Mathematical analysis and homogenization of a free boundary problem for in situ leaching of rare earths with a special periodic structure

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Abstract

We consider an initial–boundary value problem modeling in situ leaching of rare earths with a special periodic structure by an acid solution, improving some previous studies by the second author and collaborators. At the microscopic scale, fluid motion in the pore space is described by the Stokes equations for a slightly compressible fluid coupled with the deformation of the elastic skeleton, governed by the Lamé system, and the diffusion equation for the acid solution. Due to rock dissolution, the interface between liquid and solid phases is unknown (it is a free boundary) and must be determined as part of the solution. To overcome this difficulty, we introduce a family of approximate microscopic models with prescribed pore geometry and establish their well-posedness in a weak formulation. Using a priori estimates and Galerkin's method, we obtain existence results and apply the method of two-scale convergence for periodic structures to derive the corresponding homogenized macroscopic model. Finally, a fixed-point argument yields existence and uniqueness for the resulting macroscopic system.

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1 Introduction

In this paper, we study a microscopic mathematical model describing in-situ leaching of rare earths and establish the existence and uniqueness results for classical solutions of the corresponding initial–boundary value problem for the associated system of differential equations. Leaching-based extraction of rare earths (or metals) plays a crucial role of the present time. Deposits of uranium, nickel, and other rare metals are geologically complex and strongly heterogeneous. Such heterogeneity implies that the physical properties of the medium vary spatially. Investigations of wells and core samples demonstrate that key geological characteristics—such as porosity and permeability—can vary significantly even within a single ore body. In many cases, insufficient attention to these heterogeneities during the planning stage leads to serious operational problems, for example when the acid solution injected through wells migrates to locations far from those intended. Furthermore, the effectiveness of the process is strongly influenced by the concentration of the injected acid, the injection regimes, and other technological parameters.

Consequently, a detailed understanding of fluid flow in heterogeneous porous media, as well as the mechanisms governing acid-induced rock dissolution, is of fundamental importance for efficient rare earth extraction. This understanding can be achieved through the development of a hydrodynamic simulator for the ore body based on appropriate mathematical models, enabling optimization of the entire technological process. Such a simulator is a complex system comprising a hierarchy of mathematical models describing the physical processes (forming the simulator prototype), digital representations of the geometric and physical properties of the solid framework, and software tools for visualizing the processes and tracking the evolution of the main model characteristics.

At present, numerous mathematical models exist for describing rock leaching dynamics; however, these models operate exclusively at the *macroscopic scale* (see [21], [7], [37] and references therein). In contrast to *microscopic models*—where the characteristic length scale is on the order of tens of microns—macroscopic models typically involve spatial scales of meters or even tens of meters. As a result, such models do not resolve the microstructure of the medium. Instead, each spatial point is treated as containing both the solid skeleton and the pore-filling fluid. Despite their diversity, these macroscopic models are based on a common framework. Fluid motion is generally described by Darcy's law or its modifications, while the transport of acid and reaction products is postulated through equations resembling diffusion–convection relations for the relevant concentrations.

A key distinction among these models lies in the choice of coefficients in the governing equations, leading to a wide variety of formulations depending largely on the preferences of individual authors. This diversity is understandable, since the core physical mechanism—the interaction at the unknown *free boundary* between the pore space and the solid skeleton—is not explicitly represented in macroscopic descriptions. It is precisely at this interface that rock dissolution occurs, acid concentration changes, and reaction products are generated within the fluid. Moreover, the geometry of the pore space itself evolves in both time and space during the leaching process. These essential phenomena take place at the microscopic scale, corresponding to the typical size of pores and fractures, whereas macroscopic models operate at much larger scales and therefore cannot resolve free boundaries or detailed acid–rock interactions. This fundamental mismatch in scales explains the large number of competing macroscopic models.

The authors of such models lack both a rigorous framework for describing microscopic processes based on the fundamental laws of continuum mechanics and chemistry, and the means to directly incorporate microstructural features into macroscopic formulations. As a consequence, they are forced to rely on heuristic assumptions. This situation naturally raises the question of model adequacy: when several macroscopic models claim to describe the same physical process under identical conditions, how can one determine which is most accurate? Experimental validation provides little guidance, since each model contains numerous adjustable parameters that are not directly tied to reservoir geometry (such as porosity) or intrinsic physical properties (such as fluid viscosity or solid matrix characteristics). By tuning these parameters, virtually any experimental outcome can be reproduced.

R. Burridge and J. B. Keller [5], together with E. Sanchez-Palencia [38], were the first to show that an accurate macroscopic description of fluid filtration and seismic wave propagation in rocks is possible if and only if three conditions are satisfied: (a) the physical process is rigorously described at the microscopic level using the equations of classical Newtonian continuum mechanics; (b) a suitable set of small dimensionless parameters is identified; and (c) the macroscopic model emerges as the exact asymptotic limit—via homogenization—of the microscopic model as these small parameters tend to zero. Numerous special cases of exact macroscopic models for acoustics and fluid flow in rocks have since been studied by various authors (see [17]-[4]). Although different homogenization techniques were employed, their application generally required substantial analytical effort and ingenuity.

A major shift occurred with the publication of G. Nguetseng's work [33], in which the *method of two-scale convergence* for periodic structures was introduced. This development transformed homogenization from a highly specialized analytical art into a systematic and widely applicable tool. As a result, homogenization theory has largely ceased to exist as a standalone branch of mathematical analysis, with contemporary research efforts focusing instead on applications in mechanics, physics, biology, and related fields.

The organization of this paper is as follows: In Section 2, we provide a detailed exposition of the necessary preliminaries and state the three main results, after introducing the microscopic system under consideration and its homogenization, which are developed in several parts. Subsequently, three separate sections are devoted to the proofs of the aforementioned main results, with each section containing the proof of one result.

2 Preliminaries and statement of the main results.

2.1 Statement of the main and intermediate problems

As we have already noted, the derivation of macroscopic mathematical models should be based on a mathematical model faithful to the physical formulation at the microscopic level, described by the laws of Newtonian classical continuum mechanics (see, e.g., [19] and Appendix A, section A.7 in [29] which summarize the exposition made in [36]).

In what follows we will assume that elastic skeleton is stationary and that the surrounding fluid is slightly compressible. In this way we will extend some previous treatments in the mathematical literature dealing with incompressible fluids (see [30] and [32]). Notice that since the characteristic time scale for the elastic skeleton is much larger than the characteristic time for changes in the fluid, the hypothesis that the elastic model is stationary is well justified.

For now, we will postpone the details regarding the geometric hypotheses constituting the region $\Omega-\Omega_f^\varepsilon$ occupied by the elastic skeleton of the pores. We will therefore prioritize the differential equations that characterize the motion. The to compressible viscous flow motion in a pore space $\Omega_f^\varepsilon\subset\Omega$, for $t>0$, in the dimensionless variables (see subsection 2.2), is governed by the stationary Stokes equations for the compressible viscous fluid (see, e.g., [29] and [15])

$$\nabla\cdot\mathbb{P}_f^\varepsilon=\nabla p^0,\ \mathbb{P}_f^\varepsilon=\alpha_\mu^\varepsilon\mathbb{D}(x,\boldsymbol{v}_f^\varepsilon)-(p_f^\varepsilon-p^0)\mathbb{I},\ \boldsymbol{v}_f^\varepsilon=\frac{\partial\boldsymbol{w}_f^\varepsilon}{\partial t}, \tag{2.1}$$

the linearized continuity equation

$$\frac{1}{c_{f,p}^2}(p_f^\varepsilon-p^0)+\nabla\cdot\boldsymbol{w}_f^\varepsilon=0 \tag{2.2}$$

for dynamic characteristics $\boldsymbol{w}_f^\varepsilon$ (*fluid displacements*), $\boldsymbol{v}_f^\varepsilon=\dfrac{\partial\boldsymbol{w}_f^\varepsilon}{\partial t}$ (*fluid velocity*) and p_f^ε (*fluid pressure*). Here ε is a small parameter equal to $\dfrac{L}{n}$, where n is an integer number, $n\gg L$, $\mathbb{D}(x,\boldsymbol{v}^\varepsilon)=\dfrac{1}{2}\big(\nabla\boldsymbol{v}^\varepsilon+(\nabla\boldsymbol{v}^\varepsilon)^*\big)$ is the symmetric part of the gradient $\nabla\boldsymbol{v}^\varepsilon$, $\mathbb{P}_f^\varepsilon$ is a part of the stress tensor in the fluid component, $p^0(\boldsymbol{x})$ is a given pressure, $p^0(\boldsymbol{x})=p^i{=}const$ for $\boldsymbol{x}\in S^i, i=1,2$.

The motion of the compressible elastic skeleton in the domain Ω_s^ε, for $t>0$, is described by the stationary Lame equations

$$\nabla\cdot\mathbb{P}_s^\varepsilon=\nabla p^0,\ \mathbb{P}_s^\varepsilon=\lambda_0\mathbb{D}(x,\boldsymbol{w}_s^\varepsilon)-(p_s^\varepsilon-p^0)\mathbb{I},\ \boldsymbol{v}_s^\varepsilon=\frac{\partial\boldsymbol{w}_s^\varepsilon}{\partial t}, \tag{2.3}$$

and the linearized continuity equation

$$\frac{1}{c_{s,p}^2}(p_s^\varepsilon-p^0)+\nabla\cdot\boldsymbol{w}_s^\varepsilon=0 \tag{2.4}$$

for dynamic characteristic $\boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t)$ (*elastic displacements*) and p_s^ε (*elastic pressure*), where the symmetric matrix $\mathbb{D}(x,\boldsymbol{v}_f^\varepsilon)$ and other expressions and constants in this statement of the problem will be recalled later. Here $\mathbb{P}_s^\varepsilon$ is a part of the stress tensor in the elastic component and $\mathbb{P}^\varepsilon=\chi^\varepsilon\mathbb{P}_f^\varepsilon+(1-\chi^\varepsilon)\mathbb{P}_s^\varepsilon$ is a part of the *stress tensor*, with χ^ε the characteristic function of the pore space: that is, $\chi^\varepsilon=1$ in the domain occupied by the fluid and $\chi^\varepsilon=0$ in the elastic skeleton.

Diffusion of the acid and products of chemical reactions in the pore space, for $t>0$, are described by diffusion equation

$$\frac{\partial c^\varepsilon}{\partial t}=\nabla\cdot(\alpha_0\,\nabla c^\varepsilon) \tag{2.5}$$

for the *acid concentration* c^ε and by the transport equations

$$\frac{\partial c_j^\varepsilon}{\partial t}+\boldsymbol{v}_f^\varepsilon\cdot\nabla c_j^\varepsilon=0,\ j=1,...,k, \tag{2.6}$$

for *concentrations of products of chemical reactions* $c_j^\varepsilon(\boldsymbol{x},t),\ j=1,...,k$.

At the free boundary $\Gamma^{\,\varepsilon}$ (sometimes called as *moving free boundary*, since it depends of time) between the fluid and elastic components it is assumed the following boundary conditions, the first two of them are typical of *fluid-structure interaction modeling* (see, e.g., [29], [14] and their references)

$$\boldsymbol{w}_f^\varepsilon = \boldsymbol{w}_s^\varepsilon, \tag{2.7}$$

$$\mathbb{P}_f^\varepsilon < \mathbf{N}^\varepsilon >= \mathbb{P}_s^\varepsilon < \mathbf{N}^\varepsilon >, \tag{2.8}$$

$$(D_N^\varepsilon + \beta^\varepsilon)\, c^\varepsilon + \alpha_0\, \frac{\partial c^\varepsilon}{\partial\, N} = 0, \tag{2.9}$$

$$(D_N^\varepsilon - v_{f,N}^\varepsilon)\, c_j^\varepsilon = 0,\ j=1,...,k, \tag{2.10}$$

where $\mathbf{N}^\varepsilon$ is the normal vector to the free boundary (here assumed to be smooth enough), D_N^ε is the *normal velocity* of the boundary Γ^ε in the direction of the unit normal $\boldsymbol{N}^\varepsilon$ to the boundary Γ^ε, outward to the domain Ω_f^ε, and $v_{f,N}^\varepsilon = \chi^{\,\varepsilon}(\boldsymbol{v}_f^\varepsilon \cdot \boldsymbol{N}^\varepsilon)$ is the normal component of the fluid velocity at the free boundary. Such conditions express the laws of conservation of mass and momentum (see Appendix A, section A7 in [29]).

Finally, an additional boundary condition, that should allow us to characterize the free boundary, must be postulated as a decreasing law of the diameter of the elastic particles. We will use such *constitutive law* in our formulation by similarity with a condition used in [34] (where the growth of biological tissue particles in a nutrient medium is considered, but with the opposite sign):

$$D_N^\varepsilon(\boldsymbol{x},t) = \alpha^\varepsilon\, c^\varepsilon(\boldsymbol{x},t),\ \ \boldsymbol{x}\in\,\Gamma^\varepsilon,\ \ t>0. \tag{2.11}$$

At the given boundaries, the *injection wells* S^1, *producing wells* S^2, and the impermeable boundary S^0, the following auxiliary conditions are given

$$\mathbb{P}_f^\varepsilon < \boldsymbol{n} >= 0,\ \boldsymbol{x}\in\, S^1\cup S^2,\ t>0, \tag{2.12}$$

$$\frac{\partial}{\partial n}(c^\varepsilon - c^0)(\boldsymbol{x},t) = 0,\ \ \boldsymbol{x}\in S^0,\ \ t>0, \tag{2.13}$$

$$\boldsymbol{w}_f^\varepsilon(\boldsymbol{x},t) = \boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t) = 0,\ \ \boldsymbol{x}\in S^0,\ \ t>0, \tag{2.14}$$

$$c(\boldsymbol{x},t) = c_0(\boldsymbol{x}),\ \ \boldsymbol{x}\in S^1\cup S^2,\ \ t>0, \tag{2.15}$$

$$c_j(\boldsymbol{x},t) = 0,\ j=1,...,k,\ \ \boldsymbol{x}\in S^1,\ \ t>0. \tag{2.16}$$

The formulation of the problem ends with the given initial conditions

$$c(\boldsymbol{x},0) = c_0(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega_f^0, \tag{2.17}$$

$$\Gamma_{t=0}^\varepsilon = \Gamma^0, \tag{2.18}$$

$$c_j(\boldsymbol{x},0) = 0,\ j=1,...,k,\ \ \boldsymbol{x}\in\Omega_f^0. \tag{2.19}$$

In (2.1) – (2.19) the positive constants p^1, p^2, α_0, λ, $c_{f,p}$ and $c_{s,p}$ are supposed to be given.

Here l is characteristic pore size and L is the characteristic size of the physical domain under consideration, τ is the characteristic duration time of the physical process, $\rho_{\,0}$ is the density of water, g is the acceleration of gravity and μ is the dynamic viscosity of the liquid, ϱ_s is the dimensionless density of the solid skeleton, related to the density of water ρ_0 and ϱ_f is the dimensionless density of the liquid component related to the density of water ρ_0, $c_{p,f}$ is the speed of sound in the liquid component and $c_{p,s}$ is the speed of sound in the solid component and D is the acid diffusion coefficient. Parameters α_c^ε and α_μ^ε may depend on the small parameter ε and parameters α_0, β and β_j $j=1,...,k$, are given positive constants that do not depend on the small parameter ε.

The case of an absolutely rigid solid skeleton was considered in [30] and, as in that occasion, we also will use a functional set which is the key in the study of the microscopic description given by a function $r(\boldsymbol{x},t)$. We define

$$\mathfrak{M}_{(0,T)} = \{r \in \mathrm{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T),\ 0 < r_0(\boldsymbol{x}) < \frac{1}{2},\ -\theta \leqslant \frac{\partial r}{\partial t}(\boldsymbol{x},t) \leqslant 0,$$
$$0<\gamma<1,\ \theta = \text{const} > 0;\ |r|_{\Omega_T}^{(2+\gamma,\frac{2+\gamma}{2})} \leqslant M_0\}, \quad (2.20)$$

which determined the structure of the pore space. Here we are following the notation used in [24] for the functional space $\mathrm{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$: the Banach space of functions $r(\boldsymbol{x},t)$ that are Hölder continuous in $\overline{\Omega}_T = [0,T]\times\overline{\Omega}$ together with all derivatives of the form $D_t^b D_x^s$ for $2b+s<2+\gamma$, and have a finite vale for the associated norm. It can be proved (see expression in (1,10) of [24]), that some norms for the Hölder spaces $\mathrm{H}^{\alpha}(\overline{\Omega})$ and $\mathrm{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_T)$ are given by

$$|u|_{\Omega}^{(\alpha)} = \max\frac{|u(\mathbf{x}+\mathbf{h})-u(\mathbf{x})|}{|\mathbf{h}|^{\alpha}}$$

and

$$|u|_{\Omega_T}^{(\alpha,\frac{\alpha}{2})} = \max\frac{\left|u(\mathbf{x}+\mathbf{h},t+\frac{|\mathbf{h}|}{2})-u(\mathbf{x},t)\right|}{|\mathbf{h}|^{\alpha}},$$

respectively. Moreover, for any natural number k and any $t_0\in[0,T]$, the space $\mathrm{H}^{k+\gamma,\frac{k+\gamma}{2}}(\overline{\Omega}_{t_0})$ is a Banach space with the norm

$$|u|_{\Omega_{t_0}}^{(k+\alpha,\frac{k+\alpha}{2})} = \max\frac{\left|D^k u(\mathbf{x}+\mathbf{h},t+\frac{|\mathbf{h}|}{2})-D^k u(\mathbf{x},t)\right|}{|\mathbf{h}|^{\alpha}} + |u|_{\Omega_{t_0}}^{(\alpha,\frac{\alpha}{2})}.$$

Now we will define the different auxiliary problems that we will consider gradually until we conclude with the final problem that contemplates the most general situation considered in this article. We call $\mathbb{A}^{\varepsilon}$ to the problem (2.1) – (2.5), (2.7) – (2.9), (2.11) – (2.15), (2.17), (2.18) (i.e., without including the conditions relating to the products of the chemical reactions $c_j(\boldsymbol{x},t)$).

We call $\mathbb{B}^{\varepsilon}(r)$ to the problem $\mathbb{A}^{\varepsilon}$, without the boundary condition (2.11) at the free boundary, but assuming known the structure of the pore space, given by the function $r(\boldsymbol{x},t)\in\mathfrak{M}_{(0,T)}$ and assuming an additional term in the dynamic equation for the fluid component in the form

$$\nabla\cdot\mathbb{P}_f^{\varepsilon} - \varepsilon\frac{\partial \boldsymbol{w}_f^{\varepsilon}}{\partial t} = \nabla p^0. \quad (2.21)$$

Notice that this equation corresponds to the so-called parabolic regularization and will allow to get easily better regularity of solutions. Moreover, the extra term will disappear in the homogenization process, as $\varepsilon\to 0$.

As we already mentioned in the abstract, we first consider the problem $\mathbb{B}^{\varepsilon}(r)$ with a given structure on the pore space.

In this problem, $\mathbb{B}^{\varepsilon}(r)$, for a fixed $\varepsilon>0$, the elastic skeleton is the union of some disjoint sets, sufficiently close to balls of radius $\varepsilon\, r$, slowly decreasing in volume, which simplifies the geometry of the original pore space. This will allows us to prove the existence of approximate solutions.

As usual, subsequent intermediate problems have multiple choices. For example, for our case we may consider non stationary Stokes equations, but then we somehow must find *a priori* estimates for the fluid velocity, keeping in mind the difficulties with free boundary separating liquid and solid components.

We call *dynamic problem* $\mathbb{B}^{\varepsilon}_{dyn}(r)$, to the problem (2.1)–(2.4), (2.7), (2.8), (2.12), (2.14), (2.18), (2.21), and we call *diffusion problem* $\mathbb{B}^{\varepsilon}_{diff}(r)$ to the problem (2.5), (2.9), (2.13), (2.15), (2.17).

We call homogenized dynamic problem $\mathbb{H}_{dyn}(r)$ to the homogenization of the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$, and we call the homogenized diffusion problem $\mathbb{H}_{diff}(r)$ to the homogenization of the diffusion problem $\mathbb{B}^{\varepsilon}_{diff}(r)$.

Finally, we will prove that the homogenization of the *constitutive law* of the free boundary (2.11) allow us to define an operator $\mathbb{F}$, that transforms the set $\mathfrak{M}_{(0,T)}$ into itself. Moreover we will prove that $\mathbb{F}$ has a unique fixed point r^* which will determine the desired unique homogenization $\mathbb{H}=\mathbb{H}(r^*)$ of the problem $\mathbb{A}^{\varepsilon}$.

To homogenize the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$ for the fluid component, the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$ for the elastic component and the diffusion problem $\mathbb{B}^{\varepsilon}_{diff}(r)$, we will use a modification of the Nguetseng's two-scale convergent method ([33]) adapted to *structures with special periodicity (see* [30]).

The physical process we are considering has a rather long duration (the filtration rate of the fluid is several meters per year). Therefore, the most interesting mathematical results are the theorems about the existence of solutions, globally in time, to the corresponding initial - boundary value problems. On the other hand, due to the strong nonlinearity typical of free boundary problems (see, e.g., [30]) it is usually not possible to prove any result globally in time for mathematical models at the microscopic level. That is, the possible results can be theorems on the existence of a generalized or classical solution to the initial-boundary value problem for a system of differential equations describing in-situ leaching at the macroscopic level locally in time.

It should be noted that the usual Stefan problem was formulated only at the macroscopic level and for a one-phase problem, where the free boundary is monotonic in time, which made it possible to prove the existence of a classical solution globally in time (see [28], [16], [11]). But in a general formulation, how can we obtain a macroscopic mathematical model if we know nothing about the existence of solutions of the associated microscopic mathematical model of which it is supposed to be the limit?

To get around these difficulties, we will follow the ideas introduced in [30] leading to the existence of some suitable fixed point by application some of the theorems in the literature. To do this, we define the *structure of the pore space*, given by the characteristic function $\chi^{\varepsilon}(\boldsymbol{x},t)=\chi\big(r(\boldsymbol{x},t);[\frac{\boldsymbol{x}}{\varepsilon}]\big)$, periodic in the variable $\boldsymbol{y}$ (see subsection 2.3). As we have already noted, in the case of a general formulation, solving the emerging problem is almost impossible. Therefore, it is reasonable to limit ourselves to the simplest cases. For example, when a non-negative function $r(\boldsymbol{x},t)$, from some set $\mathfrak{M}_{(0,T)}$, uniquely determines a characteristic function of the pore space $\chi(r;\boldsymbol{y})$.

Then, for a fixed $r\in\mathfrak{M}_{(0,T)}$, we consider the initial boundary value problem $\mathbb{B}^{\varepsilon}(r)$ in a given domain $\Omega^{\varepsilon}_{f}(r)$ occupied by the fluid component and a given domain $\Omega^{\varepsilon}_{s}(r)$ occupied by the elastic component in order to determine the main unknowns of the problem $\mathbb{A}^{\varepsilon}$ (velocities, displacements, pressers and acid concentration), still without the free boundary condition (2.11). To understand what should be the *homogenized problem* $\mathbb{H}(r)$ of the problems $\mathbb{B}^{\varepsilon}(r)$, a formal homogenization of the problem $\mathbb{A}^{\varepsilon}$ is performed beforehand. The sufficient conditions for the existence of a homogenization the boundary condition (2.11) are formulated in the Lemma 20. If $r^{\varepsilon}(\boldsymbol{x},t)$ defines the structure of the elastic skeleton and pore space in the problem $\mathbb{A}^{\varepsilon}$ and we assume that $r^{\varepsilon}\to r^*$ as $\varepsilon\to 0$, then the homogenized problem $\mathbb{H}(r^*)$ of the problem $\mathbb{B}^{\varepsilon}(r^*)=\mathbb{A}^{\varepsilon}$ should coincides with the homogenization $\mathbb{H}$ of the problem $\mathbb{A}^{\varepsilon}$ without the homogenization of the boundary condition (2.11).

It is clear that the homogenization of the free boundary condition (2.11), with a given structure of the pore space defined by a function $r\in\mathfrak{M}_{(0,T)}$, defines an operator $\mathbb{F}:\mathfrak{M}\to\mathfrak{M}$, whose unique fixed point $r^*(\boldsymbol{x},t)$ determines the required unique homogenization $\mathbb{H}$ of the problem $\mathbb{A}^{\varepsilon}$ (see subsection 5.5).

As said before, to solve problem $\mathbb{H}(r)$ we, first of all, have to solve the linear problem $\mathbb{B}^{\varepsilon}(r)$ and then find its homogenization $\mathbb{H}(r)$ as $\varepsilon\to 0$. In turns out that the linear problem $\mathbb{B}^{\varepsilon}(r)$ can be decomposed into a sequential formulation of the *dynamic problem* $\mathbb{B}^{\varepsilon}_{dyn}(r)$, defining the dynamic unknowns $\boldsymbol{w}^{\varepsilon}_{f}$, $\boldsymbol{v}^{\varepsilon}_{f}$, $\boldsymbol{w}^{\varepsilon}_{s}$, p^{ε}_{f}, p^{ε}_{s} and then the *diffusion problem* $\mathbb{B}^{\varepsilon}_{diff}(r)$ which defines the acid concentration unknown c^{ε}.

Due to the linearity of these auxiliary problems, the existence and uniqueness of a weak solution to each of them follows, for instance, by the *Galerkin's method (*see, e.g., [24], [25]) from suitable *a priori* estimates and known methods for passing to the limit and solving linear differential equations.

The next step is the homogenization the problem $\mathbb{B}^{\varepsilon}(r)$. To get a rigorous proof of the convergence, when $\varepsilon \to 0$, we will apply a modification of the Nguetseng's two-scale convergence method (following some ideas of [30]). We will get a limit formulation given by a dynamic model $\mathbb{H}_{dyn,f}(r)$ for the fluid component, a dynamic model $\mathbb{H}_{dyn,e}(r)$ for the elastic component, and a diffusion model $\mathbb{H}_{diff}(r)$ for the acid concentration. But since this method was developed only for homogenization of functionals, we will need to write down the *strong formulation of the mathematical model* in terms of a system of integral identities which under some regularity assumptions are equivalent to the original system of differential equations and boundary conditions.

The integral identities, weak formulations of the dynamic Stokes and Lame equations, as well as the integral identities, weak formulation for the diffusion equation with standard boundary conditions are well known in the literature. But the respective expressions of the differential equations in the form of integral identities are a general and rather difficult challenge for free boundary problems. One of the older examples was the Stefan problem ([22], [35]), describing phase transitions in pure (without impurities) media. The question of the existence of a classical solution globally in time to the one-phase Stefan problem remained open until 1975 [16]. The existence of a classical solution to the two-phase Stefan problem locally in time was proved in 1979 [28], provided that the modulus of the temperature gradient at the free boundary at the initial time is positive. Moreover, in [27] it was shown that if this condition is violated, the classical solution of the two-phase Stefan problem does not exist. For some other results on this important problem see, e.g., [11], [18], and their many references.

In our mathematical model on *in situ leaching*, it is very important to find a weak formulation of the problem in the form of a system of integral identities. This only will require some minimal smoothness of the solutions to the problem. Nevertheless, the peculiar geometry of the spatial domain (for instance for the acid concentration c^ε, defined only in the pore space $\Omega^\varepsilon_{f,T}(r) = \bigcup_{t=0}^{t=T} \Omega^\varepsilon_f(r)$), it is necessary to find an extension of the considered functions, from the domain of their definition onto a global domain $\Omega_T = \Omega \times (0,T)$, preserving their best differential properties. To do this, we used some results ([1], [8]) on the extension of such type functions. Many other extensions results could be also apply (see, e.g., the exposition made in [12]).

A'priori estimates on weak solutions (i.e., solutions of the corresponding integral identities) usually require a special choice of the test functions used in the integral identities and followed of suitable integration by parts. For the latter, sufficient smoothness of the boundary of the pore space $\Omega^\varepsilon_f(r)$ (the domain filled by the fluid) is necessary. We point out that the smoothness of the boundary $\partial\Omega^\varepsilon_f(r)$ is determined by the regularity of the function $r \in \mathfrak{M}_{(0,T)}$. This simple fact will be central to the derivation of a priori estimates in our case. Moreover, we will show that the operator $\mathbb{F} = \mathbb{F}(r)$ is Lipschitz continuous, with the corresponding constant bounded by some linear function of T. This property will allows us to prove the existence of a unique fixed point $r^*(\boldsymbol{x},t)$, at least locally in time. Finally, using the regularity of the solutions to the problem $\mathbb{H}(r)$ we will prove the well-posedness (existence and uniqueness of solutions) of the limit mathematical model $\mathbb{H}$, for any $T > 0$.

We will use some of the notations adopted in [24] and [25]. Nevertheless, for the sake of the reader, we recall that $\mathrm{W}^{1,0}_2(\Omega_T) = \mathrm{L}^2(0,T : \mathrm{H}^1(\Omega))$ and in the case of vectorial functions we write the space in bold case $\mathbf{W}^{1,0}_2(\Omega_T) = \mathrm{L}^2(0,T : \mathbf{H}^1(\Omega)) = \mathrm{L}^2(0,T : \mathrm{H}^1(\Omega)^3)$. We recall the notation, used in [24], on the norm in $\mathrm{W}^{1,0}_2(\Omega_T)$

$$|u|_{\Omega_T} := \operatorname*{ess\,sup}_{t\in[0,t]} \|u(.,t)\|_{L^2(\Omega)} + \|\nabla_x u(.,.)\|_{L^2(\Omega_T)} .$$

It is clear that we may find the unknowns $c_j(\boldsymbol{x},t)$, representing the concentrations of products of the chemical reactions, after finding the solutions to the problems $\mathbb{B}^\varepsilon(r)$ and $\mathbb{H}(r)$ (remember that the system of equations (2.6) where not included in the definition of these problems).

2.2 Statement of the main results.

Although many other preliminary notations will be recalled in the rest of this section (as, for instance, the detailed definition of weak solutions), we are now in conditions to state the main results of this paper:

Theorem 1. *Let* $c_0 \in \mathrm{H}^{2+\alpha}(\overline{\Omega})$ *and* $p^0 \in \mathrm{H}^{1+\alpha}(\overline{\Omega})$. *Then the problem* $\mathbb{B}^\varepsilon(r)$ *has an unique weak solution* $\boldsymbol{w}_f$, $\boldsymbol{w}_f \in \mathbf{W}_2^{1,0}(\Omega_T)$, $c \in \mathrm{W}_2^{1,0}(\Omega_T)$ *and* p_f, $p_s \in \mathrm{L}^2(\Omega_T)$.

Theorem 2. *Under conditions of the Theorem 1 the problem* $\mathrm{H}(r)$ *has an unique weak solution* $\boldsymbol{w}_f$, $\boldsymbol{w}_f \in \mathbf{W}_2^{1,0}(\Omega_T)^3$, $c \in \mathrm{W}_2^{1,0}(\Omega_T)$ *and* p_f, $p_s \in \mathrm{L}^2(\Omega_T)$.

Theorem 3. *Under conditions of the Theorem 1 the problem* $\mathbb{H}$ *has an unique classical solution* $p \in \mathrm{H}^{1+\alpha,\frac{1+\alpha}{2}}(\overline{\Omega}_T)$, $c \in \mathrm{H}^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$, $\boldsymbol{w}_f \in \mathrm{H}^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)^3$, *and* $\boldsymbol{w}_s \in \mathrm{H}^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)^3$.

2.3 Dimensionless parameters.

If l is the characteristic pore size and L is the characteristic size of the physical domain under consideration, we will use in a crucial way that the dimensionless parameter $\varepsilon = \dfrac{l}{L}$ is a very small parameter. Furthermore, without loss of generality, we will assume that $\varepsilon = \dfrac{1}{n}$, where n is an integer.

The dimensionless parameter α_μ^ε (arising in (2.1)) characterizes the viscosity of the fluid in pores. It is given by

$$\alpha_\mu^\varepsilon = \frac{2\,\mu}{L\,g\,\tau\,\rho_0},$$

where τ is the characteristic duration time of the physical process, ρ_0 is the density of the water, g is the acceleration of gravity and μ is the dynamic viscosity of the fluid. The dimensionless parameter α_0^ε characterizes the speed of dissolution of the elastic skeleton.

The diffusion of acid is characterized by the dimensionless coefficient (arising in (2.5))

$$\alpha_0 = \frac{D\,T}{L^2}.$$

We also recall that ϱ_s is the dimensionless density of the elastic skeleton, related to the density of the water ρ_0, ϱ_f is the dimensionless density of the fluid component related to the density of water ρ_0, $c_{p,f}$ is the speed of sound in the liquid component and $c_{p,s}$ is the speed of sound in the elastic component and D is the acid diffusion coefficient. Parameters α_c^ε and α_μ^ε may depend on the small parameter ε and parameters α_0, β and β_j $j = 1, ..., k$, are given positive constants that do not depend on ε.

In this paper we consider the so-called *Biot's assumption* (in honor to Maurice Anthony Biot (1905-1985)), for in-situ leaching, saying that

$$\alpha_\mu^\varepsilon = \varepsilon^2 \mu_1, \text{ and } \mu_1 = \text{const} > 0.$$

It is not too difficult to prove that without this structural condition the homogenized problems become rather trivial (see, e.g. the analysis made in [12] for some different systems).

2.4 The structure of the pore space.

In what follows all functions of the type $\varphi(\boldsymbol{y};\boldsymbol{x},t)$, where $(\boldsymbol{x},t) \in \Omega$ and $\boldsymbol{y} \in \mathbb{R}^3$, are considered 1 - periodic in the variable $\boldsymbol{y}$:

$$\varphi(\boldsymbol{y};\boldsymbol{x},t) = \varphi\big(\boldsymbol{\varsigma}(\boldsymbol{y});\boldsymbol{x},t\big), \;\; \boldsymbol{y} = [|\boldsymbol{y}|] + \varepsilon\,\boldsymbol{\varsigma}(\boldsymbol{y}), \;\; [|\boldsymbol{y}|] = ([|y_1|], [|y_2|], [|y_3|]), \tag{2.22}$$

where the number $[|a|]$ denotes the integer part of the number a.

For the problem $\mathbb{A}^\varepsilon$, before defined, for any $r^* \in (0, 1/2)$, it is convenient to introduce the sets $\mathbf{Y} = \{\boldsymbol{y} \in \mathbb{R}^3 : -\dfrac{1}{2} < y_k < \dfrac{1}{2}, \; k = 1, 2, 3\}$ and

$$\mathbf{Y}_s(r^*) = \{\boldsymbol{y} \in \mathbf{Y} : |\boldsymbol{y}| = (y_1^2 + y_2^2 + y_3^2)^{\frac{1}{2}} < r^*\}, \; \mathbf{Y}_f(r^*) = \{\boldsymbol{y} \in \mathbf{Y} : |\boldsymbol{y}| > r^*\},$$
$$\gamma(r^*) = \partial\mathbf{Y}_f(r^*) \cap \partial\mathbf{Y}_s(r^*), \tag{2.23}$$

and the auxiliary functions

$$\chi(r^*;\boldsymbol{y})=\frac{\operatorname{sgn}(|\boldsymbol{y}|-r^*)+1}{2},\ \chi^\varepsilon(\boldsymbol{x},t)=\chi\big(r(\boldsymbol{x},t);[\frac{\boldsymbol{x}}{\varepsilon}]\big). \tag{2.24}$$

By $\boldsymbol{n}(r^*)=-\dfrac{\boldsymbol{y}}{|\boldsymbol{y}|}$ we will denote the outward unit normal to the domain $Y_f(r^*)\subset Y$. The same notations will be used also for the problem $\mathbb{B}^\varepsilon(r)$, where instead of r^* we write $r\in(0,1/2)$,

$$\chi(r;\boldsymbol{y})=\frac{\operatorname{sgn}(|\boldsymbol{y}|-r)+1}{2},\ \chi^\varepsilon(\boldsymbol{x},t)=\chi\big(r(\boldsymbol{x},t);[\frac{\boldsymbol{x}}{\varepsilon}]\big). \tag{2.25}$$

2.5 Domains and boundaries.

We assume that $\Omega\subset\mathbb{R}^3$ is a bounded domain with piecewise smooth boundary $S=\partial\Omega=\bar S^0\cup\bar S^1\cup\bar S^2$. The boundary $S^0\subset\mathbb{R}^3$ is impermeable to the fluid in the pore space, the boundary $S^1\subset\mathbb{R}^3$ simulates the *injection wells* and the boundary $S^2\subset\mathbb{R}^3$ simulates the *production wells.*

As a matter of facts, we will simplify the geometry of the spatial domain by assuming that Ω is the unit origin-centered cube of $\mathbb{R}^3$, and with $S^0=\{\boldsymbol{x}:x_3=\pm\frac{1}{2},\ -\frac{1}{2}\leqslant x_1,\,x_2\leqslant\frac{1}{2}\}$, $S^1=\{\boldsymbol{x}:x_1=-\frac{1}{2},\ -\frac{1}{2}\leqslant x_2,\,x_3\leqslant\frac{1}{2}\}$, $S^2=\{\boldsymbol{x}:x_1=\frac{1}{2},\ -\frac{1}{2}\leqslant x_2,\,x_3\leqslant\frac{1}{2}\}$.

We define now the following subsets of Ω

$$\Omega^\varepsilon_f(r)=\{\boldsymbol{x}\in\Omega:\chi^\varepsilon(r;\boldsymbol{x})=1\},\qquad \Omega^\varepsilon_s(r)=\{\boldsymbol{x}\in\Omega:\chi^\varepsilon(r;\boldsymbol{x})=0\},\ \Omega^\varepsilon_{f,T}(r)=\bigcup_{t=0}^{t=T}\Omega^\varepsilon_f(r),$$

$$\Omega^\varepsilon_{s,T}(r)=\bigcup_{t=0}^{t=T}\Omega^\varepsilon_s(r),\qquad \Gamma^\varepsilon(r)=\overline{\Omega}_f(r)\cap\overline{\Omega}_s(r),\quad \Gamma^\varepsilon_T=\bigcup_{t=0}^{t=T}\Gamma^\varepsilon(r).\ \text{Thus}$$

$$\Omega=\bigcup_{\boldsymbol{k}\in\mathbb{Z}}\overline{\Omega}^{\boldsymbol{k},\varepsilon},\ \ \Omega^{\boldsymbol{k},\varepsilon}=\{\boldsymbol{x}\in\Omega:\boldsymbol{x}=\varepsilon\boldsymbol{k}+\varepsilon\boldsymbol{y}\},\ \Omega^{\boldsymbol{k},\varepsilon}_f(r)=\Omega^\varepsilon_f(r)\cap\Omega^{\boldsymbol{k},\varepsilon},$$

$$\Omega^{\boldsymbol{k},\varepsilon}_s(r)=\Omega^\varepsilon_s(r)\cap\Omega^{\boldsymbol{k},\varepsilon},\ \Omega^0_f=\Gamma^{\boldsymbol{k},\varepsilon}(r)=\Gamma^\varepsilon(r)\cap\Omega^{\boldsymbol{k},\varepsilon},$$

for all $\boldsymbol{k}=(k_1,k_2,k_3)$, $k_1,k_2,k_3\in\mathbb{Z}$ (integer numbers) and for all $\boldsymbol{y}\in Y=(-\frac{1}{2},\frac{1}{2})^3\subset\mathbb{R}^3$.

In this way,

$$\Omega^\varepsilon_f(r)=\{\boldsymbol{x}\in\Omega:\chi^\varepsilon(\boldsymbol{x},t)=1\},\ \Omega^0_f=\Omega^\varepsilon_f(r_0),$$

$$\Omega^\varepsilon_s(r)=\{\boldsymbol{x}\in\Omega:\chi^\varepsilon(\boldsymbol{x},t)=0\},\ \Omega^0_s=\Omega^\varepsilon_s(r_0),$$

$$\Omega^{\boldsymbol{k},\varepsilon}_j(r)=\Omega^{\boldsymbol{k},\varepsilon}\cap\Omega^\varepsilon_j(r),\ \ j=f,s,$$

$$\Gamma^\varepsilon(r)=\overline{\Omega}^\varepsilon_f(r)\bigcap\overline{\Omega}^\varepsilon_s(r)=\bigcup_{k=1}^{n^3}\Gamma^{\varepsilon,k}(r),\ \Gamma^{\varepsilon,k}(r)=\Omega^{\boldsymbol{k},\varepsilon}\cap\Gamma^\varepsilon(r). \tag{2.26}$$

We call the structure, defined by the formula (2.25) as *structure with special periodicity.*

For a given structure $r(\boldsymbol{x},t)$, with characteristic function $\chi(r;\boldsymbol{y})$, the function

$$m(r)=\int_Y\chi(r;\boldsymbol{y})dy=1-\frac{4}{3}\pi r^3\geq\frac{4}{3}, \tag{2.27}$$

represents the porosity of the elastic skeleton at the point $(\boldsymbol{x},t)$.

For any continuous function $u(\boldsymbol{x})$ on $\Omega_f\cup\Omega_s$ its directional limits at the points $\boldsymbol{x}_0\in\Gamma^\varepsilon(r)$ are denoted as

$$u(\boldsymbol{x}_0+0)=\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}u(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega^\varepsilon_{f,t_0}(r),\ \ \boldsymbol{x}_0\in\Gamma^\varepsilon(r),$$

$$u(\boldsymbol{x}_0-0)=\lim_{\boldsymbol{x}\to\boldsymbol{x}_0}u(\boldsymbol{x}),\ \ \boldsymbol{x}\in\Omega^\varepsilon_{s,t_0}(r),\ \ \boldsymbol{x}_0\in\Gamma^\varepsilon(r).$$

Sometimes, we will choose a small parameter $\varepsilon = \dfrac{1}{n}$, $n = 1, 2, 3, ...$, so that the boundary condition (2.12), on $S^1 \cup S^2$, makes sense.

2.6 Some notations regarding matrices and differential operators.

We assume given the standard Cartesian orthogonal basis $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ in $\mathbb{R}^3$ and consider some *tensors* (i.e., linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$) $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$. The action of the tensor $\mathbb{A}$ on a vector $\boldsymbol{b}$ is denoted by the vector $\boldsymbol{c} = \mathbb{A} < \boldsymbol{b} >$. By $(\boldsymbol{a} \cdot \boldsymbol{b})$ we denote the *scalar product* of vectors $\boldsymbol{a}$, $\boldsymbol{b}$. We recall that the product $\mathbb{C} = \mathbb{A} \cdot \mathbb{B}$ is a transformation $\mathbb{A} : \mathbb{B}(\mathbb{R}^3) \to \mathbb{R}^3$, where $\mathbb{B}(\mathbb{R}^3) = \{\boldsymbol{y} \in \mathbb{R}^3 : \boldsymbol{y} = \mathbb{B}(\boldsymbol{x}), \forall \boldsymbol{x} \in \mathbb{R}^3\}$ and that $\mathbb{I}$ denotes the unit tensor, i.e., such that $\mathbb{I} \cdot \mathbb{A} = \mathbb{A} \cdot \mathbb{I} = \mathbb{A}$ for any tensor $\mathbb{A}$.

For any vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, by $\boldsymbol{a} \otimes \boldsymbol{b}$ we denote the *diad* (second-order tensor) defined by $(\boldsymbol{a} \otimes \boldsymbol{b}) < \boldsymbol{c} >= \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c})$. By $\boldsymbol{J}_{ij}$ we denote the tensor $\dfrac{1}{2}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + \boldsymbol{e}_j \otimes \boldsymbol{e}_i)$. Then we can write $\mathbb{A} = \sum_{i,j=1}^{3} a_{ij} \boldsymbol{J}_{ij}$. In particular, a tensor $\mathbb{A}$ is *symmetric* if $(\mathbb{A} < \boldsymbol{e}_j > \cdot \boldsymbol{e}_i) = (\mathbb{A} < \boldsymbol{e}_i > \cdot \boldsymbol{e}_j)$ and more in general if $(\mathbb{A} < \boldsymbol{a} > \cdot \boldsymbol{b}) = (\mathbb{A} < \boldsymbol{b} > \cdot \boldsymbol{a})$.

Given some tensors $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$, by (A), (B) and (C) we denote the associate matrices in the chosen Cartesian coordinate system

$$(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ (B) = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}, \ (C) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix},$$

The usual operations of sum $(A) + (B)$, multiplication by scalars $\alpha(B)$ and product $(A) \cdot (B)$ are well defined in a compatible sense with the operations with tensors.

We will use the vectorial notation $\boldsymbol{u}(\boldsymbol{x}, t) = \big(u_1(\boldsymbol{x}, t), u_2(\boldsymbol{x}, t), u_3(\boldsymbol{x}, t)\big)$. The symmetric part of the gradient tensor is given by $\mathbb{D}(x, \boldsymbol{u}) = \dfrac{1}{2}\big(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^*\big)$. Then, the *symmetric gradient* of a vectorial function $\boldsymbol{u}$ is given by the second-order symmetric tensor

$$\mathbb{D}(x, \boldsymbol{u}) = \frac{1}{2} \sum_{i,j=1}^{3} d_{ij}(x, \boldsymbol{u})(\boldsymbol{e}_i \otimes \boldsymbol{e}_j + d_{ji} \boldsymbol{e}_j \otimes \boldsymbol{e}_i), \ \text{ with } d_{ji}(x, \boldsymbol{u}) = \frac{\partial u_i}{\partial x_j}, \ i, j = 1, 2, 3.$$

We will use the notation

$$\mathbb{D}(x, \boldsymbol{w}) < \boldsymbol{a} > \overset{def.}{=} \frac{1}{2}\Big(\sum_{i=1}^{3} d_{ij}(x, \boldsymbol{w})(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) + (\boldsymbol{e}_j \otimes \boldsymbol{e}_i)\Big) < \boldsymbol{a} > . \tag{2.28}$$

Then we define the some expressions which will appear later involved in suitable norms (when the vectorial function $\boldsymbol{w}$ satisfies suitable properties (see, e.g., [29])

$$\mathbb{D}(x, \boldsymbol{w}) : \mathbb{D}(x, \boldsymbol{\varphi}) = \sum_{i,j=1}^{3} d_{ij}(x, \boldsymbol{w}) d_{ij}(x, \boldsymbol{\varphi}), \ |\mathbb{D}(x, \boldsymbol{w})|^2 = \sum_{i,j=1}^{3} |d_{ij}(x, \boldsymbol{w})|^2,$$

$$\|\mathbb{D}\big(x, \boldsymbol{w}(., t)\big)\|_{2,\Omega}^2 = \sum_{i,j=1}^{3} \|d_{ij}\big(x, \boldsymbol{w}(., t)\big)\|_{2,\Omega}^2 \leqslant 3\|\frac{\partial \boldsymbol{w}}{\partial t}(., t)\|_{2,\Omega}^2,$$

$$\|\mathbb{D}(x, \boldsymbol{w})\|_{2,\Omega_T}^2 = \sum_{i,j=1}^{3} \|d_{ij}(x, \boldsymbol{w})\|_{2,\Omega_T}^2 \leqslant 3\|\frac{\partial \boldsymbol{w}}{\partial t}\|_{2,\Omega_T}^2, \tag{2.29}$$

$$\big(\mathbb{D}(x, \boldsymbol{w}) < \boldsymbol{a} > \cdot \boldsymbol{b}\big) = \big(\mathbb{D}(x, \boldsymbol{w}) < \boldsymbol{b} > \cdot \boldsymbol{a}\big) \overset{def.}{=} \mathbb{D}(x, \boldsymbol{w}) < \boldsymbol{a}, \boldsymbol{b} >, \tag{2.30}$$

$$|\mathbb{D}\big(x, \boldsymbol{w}(., t)\big)|^2 = \int_0^t |\mathbb{D}\big(x, \frac{\partial \boldsymbol{w}}{\partial \tau}(., \tau)\big)|^2 d\tau. \tag{2.31}$$

2.7 Moving boundaries and strong gaps.

Let $[A] = A_f - A_s$ and $[\mathbf{B}] = \mathbf{B}_f - \mathbf{B}_s$ be the discontinuity jumps of some given scalar and vectorial functions, A and $\mathbf{B}$, over a C^1 boundary $\Gamma^\varepsilon(r)$. We have

Lemma 4. *(Integration by parts: [29], Appendix A)*
Let a C^1 boundary $\Gamma^\varepsilon(r)$ separating Ω_T into two subdomains $\Omega_{f,T}$ and $\Omega_{f,T}$. Then, for any smooth function η, vanishing at $\partial\,\Omega$, the following integral identity holds true

$$\int_0^{t_0}\int_\Omega \eta\,\big(\frac{\partial A_f}{\partial t}\chi^\varepsilon + \frac{\partial A_s}{\partial t}(1-\chi^\varepsilon) + \nabla\cdot(\chi^\varepsilon \boldsymbol{B}_f + (1-\chi^\varepsilon)\boldsymbol{B}_f\big)dxdt =$$
$$\int_0^{t_0}\int_{\Gamma^\varepsilon(r)} \eta\,\big((A_s - A_f)\,D_N^\varepsilon + (\boldsymbol{B}_f - \boldsymbol{B}_s)\cdot\boldsymbol{N}^\varepsilon\big)\sin\psi\,d\sigma dt +$$
$$\int_\Omega \eta(\boldsymbol{x},t_0)\Big(\chi^\varepsilon(\boldsymbol{x},t_0)\big(A_f(\boldsymbol{x},t_0)\chi^\varepsilon(\boldsymbol{x},t_0) + A_s(\boldsymbol{x},t_0)(1-\chi^\varepsilon(\boldsymbol{x},t_0))\Big)dx -$$
$$\int_\Omega \eta(\boldsymbol{x},0)\Big(\chi^\varepsilon(\boldsymbol{x},0)\big(A_f(\boldsymbol{x},0)\chi^\varepsilon(\boldsymbol{x},t_0) + A_s(\boldsymbol{x},t_0)(1-\chi^\varepsilon(\boldsymbol{x},t_0))\Big)dx -$$
$$\int_0^{t_0}\int_\Omega \big(\frac{\partial\eta}{\partial t}(A_f\chi^\varepsilon + A_s(1-\chi^\varepsilon) + \big((\chi^\varepsilon \boldsymbol{B}_f + (1-\chi^\varepsilon)\boldsymbol{B}_s\big)\cdot\nabla\eta)dxdt.$$

Here $0 < t_0 < T$, $\boldsymbol{N}^\varepsilon \in \mathbb{R}^3$ is the unit normal vector to $\Gamma^\varepsilon(r)$, pointing outward to $\Omega_f^\varepsilon(r)$, D_N^ε is the normal velocity of the boundary $\Gamma^\varepsilon(r)$ in the direction of the normal $\boldsymbol{N}^\varepsilon$, and ψ is the angle between the unit vector $\boldsymbol{l}$ of the time axis and the unit normal vector $\boldsymbol{\nu} \in \mathbb{R}^4$ to Γ_T^ε, pointing outward to $\Omega_{f,T}^\varepsilon$, such that $\sin\psi = \boldsymbol{\nu}\cdot\boldsymbol{N}$ and $\cos\psi = \boldsymbol{\nu}\cdot\boldsymbol{l}$.
In particular,

$$\int_0^{t_0}\int_{\Omega_s^\varepsilon(r)} \eta\,\frac{\partial A_s}{\partial t}dxdt =$$
$$\int_0^{t_0}\int_{\Gamma^\varepsilon(r(.,t))} \eta\,A_s D_N^\varepsilon \sin\psi\,d\sigma dt - \int_0^{t_0}\int_{\Omega_s^\varepsilon(r)} A_s\frac{\partial\eta}{\partial t}dxdt,$$
$$\int_0^{t_0}\int_{\Omega_f^\varepsilon(r)} \eta\,\frac{\partial A_f}{\partial t}dxdt =$$
$$-\int_0^{t_0}\int_{\Gamma^\varepsilon(r(.,t))} \eta\,A_f D_N^\varepsilon \sin\psi\,d\sigma dt - \int_0^{t_0}\int_{\Omega_f^\varepsilon(r)} A_f\frac{\partial\eta}{\partial t}dxdt. \quad (2.32)$$

2.8 A consequence of the Poincaré inequality.

Lemma 5. *Let $Q \subset \mathbb{R}^3$ be bounded domain with Lipschitz piecewise smooth boundary. Then for any function $w \in \mathrm{H}_0^1(Q)$ we have*

$$\|w\|_{2,Q} \leqslant M_Q\|\nabla w\|_{2,Q},$$

where $M_Q < \infty$ for a bounded domain Q. In particular, if $\Omega \subset \bigcup_{|\boldsymbol{k}|=1}^{n^3}\Omega^{\boldsymbol{k},\varepsilon}$ and $w \in \mathrm{H}_0^1(\Omega^{\boldsymbol{k},\varepsilon})$ $\boldsymbol{k} = (k_1,k_2,k_3), \in \mathbb{Z}$, then

$$\int_{\Omega^{\boldsymbol{k},\varepsilon}} |\boldsymbol{w}|^2 dx \leqslant \varepsilon^2\,M_\Omega\int_{\Omega^{\boldsymbol{k},\varepsilon}} |\nabla w|^2 dx$$

and

$$\int_\Omega |w|^2 dx \leqslant \varepsilon^2\,M_\Omega\int_\Omega |\nabla w|^2 dx. \quad (2.33)$$

Remark 6. *A related result (consequence of the so called Poincaré-Wirtinger inequality: see, e.g., [2])*

$$\int_\Omega (|w - \frac{1}{|\Omega|}\int_\Omega |w|dx)dx \leqslant \varepsilon^2 M_\Omega \int_\Omega |\nabla w|^2 dx,$$

holds true for any $w \in \mathrm{H}^1(\Omega)$, when Ω is as in Lemma 5 .

We also recall one of the simpler embedding results:

Lemma 7. *Let $\Omega \subset \mathbb{R}^3$ with piecewise C^1 boundary. Then for any function $u \in \mathrm{H}^1(\Omega)$ identically equal zero on some part of the boundary $\partial\Omega$ with positive surface measure, we have the estimate*

$$\|u\|_{2,\Omega} \leqslant M\|\nabla u\|_{2,\Omega}, \tag{2.34}$$

where the constant M is bounded if, for instance, Ω is bounded.

2.9 Mollifiers.

Let $J(s) \geqslant 0$, $J(s) = 0$ for $|s| > 1$, $J(s) = J(-s)$, $J \in C^\infty(-\infty, +\infty)$, and such that

$$\int_{\mathbb{R}^3} J(|\boldsymbol{x}|)dx = 1, \;\; \boldsymbol{x} \in \mathbb{R}^3.$$

Definition 8. *The operator $\mathbf{M}_h :\ \mathrm{L}^2(\Omega) \to \mathrm{C}^\infty(\overline{\Omega})$ defined by*

$$\mathbf{M}_h(\mathbf{u})(\boldsymbol{x}) = \frac{1}{h^3}\int_{\mathbb{R}^3} J(\frac{|\boldsymbol{x}-\boldsymbol{y}|}{h})\mathbf{u}(\boldsymbol{y})dy, \tag{2.35}$$

is called a mollifier and the function $\mathbf{M}_h(\mathbf{u})$ is called the mollification of $\mathbf{u}$.

We have:

Lemma 9. *Let $\mathbf{u} \in \mathrm{L}^p(\Omega)$ and $p \geqslant 1$. Then*

$$\int_\Omega \mathbf{M}_h(\mathbf{u})\mathbf{v}dx = \int_\Omega \mathbf{u}\mathbf{M}_h(\mathbf{v})dx,$$

$$\|\mathbf{M}_h(\mathbf{u})\|_{p,\Omega} \leqslant \|\mathbf{u}\|_{p,\Omega}, \;\; \lim_{h\to 0}\|\mathbf{M}_h(\mathbf{u}) - \mathbf{u}\|_{p,\Omega} = 0 \tag{2.36}$$

For a proof see, e.g., Lemma 2.18 of [2].

2.10 Extension Lemma

It is well-known that extension results are very important in homogenization (see, e.g., the expositions made in [20], [39], [12], and their many references). For instance, very often some sequence of functions has different properties in different subdomains, but only such kind of properties of the sequence on a global domain permits to choose convergent subsequence. Therefore, we must preserve the best properties of the sequence and apply the extension from the global domain onto the mentioned subsets. Fortunately all the indicated results apply for our case (for structure with special periodicity) because in each cell of periodicity $\Omega^{\boldsymbol{k},\varepsilon}$ we may directly use the method suggested in Chapter 3 of [20].

The following lemma concerns solutions $\{\boldsymbol{w}_j^\varepsilon,\, p_j,\, j = f, s\}$ to the problem $\mathbb{B}^\varepsilon(r)$.

Lemma 10. *1) Let $\{p_f^\varepsilon\}$ and $\{p_s^\varepsilon\}$ be bounded sequences in $\mathrm{L}^2(0,T:\mathrm{L}^2(\Omega_f^\varepsilon(r)))$ and $\mathrm{L}^2(0,T:\mathrm{L}^2(\Omega_s^\varepsilon(r)))$, respectively. Then for all $\varepsilon > 0$ there exist the extensions*

$$\widetilde{p}^{\,\varepsilon} = \chi^\varepsilon(p_f^\varepsilon - p^0) + (1-\chi^\varepsilon)(p_s^\varepsilon - p^0), \;\; \|\widetilde{p}^{\,\varepsilon}\|_{2,\Omega_T} \leqslant \|\chi^\varepsilon p_f^\varepsilon\|_{2,\Omega_T} + \|(1-\chi^\varepsilon)p_s^\varepsilon\|_{2,\Omega_T}.$$

2) Let $\{\boldsymbol{w}_f^\varepsilon\}$ be a bounded sequence in $\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega_f^\varepsilon(r)))\cap\mathrm{H}^1(0,T:\mathbf{L}^2(\Omega_f^\varepsilon(r)))$. *Then, for all* $\varepsilon>0$, *there exist an extension operator* $\mathbb{E}_f:\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega_f^\varepsilon(r)))\cap\mathrm{H}^1(0,T:\mathbf{L}^2(\Omega_f^\varepsilon(r)))\to\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega))\cap\mathrm{H}^1(0,T:\mathbf{L}^2(\Omega))$, *denoted by* $\mathbb{E}_f(\boldsymbol{w}_f^\varepsilon)=\widetilde{\boldsymbol{w}}_f^\varepsilon$, *such that*

$$(\widetilde{\boldsymbol{w}}_f^\varepsilon-\boldsymbol{w}_f^\varepsilon)\chi^\varepsilon=0,\ \big(\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)-\mathbb{D}(x,\boldsymbol{w}_f^\varepsilon)\big)\chi^\varepsilon=0,\ \widetilde{\boldsymbol{w}}_f^\varepsilon(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in\overline{\Omega}_s^\varepsilon(r),$$
$$(\widetilde{\boldsymbol{w}}_f^\varepsilon-\boldsymbol{w}_f^\varepsilon)\chi^\varepsilon=0,\ (\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}-\frac{\partial\boldsymbol{w}_f^\varepsilon}{\partial t})\chi^\varepsilon=0;$$
$$\|\widetilde{\boldsymbol{w}}_f^\varepsilon\|_{2,\Omega_T}\leqslant M\|\boldsymbol{w}_f^\varepsilon\|_{2,\Omega_{f,T}(r)},\ \ \|\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)\|_{2,\Omega_T}\leqslant M\|\mathbb{D}(x,\boldsymbol{w}_f^\varepsilon)\|_{2,\Omega_{f,T}(r)},$$
$$\|\chi^\varepsilon(\varepsilon\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})\|_{2,\Omega_T}=\|\chi^\varepsilon(\varepsilon\frac{\partial\boldsymbol{w}_f^\varepsilon}{\partial t})\|_{2,\Omega_{f,T}}.$$

3) Let $\{\boldsymbol{w}_s^\varepsilon\}$ be a bounded sequence in $\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega_s^\varepsilon(r)))$. *Due to condition $\widetilde{\boldsymbol{w}}_f^\varepsilon$ on $\Gamma^\varepsilon(r)$, we set $\boldsymbol{w}_s^\varepsilon=0$ at $\Gamma^\varepsilon(r)$. Then, for all $\varepsilon>0$, there exist extension operator* $\widetilde{\boldsymbol{w}}_s^\varepsilon=\mathbb{E}_s(\boldsymbol{w}_s^\varepsilon)$, $\mathbb{E}_s:\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega_s^\varepsilon(r)))\to\mathrm{L}^2(0,T:\mathbf{H}^1(\Omega))$, *such that $\widetilde{\boldsymbol{w}}_s^\varepsilon=0$ in $\overline{\Omega}_{f,T}^\varepsilon$, and*

$$(\widetilde{\boldsymbol{w}}_s^\varepsilon-\boldsymbol{w}_s^\varepsilon)(1-\chi^\varepsilon)=0,\ \big(\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^\varepsilon)-\mathbb{D}(x,\boldsymbol{w}_s^\varepsilon)\big)(1-\chi^\varepsilon)=0,$$
$$\widetilde{\boldsymbol{w}}_s^\varepsilon(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in\overline{\Omega}_f^\varepsilon(r),\ |\widetilde{\boldsymbol{w}}_s^\varepsilon|^{(1,0)}{}_{\Omega_T}\leqslant M\|(1-\chi^\varepsilon)\boldsymbol{w}_s^\varepsilon\|^{(1,0)}{}_{2,\Omega_{s,T}(r)},\tag{2.37}$$

where M is independent of ε.

Proof. The estimates for $\widetilde{p}^\varepsilon$ are obvious. To prove the second statement we note that there are several options for extensions of $\boldsymbol{w}_f^\varepsilon$. We chose the extension

$$\widetilde{\boldsymbol{w}}_f^\varepsilon=\chi^\varepsilon\boldsymbol{w}_f^\varepsilon-(1-\chi^\varepsilon)\boldsymbol{w}_s^\varepsilon\tag{2.38}$$

for which

$$\widetilde{\boldsymbol{w}}_f^\varepsilon(\boldsymbol{x},t)=0,\ \boldsymbol{x}\in\overline{\Omega}_{s,T}^\varepsilon(r),\ \chi^\varepsilon\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)=\mathbb{D}(x,\boldsymbol{w}_f^\varepsilon)\ \ (\boldsymbol{x},t)\in\overline{\Omega}_{s,T}^\varepsilon(r).\tag{2.39}$$

Thus, to prove the statement we just take $\widetilde{\boldsymbol{w}}_f^\varepsilon(\boldsymbol{x},t)=0$ in $\overline{\Omega}_{s,T}$. Next we take $\widetilde{\boldsymbol{w}}_s^\varepsilon=0$ in $\overline{\Omega}_{f,T}$. It is easy to see that $\widetilde{\boldsymbol{w}}_s^\varepsilon$ satisfies all the conditions of the lemma. ∎

The following lemma was proved in [1] (see also [8]):

Lemma 11. *Let $\{c^\varepsilon\}$ be bounded sequence in* $\mathrm{W}_2^{1,0}\big(\Omega_{f,T}^\varepsilon(r)\big)$. *Then for all $\varepsilon>0$ there exist some extensions $\widetilde{c}^\varepsilon$, such that*

$$\|(\widetilde{c}^\varepsilon-c_0)\|_{2,\Omega_T}+\|\nabla(\widetilde{c}^\varepsilon-c_0)\|_{2,\Omega_T}\leqslant M.\tag{2.40}$$

In what follows, we will use also the notation $\widetilde{\mathbb{P}}^\varepsilon=\chi^\varepsilon\mathbb{P}_f+(1-\chi^\varepsilon)\mathbb{P}_s$.

Remark 12. *Due to the choice of function $p^0(\boldsymbol{x})$, we know that*

$$\widetilde{\mathbb{P}}^\varepsilon<\boldsymbol{n}>=0,$$

on the boundary $S^1\cup S^2$. Here $\boldsymbol{n}$ is the exterior unit normal vector to S^1 and S^2.

2.11 Two-scale convergent methods.

In the present section we consider 1–periodic in the variable $\boldsymbol{y} \in Y$ and functions $W(\boldsymbol{y};\boldsymbol{x},t)$, with $(\boldsymbol{x},t) \in \Omega_T$.

Definition 13. The sequence $\{w^{\varepsilon}\} \subset \mathrm{L}^2(\Omega_T)$, is said to be two – scale convergent to the function $W(\boldsymbol{x},t,\boldsymbol{y}) \in \mathrm{L}^2(\Omega_T \times Y)$, which is 1–periodic in the variable $\boldsymbol{y} \in Y$ (with the notation $w^{\varepsilon} \overset{2-sc.}{\to} W(\boldsymbol{x},t;\boldsymbol{y})$), if for any smooth function $\sigma = \sigma(\boldsymbol{y};\boldsymbol{x},t)$, 1–periodic in the variable $\boldsymbol{y}$, we have

$$\lim_{\varepsilon\to 0}\int\int_{\Omega_T} w^{\varepsilon}(\boldsymbol{x},t)\sigma(\boldsymbol{x},t;\frac{\boldsymbol{x}}{\varepsilon})dxdt = \int\int_{\Omega_T}\Big(\int_Y W(\boldsymbol{x},t;\boldsymbol{y})\sigma(\boldsymbol{x},t;\boldsymbol{y})dy\Big)dxdt. \tag{2.41}$$

Note that weak and two – scale convergence are connected by the relation:

if $u^{\varepsilon} \overset{2-sc.}{\to} U(\boldsymbol{x},t;\boldsymbol{y})$ (two – scale convergence),

then $u^{\varepsilon}(\boldsymbol{x},t) \rightharpoonup \int_Y U(\boldsymbol{y};\boldsymbol{x},t)d\boldsymbol{y}$ (weak convergence).

The existence and basic properties of two – scale convergent sequences are proved in the following theorem:

Theorem 14. *(Nguetseng's Theorem)* [33]

1. Any bounded in $\mathrm{L}^2(0,T:\mathbf{L}^2(\Omega))$ sequence $\{\boldsymbol{w}^{\varepsilon}\}$ contains some subsequence two – scale convergent to some function $\boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t)$, $\boldsymbol{W} \in \mathrm{L}^2(0,T:\mathbf{L}^2(\Omega\times\mathbf{Y}))$, 1–periodic in the variable $\boldsymbol{y}$.

2. Let sequences $\{\boldsymbol{w}^{\varepsilon}\}$ and $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ be uniformly bounded in $\mathrm{L}^2(0,T:\mathbf{L}^2(\Omega))$. Then, there exists a function $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t)$, 1–periodic in $\boldsymbol{y}$, and the sequence $\{\boldsymbol{w}^{\varepsilon}\}$ such that $\boldsymbol{W}$, $\nabla_y\boldsymbol{W} \in \mathrm{L}^2(0,T:\mathbf{L}^2(\Omega\times\mathbf{Y}))$, and sequences $\{\boldsymbol{w}^{\varepsilon}\}$ and $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ (for simplicity we keep the same indices for subsequences) two – scale convergent in $\mathrm{L}^2(0,T:\mathbf{L}^2(\Omega\times\mathbf{Y}))$ to $\boldsymbol{W}$ and $\mathbb{D}(y,\boldsymbol{W})$, respectively.

3. Let sequences $\{\boldsymbol{w}^{\varepsilon}\}$ and $\{D(x,\boldsymbol{w}^{\varepsilon})\}$ be bounded in $\mathrm{L}^2(0,T:\mathbf{L}^2(\Omega))$. Then there are some functions $\boldsymbol{w}(\boldsymbol{x},t)$, $\boldsymbol{w} \in \mathrm{W}^{1,0}_2(\Omega_T)$, and $\boldsymbol{W}(\boldsymbol{y};\boldsymbol{x},t)$, $\boldsymbol{W} \in \mathrm{L}^2(\Omega_T\times\mathbf{Y})\cap\mathrm{W}^{1,0}_2(\mathbf{Y})$, some subsequence from $\{\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ such that the function $\boldsymbol{W}$ is 1–periodic in $\boldsymbol{y}$, $\mathbb{D}(x,\boldsymbol{w}) \in \mathrm{L}^2(\Omega_T)$, $D(y,\boldsymbol{W}) \in \mathrm{L}^2(\Omega_T\times\mathbf{Y})$, and the sequence $\{\mathbb{D}(x,\boldsymbol{w}^{\varepsilon})\}$ is two – scale convergent to the function $\mathbb{D}(x,\boldsymbol{w}) + D(y,\boldsymbol{W})$.

2.12 Two useful compactness criteria.

We start by recalling a well-known definition:

Definition 15. *We say that a function $c(\boldsymbol{x},t)$, $c \in \mathrm{L}^2(0,T:\mathrm{L}^2(\Omega))$, possesses a time derivative with $\dfrac{\partial c}{\partial t} \in \mathrm{L}^2\big(0,T;\mathrm{H}^{-1}(\Omega)\big)$, if*

$$|\int\int_{\Omega_T} c\frac{\partial\xi}{\partial t}dxdt| \leqslant M_u\,|\int\int_{\Omega_T}|\nabla\xi|^2dxdt|^{\frac{1}{2}}$$

for all functions $\xi \in \mathrm{H}^1(0,T:\mathrm{H}^1(\Omega))$, for some positive constant M_u independent of ξ.

Remark 16. *We denote the norm of an element $\varphi \in \mathrm{L}^2\big(0,T;\mathrm{H}^{-1}(\Omega)\big)$ by $\|\varphi\|_{W_2^{-1}}$.*

The following compactness result was proved in [25]:

Lemma 17. *Assume the sequences $\{c^{\varepsilon}\}$ and $\{\nabla c^{\varepsilon}\}$ be uniformly bounded in $\mathrm{L}^2(\Omega_T)$, and the sequence of derivatives $\{\dfrac{\partial c^{\varepsilon}}{\partial t}\}$ be uniformly bounded in $\mathrm{L}^2\big(0,T;\mathrm{H}^{-1}(\Omega)\big)$. Then, there exists some subsequence of $\{c^{\varepsilon}\}$ strongly convergent in $\mathrm{L}^2(\Omega_T)$.*

The generalization of this lemma for domains with a periodic structure, of characteristic function $\chi^{\varepsilon}(\boldsymbol{x}) = \chi(\frac{\boldsymbol{x}}{\varepsilon})$, was proved in [31] .

Lemma 18. *Let* $\chi^{\varepsilon}(\boldsymbol{x}) = \chi(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon})$, *where* $\chi(\boldsymbol{x}, \boldsymbol{y})$ *is 1 – periodic in* $\boldsymbol{y}$ *function, and assume the sequences* $\{c^{\varepsilon}\}$ *and* $\{\nabla c^{\varepsilon}\}$ *be uniformly bounded in* $\mathrm{L}^2(\Omega_T)$, *and the sequence* $\{\chi^{\varepsilon}\frac{\partial c^{\varepsilon}}{\partial t}\}$ *be uniformly bounded in* $\mathrm{L}^2\big(0, T; \mathrm{H}^{-1}(\Omega)\big)$.*Then there exists some subsequence of* $\{c^{\varepsilon}\}$ *that converges strongly in* $\mathrm{L}^2(\Omega_T)$.

In our study, we will use the following extension of the above compactness result for periodic structures with a special periodicity of the space structure, obtained in Theorem 2.2 and Lemma 2.4 of [30].

Theorem 19. *Let the structure function* $\chi(r; \boldsymbol{y})$ *of the pore space be given by formula* (2.25)*, where* $r \in \mathfrak{M}_{(0,T)}$ *with*

$$\mathfrak{M}_{(0,T)} = \{r \in \mathrm{H}^{2+\gamma, \frac{2+\gamma}{2}}(\overline{\Omega}_T),\ 0 \leqslant r(\boldsymbol{x}, t) \leqslant \frac{1}{2},\ -\theta \leqslant \frac{\partial r}{\partial t}(\boldsymbol{x}, t) \leqslant 0,$$
$$0 < \gamma < 1,\ \theta = const > 0;\ |r|^{(2+\gamma)}_{\Omega_T} \leqslant M_0\}. \quad (2.42)$$

Then any uniformly bounded sequence $\{\widetilde{c}^{\varepsilon}\}$*, in the sense that*

$$\|\widetilde{c}^{\varepsilon}\|_{2,\Omega_T} + \|\nabla\widetilde{c}^{\varepsilon}\|_{2,\Omega_T} + \|\frac{\partial}{\partial t}\widetilde{c}^{\varepsilon}\|_{W_2^{-1}} \leqslant M,$$

where M *does not depend on* ε*, contains a strongly convergent in* $\mathrm{L}^2(\Omega_T)$ *subsequence. In particular, for almost* $t_0 \in (0, T)$*, the sequence* $\{\chi^{\varepsilon}(r(\boldsymbol{x}, t_0)\widetilde{c}^{\varepsilon}(\boldsymbol{x}, t_0)\}$ *converges weakly in* $\mathrm{L}^2(\Omega)$ *to* $m(\boldsymbol{x}, t_0)c(\boldsymbol{x}, t_0)$*, with* $m(\boldsymbol{x}, t_0)$ *the porosity given by (2.27).*

2.13 Weak formulation of the partial differential equations in problem $\mathbb{A}^{\varepsilon}$

2.13.1 Weak formulation of the structural free boundary condition (2.11).

The structural free boundary condition (2.11), in the case of a slightly compressible fluid can be analyzed by generalizing the treatment made for the case of an incompressible fluid with a rigid skeleton. In this way, the study made in Lemma 4.2 of [30] remains valid also in our framework.

Lemma 20. *Under conditions*

$$\alpha^{\varepsilon} = \varepsilon\,\theta,\ \ \beta^{\varepsilon} = \varepsilon,$$

where θ *is a given positive constant, the structural free boundary condition* (2.11) *can be weakly formulated by the integral identity*

$$\int_0^{t_0}\int_{\Omega} \chi^{*,\varepsilon}\Big(-\frac{\partial}{\partial t}\big((\zeta\,\boldsymbol{a}^{\varepsilon}) \cdot \boldsymbol{\xi}_0^{\varepsilon})\big) + \varepsilon\,\nabla \cdot (\zeta(\widetilde{c}^{\varepsilon} - c^0)\boldsymbol{\xi}_0^{\varepsilon})\Big)dxdt = 0, \quad (2.43)$$

which is valid for any smooth function $\boldsymbol{\xi}_c^{\varepsilon}(r, \boldsymbol{x}) = \boldsymbol{\xi}_c(r, \frac{\boldsymbol{x}}{\delta})$*, function* ζ*, vanishing at* $t = 0$ *and at* $t = t_0$ *and at boundary* $\partial\Omega$*, and function* $\boldsymbol{a}_c^{\varepsilon}(r, \boldsymbol{x}) = \boldsymbol{a}_c(r, \frac{\boldsymbol{x}}{\delta})$*, such that* $\boldsymbol{a}_c$ *vanishes outside of some small neighborhood of* $\gamma_c(r)$ *and* $\boldsymbol{a}_c(r, \boldsymbol{y}) = \boldsymbol{n}_c(r)$*, where* $\boldsymbol{n}_c(r)$ *is the unit normal to the surface* $\gamma_c(r) = \{\boldsymbol{y} \in \mathbf{Y} : |\boldsymbol{y}| = r\}$*, outward to the domain* $\mathbf{Y}_f(r)$*. Here* $\chi^{*,\varepsilon}$ *denotes the structure of the pore space* $\Omega^{\varepsilon}_{f,T}(r^*)$ *which is supposed be given for a function* $r^* \in \mathfrak{M}_{(0,T)}$.

2.13.2 Weak formulation of the dynamic problem $\mathbb{A}^{\varepsilon}_{dyn}$.

We assume that $p^0(\boldsymbol{x})$ is given bounded function, $p^0 \in \mathrm{C}^1(\overline{\Omega})$ and $p^0(\boldsymbol{x}) = p^j =$*const* for $\boldsymbol{x} \in S^j, j = 1, 2$.

Definition 21. *Let the structure $\chi^{*,\varepsilon}$ of the pore space $\Omega^{\varepsilon}_{f,T}(r^*)$ be given by a function $r^* \in \mathfrak{M}_{(0,T)}$, and let $p^{\varepsilon} = \chi^{*,\varepsilon}(p^{\varepsilon}_f - p^0) + (1-\chi^{*,\varepsilon})(p^{\varepsilon}_s - p^0)$, $\mathbb{P}^{\varepsilon} = \chi^{*,\varepsilon}\mathbb{P}^{\varepsilon}_f + (1-\chi^{*,\varepsilon})\mathbb{P}^{\varepsilon}_s$, $\mathbb{P}^{\varepsilon}_f = \varepsilon^2\mu_1\mathbb{D}(x, \varepsilon\dfrac{\partial \boldsymbol{w}^{\varepsilon}_f}{\partial t}) - (p^{\varepsilon}_f - p^0)\mathbb{I}$, $\mathbb{P}^{\varepsilon}_s = \lambda_0\mathbb{D}(x, \boldsymbol{w}^{\varepsilon}_s) - (p^{\varepsilon}_s - p^0)\mathbb{I}$. We say that functions $\boldsymbol{w}^{\varepsilon}_f \in \mathrm{W}^{1,0}_2\big(\Omega_{f,T}(r^*)\big)$, $\boldsymbol{w}^{\varepsilon}_s \in \mathrm{W}^{1,0}_2\big(\Omega_{s,T}(r^*)\big)$, p^{ε}_f and p^{ε}_f, define a weak solution to the dynamic problem $\mathbb{A}^{\varepsilon}$ if equations (2.2), (2.4) hold true and we have the integral identity*

$$-\int_0^{t_0}\int_{\Omega}(\nabla p^0 \cdot \boldsymbol{\varphi})dxdt =$$
$$\int_0^{t_0}\int_{\Omega}\Big(\chi^{*,\varepsilon}\varepsilon^2\mu_1\mathbb{D}(x, \frac{\partial \boldsymbol{w}^{\varepsilon}_f}{\partial t}) + (1-\chi^{*,\varepsilon})\lambda_0\mathbb{D}(x, \boldsymbol{w}^{\varepsilon}_s) -$$
$$\big(\chi^{*,\varepsilon}(p^{\varepsilon}_f - p^0) + (1-\chi^{*,\varepsilon})(p^{\varepsilon}_s - p^0)\mathbb{I}\big)\Big) : \mathbb{D}(x, \boldsymbol{\varphi})dxdt \quad (2.44)$$

for any arbitrary smooth functions $\boldsymbol{\varphi}$, vanishing at the boundary $\big(S^1 \cup S^2\big) \times (0,T)$ and satisfying the following conditions on the free boundary $\Gamma^{\varepsilon}(r^*)$

$$\boldsymbol{\varphi}(\boldsymbol{x}_0 + 0) = \boldsymbol{\varphi}(\boldsymbol{x}_0 - 0),\ \ \boldsymbol{x}_0 \in \Gamma^{\varepsilon}(r^*)$$
$$\boldsymbol{\varphi}(\boldsymbol{x}_0 + 0) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\boldsymbol{\varphi}(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega^{\varepsilon}_{f,t_0}(r^*),\ \ \boldsymbol{x}_0 \in \Gamma^{\varepsilon}(r^*),$$
$$\boldsymbol{\varphi}(\boldsymbol{x}_0 - 0) = \lim_{\boldsymbol{x}\to\boldsymbol{x}_0}\boldsymbol{\varphi}(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega^{\varepsilon}_{s,t_0}(r^*),\ \ \boldsymbol{x}_0 \in \Gamma^{\varepsilon}(r^*). \quad (2.45)$$

2.13.3 Weak formulation of the diffusion problem $\mathbb{A}^{\varepsilon}_{diff}$.

Definition 22. *Let the structure $\chi^{*,\varepsilon}$ of the pore space $\Omega^{\varepsilon}_{f,T}(r^*)$ be given by a function $r^* \in \mathfrak{M}_{(0,T)}$. We say that function c^{ε} is a weak solution to the diffusion problem $\mathbb{A}^{\varepsilon}_{diff}$, if the following integral identity holds true*

$$\int_{\Omega}\chi^{*,\varepsilon}(.,t_0)\big(c^{\varepsilon}(.,t_0) + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}}\big)\xi\chi^{*,\varepsilon}(.,t_0)dx - \int_{\Omega}\chi^{*,\varepsilon}(.,0)\big(c^0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}}\big)\,\xi\chi^{*,\varepsilon}(.,0)\,dx +$$
$$\int_0^{t_0}\int_{\Omega}\chi^{*,\varepsilon}\big(-(c^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})\,\frac{\partial \xi}{\partial t} + \nabla\xi\cdot(\alpha_c\nabla c^{\varepsilon}\big)dxdt = 0, \quad (2.46)$$

for any arbitrary smooth function ξ, vanishing at the boundary $\big(S^1 \cup S^2\big) \times (0,T)$.

Remark 23. *In deriving the integral identity (2.46), we used the boundary condition (2.11) on the free boundary, so that the term containing the integral over this boundary vanishes.*

2.13.4 Weak formulation of the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$.

By introduce the *antiderivative* of the function $(p^{\varepsilon} - p^0 t)$ by means of the following function

$$\widetilde{\pi}^{\varepsilon}(\boldsymbol{x},t) - p^0 t = \int_0^t \chi^{\varepsilon}(\boldsymbol{x},\tau)\big(\widetilde{p}^{\varepsilon}_f(\boldsymbol{x},\tau) - p^0\big)d\tau,\ \ \frac{\partial}{\partial t}\widetilde{\pi}^{\varepsilon}(\boldsymbol{x},t) = \chi^{\varepsilon}\widetilde{p}^{\varepsilon}_f.$$

Definition 24. *Let the structure χ^{ε} of the pore space $\Omega^{\varepsilon}_{f,T}(r)$ be given by the function $r \in \mathfrak{M}_{(0,T)}$. We say that functions $\widetilde{\boldsymbol{w}}^{\varepsilon}_f \in \mathrm{W}^{1,0}_2\big(\Omega_{f,T}(r)\big)$, $\widetilde{p}^{\varepsilon}$, $\dfrac{\partial \pi^{\varepsilon}}{\partial t} \in \mathrm{L}^2\big(\Omega_T(r)\big)$ define a weak solution to the dynamic* *problem* $\mathbb{B}^{\varepsilon}_{dyn}(r)$ *for the fluid component, if the following conditions hold: the continuity equation (2.2) and the integral identity for the fluid component*

$$0=\int_0^{t_0}\int_\Omega \chi^\varepsilon\Big((\nabla p^0+\varepsilon\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})\cdot\boldsymbol{\varphi}\Big)+\varepsilon^2\mu_1\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})-(\frac{\partial}{\partial t}(\widetilde{\pi}^\varepsilon-p^0t)\mathbb{I}\Big):\mathbb{D}(x,\boldsymbol{\varphi})dxdt=$$
$$-\int_0^{t_0}\int_\Omega\Big(\big(\chi^\varepsilon(\nabla p^0t+\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}\big)+\big(\varepsilon^2\mu_1\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)-$$
$$(\widetilde{\pi}^\varepsilon-p^0t)\mathbb{I}\big):\mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})\Big)dxdt=\mathbb{I}_f^\varepsilon,\quad(2.47)$$

for any test function $\boldsymbol{\varphi}$*, vanishing at the boundary* $S^0\times(0,T)$ *and satisfying condition (2.45) at the boundary* $\Gamma^\varepsilon(r)$.

Definition 25. *Let the structure* χ^ε *of the pore space* $\Omega^\varepsilon_{f,T}(r)$ *be given by the function* $r\in\mathfrak{M}_{(0,T)}$. *We say that functions* $\widetilde{\boldsymbol{w}}_s^\varepsilon\in \mathrm{W}_2^{1,0}\big(\Omega_{f,T}(r)\big)$, $\widetilde{p}_s^\varepsilon\in\mathrm{L}^2\big(\Omega_T(r)\big)$ *define a weak solution to the dynamic problem* $\mathbb{B}^\varepsilon_{dyn}(r)$ *for the solid component, if the following conditions hold: the continuity equation (2.4) and the integral identity*

$$\int_0^{t_0}\int_\Omega\Big(\big((1-\chi^\varepsilon)(\nabla p^0\cdot\boldsymbol{\varphi})+\lambda_0\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^\varepsilon)+c_s^2(\nabla\cdot\widetilde{\boldsymbol{w}}_s^\varepsilon)\mathbb{I}\big):\mathbb{D}(x,\boldsymbol{\varphi})\Big)dxdt=0,\quad(2.48)$$

for the solid component, for any test function $\boldsymbol{\varphi}$*, satisfying conditions (2.45) at the boundary* $\Gamma^\varepsilon(r)$.

Notice that we used the continuity equation (2.4) in the identity (2.48). We recall that according to the boundary conditions (2.12)

$$\mathbb{P}^\varepsilon_f<\boldsymbol{n}>=0,\ \text{ for } \boldsymbol{x}\in S^1\cup S^2.\quad(2.49)$$

2.13.5 Weak formulation of the diffusion problem $\mathbb{B}^\varepsilon_{diff}(r)$.

Definition 26. *Let the structure* χ^ε *of the pore space* $\Omega^\varepsilon_{f,T}(r)$ *be given by the function* $r\in\mathfrak{M}_{(0,T)}$. *We say that function* $\widetilde{c}^\varepsilon$ *is a weak solution to the diffusion problem* $\mathbb{B}^\varepsilon_{diff}(r)$*, if the integral identity*

$$\int_\Omega\chi^\varepsilon(.,t_0)\big(\widetilde{c}^\varepsilon(.,t_0)+\frac{\beta^\varepsilon}{\alpha^\varepsilon}\big)\xi\chi^\varepsilon(.,t_0)dx-$$
$$\int_0^{t_0}\int_\Omega\chi^\varepsilon\big(-(\widetilde{c}^\varepsilon-c^0+\frac{\beta^\varepsilon}{\alpha^\varepsilon})\frac{\partial\xi}{\partial t}+\nabla\xi\cdot(\alpha_c\nabla\widetilde{c}^\varepsilon\big)dxdt=0\quad(2.50)$$

holds true, for any arbitrary smooth function ξ*, vanishing at the boundary* $\big(S^1\cup S^2\big)\times(0,T)$ *and at* $t=0$.

2.14 Formal homogenization of the problem $\mathbb{A}^\varepsilon$.

As in [30], under the conditions of Theorem 14, the formal homogenization $\mathbb{H}$ of the problem $\mathbb{A}^\varepsilon$ consists of, i) the Darcy law of filtration

$$\boldsymbol{w}_f=-\frac{1}{\mu_1}\mathbb{B}^{(w)}(r)<\nabla(\pi-p_0t)>,\ \nabla_x\cdot\boldsymbol{w}_f=0,\quad(2.51)$$

for the fluid displacements $\boldsymbol{w}_f$ and the antiderivative π of the fluid pressure p_f in the domain Ω_T, for some symmetric matrix $\mathbb{B}^{(w)}(r)$, ii) the homogenized Lame system

$$\nabla\cdot\big(\lambda_0\mathfrak{N}_1^{(s)}:\mathbb{D}(x,\boldsymbol{w}_s)+c_s^2(\nabla\cdot\boldsymbol{w}_s)\mathbb{I}\big)=\nabla p_0, in\quad(2.52)$$

$$\boldsymbol{w}_s=0,\text{on the boundary},\quad(2.53)$$

for the solid displacements $\boldsymbol{w}_s$ and solid pressure p_s, for some tensor $\mathfrak{N}_1^{(s)}$, and iii) the homogenized system, describing the diffusion of the acid

$$\frac{\partial}{\partial t}\big(m(r)c\big) = \nabla \cdot (\alpha_c \mathbb{B}^{(c)}(r) < \nabla(c - c^0) > \tag{2.54}$$

in the domain Ω_T. Here, $m(r)$ is the porosity given by (2.27). Moreover, the above differential equations are completed with the boundary and initial conditions

$$\pi(\boldsymbol{x},t) - p^0 t = 0,\ \ \boldsymbol{x} \in S^1 \cup S^2,\ \ 0 < t < T, \tag{2.55}$$

$$\boldsymbol{w}_f \cdot \boldsymbol{n} = 0,\ \ \boldsymbol{x} \in S^0,\ \ 0 < t < T, \tag{2.56}$$

where $\boldsymbol{n}$ is the normal unit vector to the boundary S^0,

$$\boldsymbol{w}_s(\boldsymbol{x},t) = 0,\ \ \boldsymbol{x} \in \partial\Omega,\ \ 0 < t < T \tag{2.57}$$

$$\big(\lambda_0 \mathbb{D}(x,\boldsymbol{w}_s) - (p_s - p_0)\mathbb{I}\big) < \boldsymbol{n} >= 0,\ \ \boldsymbol{x} \in S^1 \cup S^1,\ \ 0 < t < T, \tag{2.58}$$

$$c(\boldsymbol{x},t) = c^0(\boldsymbol{x}),\ \ \boldsymbol{x} \in S^1 \cup S^2,\ \ t > 0, \tag{2.59}$$

$$\frac{\partial c}{\partial n}(\boldsymbol{x},t) = 0,\ \boldsymbol{x} \in S^0,\ \ t > 0, \tag{2.60}$$

$$c(\boldsymbol{x},0) = c^0(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega. \tag{2.61}$$

Remark 27. *In (2.51), the symmetric matrix $\mathbb{B}^{(c)}$ is given by formula (1.1.27) of [29]. The tensor $\mathfrak{N}_1^{(s)}$, in (2.52), is given by formula (1.2.38) of [29]. We also point out that the homogenization process for an ideal compressible fluid ([15]) in a porous medium, leading to the popular "porous medium equation" was proposed in [10] and then rigorously proved, under suitable conditions, in [26]. It would be interesting to study the presence of some global free boundaries, for the homogenized problems, to the light of local energy methods ([3]), for instance, for some compressible flows in transitory regime, or for the case in which there is a chemical reaction in the microscopic free boundary, as in [13], [9] and [12].*

3 Proof of Theorem 1: existence of the weak solution to problem $\mathbb{B}^{\varepsilon}(r)$

It will be a consequence of the following two subsections.

3.1 Existence of the weak solution to the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$.

Thanks to the linearity of the problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$, it is sufficient to derive some a priori estimates. Here, we extend the previous studies made for the cases of an incompressible fluid and a rigid solid skeleton ([30]), or an elastic skeleton ([32]).

Lemma 28. *Under conditions of Theorem 14 the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$ has a unique weak solution such that*

$$\max_{0<t<T} \big(\|\chi^{\varepsilon}(.,t)\big(\widetilde{\boldsymbol{w}}_f^{\varepsilon}(.,t)\|_{2,\Omega} + \|\sqrt{\varepsilon}\frac{\partial \widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t} + \|_{2,\Omega_T} +$$
$$\max_{0<t<T} \varepsilon\|\chi^{\varepsilon}(.,t)\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^{\varepsilon}(.,t)\big)\|_{2,\Omega} + \|\mathbb{D}(x,\varepsilon\frac{\partial \widetilde{\boldsymbol{w}}_f^{\varepsilon}}{\partial t})\|_{2,\Omega_T} \leqslant M, \tag{3.62}$$

$$\max_{0<t<T} \Big(\|\big(1-\chi^{\varepsilon}(.,t)\widetilde{\boldsymbol{w}}_s^{\varepsilon}(.,t)\big)\|_{2,\Omega} + \|(1-\chi^{\varepsilon}(.,t)\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^{\varepsilon}(.,t)\big)\|_{2,\Omega} \leqslant M, \tag{3.63}$$

$$\max_{0<t<T} (\|\chi^{\varepsilon}\nabla \cdot \widetilde{\boldsymbol{w}}_f^{\varepsilon}(.,t_0)\|_{2,\Omega} + \|(1-\chi^{\varepsilon})\nabla \cdot \widetilde{\boldsymbol{w}}_s^{\varepsilon}(.,t_0)\|_{2,\Omega} \leqslant M, \tag{3.64}$$

where M does not depend on ε.

Proof. Let in (2.48) $\boldsymbol{\varphi} = (1-\chi^\varepsilon)\widetilde{\boldsymbol{w}}_s^\varepsilon$. Then, using the continuity equation (2.4), the simplest embedding theorem (Lemma 7) and the usual Holder's inequality, we obtain

$$\lambda_0 \int_0^{t_0}\int_\Omega \big((1-\chi^\varepsilon)|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon\big)|^2 dxdt + \int_0^{t_0}\int_\Omega \big((1-\chi^\varepsilon)\big)c_f^2\varrho_f^0|\nabla\cdot\widetilde{\boldsymbol{w}}_s^\varepsilon|^2\big)dx =$$
$$|\int_0^{t_0}\int_\Omega (1-\chi^\varepsilon)(\nabla p_0\cdot\widetilde{\boldsymbol{w}}_s^\varepsilon)dxdt| \leqslant$$
$$\frac{\delta}{2}\int_\Omega \big((1-\chi^\varepsilon)\big)|\widetilde{\boldsymbol{w}}_s^\varepsilon(.,)|^2 dx + \frac{2}{\delta}\int_\Omega |\nabla p_0|^2 dx \leqslant$$
$$\frac{\delta}{2}\int_\Omega \big((1-\chi^\varepsilon)(.,t_0)\big)|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_s^\varepsilon(.,t_0)|^2 dx + \frac{M}{\delta}.$$

To estimate the fluid displacements we put in (2.47) $\boldsymbol{\varphi} = \chi^\varepsilon\dfrac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}$. Using the continuity equation (2.2), relation (2.29) and the integration by parts formula we arrive at

$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\varepsilon|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2 dxdt + \int_0^{t_0}\int_\Omega \chi^\varepsilon\varepsilon^2\mu_1|\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})|^2 dxdt + \varrho_f^0 c_f^0\int_0^{t_0}\int_\Omega \chi^\varepsilon|\nabla\cdot|\widetilde{\boldsymbol{w}}_f^\varepsilon|^2 dxdt =$$
$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\varepsilon|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2 dxdt + \int_\Omega \chi^\varepsilon(.,t_0)\varepsilon^2\mu_1|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)\big)|^2 dx +$$
$$\varrho_f^0 c_f^0\int_\Omega \chi^\varepsilon(.,t_0)|\nabla\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)|^2 dx = |\int_\Omega \chi^\varepsilon(.,t_0)\big(\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)\cdot\nabla p_0\big)dx| \leqslant$$
$$\frac{\delta}{2}\int_\Omega \chi^\varepsilon(.,t_0)|\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)|^2 dx + \frac{1}{2\delta}\int_\Omega |\nabla p^0|^2 dx.$$

Next we apply Poincaré inequality (2.33)

$$\int_\Omega \chi^\varepsilon(.,t)|\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)|^2 dx \leqslant M_\Omega\mu_1^{-1}\int_\Omega \chi^\varepsilon(.,t)\varepsilon^2\mu_1\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t)\big)|^2 dx,$$

and obtain

$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\varepsilon|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2 dxdt + \int_0^{t_0}\int_\Omega \chi^\varepsilon\varepsilon^2\mu_1|\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})|^2 dxdt +$$
$$\int_\Omega \chi^\varepsilon(.,t_0)\varepsilon^2\mu_1|\mathbb{D}\big(x,\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)\big)|^2 dx + \int_\Omega \chi^\varepsilon|\nabla\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon(.,t_0)|^2 dx \leqslant$$
$$\frac{\delta}{2}M_\Omega\mu_1^{-1}\int_\Omega \chi^\varepsilon\varepsilon^2\mu_1|\mathbb{D}(x,\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t})|^2 dx + \frac{1}{2\delta}\int_\Omega |\nabla p^0|^2 dx.$$

The desired estimates follows from the last inequality for $\delta = M_\Omega^{-1}\mu_1$. Once we have such an a priori estimate, the passing to the limit, giving the existence of a weak solution, is standard. The uniqueness of solutions is also a classical property obtained from the linearity of the problem. ∎

3.2 Existence of the weak solution to the diffusion problem $\mathbb{B}^\varepsilon_{diff}(r)$.

Again, the key stone is the following *a priori* estimate, generalizing the ones obtained in ([30]) and ([32]).

Lemma 29. *Under conditions of Theorem 14 the diffusion problem $\mathbb{B}^\varepsilon_{diff}(r)$ has a unique weak solution $\widetilde{c}^\varepsilon$, such that*

$$\|(\widetilde{c}^\varepsilon - c^0)\|_{2,\Omega_T} + \|\nabla(\widetilde{c}^\varepsilon - c^0)\|_{2,\Omega_T} \leqslant M\|\nabla c^0\|_{2,\Omega}, \tag{3.65}$$

where M does not depend on ε.

Proof. To prove it we only need to obtain *a priori* estimates to the solution of the diffusion problem $\mathbb{B}^{\varepsilon}_{diff}(r)$, written in the weak form (2.50). To do this we repeat the proof of the Lemma 2.1 in §2, chapter III [24] with test function $\xi = \widetilde{c}^{\varepsilon} - c^0$ using the Young inequality $|ab| \leqslant \delta a^2 + \dfrac{b^2}{4\delta}$ for any $\delta > 0$, the Holder inequality and integrating by parts, we obtain the chain of inequalities

$$
\begin{aligned}
0 = \int_{\Omega} \chi^{\varepsilon}(.,t_0)\big(\widetilde{c}^{\varepsilon}(\boldsymbol{x},t_0) - c^0(\boldsymbol{x}) + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}} + c^0(\boldsymbol{x})\big)\big(\widetilde{c}^{\varepsilon}(\boldsymbol{x},t_0) - c^0(\boldsymbol{x})\big)dx - \\
\int_0^{t_0}\int_{\Omega} \chi^{\varepsilon}(.,t_0)(\widetilde{c}^{\varepsilon} - c^0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}} + c^0)\frac{\partial}{\partial t}(\widetilde{c}^{\varepsilon} - c^0 + c^0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})dxdt + \\
\alpha_0 \int_0^{t_0}\int_{\Omega} \chi^{\varepsilon}\big(\nabla(\widetilde{c}^{\varepsilon} - c^0)\cdot\nabla(\widetilde{c}^{\varepsilon} - c^0 + c^0)\big)dxdt = \\
\int_{\Omega} \chi^{\varepsilon}(.,t_0)\Big(\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big)^2 + (\frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}} + c^0)\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big)\Big)dx - \\
-\frac{1}{2}\int_0^{t_0}\int_{\Omega} \chi^{\varepsilon}\frac{\partial}{\partial t}(\widetilde{c}^{\varepsilon}(.,t_0) - c^0 + c^0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})^2 dxdt + \alpha_c \int_0^{t_0}\int_{\Omega^{\varepsilon}_f(r(.,t))} |\nabla(\widetilde{c}^{\varepsilon} - c_0)|^2 dxdt + \\
\alpha_c \int_0^{t_0}\int_{\Omega} \chi^{\varepsilon}\big(\nabla(\widetilde{c}^{\varepsilon} - c^0)\cdot\nabla c^0\big)dxdt = \\
\int_{\Omega} \chi^{\varepsilon}(.,t_0)\Big(\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big)^2 + (\frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}} + c^0)\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big) + \frac{1}{2}(\widetilde{c}_0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})^2)\Big)dx + \\
\frac{1}{2}\int_0^{t_0}\int_{\Gamma^{\varepsilon}(r(.,t))} (\widetilde{c}^{\varepsilon} + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})^2 D^{\varepsilon}_N \sin\psi\, d\sigma dt + \alpha_c \int_0^{t_0}\int_{\Omega^{\varepsilon}_f(r(.,t))} |\nabla(\widetilde{c}^{\varepsilon} - c^0|^2 dxdt + \\
\alpha_c \int_0^{t_0}\int_{\Omega} \big(\nabla(\widetilde{c}^{\varepsilon} - c^0)\cdot\nabla c^0\big)dxdt \geqslant \\
\int_{\Omega} \chi^{\varepsilon}(.,t_0)\Big(\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big)^2 + (\frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big) + \frac{1}{2}(c^0 + \frac{\beta^{\varepsilon}}{\alpha^{\varepsilon}})^2)\Big)dx + \\
\alpha_c \int_0^{t_0}\int_{\Omega} \chi^{\varepsilon}|\nabla(\widetilde{c}^{\varepsilon} - c^0)|^2 dxdt + \alpha_c \int_0^{t_0}\int_{\Omega^{\varepsilon}_f(r(.,t))} \big(\nabla(\widetilde{c}^{\varepsilon} - c^0)\cdot\nabla c^0\big)dxdt \geqslant \\
\int_{\Omega} \chi^{\varepsilon}(.,t_0)\Big(\big(\widetilde{c}^{\varepsilon}(.,t_0) - c^0\big)^2 dxdt + \frac{\alpha_c}{2}\int_0^{t_0}\int_{\Omega^{\varepsilon}_f(r(.,t))} |\nabla(\widetilde{c}^{\varepsilon} - c^0)|^2 dxdt - \\
\frac{\alpha_c}{2}\int_{\Omega} |\nabla c^0|^2 dxdt \qquad (3.66)
\end{aligned}
$$

which proves the *a priori* of the statement of this Lemma. ∎

4 Proof of Theorem 2: homogenization of the problem $\mathbb{B}^{\varepsilon}(r)$.

The homogenization procedure itself is well explained in many publications ([17]-[4], [29], [31]-[12]). For the dynamic problem the reader can follow the proof of Theorem 1 in chapter I, section 1.3 in [29], and for diffusion problem-chapter I0 in [29].

Lemma 30. *Under the conditions of the Lemma 28 there exist functions $\boldsymbol{w}_f$, $\boldsymbol{w}_s$, p, π, c and 1-periodic in the variable $\boldsymbol{y}$ functions $\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)$, $\mathbb{D}\big(y, \boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)\big)$, $\boldsymbol{W}_s(\boldsymbol{y};\boldsymbol{x},t)$, $\mathbb{D}\big(y, \boldsymbol{W}_s(\boldsymbol{y};\boldsymbol{x},t)\big)$, $\Pi(\boldsymbol{y};\boldsymbol{x},t)$ and $C(\boldsymbol{y};\boldsymbol{x},t)$ such that $\boldsymbol{w}_f \in \mathrm{L}^2(\Omega_T)$, $\boldsymbol{w}_s$, π, $c \in \mathrm{W}^{1,0}_2(\Omega_T)$, $\boldsymbol{W}_f$, $C \in \mathrm{L}^2\big(0,T;\mathbb{W}^1_2(\mathbf{Y})\big)$ and $\boldsymbol{W}_s \in \mathrm{L}^2(\mathbf{Y}_s)$.*

1) The sequence $\{\widetilde{\boldsymbol{w}}^{\varepsilon}_f\}$ converges weakly to the function $\boldsymbol{w}_f$ and two-scale to the function $\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)$.

2) The sequences $\{\varepsilon\mathbb{D}(x,\widetilde{\boldsymbol{w}}^{\varepsilon}_f)\}$ and $\{\varepsilon\nabla_x\cdot\widetilde{\boldsymbol{w}}^{\varepsilon}_f)\}$ converge two-scale to the functions $\mathbb{D}(y,\boldsymbol{W}_f)$ and $\nabla_y\cdot\boldsymbol{W}_f$ respectively.

3) The sequences $\{\widetilde{\boldsymbol{w}}_s^\varepsilon\}$, converge two-scale and weakly to the function $\boldsymbol{w}_s \in \mathrm{L}^2(\Omega_T)$.
4) The sequence $\{\mathbb{D}(x, \widetilde{\boldsymbol{w}}_s^\varepsilon)\}$ converges two-scale to the function $\mathbb{D}(x, \boldsymbol{w}_s) + D(y, \boldsymbol{W}_s)$.
5) The sequence $\{\widetilde{c}^\varepsilon\}$ converges weakly and two-scale to the function $c \in \mathrm{W}_2^{1,0}(\Omega_T)$.
6) The sequence $\{\nabla\widetilde{c}^\varepsilon\}$ converges two-scale to the function $\nabla c + \nabla_y C$.
Here $\boldsymbol{W}_s \in \mathrm{L}^2(\Omega_T) \cap \mathrm{W}_2^{1,0}(\mathbf{Y}_s)$, $\boldsymbol{W}_f$, C, $\Pi \in \mathrm{L}^2(\Omega_T) \cap \mathrm{W}_2^{1,0}(\mathbf{Y}_f)$.
7) The following a priori estimates hold true

$$\|\boldsymbol{w}_f\|_{2,\Omega_T} + \|\boldsymbol{w}_s\|^{(1,0)}_{2,\Omega_T} + \|\boldsymbol{W}_f\|_{2,\mathbf{Y}_f\times\Omega_T} + \\ \|\mathbb{D}\big(y, \boldsymbol{W}_s\big)\|_{2,\mathbf{Y}_s\times\Omega_T} + \|\mathbb{D}\big(y, \boldsymbol{W}_f\big)\|_{2,Y_f\times\Omega_T} \leqslant M, \tag{4.67}$$

$$\|(C - c_0)\|^{(1,0)}_{2,Y\times\Omega_T} + \|(c - c_0)\|^{(1,0)}_{2,\Omega_T} \leqslant M, \tag{4.68}$$

$$\|(p_f - p^0)\|_{2,\Omega_T} + \|(p_s - p^0)\|_{2,\Omega_T} + \|(\pi - p^0 t)\|_{2,\Omega_T} + |\frac{\partial\pi}{\partial t}\|_{2,\Omega_T} \leqslant M, \tag{4.69}$$

where M do not depend on ε.

The proof is an easy modification of the ones obtained in ([30]) and ([31]), once we have the *a priori* estimates (3.62)-(3.65). We only point out that

$$\varepsilon\int_0^{t_0}\int_\Omega \chi^\varepsilon|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|^2 dxdt \leqslant M$$

and

$$\lim_{\varepsilon\to 0}\varepsilon\int_0^{t_0}\int_\Omega \chi^\varepsilon|\frac{\partial\widetilde{\boldsymbol{w}}_f^\varepsilon}{\partial t}|dxdt = 0.$$

We recall that

$$\pi(\boldsymbol{x}, t) = \lim_{\varepsilon\to 0}\widetilde{\pi}^\varepsilon(\boldsymbol{x}, t) = \lim_{\varepsilon\to 0}\int_0^t \widetilde{p}_f^\varepsilon(\boldsymbol{x}, \tau)d\tau \tag{4.70}$$

denotes an antiderivative of the pressure p_f.

4.1 Homogenization of the dynamic problem $\mathbb{B}^\varepsilon_{dyn}(r)$ for the fluid component.

As usual in homogenization, we need to introduce some auxiliary problems. To derive the continuity equation for unknown functions $\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)$ (fluid displacements) and $\boldsymbol{\Pi}^{(f)}(\boldsymbol{y};\boldsymbol{x},t)$ (the fluid pressure) we consider integral identity (4.79) with arbitrary test functions $\xi = \varepsilon\eta(\boldsymbol{x},t)\phi(\frac{\boldsymbol{x}}{\varepsilon})$, where $\eta(\boldsymbol{x},t)$ is an arbitrary function, vanishing at $S^1\cup S^2$ and $\phi(\boldsymbol{y})$ is a 1-periodic in $\boldsymbol{y}$ function. Using relations 1) and 2) of the Lemma 30 we obtain:

$$0 = \lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega \eta\phi\chi^\varepsilon\varepsilon\nabla\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon dxdt = \int_0^{t_0}\int_\Omega\eta\int_{Y_f(r)}\big(\phi\nabla_y\cdot\boldsymbol{W}_f\big)dydxdt. \tag{4.71}$$

Due to arbitrary choice of functions η and ϕ, the last relation implies the continuity equation

$$\nabla_y\cdot\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t) = 0,\ \ (\boldsymbol{y};\boldsymbol{x},t)\in Y_f\times\Omega_{t_0}, \tag{4.72}$$

and the corresponding boundary condition, together with the normalization condition

$$\big(\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)\cdot\boldsymbol{N}\big) = 0,\ \ (\boldsymbol{y};\boldsymbol{x},t)\in\gamma(r)\times\Omega_{t_0},\ \int_{Y_f}\boldsymbol{W}_f(\boldsymbol{y};\boldsymbol{x},t)dy = 0. \tag{4.73}$$

Moreover, if we take, in (2.47), $(1-\chi^\varepsilon)\boldsymbol{\varphi}=0$ and $\boldsymbol{\varphi}=0$ at $t=0$ and $t=t_0$ $\dfrac{\partial\boldsymbol{\varphi}}{\partial t}=\eta(\boldsymbol{x},t)\boldsymbol{\psi}(\dfrac{\boldsymbol{x}}{\varepsilon})$, where $\eta\in W_2^{1,1}(\Omega_T)$, $\eta(\boldsymbol{x},t)=0$ for $\boldsymbol{x}\in S^0$, $0<t<T$ and $\boldsymbol{\psi}\in W_2^1(Y_f)$, *supp*$\boldsymbol{\psi}\subset Y_f$, $\nabla_y\cdot\boldsymbol{\psi}=0$, we get

$$\mathbb{D}(x,\eta\boldsymbol{\psi})=\sum_{i,j=1}^{3}d_{ij}\big(x,\eta\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)\boldsymbol{e}^i\otimes\boldsymbol{e}^j,\;\; d_{ij}\big(x,\eta\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)=\frac{1}{2}\big(\frac{\partial}{\partial x_i}(\eta\psi_j(\frac{\boldsymbol{x}}{\varepsilon}))+\frac{\partial}{\partial x_j}(\eta\psi_i(\frac{\boldsymbol{x}}{\varepsilon}))=$$

$$\frac{1}{2\varepsilon}\eta\big(\frac{\partial\psi_j}{\partial y_i}(\frac{\boldsymbol{x}}{\varepsilon})+\frac{\partial\psi_i}{\partial y_j}(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{1}{2}\big(\frac{\partial\eta}{\partial x_i}\psi_j(\frac{\boldsymbol{x}}{\varepsilon})+\frac{\partial\eta}{\partial x_j}\psi_i(\frac{\boldsymbol{x}}{\varepsilon})\big),$$

$$\varepsilon^2\mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})=\eta\varepsilon\mathbb{D}\big(y,\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)+\frac{\varepsilon^2}{2}(\nabla\eta\otimes\boldsymbol{\psi}+\boldsymbol{\psi}\otimes\nabla\eta),\;\;\nabla\cdot(\eta\boldsymbol{\psi})=(\nabla\eta\cdot\boldsymbol{\psi}).$$

Next we consider the functions

$$A_f(\boldsymbol{x},t)=\int_{Y_f}\big(\nabla(p^0t))\cdot\boldsymbol{\psi}\big)-\nabla_y\cdot\big(\mu_1\mathbb{D}(y,\boldsymbol{W}_f)-\nabla\Pi^{(f)}\mathbb{I}\big)dy$$

$$\boldsymbol{B}_f(\boldsymbol{x},t)=\big(\pi(\boldsymbol{x},t)-p^0t)\big)\int_{Y_f}\boldsymbol{\psi}dy \quad (4.74)$$

and the integral identity

$$I_f^\varepsilon(\eta\boldsymbol{\psi})=\int_0^{t_0}\int_\Omega\chi^\varepsilon\Big(\big((\nabla p^0t+\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}\big)+\big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)-(\widetilde{\pi}^\varepsilon-p^0t)\mathbb{I}\big):\mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})\Big)dxdt.$$

Lemma 31. *Under the conditions of the Theorem 14 the limiting procedure in the equations (2.1) and (2.2) and the integral identity (2.47) results in the following dynamic problem* $\mathbb{H}(r)$ *for displacements and pressure of the liquid component consisting of Darcy law of filtration*

$$\boldsymbol{w}_f=-\frac{1}{\mu_1}\mathbb{B}^{(w)}(r)<\nabla(\pi-p_0t)>,\;(\pi-p^0t)+c_f^2\varrho_f^0(\nabla_x\cdot\boldsymbol{w}_f)=0, \quad (4.75)$$

for the liquid displacements $\boldsymbol{w}_f$ *and the antiderivative* π *of the pressure* p *in the domain* Ω_T, *completed with the boundary conditions*

$$\pi(\boldsymbol{x},t)-p^0t=0,\;\;\boldsymbol{x}\in S^1\cup S^2,\;\;0<t<T, \quad (4.76)$$

$$\boldsymbol{w}_f\cdot\boldsymbol{n}=0,\;\;\boldsymbol{x}\in S^0,\;\;0<t<T, \quad (4.77)$$

where $\boldsymbol{n}$ *is a normal vector to the boundary* S^0. *Moreover, the symmetric strictly positive definite matrix* $\mathbb{B}^{(w)}(r)$ *is defined by formula (4.87), i.e.,*

$$\mathbb{B}^{(w)}(r)=\frac{1}{2\mu_1}\sum_{i=1,j}^{3}\int_{Y_f}(\boldsymbol{W}_f^{(i)}\otimes\boldsymbol{e}^j+\boldsymbol{e}^i\otimes\boldsymbol{W}_f^{(j)})dy. \quad (4.78)$$

Proof. First, we derive the continuity equations for functions p_f, $\boldsymbol{w}_f$, and we will end our study of $\boldsymbol{W}_f$. To do that we consider the integral identity

$$\int_0^{t_0}\int_\Omega\chi^\varepsilon\big(\eta(\widetilde{\pi}^\varepsilon-p^0t)+c_f^2\varrho_f^0\nabla\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon\big)dxdt=0, \quad (4.79)$$

which is a result of the multiplication of the equation (2.2) by an arbitrary function η, vanishing at $S^1\cup S^2$, integration by parts and passage to the limit, as $\varepsilon\to 0$.

One has the chain of equalities

$$
\begin{aligned}
0=\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega \eta\chi^\varepsilon(\widetilde{\pi}^\varepsilon-p^0t)+c_f^2\varrho_f^0(\nabla\cdot\widetilde{\boldsymbol{w}}_f^\varepsilon)dxdt=\\
\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega \chi^\varepsilon\big(\eta(\widetilde{\pi}^\varepsilon-p^0t)-c_f^2\varrho_f^0)-(\widetilde{\boldsymbol{w}}_f^\varepsilon\cdot\nabla\eta)\big)dxdt=\\
\int_0^{t_0}\int_\Omega \big(\eta(\pi-p^0t)-c_f^2\varrho_f^0(\boldsymbol{w}_f\cdot\nabla\eta)\big)dxdt=\\
\int_0^{t_0}\int_\Omega \eta\big((\pi-p^0t)+c_f^2\varrho_f^0\nabla\cdot\boldsymbol{w}_f\big)dxdt-\int_0^{t_0}\int_\Omega\nabla\cdot(\boldsymbol{w}_f\eta)dxdt=\\
\int_0^{t_0}\int_\Omega \eta\big((\pi-p^0t)+c_f^2\varrho_f^0\nabla\cdot\boldsymbol{w}_f\big)dxdt-\int_0^{t_0}\int_{S^1\cup S^2}\eta(\boldsymbol{w}_f\cdot\boldsymbol{e}^2)d\sigma dt=0,
\end{aligned}
$$

which implies the identity

$$
\int_0^{t_0}\int_\Omega \eta\big((\pi-p^0t)+c_f^2\varrho_f^0(\nabla\cdot\boldsymbol{w}_f)\big)dxdt=0,
$$

where $\boldsymbol{n}$ is the unit normal to the boundary $S^1\cup S^1$ and

$$
\int_0^{t_0}\int_{S^0}\eta\big(\boldsymbol{w}_f\cdot\boldsymbol{n}\big)d\sigma dt=0.
$$

Last identity obviously proves the continuity equation in (4.75) and the boundary condition (4.77). To end our study of function $\boldsymbol{W}_f$, in accordance with Lemma 30, we get

$$
\begin{aligned}
0=I_f^0(\eta\boldsymbol{\psi})=\lim_{\varepsilon\to 0}I_f^\varepsilon(\eta\boldsymbol{\psi})=\\
-\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega \chi^\varepsilon\Big(\big((\nabla p^0t+\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon)\cdot\frac{\partial\boldsymbol{\varphi}}{\partial t}\big)+\big(\mu_1\varepsilon^2\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon)-(\widetilde{\pi}^\varepsilon-p^0t)\mathbb{I}\big):\mathbb{D}(x,\frac{\partial\boldsymbol{\varphi}}{\partial t})\Big)dxdt=\\
-\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega \eta\chi^\varepsilon\Big(\big(\nabla(p^0t)+\varepsilon\widetilde{\boldsymbol{w}}_f^\varepsilon)\cdot\boldsymbol{\psi}\big)+\mu_1\varepsilon\mathbb{D}(x,\widetilde{\boldsymbol{w}}_f^\varepsilon):\mathbb{D}\big(y,\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)-(\widetilde{\pi}^\varepsilon-p^0t)\big(\nabla\eta\cdot\boldsymbol{\psi}(\frac{\boldsymbol{x}}{\varepsilon})\big)\Big)dxdt=\\
-\int_0^{t_0}\int_\Omega\Big(\eta\big(\int_{Y_f}\big(\nabla(p^0t)\big)\cdot\boldsymbol{\psi}\big)+\big(\mu_1\mathbb{D}(y,\boldsymbol{W}_f):\mathbb{D}(y,\boldsymbol{\psi})dy\big)+\big(\int_{Y_f}(\boldsymbol{\psi}dy)(\pi-p^0t)\cdot\nabla\eta\Big)dxdt=\\
-\int_0^{t_0}\int_\Omega\Big(\eta\big(\int_{Y_f}\big(\nabla(p^0t)\big)\cdot\boldsymbol{\psi}\big)-\nabla_y\cdot\big(\mu_1\mathbb{D}(y,\boldsymbol{W}_f)-\Pi^{(f)}\mathbb{I}\big)dy+(\pi-p^0t)\int_{Y_f}\boldsymbol{\psi}dy\cdot\nabla\eta\Big)dxdt=\\
\int_0^{t_0}\int_\Omega\big((\boldsymbol{B}_f\cdot\nabla\eta)-A_f\eta\big)dxdt=0.
\end{aligned}
\tag{4.80}
$$

The last identity in (4.80)

$$
\int_0^{t_0}\int_\Omega\big((\boldsymbol{B}_f\cdot\nabla\eta)-A_f\eta\big)dxdt=0 \tag{4.81}
$$

means that function $\pi\in\mathrm{W}_2^{1,0}(\Omega_T)$ and identity (4.74) takes the form of the differential equation

$$
\begin{aligned}
\nabla_y\cdot\big(\mu_1\mathbb{D}(y,\boldsymbol{W}_f)-\nabla_y\Pi^{(f)}\mathbb{I}\big)\big)=-\nabla_x(\pi-p^0t)(\boldsymbol{x},t)=\\
-\sum_{i=1}^3\frac{\partial}{\partial x_i}(\pi-p^0t)(\boldsymbol{x},t)\boldsymbol{e}^i,
\end{aligned}
\tag{4.82}
$$

completed with the continuity equation (4.72), the boundary condition (4.73) and boundary condition (4.76)

$$\pi(\boldsymbol{x},t) - p^0 t = 0, \ \ (\boldsymbol{x},t) \in S^1 \cup S^2, \tag{4.83}$$

which is a consequence of the identity (4.81). To solve the periodic boundary value problem (4.72), (4.73), (4.82), we use the decomposition

$$\boldsymbol{W}_f\big(\boldsymbol{y};\boldsymbol{x},t\big) = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\pi - p^0 t)(\boldsymbol{x},t)\boldsymbol{W}_f^{(i)}(\boldsymbol{y}) =$$

$$-\frac{1}{2}\sum_{i,j=1}^{3}(\boldsymbol{W}_f^{(i)} \otimes \boldsymbol{e}^i + \boldsymbol{W}_f^{(j)} \otimes \boldsymbol{e}^j) < \nabla(\pi - p^0 t) >, \tag{4.84}$$

where

$$\left.\begin{array}{l} \nabla_y \cdot \big(\mu_1 \mathbb{D}(y, \boldsymbol{W}_f^{(i)})\big)\big) = \boldsymbol{e}^i, \\ \nabla_y \cdot \boldsymbol{W}_f^{(i)} = 0, \ \ (\boldsymbol{y};(\boldsymbol{x},t) \in Y_f \times \Omega_T, \\ \int_{Y_f} \boldsymbol{W}_f^{(i)} dy = 0, \ \ (\boldsymbol{W}_f^{(i)} \cdot \boldsymbol{N}) = 0, \ \ \boldsymbol{y} \in \gamma(r), \ \ i=1,2,3. \end{array}\right\} \tag{4.85}$$

The proof of the existence and uniqueness of the solutions results for the problem (4.85) is standard and follows from the energy estimates

$$\int_{Y_f} (|\boldsymbol{W}_f^{(i)}|^2 + |\mathbb{D}(y, \boldsymbol{W}_f^{(i)})|^2) dy \leqslant M, \ \ i=1,2,3, \tag{4.86}$$

which are the result of multiplying equation in (4.85) by $\boldsymbol{W}_f^{(i)}$ summing over i from 1 to 3 integrating by parts, and the application of the Poincaré-Wirtinger inequality (see Remark 2). Next, we define the matrix $\mathbb{B}^{(w)}(r)$ as

$$\mathbb{B}^{(w)}(r) = \frac{1}{2\mu_1}\sum_{i=1,j}^{3}\int_{Y_f}(\boldsymbol{W}_f^{(i)} \otimes \boldsymbol{e}^j + \boldsymbol{e}^i \otimes \boldsymbol{W}_f^{(j)}) dy. \tag{4.87}$$

Then, taking into account (4.84), we obtain

$$\boldsymbol{W}_f = -\frac{1}{2\mu_1}\sum_{i,j=1}^{3}\int_{Y_f}(\boldsymbol{W}_f^{(i)} \otimes \boldsymbol{e}^j + \boldsymbol{e}^j \otimes \boldsymbol{W}_f^{(i)} dy) < \nabla_x(\pi - p^0 t) >=$$

$$-\frac{1}{\mu_1}\mathbb{B}^{(w)}(r) < \nabla_x(\pi - p^0 t) > . \tag{4.88}$$

The matrix $\mathbb{B}^{(w)}(r)$ is obviously symmetric and strictly positively defined. ∎

4.2 Homogenization of the dynamic problem $\mathbb{B}^{\varepsilon}_{dyn}(r)$ for the elastic component.

Again, we will extend some related previous results obtained in ([30]) and ([32]).

Lemma 32. *Under the conditions of the Theorem 14 the limiting procedure in the integral identity (2.48) results the following dynamic problem $\mathbb{H}(r)$ for displacements and pressure, consisting of the homogenized Lame system*

$$\nabla \cdot \big(\lambda_0 \mathfrak{N}_2^{(s)} : \mathbb{D}(x, \boldsymbol{w}_s) - (p_s - p_0)\mathbb{I}\big) = \nabla p_0, \tag{4.89}$$

$$\boldsymbol{w}_s(\boldsymbol{x},t) = 0, \ \ \boldsymbol{x} \in \partial\Omega, \ \ 0 < t < T, \tag{4.90}$$

$$\big(\lambda_0\mathfrak{N}_2^{(s)}:\mathbb{D}(x,\boldsymbol{w}_s)-(p_s-p_0)\mathbb{I}\big)<\boldsymbol{n}>=0,\ \ \boldsymbol{x}\in S^1\cup S^1,\ \ 0<t<T, \tag{4.91}$$

In (4.91) $\boldsymbol{n}$ is the unit normal vector to $S=\partial\Omega$, the symmetric strictly positively definite tensor $\mathfrak{N}_2^{(s)}$ is given by formula (1.2.38) of [29].

Proof. The continuity equation (4.90) and the corresponding boundary condition

$$\nabla\cdot\boldsymbol{W}_s=0,\ \boldsymbol{y}\in Y_s(r),\ (\boldsymbol{W}_s\cdot\boldsymbol{N})=0,\ \boldsymbol{y}\in\gamma(r), \tag{4.92}$$

are derived in the same way as the continuity equation (4.72) and boundary condition (4.73) for the fluid component.
To derive the homogenized Lame equation, we consider the notion of weak solution for the elastic component with a test function $\boldsymbol{\varphi}=\boldsymbol{\varphi}(\boldsymbol{x},t)$. The limit, as $\varepsilon\to 0$, according to Lemma 30, gives us

$$0=\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega\Big((1-\chi^\varepsilon)(\nabla p^0\cdot\boldsymbol{\varphi})+\big(\lambda_0\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^\varepsilon)-(\widetilde{p}_s^\varepsilon-p^0):\mathbb{D}(x,\boldsymbol{\varphi})\big)\Big)dxdt=$$
$$\int_0^{t_0}\int_\Omega\int_{Y_s}\Big(\nabla p^0\cdot\boldsymbol{\varphi})+\lambda_0\mathbb{D}(x,\boldsymbol{w}_s)+\lambda_0\mathbb{D}(y,\boldsymbol{W}_s)dy-(p_s-p^0)\mathbb{I}\big):\mathbb{D}(x,\boldsymbol{\varphi})\big)\Big)dxdt.$$

After the reintegration of the last identity, we obtain

$$\nabla\cdot\lambda_0\big(\mathbb{D}(x,\boldsymbol{w}_s)+\lambda_0\int_{Y_s}\mathbb{D}(y,\boldsymbol{W}_s)dy-(p_s-p^0)\mathbb{I}\big)=\nabla p^0. \tag{4.93}$$

To calculate the integral $\displaystyle\int_{Y_s}\mathbb{D}(y,\boldsymbol{W}_s)dy$, we consider the notion of a weak solution for arbitrary test functions $\boldsymbol{\varphi}=\varepsilon\eta(\boldsymbol{x},t)\boldsymbol{\phi}(\dfrac{\boldsymbol{x}}{\varepsilon})$, such that

$$\nabla_y\cdot\boldsymbol{\phi}=0,\ \ \varepsilon\mathbb{D}(y,\eta\boldsymbol{\phi})=\eta\mathbb{D}(y,\boldsymbol{\phi})+\frac{\varepsilon}{2}(\nabla\eta\otimes\boldsymbol{\phi}+\boldsymbol{\phi}\otimes\nabla\eta)$$

and pass to the limit as $\varepsilon\to 0$ (for details, see the proof of Lemma 31):

$$0=\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega\varepsilon\Big((1-\chi^\varepsilon)(\nabla p^0\cdot\boldsymbol{\varphi})+\big(\lambda_0\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^\varepsilon)-(\widetilde{p}_s^\varepsilon-p^0)\mathbb{I}\big):\mathbb{D}(x,\boldsymbol{\varphi})\big)\Big)dxdt=$$
$$\lim_{\varepsilon\to 0}\int_0^{t_0}\int_\Omega\Big(\varepsilon(1-\chi^\varepsilon)\big(\nabla p^0\cdot(\eta\boldsymbol{\phi})\big)+\eta\big(\lambda_0\mathbb{D}(x,\widetilde{\boldsymbol{w}}_s^\varepsilon)-(\widetilde{p}_s^\varepsilon-p^0)\mathbb{I}\big):\mathbb{D}(x,\boldsymbol{\phi})\big)\Big)dxdt=$$
$$\int_0^{t_0}\int_\Omega\eta\int_{Y_s}\boldsymbol{\phi}\Big(\nabla\cdot\big(\lambda_0\mathbb{D}(y,\boldsymbol{W}_s)-\Pi_s\mathbb{I}\big)+\lambda_0\mathbb{D}(x,\boldsymbol{w}_s-(p_s-p^0)\mathbb{I}\Big)dxdtdy=0.$$

The integration of the last identity leads to

$$\nabla_y\Big(\lambda_0\int_{Y_s}\big(\mathbb{D}(y,\boldsymbol{W}_s)dy\big)dy+\lambda_0\mathbb{D}(x,\boldsymbol{w}_s-(p_s-p^0)\mathbb{I}\Big)=0. \tag{4.94}$$

The differential equation (4.94) completed with the continuity equation (4.92) and the boundary and normalization conditions imply that

$$\left.\begin{array}{l}\lambda_0\triangle\boldsymbol{W}_s^{(i)}=\boldsymbol{e}_i,\\ \nabla_y\cdot\boldsymbol{W}_s^{(i)}=0,\ \ (\boldsymbol{y};(\boldsymbol{x},t)\in Y_s\times\Omega_T,\\ \int_{Y_s}\boldsymbol{W}_s^{(i)}dy=0,\ \ (\boldsymbol{W}_s^{(i)}\cdot\boldsymbol{N})=0,\ \ \boldsymbol{y}\in\gamma(r),\ \ i=1,2,3,\end{array}\right\} \tag{4.95}$$

where

$$\boldsymbol{W}_s(\boldsymbol{y};\boldsymbol{x},t)=\sum_{i=1}^3\big(\mathbb{D}(x,\boldsymbol{w}_s)-(p_s-p^0)\mathbb{I}\big)<\boldsymbol{W}_s^{(i)}>.$$

The well-possedness of the problem (4.95) is proven in the same way as for problem (4.85).
By assumption,

$$\int_{Y_s} \mathbb{D}(y, \boldsymbol{W}_s)dy = \frac{1}{2}\sum_{i,j=1}^{3}\int_{Y_s} \big(d_{ij}(\boldsymbol{W}_s^{(i)})\boldsymbol{e}_i \otimes \boldsymbol{e}_j\big) + \big(d_{ji}(\boldsymbol{W}_s^{(j)})\boldsymbol{e}_j \otimes \boldsymbol{e}_i\big)dy = 0.$$

Thus,

$$\int_{Y_s} \mathbb{D}(y, \boldsymbol{W}_s)dy = 0. \tag{4.96}$$

Finally, we obtain the desired homogenization equation for the elastic component

$$\nabla \cdot \big(\lambda_0 \mathbb{D}(x, \boldsymbol{w}_s) - (p_s - p_0)\mathbb{I}\big) = \nabla p_0. \tag{4.97}$$

∎

4.3 Homogenization of the diffusion problem $\mathbb{B}^{\varepsilon}_{diff}(r)$.

We extend now some related previous results obtained in ([30]) and ([32]) for the diffusion problem.

Lemma 33. *Under the conditions of Theorem 14 the limiting procedure in the integral identity (2.50) results the following homogenized diffusion problem $\mathbb{H}_{diff}(r)$ for the concentration of the acid, consisting of the partial differential equation*

$$\frac{\partial}{\partial t}\big(m(r)c\big) = \nabla \cdot (\alpha_c \mathbb{B}^{(c)}(r) < \nabla(c - c^0) > \tag{4.98}$$

in the domain Ω_T, and the boundary and initial conditions

$$c(\boldsymbol{x}, t) = c^0(\boldsymbol{x}),\ \ \boldsymbol{x} \in S^1 \cup S^2,\ \ t > 0, \tag{4.99}$$

$$\frac{\partial c}{\partial n}(\boldsymbol{x}, t) = 0,\ \ \boldsymbol{x} \in S^0,\ \ t > 0, \tag{4.100}$$

$$c(\boldsymbol{x}, 0) = c^0(\boldsymbol{x}),\ \ \boldsymbol{x} \in \Omega. \tag{4.101}$$

As before, the symmetric strictly positively definite matrix $\mathbb{B}^{(c)}(r)$ is given by formula

$$\mathbb{B}^{(c)}(r) = \frac{1}{2}(\sum_{i=1}^{i=3}(\nabla_y C^i \otimes \boldsymbol{e}^j + \nabla_y C^j \otimes \boldsymbol{e}^i). \tag{4.102}$$

Proof. By definition

$$\int_0^{t_0}\int_\Omega \chi^\varepsilon\big(- (\widetilde{c}^\varepsilon + \frac{\beta^\varepsilon}{\alpha^\varepsilon})\frac{\partial \xi}{\partial t} + \nabla \xi \cdot (\alpha_c \nabla \widetilde{c}^\varepsilon\big)dxdt = 0 \tag{4.103}$$

for any arbitrary smooth functions ξ, vanishing at the boundary $\big(S^1 \cup S^2\big)$, at $t = 0$ and at $t = t_0$.
In accordance with Lemma 30, we get

1) the sequence $\{\widetilde{c}^\varepsilon\}$ converges weakly and two-scale to the function $c \in \mathrm{W}_2^{1,0}(\Omega_T)$;

2) the sequence $\{\nabla\widetilde{c}^\varepsilon\}$ converges two-scale to the function $\nabla c + \nabla_y C$.

Next, taking into account Lemma 30 we pass to the limit as $\varepsilon \to 0$ and obtain

$$\int_\Omega\int_0^{t_0} \big((-c + \frac{1}{\theta})\frac{\partial \xi}{\partial t} + \nabla \xi \cdot (\nabla c + \int_{Y_f} \nabla_y C dy)\big)dxdt = 0. \tag{4.104}$$

To calculate the integral $\int_{Y_f} \nabla_y C dy$, we will again consider the first integral identity of this proof, with test functions $\xi(\boldsymbol{y};\boldsymbol{x},t) = \varepsilon\eta(\boldsymbol{x},t)\phi(\frac{\boldsymbol{x}}{\varepsilon})$, take the limit as $\varepsilon \to 0$ and get the identity

$$\int_\Omega \int_0^{t_0} \big(\eta(\boldsymbol{x},t) \int_{Y_f} \nabla_y \phi(\boldsymbol{y}) \cdot (\nabla_x c + \nabla_y C dy)\big) dx dt = 0,$$

which leads to the differential equation

$$\nabla_y \cdot (\nabla_x c + \nabla_y C) = 0, \;\; \boldsymbol{y} \in Y_f, \tag{4.105}$$

and the boundary condition

$$\big((\nabla_x c + \nabla_y C) \cdot \boldsymbol{n}\big) = 0, \;\; \boldsymbol{y} \in \partial Y_f, \tag{4.106}$$

where $\boldsymbol{n}$ is the unit normal vector to the boundary ∂Y_f. To solve the last equation, we use the decomposition

$$C(\boldsymbol{y};\boldsymbol{x},t) = \sum_{i=1}^{3} C^i(\boldsymbol{y}) \frac{\partial c}{\partial x_i}(\boldsymbol{x},t), \tag{4.107}$$

in a similar way as in the proof of Lemma 32.

As a consequence of the maximum principle and the regularity theory for linear diffusion equations (see Theorem 10.1, Chapter IV of [24], and the technique of proof used in [6]), we have:

Corollary 34. *Let $c_0 \in \mathrm{H}^{2+\alpha}(\overline{\Omega})$ and $p^0 \in \mathrm{H}^{1+\alpha}(\overline{\Omega})$. Then the homogenized problem* $\mathrm{H}(r)$ *has a unique classical solution. In particular $c \in \mathrm{H}^{2+\alpha,\frac{2+\alpha}{2}}(\overline{\Omega}_T)$.*

4.4 Homogenization of the structural free boundary condition (2.11).

This time, we can use Lemma 4.2 of [30] since no important modification is needed to prove the following result:

Lemma 35. *Let $r \in \mathfrak{M}_{(0,T)}$ and*

$$\alpha^\varepsilon = \varepsilon\theta, \;\; \beta^\varepsilon = \varepsilon, \tag{4.108}$$

where θ is a given positive constant. Then the velocity of the homogenized free boundary $d_{\mathbf{n}}(\boldsymbol{x},t)$, with respect to its unit normal vector $\mathbf{n}$, is given by the homogenization of the boundary condition (2.11), and it satisfies

$$d_n(\boldsymbol{x},t) := \frac{\partial r}{\partial t}(\boldsymbol{x},t) = \theta\, c(\boldsymbol{x},t), \tag{4.109}$$

with $r(\boldsymbol{x},0) = r_0(\boldsymbol{x})$.

5 Proof of Theorem 3: the existence of a fixed point for the operator $\mathbb{F}(r)$

As mentioned in Section 2, given $T > 0$, and the structure of pore space, we define the operator

$$\mathbb{F} : \mathfrak{M}_{(0,T)} \to \mathfrak{M}_{(0,T)}$$

by the expression

$$\mathbb{F}(r)(\mathbf{x},t) = r_0(\mathbf{x}) - \theta \int_0^t c(\mathbf{x},\tau)d\tau,$$

for $r \in \mathfrak{M}_{(0,T)}$. This function $\mathbb{F}(r)$ determines (by (2.25) a new structure of the pore space. An easy modification of Lemma 4.3 of [30] allows to see that $\mathbb{F}$ is well defined (in the sense that $\mathbb{F}(\mathfrak{M}_{(0,T)}) \subset \mathfrak{M}_{(0,T)}$).
From Lemma 35 and Corollary 34 we get, from estimate (4.109), that if $T_1 \in (0,T]$ then

$$|\mathbb{F}(r_1) - \mathbb{F}(r_2)|_{\Omega_T}^{(2+\gamma,\frac{2+\gamma}{2})} \leq T_1 M_c \, |r_1 - r_2|_{\Omega_T}^{(2+\gamma,\frac{2+\gamma}{2})} \tag{5.110}$$

where $M_c > 0$ is given by

$$|c|_{\Omega_T}^{(2+\gamma,\frac{2+\gamma}{2})} \leq M_c.$$

Then, if

$$T_1 < \min\left\{\frac{M_c}{2}, M_{c,\infty}\right\},$$

with

$$0 \leq c(\mathbf{x},t) \leq M_{c,\infty},$$

from (5.110) we deduce that $\mathbb{F} = \mathbb{F}(r)$ is Lipschitz continuous. Then, from the well-known Banach Theorem, we get the existence and uniqueness of a fixed point element $r^* \in \mathfrak{M}_{(0,T)}$ and the conclusion of Theorem 3 holds on the time interval $[0,T_1]$. If we repeat the analysis but now replacing Ω_T by $\Omega_{(T_1,T)}$ and $\mathfrak{M}_{(0,T)}$ by

$$\mathfrak{M}_{(T_1,T)} = \{r \in \mathrm{H}^{2+\gamma,\frac{2+\gamma}{2}}(\overline{\Omega}_{(T_1,T)}),\ 0 \leqslant r_1(\boldsymbol{x},t) \leqslant \frac{1}{2},\ -\theta \leqslant \frac{\partial r_1}{\partial t}(\boldsymbol{x},t) \leqslant 0,$$
$$0 < \gamma < 1,\ \theta = \text{const} > 0;\ |r_1|_{\Omega_{(T_1,T)}}^{(2+\gamma,\frac{2+\gamma}{2})} \leqslant M_0\}, \tag{5.111}$$

with $r_1(\boldsymbol{x},t) := \max\{0, r^*(\boldsymbol{x},T_1) - r(\boldsymbol{x},t)\}$, if $t \in [T_1,T]$, we obtain, again, a fixed point element $r_1^* \in \mathfrak{M}_{(T_1,T)}$. We iterate this process and it ends only in a time $T^* > 0$ if the last fixed point $r^*(\boldsymbol{x},T^*)$ vanishes (this represents the case in which the fluid fills the spatial domain Ω). This completes the proof of Theorem 3.

Remark 36. *We will end this paper by pointing out an open problem (in the spirit of the suggestions made in Remark 27). The pore structure function $r^*(\mathbf{x},t)$ is solution of the non-local double obstacle problem*

$$\begin{cases} \mathbb{F}(r^*) = r^* \\ 0 \leq r^*(\mathbf{x},t) \leq \frac{1}{2}. \end{cases}$$

For any fixed $t \in (0,T]$ the spatial "extinction set at time t"

$$\Omega^{ext}(t) := \{\mathbf{x} \in \Omega \text{ such that } 0 = r^*(\mathbf{x},t)\},$$

has an important meaning in mining applications. Location estimates (in terms of the initial and boundary conditions and constitutive parameters), the regularity of its boundary (a global free boundary), and its geometric properties are unknown at the time of writing this paper. Intuitively, the solid-phase rare earths (the resource contained in the ore) are expected to disappear first near the injection wells, in contrast to the production wells. The depletion of rare earths occurs along a reaction front that propagates from the injectors toward the producers. Behind this front (near the injectors), the solid-phase rare earths have already been leached and therefore disappear at early times. Ahead of the front (toward the producers), rare earths remain untouched until a breakthrough. This behavior is characteristic of advection-dominated reactive transport with a moving leaching front. The open problem consists of finding a rigorous mathematical proof of these intuitive observations

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