu}\to\mathrm{H}$ equipped with the norm

$$\|\mathcal{D}\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}=\sum_{n\ge 0}\|\mathcal{D}\mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{e}_n\|^2=\sum_{n\ge 0}\|\mathcal{D}\mathcal{Q}_\mu^{\frac{1}{2}}\mathcal{Q}_\mu^{\frac{1}{2}}\mathcal{D}^*\mathrm{e}_n\|^2.$$

In fact, one has

$$\|\mathcal{D}\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}=\sum_{n\ge 0}\|\mathcal{D}\mathcal{Q}_\mu^{\frac{1}{2}}\mathrm{e}_n\|^2=\mathrm{Trace}\ \mathcal{D}\mathcal{Q}_\mu\mathcal{D}^*=\sum_{n\ge 0}\frac{1}{\widehat{\mu}_n}\|\mathcal{D}\mathrm{e}_n\|^2.$$

So that, one defines the $\mathcal{Q}_\mu$-predictable process as the process $\mathrm{B}:\Omega_\mathrm{T}\to\mathcal{L}_{\mathcal{Q}_\mu}$, $\mathrm{B}_t(\cdot)(\omega)\in\mathcal{L}_{\mathcal{Q}_\mu}$ for which

$$|\!|\!|\mathrm{B}|\!|\!|_{\mathcal{P}_\mathrm{T}}\doteq\left(\mathbb{E}\left[\int_0^\mathrm{T}\mathrm{Trace}\ \mathrm{B}_t\mathcal{Q}_\mu\mathrm{B}_t^*dt\right]\right)^{\frac{1}{2}}=\left(\mathbb{E}\left[\int_0^\mathrm{T}\|\mathrm{B}_t\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}dt\right]\right)^{\frac{1}{2}}<\infty.$$

Among the $\mathcal{Q}_\mu$-predictable processes, we focus on the stochastic integral

$$\mathrm{B}\cdot\mathrm{W} \doteq \int_0^{\mathrm{T}} \mathrm{B}_s d\mathrm{W}_t,$$

where $\{\mathrm{B}_t\}_{t\geq 0} \in \mathcal{P}_\mathrm{T}$ is a $\{\mathcal{F}_t\}_{t\geq 0}$ adapted process $\mathcal{L}_{\mathcal{Q}_\mu}$ valued posed in $\mathrm{H}_{\mathcal{Q}_\mu}$ (see [11, 13, 42] for definition) for which one introduces the processes $\{(\mathrm{B}\cdot\mathrm{W})_t\}_{t\in[0,\mathrm{T}]}$ given by

$$(\mathrm{B}\cdot\mathrm{W})_t \doteq 1\!\!\mathrm{I}_{[0,t]}\mathrm{B}\cdot\mathrm{W}$$

(see [11, 42] again). Thus $(\mathrm{B}\cdot\mathrm{W})_t(\omega) \in \mathcal{L}_{\mathcal{Q}_\mu},\ \omega\in\Omega$, satisfies the Ito isometry

$$\mathbb{E}\big[\|(\mathrm{B}\cdot\mathrm{W})_\mathrm{T}\|^2\big] = \left(\mathbb{E}\left[\int_0^\mathrm{T} \mathrm{Trace}\ \mathrm{B}_t\mathcal{Q}_\mu\mathrm{B}_t^* dt\right]\right)^{\frac{1}{2}} = \left(\mathbb{E}\left[\int_0^\mathrm{T} \|\mathrm{B}_t\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}} dt\right]\right)^{\frac{1}{2}} = \|\!|\mathrm{B}|\!\|^2_{\mathcal{P}_\mathrm{T}} < \infty. \tag{36}$$

Relative to the other terms of the general semilinear problem

$$\begin{cases} d\mathrm{X}_t + \mathrm{A}_\mu\mathrm{X}_t dt = \mathrm{F}_t(\mathrm{X}_t)dt + \mathrm{B}_t(\mathrm{X}_t)d\mathrm{W}_t, \\ \mathrm{X}_0 = \xi, \end{cases} \tag{37}$$

on the Hilbert space H (see (25)), we will require that $\mathrm{F}_t(\mathrm{u})(\omega)\in\mathrm{H},\ (t,\omega,\mathrm{u})\in\Omega_\mathrm{T}\times\mathrm{H}$ for which the Bochner integral

$$\int_0^\mathrm{T} \mathrm{F}_s ds$$

posed in H (see [42]) is well defined. Finally, once again, we recall that A_μ is the infinitesimal generator of a strongly linear semigroup of contractions $\{\mathrm{S}_t^\mu = e^{-\mu t}\mathrm{S}_t\}_{t\geq 0}$ in $\overline{\mathrm{D}(\mathrm{A}_\mu)} = \overline{\mathrm{D}(\mathrm{A})} = \mathrm{H}$.

From the constants variation formula we get that a solution of (37) is a measurable process X from $(\Omega_\mathrm{T},\mathcal{P}_\mathrm{T})$ into $(\mathrm{H},\mathcal{B}_\mathrm{H})$ satisfying for arbitrary $t\in[0,\mathrm{T}]$

$$\mathrm{X}_t = \mathrm{S}_t^\mu\xi + \int_0^t \mathrm{S}_{t-s}^\mu\big(\mathrm{F}_s(\mathrm{X}_s)\big)ds + \int_0^t \mathrm{S}_{t-s}^\mu\mathrm{B}_s(\mathrm{X}_s)d\mathrm{W}_s, \quad \mathbb{P}\ a.e.$$

where ξ is a H-valued random variable. We will get later that the following property holds

$$\mathbb{P}\left(\int_0^t\left(\big\|\mathrm{S}_{t-s}^\mu\mathrm{F}_s(\mathrm{X}_s)\big\|_\mathrm{H} + \big\|\mathrm{S}_{t-s}^\mu\mathrm{B}_s(\mathrm{X}_s)\big\|^2_{\mathcal{L}_\mathcal{Q}}\right)ds < +\infty\right) = 1,$$

in order to the above equality is well defined. We call such a type of function X a *mild solution.*

So, we will solve the semilinear integral equation by proving the existence of a process X which is a fixed point of the operator $\mathcal{G}$

$$\mathrm{X} = \mathcal{G}^\xi\mathrm{X},$$

where

$$\big(\mathcal{G}^\xi\mathrm{X}\big)_t \doteq \mathrm{S}_t^\mu\xi + \int_0^t \mathrm{S}_{t-s}^\mu\mathrm{F}_s\big(\mathrm{X}_s\big)ds + \int_0^t \mathrm{S}_{t-s}^\mu\mathrm{B}_s\big(\mathrm{X}_s\big)d\mathrm{W}_s.$$

Certainly, any such fixed point X solves the semilinear integral equation. Among others possibilities, a way to prove the existence of a fixed point X is to find some suitable topology in which the relative Picard type of successive approximations

$$\big(\mathrm{X}_0\big)_t = \mathrm{S}_t^\mu\xi, \qquad \mathrm{X}_{n+1} = \mathcal{G}^\xi\mathrm{X}_n,\ n\geq 0,$$

converge to X. This approximation problem will be used in the next Section 4.

Remark 11 (Doss-Sussman type transformations) There are some other ways to study semilinear stochastic equations. This is the case of some useful transformations as the one due to H. Doss and H.J. Sussman [17, 53] (see also the pioneering transformation made in [6]). In particular, in the multiplicative noise case, we may consider the transformation $\mathrm{X}_t = \Psi(\mathrm{Y}_t, \mathrm{W}_t)$, for which Ito' Rule gives

$$\begin{aligned} d\mathrm{X}_t - \mathrm{B}(\mathrm{X}_t)d\mathrm{W}_t &= \big[\Psi_w(\mathrm{Y}_t,\mathrm{W}_t) - \mathrm{B}\big(\Psi(\mathrm{Y}_t,\mathrm{W}_t)\big]d\mathrm{W}_t \\ &+\Psi_y(\mathrm{Y}_t,\mathrm{W}_t)d\mathrm{Y}_t + \frac{1}{2}\Psi_{ww}(\mathrm{Y}_t,\mathrm{W}_t)\sigma(\mathrm{W}_t)\sigma^*(\mathrm{W}_t)dt, \end{aligned} \tag{38}$$

whence

$$d\mathrm{X}_t - \mathrm{B}_t(\mathrm{X}_t)d\mathrm{W}_t = \Psi_y(\mathrm{Y}_t,\mathrm{W}_t)d\mathrm{Y}_t + \frac{1}{2}\Psi_{ww}(\mathrm{Y}_t,\mathrm{W}_t)\sigma(\mathrm{W}_t)\sigma^*(\mathrm{W}_t)dt, \tag{39}$$

provided

$$\Psi_w(y,w) = \mathrm{B}\big(\Psi(y,w)\big). \tag{40}$$

Thus, by means of (40) the Doss-Sussman transfomation $\mathrm{X}_t = \Psi(\mathrm{Y}_t, \mathrm{W}_t)$ converts an stochastic integral in a random integral (see (39)). In particular,

$$d\mathrm{X}_t = \big(\mathcal{A}\mathrm{X}_t + \mathrm{F}_t(\mathrm{X}_t) + \mathrm{f}_t\big)dt + \mathrm{B}(\mathrm{X}_t)d\mathrm{W}_t, \tag{41}$$

becomes the random differential equation (which can be considered as a kind of deterministic nonlinear equation)

$$\frac{d\mathrm{Y}_t}{dt} = \frac{1}{\Psi_y(\mathrm{Y}_t,\mathrm{W}_t)}\left[\mathcal{A}\Psi(\mathrm{Y}_t,\mathrm{W}_t) + \mathrm{F}_t\big(\Psi(\mathrm{Y}_t,\mathrm{W}_t)\big) + \mathrm{f}_t + \frac{1}{2}\Psi_{ww}(\mathrm{Y}_t,\mathrm{W}_t)\sigma(\mathrm{W}_t)\sigma^*(\mathrm{W}_t)\right].$$

The study of (41) by the Doss-Sussman transformation is very tedious whenever $\mathrm{B}(\mathrm{X}_t)$ is a general nonlinear diffusion term.
Nevertheless, as mentioned in the Introduction, in the multiplicative linear case $\mathrm{B}_t(\mathrm{X}_t) = a\mathrm{X}_t$, we may take $\Psi(y,w) = y\Phi(w)$. Then the Doss-Sussman equation (40) becomes

$$\Phi'(w) = a\Phi(w) \quad \Rightarrow \quad \Phi(w) = e^{aw}$$

and we obtain the random differential equation

$$\frac{d\mathrm{Y}_t}{dt} = e^{-a\mathrm{W}_t}\mathcal{A}e^{a\mathrm{W}_t}\mathrm{Y}_t + e^{-a\mathrm{W}_t}\big(\mathrm{F}_t\big(e^{a\mathrm{W}_t}\mathrm{Y}_t\big) + \mathrm{f}_t\big) + \frac{1}{2}\mathrm{Y}_t\sigma(\mathrm{W}_t)\sigma^*(\mathrm{W}_t).$$

On the other hand, in the additive noise case $\mathrm{B}_t(\mathrm{X}_t) \equiv \mathrm{B}_t$ this type of Doss-Sussman transformation is quite simpler (see e.g., [20])

$$d\mathrm{X}_t - \mathrm{B}_t d\mathrm{W}_t = d\mathrm{Y}_t \quad \Rightarrow \quad \mathrm{X}_t = \mathrm{Y}_t + (\mathrm{B}\cdot\mathrm{W})_t,$$

and then

$$d\mathrm{Y}_t = d\mathrm{X}_t - \mathrm{B}d\mathrm{W}_t = \big(\mathcal{A}\mathrm{X}_t + \mathrm{F}_t(\mathrm{X}_t)\big)dt,$$

which becomes the random differential equation

$$\frac{d\mathrm{Y}_t}{dt} = \mathcal{A}\mathrm{Y}_t + \mathrm{F}_t(\mathrm{Y}_t + (\mathrm{B}\cdot\mathrm{W})_t) + \mathrm{f}_t + \mathcal{A}(\mathrm{B}\cdot\mathrm{W})_t.$$

□

4 An application of the successive approximation method to the stochastic energy balance model with a non-Lipschitz co-albedo

In what follows, we assume the hypotheses $(\mathbf{H}_g)$ and $(\mathbf{H}_s)$, with β given by (1), as in the Introduction. We will apply the stochastic framework of the Section 3. In particular, we will use the notations and comments made there when we look at problem $(\mathrm{E}_{\beta,\varepsilon})$ as a special case of the abstract stochastic equation

$$d\mathrm{u}_t + \big(\mathrm{A}\mathrm{u}_t + \mathrm{R}_e\big)dt = \mathrm{R}_a\big(dt + \varepsilon d\mathrm{W}_t\big), \quad t > 0, \tag{42}$$

prescribing the initial datum $\mathrm{u}_0 \in \mathrm{H}$ (see (25) where $\{\mathrm{W}_t\}_{t\geq 0}$ was given in (34)).

As it was pointed out, different kind of notions of solutions are possible, and then it is crucial to formulate correctly the assumptions on the data. At least formally, the problem (42) is equivalent to the integral identity

$$\mathrm{u}_t = \mathrm{u}_0 + \int_0^t (-\mathrm{A}\mathrm{u}_s + \big(\mathrm{R}_e - \mathrm{R}_a\big)ds + \varepsilon \int_0^t \mathrm{R}_a d\mathrm{W}_s, \quad \forall t > 0, \tag{43}$$

where $\{\mathrm{u}_t\}_{t\geq 0}$ must be an adapted random process to the filtration satisfying, in some sense, the integral representation (43).
From Section 3 we recall that the operator $\mathrm{A} : \mathrm{D(A)} \to \mathrm{H}$ defined in (26) generates a semi-group $\{\mathrm{S}_t\}_{t\geq 0}$ of contractions on H. In fact, since A is not invertible, given $\mu > 0$, we replace it by the operator $\mathrm{A}_\mu \mathrm{u} = \mathrm{A}\mathrm{u} + \mu \mathrm{u}$, and consider the equation

$$d\mathrm{u}_t + \big(\mathrm{A}_\mu \mathrm{u}_t - \mu \mathrm{X}_t + \mathrm{R}_e\big)dt = \mathrm{R}_a\big(dt + \varepsilon d\mathrm{W}_t\big), \quad t > 0. \tag{44}$$

On the other hand, as in the Budyko proposal, we assume that the Earth's radiation is of the type

$$\mathrm{R}_e(x,t,u) = g\big(u(x,t)\big)$$

where $g : \mathbb{R} \to \mathbb{R}$ is a continuous increasing (see $(\mathbf{H}_g)$) (see the Introduction). The above change in the diffusion operator implies that now we include in (42) an artificial term

$$\widehat{\mathrm{R}}_e = g\big(\mathrm{u}_t\big) - \mu \mathrm{u}_t.$$

The absorbed radiation energy is given by

$$\mathrm{R}_a(x,t,u) = \mathrm{QS}(x)\beta_{-10}\big(u(x,t)\big),$$

under the assumption $\big(\mathbf{H}_s\big)$ and involving the hybrid co-albedo function $\beta_{-10}(u) = \beta(u - 10)$, where the function β is the coalbedo profile given in (23).

As before, we simplify the exposition by making the change of unknown $\mathrm{X}_t = \mathrm{u}_t - 10$. Then, (42) becomes

$$\begin{cases} d\mathrm{X}_t + \mathrm{A}_\mu \mathrm{X}_t dt = \mathrm{F}(\mathrm{X}_t)dt + \mathrm{B}(\mathrm{X}_t)d\mathrm{W}_t, \quad t > 0, \\ \mathrm{X}_0 = \mathrm{u}_0 - 10 \in \mathrm{H}, \end{cases} \tag{45}$$

where

$$\begin{cases} \mathrm{F}(\mathrm{X}_t) = \mu\big(\mathrm{X}_t + 10\big) - g\big(\mathrm{X}_t + 10\big) + \mathrm{QS}\theta(\mathrm{X}_t), \\ \mathrm{B}(\mathrm{X}_t) = \varepsilon \mathrm{QS}\theta(\mathrm{X}_t) \in \mathcal{L}^2_{\mathcal{Q}_\mu}, \end{cases} \tag{46}$$

(see Section 3).

Remark 12 For the condition $\mathrm{B(X)} \in \mathcal{L}^2_{\mathcal{Q}_\mu}$ we mean that it is given by

$$\big(\mathrm{B(X)}u\big)(x) \doteq \varepsilon \mathrm{QS}(x)\theta(\mathrm{X})u(x), \quad x \in \mathrm{I}, \quad \mathrm{u} \in \mathrm{H}.$$

Since $\mathrm{S} \in \mathrm{L}^\infty(\mathrm{I})$ one has that $\mathrm{B(X)u} \in \mathrm{H}$ for $\mathrm{u} \in \mathrm{H}$. Moreover

$$\|\mathrm{B(X)}\|^2_{\mathcal{L}^2_{\mathcal{Q}_\mu}} \leq \varepsilon^2 \mathrm{Q}^2 \big(\theta(\mathrm{X})\big)^2 \|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})} \mathrm{Trace}\ \mathcal{Q}_\mu. \tag{47}$$

□

From the constants variation formula we have that any solution of (45) must be a measurable process X from $(\Omega_{\mathrm{T}}, \mathcal{P}_{\mathrm{T}})$ into $(\mathrm{H}, \mathcal{B}_{\mathrm{H}})$ satisfying, for arbitrary $t \in [0, \mathrm{T}]$,

$$\mathrm{X}_t = e^{-\mu t}\mathrm{S}_t\mathrm{X}_0 + \int_0^t e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{F}\big(\mathrm{X}_s\big)ds + \int_0^t e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{B}(\mathrm{X}_s)d\mathrm{W}_s, \quad t \geq 0. \tag{48}$$

Notice that here $\mathrm{M} \doteq \sup_{0\leq t\leq \mathrm{T}} e^{-\mu t}\|\mathrm{S}_t\|_{\mathrm{L}(\mathbb{H})} \leq 1$. We will also check that the following property holds

$$\mathbb{P}\left(\int_0^t \left(\left\|e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{F}(\mathrm{X}_s)\right\|_{\mathrm{H}} + \left\|e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{B}(\mathrm{X}_s)\right\|^2_{\mathcal{L}^2_{\mathcal{Q}_\mu}}\right)ds < +\infty\right) = 1. \tag{49}$$

When (48) and (49) hold, we say that the process X is a mild solution.

Due to the presence of nonlinear terms in (45), our goal is to prove the existence of processes $\mathrm{X}^{\mathrm{u}_0}$ which are fixed points of $\mathcal{G}^{\mathrm{u}_0}$,

$$\mathrm{X}^{\mathrm{u}_0} = \mathcal{G}^{\mathrm{u}_0}\mathrm{X}^{\mathrm{u}_0},$$

for the operator

$$\big(\mathcal{G}^{\mathrm{u}_0}\mathrm{X}\big)_t \doteq e^{-\mu t}\mathrm{S}_t\mathrm{X}_0 + \int_0^t e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{F}\big(\mathrm{X}_s\big)ds + \int_0^t e^{-\mu(t-s)}\mathrm{S}_{t-s}\mathrm{B}(\mathrm{X}_s)d\mathrm{W}_s. \tag{50}$$

Among other possible arguments, a way to prove the existence of a fixed point $\mathrm{X}^{\mathrm{u}_0}$ is to find some suitable topology in which the relative Picard type of successive approximations

$$\big(\mathrm{X}_0\big)_t = e^{-\mu t}\mathrm{S}_t(\mathrm{u}_0 - 10), \qquad \mathrm{X}_{n+1} = \mathcal{G}^{\mathrm{u}_0}\mathrm{X}_n, \ n \geq 0 \tag{51}$$

converge to $\mathrm{X}^{\mathrm{u}_0}$. Some reasons (see (53) below) advise introducing the Banach space (where we will solve the fixed-point problem) given by

$$\mathbb{B}_{\mathrm{T}} \doteq \left\{\mathrm{X} \in \mathcal{P}_{\mathrm{T}} : \ \mathbb{E}\left[\sup_{0\leq s\leq \mathrm{T}} \|\mathrm{X}_s\|^2_{\mathrm{H}}\right] < \infty\right\} \tag{52}$$

endowed with the norm

$$\|\mathrm{X}\|_{\mathbb{B}_{\mathrm{T}}} \doteq \left(\mathbb{E}\left[\sup_{0\leq s\leq \mathrm{T}} \|\mathrm{X}_s\|^2_{\mathrm{H}}\right]\right)^{\frac{1}{2}}.$$

Certainly, we may then consider the subspaces $\mathbb{B}_t \subset \mathbb{B}_{\mathrm{T}}$ equipped with

$$\|\mathrm{X}\|_{\mathbb{B}_t} \doteq \left(\mathbb{E}\left[\sup_{0\leq s\leq t} \|\mathrm{X}_s\|^2_{\mathrm{H}}\right]\right)^{\frac{1}{2}} \leq \|\mathrm{X}\|_{\mathbb{B}_{\mathrm{T}}},$$

for each $t \in [0, \mathrm{T}]$.

Under Lipschitz assumptions on the nonlinear terms F and B the applicability of the classical successive approximation method is well known in the literature (see, e.g., [13] and [59]). When Lipchitz assumptions do not hold, other assumptions are needed. We send to [4, 54] where the adaptation of the method under non Lipschitz conditions is considered.

A key stone in our reasoning is based on an extension of the maximal sub-martingale inequality (see [13, Lemma 7.2]) that we apply to the stochastic convolution term

$$\big(\mathrm{W}^{\mathrm{B(X)}}_{\mathrm{A}}\big)_t \doteq \int_0^t \mathrm{S}_{t-s}\mathrm{B}_s\big(\mathrm{X}_s\big)d\mathrm{W}_s.$$

More precisely, we will use a suitable simple consequence of [13, Proposition 7.3] (see also [56, Theorem 1]).

Proposition 1 ([22]) . *There exists a positive constant* c_{T} *such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq \mathrm{T}}\left\|\int_0^t \mathrm{S}_{t-s}\mathrm{B}\big(\mathrm{X}_s\big)d\mathrm{W}_s\right\|^2_{\mathrm{H}}\right] \leq \mathrm{c}_{\mathrm{T}}\int_0^{\mathrm{T}} \mathbb{E}\big[\big\|\mathrm{B}\big(\mathrm{X}_s\big)\big\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}\big]ds. \tag{53}$$

holds. □

Remark 13 The general estimate

$$\mathbb{E}\left[\sup_{0\le t\le \mathrm{T}}\left\|\int_0^t \mathrm{S}_{t-s}\mathrm{B}(\mathrm{X}_s)d\mathrm{W}_s\right\|_{\mathrm{H}}^p\right]\le c_{\mathrm{T}}\mathbb{E}\left[\int_0^{\mathrm{T}}\|\mathrm{B}(\mathrm{X}_s)\|^p_{\mathcal{L}_{\mathcal{Q}_\mu}}ds\right], \tag{54}$$

only holds for some power $p>2$ (see [13, Proposition 7.3]). However, when $\{\mathrm{S}_t\}_{t\ge 0}$ is a semigroup of contractions, the case $p=2$

$$\mathbb{E}\left[\sup_{0\le t\le \mathrm{T}}\left\|\int_0^t \mathrm{S}_{t-s}\mathrm{B}(\mathrm{X}_s)d\mathrm{W}_s\right\|_{\mathrm{H}}^2\right]\le c_{\mathrm{T}}\mathbb{E}\left[\int_0^{\mathrm{T}}\|\mathrm{B}(\mathrm{X}_s)\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}ds\right] \tag{55}$$

is valid (see [56, Theorem 1]). In fact, arguing as in [13, Theorem 4.7] we also arrive to (53) □

The approximation (51) will be studied, in a more general framework, in [22]. As in [4], this idea was motivated by [13, Theorem 7.2].

Remark 14 Since our datum u_0 will be assumed to be a bounded function, it is natural to search for bounded solutions of the deterministic associated problem

$$(\mathrm{E}_{\beta,0})\begin{cases}\dfrac{\partial u}{\partial t}(x,t)-\dfrac{\partial}{\partial x}\left((1-x^2)\dfrac{\partial u}{\partial x}\right)(x,t)+g(u(x,t))=\mathrm{QS}(x)\beta_{-10}(u(x,t)),\\ u(x,0)=u_0(x),\end{cases}$$

when $g(u)$ is assumed to be locally Lipschitz continuous and increasing. In short, if we try to build a constant supersolution

$$\overline{u}(x,t)=\mathrm{K},$$

then, we get that it is enough to assume

$$\mathrm{K}\ge\max\left\{\|u_0\|_{\mathrm{L}^\infty(\mathrm{I})},g^{-1}\left(\mathrm{Q}\|\mathrm{S}\|_{\mathrm{L}^\infty(\mathrm{I})}\beta_w\right)\right\}.$$

(see (23)). This explains that we can assume that $g(u)$ is globally Lipschitz continuous, since we may replace it by the truncated function

$$\widehat{g}(u)=\begin{cases}g(-\mathrm{K})+g'(-\mathrm{K})(u-\mathrm{M}) & \text{if } u<\mathrm{K},\\ g(u) & \text{if } -\mathrm{K}\le u\le\mathrm{K},\\ g(\mathrm{K})+g'(\mathrm{K})(u+\mathrm{K}) & \text{if } \mathrm{K}<u,\end{cases} \tag{56}$$

(see Figure 3). □

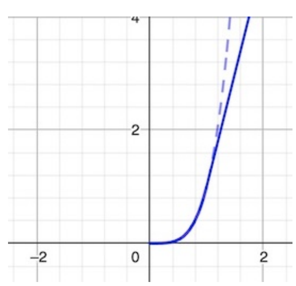

Figure 3: Lipschitz truncation

Here we will apply the successive approximation method to the case of the hybrid co-albedo function (7). By using Theorem 2 and Lemma 1 below we will prove the following result:

Proposition 2 *Let* $\mathrm{u}_0,\ \widehat{\mathrm{u}}_0 \in \mathrm{H}$. *Then the operator* $\mathcal{G}^{\mathrm{u}_0} : \mathbb{B}_\mathrm{T} \to \mathbb{B}_\mathrm{T}$ *given by* (50) *is well defined and continuous. Moreover, one has the estimate*

$$\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{X} - \mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{Y}\right\|^2_{\mathbb{B}_t} \le 4\mathrm{M}^2\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|^2_\mathrm{H} + \widehat{\mathrm{C}}_\mathrm{T}\int_0^\mathrm{T} \theta_{\mathrm{F,B}}\big(\left\|\mathrm{X}-\mathrm{Y}\right\|^2_{\mathbb{B}_s}\big)ds, \quad t\in[0,\mathrm{T}]. \tag{57}$$

Here $\mathrm{M} \doteq \sup_{0\le s\le \mathrm{T}} \|\mathrm{S}_t\|_{\mathrm{L(H)}}$ *and* $\widehat{\mathrm{C}}_\mathrm{T} = \max\{16\mathrm{M}^2\mathrm{T}, \mathrm{C}_\mathrm{T}\}$, *where* $\mathrm{C}_\mathrm{T} = \varepsilon^2\mathrm{Q}^2\|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})}\mathrm{c}_\mathrm{T}\mathrm{Trace}\ \mathcal{Q}_\mu$, *with* c_T *the positive constant of Proposition 1. Here we are using the notation* $\theta_{\mathrm{F,B}}(s) = \theta_\mathrm{F}(s)+\theta(s)$ *with* $\theta_\mathrm{F}(s) = (\mathrm{L}_g+\mu)s + \mathrm{QS}_\infty\theta(s)$ *where* $\mathrm{S}_\infty = \|\mathrm{S}\|_{\mathrm{L}^\infty}$ *and* L_g *is a positive constant assuming that* g *is a Lispchitz continuous function (see Remark 14 below) and* θ *is the function given in* (7).

First, we obtain a technical estimate:

Lemma 1 *With the notations* (46) *or* (60) *one verifies the inequality*

$$\mathbb{E}\left[\sup_{0\le t\le \mathrm{T}}\left\|\int_0^t \mathrm{S}_{t-s}\big(\mathrm{B}(\mathrm{X}_s) - \mathrm{B}((\mathrm{Y}_s)\big)d\mathrm{W}_s\right\|^2_\mathrm{H}\right] \le \mathrm{C}_\mathrm{T}\int_0^\mathrm{T}\theta\big(\left\|\mathrm{X}-\mathrm{Y}\right\|^2_{\mathbb{B}_s}\big)ds, \tag{58}$$

where C_T *is the positive constant given in Proposition 2.*

Proof From (53) we deduce

$$\mathbb{E}\left[\sup_{0\le t\le \mathrm{T}}\left\|\int_0^t \mathrm{S}_{t-s}\big(\mathrm{B}(\mathrm{X}_s) - \mathrm{B}((\mathrm{Y}_s)\big)d\mathrm{W}_s\right\|^2_\mathrm{H}\right] \le \mathrm{C}_\mathrm{T}\int_0^\mathrm{T}\mathbb{E}\big[\left\|\mathrm{B}(\mathrm{X}_s)-\mathrm{B}(\mathrm{Y}_s)\right\|^2_{\mathcal{L}_{\mathcal{Q}_\mu}}\big]ds. \tag{59}$$

On the other hand, by definition of the functional space $\mathcal{L}^2_{\widehat{\mathcal{Q}}}$ and (24), one has

$$\begin{aligned}\left\|\mathrm{B}(\mathrm{X}_s)-\mathrm{B}(\mathrm{Y}_s)\right\|^2_{\mathcal{L}_{\widehat{\mathcal{Q}}}} &\le \varepsilon^2\mathrm{Q}^2\|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})}\big(\beta(\mathrm{X}_s)-\beta(\mathrm{Y}_s)\big)^2\mathrm{Trace}\ \mathcal{Q}_\mu\\ &\le \varepsilon^2\mathrm{Q}^2\|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})}\big(\theta(\mathrm{X}_s-\mathrm{Y}_s)\big)^2\mathrm{Trace}\ \mathcal{Q}_\mu,\end{aligned}$$

(see (47)), and then

$$\begin{aligned}\mathbb{E}\big[\left\|\mathrm{B}(\mathrm{X}_s)-\mathrm{B}(\mathrm{Y}_s)\right\|^2_{\mathcal{L}_\mathcal{Q}}\big] &\le \varepsilon^2\mathrm{Q}^2\|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})}\mathbb{E}\big[\left\|\theta(\mathrm{X}_s-\mathrm{Y}_s)\right\|^2_\mathrm{H}\big]\mathrm{Trace}\ \mathcal{Q}_\mu\\ &\le \varepsilon^2\mathrm{Q}^2\|\mathrm{S}\|^2_{\mathrm{L}^\infty(\mathrm{I})}\theta\big(\mathbb{E}\big[\left\|\mathrm{X}_s-\mathrm{Y}_s)\right\|^2_\mathrm{H}\big]\big)\mathrm{Trace}\ \mathcal{Q}_\mu,\end{aligned}$$

(see in Section 2 the concavity and other properties of the function θ). Therefore, from (59) the proof ends. □

Proof of Proposition 2. Estimate (57) follows from

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left\|\big(\mathcal{G}^{\mathrm{u}_0}\mathrm{X}\big)_s - \big(\mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{Y}\big)_s\right\|^2_\mathrm{H}\right] &\le 4\sup_{0\le s\le t}\|\mathrm{S}_s(\mathrm{u}_0-\widehat{\mathrm{u}}_0)\|^2_\mathrm{H}\\ &\quad +16t\left(\mathbb{E}\left[\sup_{0\le s\le t}\int_0^s\left\|\mathrm{S}_{s-\tau}\big(\mathrm{F}(\mathrm{X}_\tau)-\mathrm{F}(\mathrm{Y}_\tau)\big)\right\|^2_\mathrm{H}d\tau\right]\right.\\ &\quad \left.+\mathbb{E}\left[\sup_{0\le s\le t}\int_0^s\left\|\mathrm{S}_{s-\tau}\big(\mathrm{B}(\mathrm{X}_\tau)-\mathrm{B}(\mathrm{Y}_\tau)\big)\right\|^2_\mathrm{H}d\mathrm{W}_\tau\right]\right)\\ &\le 4\mathrm{M}^2\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|^2_\mathrm{H} + 16\mathrm{M}^2t\mathbb{E}\left[\int_0^\mathrm{T}\left\|\mathrm{F}(\mathrm{X})-\mathrm{F}(\mathrm{Y})\right\|^2_{\mathbb{B}_s}ds\right]\\ &\quad +\mathrm{C}_\mathrm{T}\int_0^\mathrm{T}\theta\big(\left\|\mathrm{X}-\mathrm{Y}\right\|^2_{\mathbb{B}_s}\big)ds\\ &\le 4\mathrm{M}^2\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|^2_\mathrm{H} + 16\mathrm{M}^2t\int_0^\mathrm{T}\theta_\mathrm{F}\big(\left\|\mathrm{X}-\mathrm{Y}\right\|^2_{\mathbb{B}_s}\big)ds\\ &\quad +\mathrm{C}_\mathrm{T}\int_0^\mathrm{T}\theta\big(\left\|\mathrm{X}-\mathrm{Y}\right\|^2_{\mathbb{B}_s}ds\end{aligned}$$

(see (24) and (58)). □

Now we are in a position to give the proof of the main theorem of this paper.
Proof of Theorem 1 As said before (see Remark 14), without loss of generality we may assume that $g(u)$ is globally Lipschitz and increasing function. We may adapt the reasoning of the proof of Proposition 2 to the functions

$$\begin{cases} \mathrm{F}(\mathrm{X}_t) = \mu\big(\mathrm{X}_t + 10\big) - \widehat{g}\big(\mathrm{X}_t + 10\big) + \mathrm{QS}\theta(\mathrm{X}_t), \\ \mathrm{B}(\mathrm{X}_t) = \varepsilon \mathrm{QS}\theta(\mathrm{X}_t). \end{cases} \tag{60}$$

(see (56)). More precisely, since (57) implies

$$\begin{cases} \left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{X}\right\|^2_{\mathbb{B}_t} & \leq 2^2\left(\left\|\mathcal{G}^{\widehat{\mathrm{u}}_0}0\right\|^2_{\mathbb{B}_t} + \left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{X} - \mathcal{G}^{\widehat{\mathrm{u}}_0}0\right\|^2_{\mathbb{B}_t}\right) \\ & \leq 2^2\left(\left\|\mathcal{G}^{\widehat{\mathrm{u}}_0}0\right\|^2_{\mathbb{B}_t} + 4\mathrm{M}^2\|\mathrm{u}_0\|^2_{\mathrm{H}} + \widehat{\mathrm{C}}_{\mathrm{T}}\displaystyle\int_0^{\mathrm{T}} \theta_{\mathrm{F,B}}\big(\|\mathrm{X}\|^2_{\mathbb{B}_s}\big)ds\right), \quad t \in [0,\mathrm{T}], \end{cases}$$

we deduce

$$\|\mathrm{X}_{n+1}\|^2_{\mathbb{B}_t} \leq \mathrm{v}_0 + \alpha\int_0^t \theta_{\mathrm{F,B}}\big(\|\mathrm{X}_n\|^2_{\mathbb{B}_s}\big)ds, \quad n \geq 0, \tag{61}$$

where $\mathrm{v}_0 = 4\big(\max\{4\mathrm{M}^2, 1\}\|\mathrm{X}_0\|^2_{\mathrm{H}} + \left\|\mathcal{G}^{\widehat{\mathrm{u}}_0}0\right\|^2_{\mathbb{B}_t}\big)$ and $\alpha = 4\widehat{\mathrm{C}}_{\mathrm{T}} > 0$. Next, from Theorem 2, we consider a global solution v on $[0,\mathrm{T}]$ of the simple integral equation

$$\mathrm{v}(t) = \mathrm{v}_0 + \alpha\int_0^t \theta_{\mathrm{F,B}}\big(\mathrm{v}(s)\big)ds, \quad t \in [0,\mathrm{T}]. \tag{62}$$

From (62) we have

$$\mathrm{v}(t) - \|\mathrm{X}_{n+1}\|^2_{\mathbb{B}_t} \geq \alpha\int_0^t \left(\theta_{\mathrm{F,B}}\big(\mathrm{v}(s)\big) - \theta_{\mathrm{F,B}}\big(\|\mathrm{X}_n\|^2_{\mathbb{B}_s}\big)\right)ds.$$

Certainly $\|\mathrm{X}_0\|^2_{\mathrm{H}} \leq \mathrm{v}_0$ (see (51)). Then, the monotonicity of the function $\theta_{\mathrm{F,B}}(\mathrm{U})$ implies, by induction, the inequality

$$\|\mathrm{X}_n\|^2_{\mathbb{B}_t} \leq \mathrm{v}(t) \quad \text{for } t \in [0,\mathrm{T}],$$

where the function $\mathrm{v}(t)$ is independent on n (see (62)). Thus, $\{\mathrm{X}_n\}_{n\geq 0}$ is a bounded sequence in $\mathbb{B}_{\mathrm{T}}$. Therefore, for each $n \geq 0$

$$r_n(t) \doteq \sup_{m\geq n}\|\mathrm{X}_m - \mathrm{X}_n\|^2_{\mathbb{B}_t}$$

is a nonnegative, uniformly bounded, and nondecreasing function on $t \in [0,\mathrm{T}]$. By construction, for each $t \in [0,\mathrm{T}]$, we may consider the nonincreasing sequence $\{r_n(t)\}_{n\geq 0}$. It implies the existence of a nonnegative, and nondecreasing function given by

$$r(t) = \lim_{n\to\infty} r_n(t), \quad t \in [0,\mathrm{T}].$$

On the other hand, a similar reasoning as in the proof of Proposition 2 leads to

$$\|\mathrm{X}_m - \mathrm{X}_n\|^2_{\mathbb{B}_t} \leq \alpha\int_0^t \theta_{\mathrm{F,B}}\big(\|\mathrm{X}_{m-1}(s) - \mathrm{X}_{n-1}(s)\|^2_{\mathbb{B}_s}\big)ds.$$

Therefore we obtain

$$r(t) \leq r_n(t) \leq \alpha\int_0^r \theta_{\mathrm{F,B}}\big(r_{n-1}(s)\big)ds, \quad t \in [0,\mathrm{T}]$$

whence, by the Lebesgue Convergence Theorem,

$$r(t) \leq \alpha\int_0^r \theta_{\mathrm{F,B}}\big(r(s)\big)ds, \quad t \in [0,\mathrm{T}].$$

Now, we deduce that $r(t) \equiv 0$ for $t \in [0, \mathrm{T}]$ thanks to Theorem 2. Since

$$\|\mathrm{X}_m - \mathrm{X}_n\|_{\mathbb{B}_\mathrm{T}}^p \leq r_n(\mathrm{T})$$

we conclude that

$$\|\mathrm{X}_m - \mathrm{X}_n\|_{\mathbb{B}_\mathrm{T}}^2 \to 0 \quad \text{as } m, n \to \infty.$$

Therefore, the Picard type approximation, $\{\mathrm{X}_n\}_{n \geq 0} \subset \mathbb{B}_\mathrm{T}$ given by (51), is a Cauchy sequence in $\mathbb{B}_\mathrm{T}$ and then we have proved the existence of a point fixed, X, of the operator $\mathcal{G}^{\mathrm{u}_0}$. So, we get the existence and uniqueness of the solution $\mathrm{X}^{\mathrm{u}_0}$. Moreover, the inequality (57) becomes

$$\|\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\|_{\mathbb{B}_t}^2 \leq 4\mathrm{M}^2 \|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_\mathrm{H}^p + \widehat{\mathrm{C}}_\mathrm{T} \int_0^t \theta_{\mathrm{F},\mathrm{B}}\big(\|\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\|_{\mathbb{B}_s}^2\big) ds, \tag{63}$$

for $\in [0, \mathrm{T}]$. Therefore, we may rewrite (63) using the integral inequality

$$\int_{4\mathrm{M}^2\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_\mathrm{H}^2}^{\|\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\|_{\mathbb{B}_t}^2} \frac{ds}{\theta_{\mathrm{F},\mathrm{B}}(s)} \leq \widehat{\mathrm{C}}_\mathrm{T} t, \quad 0 \leq t \leq \mathrm{T}.$$

Finally, since we are working in $\mathrm{H} = \mathrm{L}^2(\mathrm{I})$, we have

$$\big((\mathcal{G}^{\mathrm{u}_0}\mathrm{X})_s - (\mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{X})_s\big)_+ \leq \mathrm{S}_s\big(\mathrm{u}_0 - \widehat{\mathrm{u}}_0\big)_+.$$

Then, by the reasoning of the proof of Proposition 2, we deduce

$$\|\big(\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\big)_+\|_{\mathbb{B}_t}^2 \leq 4\mathrm{M}^2 \left(\|\big(\mathrm{u}_0 - \widehat{\mathrm{u}}_0\big)_+\|_\mathrm{H}^2 + \widehat{\mathrm{C}}_\mathrm{T} \int_0^t \theta_{\mathrm{F},\mathrm{B}}\big(\|\big(\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\big)_+\|_{\mathbb{B}_s}^2\big) ds \right) \tag{64}$$

for which the conclusion holds. The proof of the comparison part of the Theorem is as follows: From (64) we get

$$\|\big(\mathrm{X}^{\mathrm{u}_0} - \mathrm{X}^{\widehat{\mathrm{u}}_0}\big)_+\|_{\mathbb{B}_t}^2 \leq \widehat{\mathrm{C}}_\mathrm{T} \int_0^t \theta_{\mathrm{F},\mathrm{B}}\big(\|\big(\big(\mathrm{u}^{\mathrm{u}_0} - \mathrm{u}^{\widehat{\mathrm{u}}_0}\big)_+\|_{\mathbb{B}_s}^2\big) ds, \quad t \in [0, \mathrm{T}].$$

Then, from the above conclusions of the Theorem we find that $\|\big(\mathrm{u}_t^{\mathrm{u}_0} - \mathrm{u}_t^{\widehat{\mathrm{u}}_0}\big)_+\|_\mathrm{H}^2 = 0.$ □

Remark 15 Many extensions of the main result of this paper seem to be possible. For instance, we can consider the case of function $c(x)$ taking into account the different heat capacity of continents and seas, we can replace the emitted radiative energy by a more general term of the form $g(t, x, u)$, we can consider a more general diffusion operator, and , finally, we can also consider the noise corresponding to a time-periodic Solar function $Q(t)$ (whose case deterministic was treated in [3]). □

Remark 16 Other types of generalizations will be presented in the paper [22] where, for instance, we prove the uniqueness of solutions of some abstract stochastic differential equations, under Nagumo's type conditions, which are especially useful when there are some singularities in terms that depend on time. □

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