Non-negative and positive solutions for some indefinite sublinear elliptic problems

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Abstract

Abstract We study the existence of non-negative and positive solutions to the indefinite sublinear elliptic problem $-\Delta u = \lambda u + m(x)\,|u|^{\alpha-1}\,u$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < \alpha < 1$, m is a bounded changing sign weight and λ is a real parameter. Existence of non-negative solutions was considered by Brown. When $\lambda = 0$ existence of positive solutions was studied by Hernández-Mancebo-Vega and Godoy-Kaufmann. We extend and improve these results, obtaining linear stability results as well. We use variational methods (Nehari manifold) for the existence of non-negative solutions and bifurcation at infinity for the existence of positive solutions. In addition, we prove the existence of solutions with compact support under suitable additional assumptions: mainly by assuming that $m(x) < 0$ in a large region near the boundary $\partial\Omega$. We apply the method of local super and subsolutions to obtain suitable barrier functions, which now have some constraint on the radius of the ball $B_R(x_0)$ and on the maximum height when $\lambda \geq \lambda_1$, the first eigenvalue for the Laplacian operator on Ω with zero Dirichlet boundary conditions. We also construct some global super and subsolutions with compact support for the case in which Ω is a ball. Finally, we analyze some applications of the Pohozaev identity to determine the non-existence of such solutions.

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1 Introduction

In this paper we study the existence and qualitative properties of nonnegative solutions of the problem

$$\begin{cases} -\Delta u = \lambda u + m(x)\,|u|^{\alpha-1}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < \alpha < 1$, $m \in L^\infty(\Omega)$ changes sign on Ω and λ is a real parameter. Problems of this type arise in the study of problems with nonlinear diffusions in mathematical biology and porous media. See, e.g., [37], [5], [16] and their references, in which many different choices of the local coefficient $m(x)$ can be found.

If $1 < \alpha < \frac{N+2}{N-2}$ and $\lambda > 0$, this problem was studied using the super-sub solutions and constrained minimization method by Ouyang [53, 54], using different techniques of Nonlinear Analysis and a priori estimates by Alama and Tarantello [2] and Berestycki, Capuzzo-Dolcetta, Nirenberg [7]. Drabek and Pohozaev, in their renowned work [34], were pioneers in applying the Nehari manifold and fibering methods to this equation. This approach was further refined in the works of Ilyasov [45, 48], Ilyasov and Runst [47], Brown and Zhang in [15], and several other contributions. The case of $f \in C^1(\mathbb{R})$ such that $f(0) = f'(0) = 0$ with sublinear growth was treated in [60] using

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global bifurcation to get positive solutions. The case $0 < \alpha < 1$ was considered in [13], where only non-negative solutions were obtained. In [3], Alama and Lu obtained dead core and compactly supported solutions of (1) in case $\lambda \leq 0$ and $\Omega = \mathbb{R}^N$, $N \geq 3$.

When $\lambda = 0$, (1) was studied by Bandle, Pozio and Tesei [5], where a very careful study of the existence of solutions with compact support was developed using super and subsolutions and variational methods, but the existence of positive solutions was not considered. In [41] the authors proved existence (and uniqueness) of a positive solution under the assumptions (6)-(8) below. More results in this direction can be found in the survey [55], in particular for the one-dimensional problem.

In spite of the diversity of potential solutions, according the spatial value of $m(x)$, as we shall see, there are two different cases which determine a qualitatively different set of solutions according the values of parameter λ. One of them, case (A), arises when

$$\int_\Omega m(x)\varphi_1^{\alpha+1} > 0, \tag{2}$$

where $\varphi_1 > 0$ (normalized by $\|\varphi_1\|_{L^\infty} = 1$) is the first eigenfunction associated to λ_1, the first eigenvalue of the problem $-\Delta w = \lambda w$ in Ω, $w = 0$ on $\partial\Omega$. We can call this case as the "mainly favourable interior medium case" (since φ_1 attains its maximum at some interior point of Ω). This case is in contrast to the opposite case, case (B) where

$$\int_\Omega m(x)\varphi_1^{\alpha+1} < 0, \tag{3}$$

which we call as the "mainly hostile interior medium case".

A simple illustration of case (B) arises if $m \equiv -1$. For instance, if $\alpha > 1$, equation (1) is the very well-known "logistic equation" arising in population dynamics and in this case there exists a unique positive solution $u_\lambda > 0$ for any $\lambda > \lambda_1$, where λ_1 is the first eigenvalue of the problem $-\Delta w = \lambda w$ in Ω, $w = 0$ on $\partial\Omega$, with eigenfunction $\varphi_1 > 0$ (normalized by $\|\varphi_1\|_{L^\infty} = 1$). The Strong Maximum Principle tells us that if $u \geq 0$ is a solution, then $u > 0$ in Ω and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$, something which is still true (even if not noticed) in the work by Brown and Zhang [15] since $\alpha > 1$ (see Remark 2 below).

But this is not necessarily the case for $0 < \alpha < 1$, where the nonlinear term is not Lipschitz any more at the origin. This was shown in [25] in the one-dimensional case for $m \equiv -1$ and in [28] for some $m(x) \leq 0$.

A simple illustration of case (A) arises when $m \equiv 1$. We will recall some preliminary results for both simple illustrations in the next section. We are mainly interested in the interaction between positive and flat or compact support solutions, see [23], [24], [25], [28], [29]. In particular we study in [28] the problem (1) when $m(x) \leq 0$ finding sufficient conditions for existence of both kinds of solutions, and here we try to extend our study to the case of indefinite $m(x)$. Our results can be summarized in the following figure:

	$\int_\Omega m(x)\varphi_1^{\alpha+1} > 0,$	$\int_\Omega m(x)\varphi_1^{\alpha+1} < 0,$
$\lambda < \lambda_1$	$\left\{ \begin{array}{c} \exists \text{ sol. } u \geq 0 \\ \exists \text{ sol. } u > 0 \text{ in } [0, \lambda_1) \text{ with (7)} \end{array} \right.$	$\left\{ \begin{array}{c} \exists \text{ sol. } u \geq 0 \\ \exists \text{ sol. } u > 0 \text{ in } [0, \lambda_1) \text{ with (7)} \end{array} \right.$
$\lambda > \lambda_1$	$\nexists$ sol. $u > 0$	$\left\{ \begin{array}{c} \text{at least two sol. in } (\lambda_1, \lambda^*) \\ \exists \text{ compact support sol. if } \lambda \in (\lambda^* - \varepsilon, \lambda^*], \\ \text{for some } \lambda^*, \text{ under additional conditions.} \end{array} \right.$

The corresponding problems for the case of the p-Laplacian diffusion were studied much in detail by Bobkov-Tanaka [9] and Kaufmann, Ramos Quoirin and Umezu [51] by using mainly variational arguments. They are particularly interested on non-negative solutions u such that $u > 0$ on $\Omega^+ = \{x \in \Omega \ |m(x) > 0\}$ (as in [5]).

In this paper we are, on the contrary, mainly interested in positive solutions and also in solutions with compact support in Ω. For this we use different tools; mainly

i) the assumption (6), (8) below, already used in [41] to obtain positive solutions,

ii) the full strength of a global bifurcation theorem providing "large" solutions $u > 0$ (with $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$) for λ close to λ_1,

iii) the linearized stability results in [41], [26] (see also [8]), in order to study stability of positive solutions,

iv) the "hidden convexity" provided by the monotonicity tools in [22] giving uniqueness of positive solutions if $\lambda \leq 0$,

v) the sufficient condition obtained in [28] for the existence of solutions with zero normal derivative (flat solutions with $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ and compact support solutions as well) obtained from a Pohozaev identity and if $N = 1$ under additional conditions on $m(x)$.

The organization of this paper is as follows: in Section 2 we introduce the problems showing what happens in the "limit" cases $m = 1$ and $m = -1$ (where for $N = 1$ we provide a complete description of the solution set [25]). Our results here should be "somewhere in between", so as to say. We also exhibit the relevant condition in [41], giving positive solutions and extend the result to the interval $[0, \lambda_1)$. In Section 3 we study the existence of non-negative solutions for $\lambda < \lambda_1$ under assumption (2) with the method of the Nehari manifold. In Section 4 we prove the uniqueness of positive solutions for $\lambda < \lambda_1$ extending the result in [41] and studying the linearized stability of the positive solutions, a delicate question. In Section 5 we consider the case of $\lambda > \lambda_1$, now under condition (3). We use again the Nehari manifold, getting multiplicity for some interval of λ' s, and also for $\lambda = \lambda_1$, which is new in this context. Section 6 is devoted to the consideration of positive and compact support solutions. We prove the existence of solutions with compact support under suitable additional assumptions: mainly by assuming that $m(x) < 0$ in a large region near the boundary $\partial\Omega$. We apply the method of local super and subsolutions to obtain suitable barrier functions which now have some constraint on the radius of the ball $B_R(x_0)$ and on the maximum height when $\lambda \geq \lambda_1$, the first eigenvalue for the Laplacian operator on Ω with zero Dirichlet boundary conditions. We also construct some global super and subsolutions with compact support for the case in which Ω is a ball. Finally, Section 7 deals with some applications of Pohozaev's identity to determine the non-existence of such solutions.

2 Preliminaries

We consider the problem

$$\begin{cases} -\Delta u = \lambda u + m(x) \left|u\right|^{\alpha-1} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < \alpha < 1$, $m \in L^\infty(\Omega)$ changes sign in Ω or it becomes *degenerate* ($m \leq 0$ and the set where $m = 0$ has positive measure) and λ is a real parameter.

As explained at the Introduction, it is convenient to recall previously some of the results available in the literature concerning the two main cases associated to m constant.

(A) Case $m \equiv 1$. First of all, there are only solutions $u \geq 0$ if $\lambda < \lambda_1$. (Here, and along the paper λ_1 and φ_1 are the first eigenvalue and eigenfunction of $-\Delta w = \lambda w$ in Ω, $w = 0$ on $\partial\Omega$, such that $\varphi_1 > 0$ and $\|\varphi_1\|_{L^\infty} = 1$).

Indeed, if we multiply (4) by φ_1 and integrate by parts on Ω, we obtain

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega u\varphi_1 + \int_\Omega u^\alpha \varphi_1$$

and $\lambda < \lambda_1$. Moreover, from $\lambda < \lambda_1$ and the Strong Maximum Principle, if $u \geq 0$ then $u > 0$ in Ω and $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. This means that we don't have flat or compact support solutions (we call "flat solution" a solution $u > 0$ in Ω such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$).

In order to prove existence of solutions we look for subsolutions of the form $u_0 \equiv c\varphi_1$, $c > 0$. We have

$$\begin{aligned} -\Delta u_0 - \lambda u_0 - u_0^\alpha &= \lambda_1 c\varphi_1 - \lambda c\varphi_1 - c^\alpha \varphi_1^\alpha \\ &= c^\alpha \varphi_1^\alpha [(\lambda_1 - \lambda) c^{1-\alpha} \varphi_1^{1-\alpha} - 1] < 0 \end{aligned}$$

for $c > 0$ "small enough. As a supersolution we can try $u^0 \equiv C\psi$, where $\psi > 0$ is the unique solution of the linear problem $-\Delta\psi = \lambda\psi + 1$ in Ω, $\psi = 0$ on $\partial\Omega$, for $\lambda < \lambda_1$. We have

$$-\Delta u^0 - \lambda u^0 - (u^0)^\alpha = C^\alpha (C^{1-\alpha} - \psi^\alpha) > 0,$$

if $C > \|\psi\|_{L^\infty}^{\alpha/(1-\alpha)}$. We can always get $u_0 < u^0$ and this gives existence. Uniqueness follows from the well-known "concavity" argument (see, e.g., [12] or [11]), since $\lambda + |u|^{\alpha-1}$ is strictly decreasing for $u > 0$.

From the uniqueness and the method of sub and supersolutions follows that the branch u_λ is increasing in λ and the linearization results [41], [42] tell us that u_λ is asymptotically stable and $\lambda \to u_\lambda$ is C^∞ (as a map into $C_0^1(\overline{\Omega})$). Below we show in a more general framework that there is asymptotic bifurcation at $\lambda = \lambda_1$. The diagram is

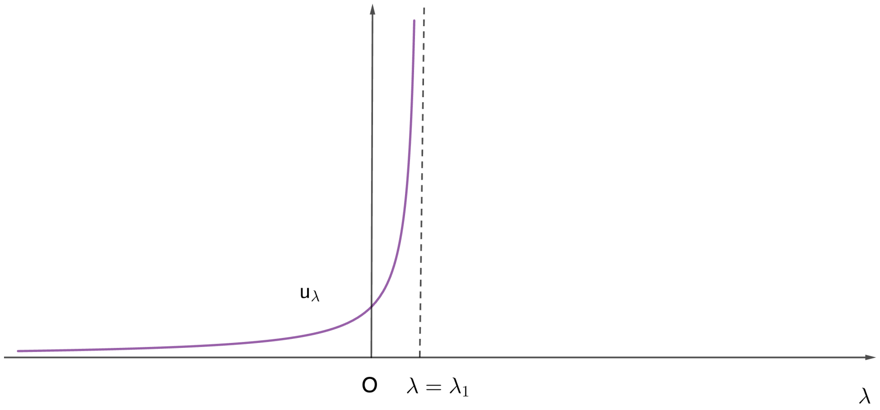

Figure 1: $m = 1$

We have thus proved the

Theorem 1 *(A) Let $m = 1$. For any $\lambda < \lambda_1$ there is a unique positive solution $u_\lambda > 0$ which is linearly stable. The branch $\lambda \to u_\lambda$ is a smooth (C^∞) increasing curve into $C_0^1(\overline{\Omega})$. There is no non-negative solution if $\lambda \geq \lambda_1$.*

(B) Case $m \equiv -1$. For $N = 1$, $\Omega = (-1, 1)$ a complete description of the solution set was obtained in [25] which is exhibited in Figure 2, by using energy methods.

We show as above that now there are non-negative solutions only for $\lambda > \lambda_1$. There exists a unique $\lambda^* > \lambda_1$ (calculated explicitly) such that for any $\lambda_1 < \lambda < \lambda^*$ there is a unique solution $u_\lambda > 0$ such that $u_\lambda(\pm 1) = 0$, $(u_\lambda)'(\pm 1) \neq 0$, which is linearly unstable and bifurcates at infinity from $\lambda = \lambda_1$. For $\lambda = \lambda^*$ there is a unique flat solution $u_{\lambda^*} > 0$ with $u_{\lambda^*}(\pm 1) = 0$, $(u_{\lambda^*})'(\pm 1) = 0$. For $\lambda > \lambda^*$ there are continua of compact support solutions constructed from u_{λ^*} (see [25] and [23] for the details).

We have thus the following

Theorem 2 *Let $m \equiv -1$, $N = 1$and $\Omega = (-1, 1)$. Then, for any $\lambda > \lambda_1$ there is a solution $u_\lambda \geq 0$ to (4). There exists $\lambda^* > \lambda_1$ such that there is a unique solution $u_\lambda > 0$ if $\lambda_1 < \lambda < \lambda^*$ and there is a unique flat solution $u_{\lambda^*} > 0$ for $\lambda = \lambda^*$, and there are continua of symmetric connected compact support solutions for $\lambda > \lambda^*$ forming a decreasing branch of solutions when λ increases.*

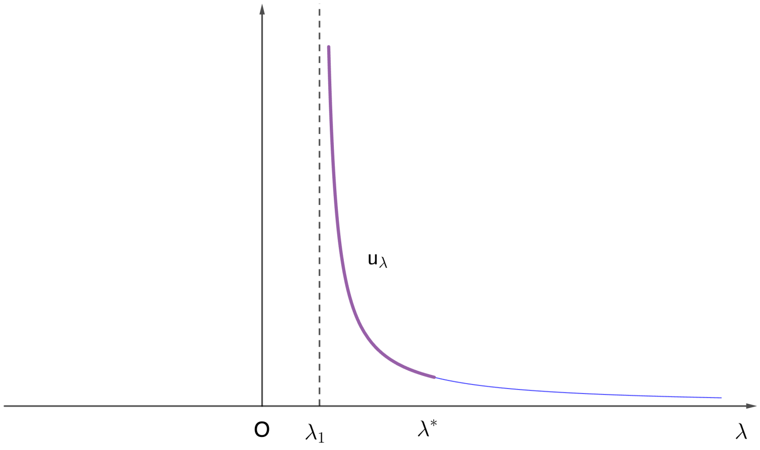

Figure 2: $m = -1$

For $N > 1$ there are less complete results in [28], [29], where $m(x) \leq 0$ non-constant is allowed.

The main goal of this paper is the study of the so called "indefinite case" in which $m \in L^\infty(\Omega)$ changes sign in Ω. So, if we define

$$\Omega^+ = \{x \in \Omega \ : m(x) = m^+(x) > 0\},$$
$$\Omega^- = \{x \in \Omega \ : m(x) = -m^-(x) < 0\},$$
$$\Omega^0 = \Omega \diagdown (\overline{\Omega^+} \cup \overline{\Omega^-})$$

and if $|A|$ denotes the Lebesgue measure of a set A of $\mathbb{R}^N$, then except we indicate other thing we assume that

$$m \in L^\infty(\Omega), \text{ and } \left|\Omega^+\right|, \left|\Omega^-\right| > 0, \tag{5}$$

and that the sets Ω^+, Ω^- and Ω^0 are regular.

It was already noticed above that (4) was studied for $\lambda = 0$ in [5], where a very complete study was carried out concerning compact support solutions related to the positive sets of $m(x)$. But no result was given for existence of *positive* solutions. Another study of the case $\lambda = 0$ was carried out in [39].

Our main interest in this paper is to study the qualitative bifurcation diagram of non-negative and positive solutions of (1) according the parameter $\lambda \in \mathbb{R}$ in the indefinite case (5). As mentioned in the Introduction, one of our main tools for the existence of solutions will be the method of the *Nehari manifold*; nevertheless, a first result on the existence of positive solutions can be obtained via the super and subsolutions method. A result of this kind was proved in Lemma 4.2 in [41] under the condition that if U is the unique solution of the linear problem

$$\begin{cases} -\Delta U = m(x) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

then

$$U > 0 \text{ in } \Omega, \tag{7}$$

and

$$\frac{\partial U}{\partial n} < 0 \text{ on } \partial\Omega. \tag{8}$$

As a matter of fact, the condition (8) is not needed for some purposes and thus U can be a "flat solution" (i.e., such that $\frac{\partial U}{\partial n} = 0$ on $\partial\Omega$): see [27]. It is clear that (7) follows from $m(x) \geq 0$, but it is also satisfied for some changing sign m's ([27]).

More precisely, we have:

Theorem 3 *Assume (5), (7) and let* $0 \leq \lambda < \lambda_1$. *Then there is a solution* $u_\lambda > 0$ *to (4). If, in addition (8) holds then there is uniqueness of positive solutions to problem (4).*

PROOF. We have $u_0 = [(1-\alpha)U]^{1/(1-\alpha)}$ as a subsolution. Indeed

$$\nabla u_0 = [(1-\alpha)U]^{\alpha/(1-\alpha)} \nabla U$$

and

$$\operatorname{div} \nabla u_0 = \alpha \left[(1-\alpha)U\right]^{(2\alpha-1)/(1-\alpha)} |\nabla U|^2 - m(x) \left[(1-\alpha)U\right]^{\alpha/(1-\alpha)},$$

which gives

$$\begin{array}{rcl} -\Delta u_0 - \lambda u_0 - m(x)(u_0)^\alpha & = & -\alpha \left[(1-\alpha)U\right]^{\frac{(2\alpha-1)}{(1-\alpha)}} |\nabla U|^2 \\ -\lambda \left[(1-\alpha)U\right]^{1/(1-\alpha)} & \leq & 0 \end{array}$$

if $\lambda \geq 0$. As a supersolution we pick again $u^0 = C\psi$ with $-\Delta\psi = \lambda\psi + 1$ in Ω, $\psi = 0$ on $\partial\Omega$, $\psi > 0$. Then

$$\begin{aligned} -\Delta u^0 - \lambda u^0 - m(x)(u^0)^\alpha &= C - C^\alpha m(x)\psi^\alpha \\ &= C^\alpha (C^{1-\alpha} - m(x)\psi^\alpha) > 0 \end{aligned}$$

if $C > (\|m\|_{L^\infty} \|\psi\|_{L^\infty}^\alpha)^{1/(1-\alpha)}$. As we will prove in Theorem 7 below, the uniqueness argument in Theorem 4.4 of [41] can be extended to the case $0 \leq \lambda < \lambda_1$ and this completes the proof of the result. □

In the following we prove some useful and basic auxiliary results.

Lemma 1 *If* $u \geq 0$ *is a solution of (4) for* $\lambda < \lambda_1$ *then we have*

$$\int_\Omega m(x) u^{1+\alpha} > 0. \tag{9}$$

PROOF. Multiplying (4) by u and integrating by parts on Ω we obtain

$$\lambda_1 \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 + \int_\Omega m(x) u^{1+\alpha}$$

and hence

$$0 < (\lambda_1 - \lambda) \int_\Omega u^2 \leq \int_\Omega m(x) u^{1+\alpha}.$$

□

Lemma 2 *If* $u > 0$ *is a solution of (4) for* $\lambda < \lambda_1$ *then we have*

$$\int_\Omega m(x) u^\alpha \varphi_1 > 0. \tag{10}$$

PROOF. Multiplying (4) by φ_1 and integrating by parts on Ω we get

$$\lambda_1 \int_\Omega u\varphi_1 = \lambda \int_\Omega u\varphi_1 + \int_\Omega m(x) u^\alpha \varphi_1$$

which gives (10). □

Lemma 3 *If* U *satisfies (7) then*

$$\int_\Omega m(x)\varphi_1 > 0. \tag{11}$$

Proof. Multiplying (6) by φ_1 and integrating we obtain

$$-\int_\Omega \Delta U \cdot \varphi_1 = \lambda \int_\Omega U\varphi_1 = \int_\Omega m(x)\varphi_1 > 0$$

by (7). □

Remark 1 *Notice that we do not need assumption (8) for suitable purposes. Condition (11) is more similar to the ones arising in the problems for* $\alpha > 1$ *([2], [7]). The converse is not valid (see the counterexamples given in [39] and [27]; see also [31]).*

Remark 2 *It is easy to show that when* $\alpha > 1$ *non-negative solutions to (1) are actually positive (notice that in [15] and [13] "positive" means actually "non-negative"). Indeed, if* $u \geq 0$ *is a solution to (1), it is a solution to the linear problem*

$$\left\{ \begin{array}{ll} -\Delta w + m^{-}(x) u^{\alpha-1} w = \lambda u + m^{+}(x) u^{\alpha} \geq 0 & \textit{in } \Omega, \\ w = 0 & \textit{on } \partial\Omega, \end{array} \right. \tag{12}$$

and the Strong Maximum Principle gives $u > 0$ *(notice that* $m^{-}(x)u^{\alpha-1} \geq 0$*).*

Remark 3 *This argument does not work if only* $u \geq 0$*. This is important since it leaves room for existence of compact support solutions (see below).*

Proposition 1 *If* $\int m|\varphi_1|^{\alpha+1}dx \geq 0$*, then equation (4) has no positive solutions for any* $\lambda > \lambda_1$*.*

3 Existence results: case $\lambda < \lambda_1$

Independently to the above Theorem 3 obtained by the method of super and subsolutions, in this section we will use *the Nehari manifold method* under the condition (5), as used in [15] for $\alpha > 1$ (see [13]), to get the existence of non-negative solutions of the problem

$$\left\{ \begin{array}{ll} -\Delta u = \lambda u + m(x) \left|u\right|^{\alpha-1} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{13}$$

We recall that Ω is a smooth bounded domain in $\mathbb{R}^N$, $0 < \alpha < 1$, λ is a real parameter and that we assume $m \in L^\infty(\Omega)$ changing sign in Ω with (5). We emphasize that we will not need to assume condition (7).

The functional $E_\lambda : H_0^1(\Omega) \to \mathbb{R}$ associated with equation (4) is defined by

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(\left|\nabla u\right|^2 - \lambda u^2\right) - \frac{1}{\alpha+1}\int_\Omega m(x)\left|u\right|^{\alpha+1}.$$

In what follows

$$\|u\| = \left(\int_\Omega \left|\nabla u\right|^2\right)^{1/2}$$

denotes the usual norm in $H_0^1(\Omega)$.

The corresponding Nehari manifold containing all critical points of the functional E_λ is defined by the set

$$N = \left\{ u \mid \int_\Omega \left(\left|\nabla u\right|^2 - \lambda u^2\right) - \int_\Omega m(x)\left|u\right|^{\alpha+1} = 0 \right\}. \tag{14}$$

The fibering functions are defined as usual by

$$\phi_u(t) = \frac{t^2}{2}\int_\Omega \left(\left|\nabla u\right|^2 - \lambda u^2\right) - \frac{t^{\alpha+1}}{\alpha+1}\int_\Omega m(x)\left|u\right|^{\alpha+1} \tag{15}$$

and by differentiating twice we get

$$\phi_u'(t) = t\int_\Omega \left(\left|\nabla u\right|^2 - \lambda u^2\right) - t^{\alpha}\int_\Omega m(x)\left|u\right|^{\alpha+1} \tag{16}$$

and

$$\phi_u''(t) = \int_\Omega \left(\left|\nabla u\right|^2 - \lambda u^2\right) - \alpha t^{\alpha-1}\int_\Omega m(x)\left|u\right|^{\alpha+1}. \tag{17}$$

If both integrals, in (16), have the same sign, then there is a unique $t > 0$ such that $\phi'_u(t) = 0$ given by

$$t(u) = t_\lambda(u) := \left(\frac{\displaystyle\int_\Omega m(x) |u|^{\alpha+1}}{\displaystyle\int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right)} \right)^{\frac{1}{1-\alpha}} . \tag{18}$$

On the Nehari manifold the functional takes the values

$$E_\lambda(u) = \left(\frac{1}{2} - \frac{1}{\alpha+1} \right) \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) = \left(\frac{1}{2} - \frac{1}{\alpha+1} \right) \int_\Omega m(x) |u|^{\alpha+1} \tag{19}$$

and the different components of N are defined as

$$\begin{aligned} N^+ &= \left\{ u \in N : \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) - \alpha \int_\Omega m(x) |u|^{\alpha+1} > 0 \right\} &(20)\\ &= \left\{ u \in N : (1-\alpha) \int_\Omega m(x) |u|^{\alpha+1} > 0 \right\} = \left\{ u \in N : \int_\Omega m(x) |u|^{\alpha+1} > 0 \right\}, &(21) \end{aligned}$$

and similarly

$$N^- = \left\{ u \in N : \int_\Omega m(x) |u|^{\alpha+1} < 0 \right\} \tag{22}$$

and

$$N^0 = \left\{ u \in N : \int_\Omega m(x) |u|^{\alpha+1} = 0 \right\}. \tag{23}$$

Next we define the subsets

$$L^+ = \left\{ u : \|u\| = 1, \ \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) > 0 \right\},$$

$$L^- = \left\{ u : \|u\| = 1, \ \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) < 0 \right\}$$

$$L^0 = \left\{ u : \|u\| = 1, \ \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) = 0 \right\}$$

and analogously

$$B^+ = \left\{ u : \|u\| = 1, \ \int_\Omega m(x) |u|^{\alpha+1} > 0 \right\}$$

and B^- and B^0.
It is easy to see that if $u \in L^+ \cap B^+$ (resp. $u \in L^- \cap B^-$) then $t(u)u \in N^+$(resp. $t(u)u \in N^-$).

We study first the

Case $\lambda < \lambda_1$: We start with an auxiliary result.

Lemma 4 *If $\lambda < \lambda_1$ there exists $\delta > 0$ such that for any $u \in H_0^1(\Omega)$ we have*

$$\int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) \geq \delta \|u\|^2 . \tag{24}$$

Proof. Assume that (24) does not hold. Then there exist $\delta_n \to 0$ and $\|u_n\| = 1$ such that

$$0 < (\lambda_1 - \lambda) \int_\Omega u_n^2 \leq \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) < \delta_n$$

and then $\int_\Omega u_n^2 \to 0$. Now, from $1 - \lambda \int_\Omega u_n^2 < \delta_n$ we get a contradiction. □

In particular, Lemma 4 implies that $L^- = L^0 = \emptyset$ and $N^- = \emptyset$, $N^0 = \{0\}$ and hence $N = N^+ \cup \{0\}$. Notice that this means that solutions to (4) "should be" in N^+ (if $N \neq \{0\}$!), which fits well with (9).

To show that $N \neq \{0\}$ we fix $\overline{u} \neq 0$ in $H_0^1(\Omega)$ and look for $t > 0$ such that $t\overline{u} \in N$. In this case

$$t^2 \int_\Omega \left(|\nabla \overline{u}|^2 - \lambda |\overline{u}|^2 \right) = t^{\alpha+1} \int_\Omega m(x) |\overline{u}|^{\alpha+1}$$

or either

$$t^{1-\alpha} \int_\Omega \left(|\nabla \overline{u}|^2 - \lambda |\overline{u}|^2 \right) = \int_\Omega m(x) |\overline{u}|^{\alpha+1}.$$

This happens if both integrals have the same sign and by Lemma 4 we need $\int_\Omega m(x) |\overline{u}|^{\alpha+1} > 0$. For that we pick $\overline{u}$ such that supp $\overline{u} = D \subset \Omega^+$, where D is a smooth domain, $\overline{u} > 0$ on D from (5). Then

$$\int_\Omega m(x) |\overline{u}|^{\alpha+1} = \int_D m(x) |\overline{u}|^{\alpha+1} > 0.$$

We have thus proved the

Lemma 5 *If $\lambda < \lambda_1$ then $N \neq \{0\}$.* □

If $u \in L^+ \cap B^+$, the fibering function is as in Figure 3.

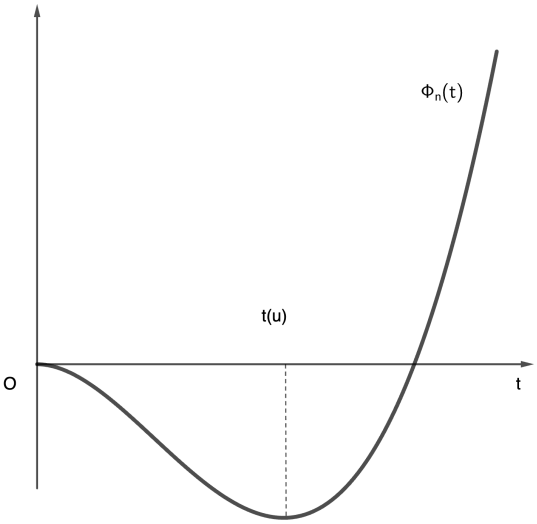

Figure 3: $u \in L^+ \cap B^+$

It turns out that the functional takes the value

$$E_\lambda(u) = \frac{\alpha - 1}{2(\alpha + 1)} \int_\Omega m(x) |u|^{\alpha+1} = \frac{\alpha - 1}{2(\alpha + 1)} \int_\Omega \left(|\nabla u|^2 - \lambda u^2 \right) \leq 0$$

on N^+. We will show that $\inf_{N^+} E_\lambda > -\infty$, i.e., that E_λ is bounded below on N^+, otherwise stated, that $|E_\lambda(u)| \leq C$ for some $C > 0$. Since $u \in N^+$, we can write $u = t(v)v$ with $v = \frac{u}{\|u\|}$ and get

$$\begin{aligned}
|E_\lambda(u)| &= \frac{1-\alpha}{2(\alpha+1)} t(v)^2 \int_\Omega \left(|\nabla v|^2 - \lambda v^2 \right) \\
&= \frac{1-\alpha}{2(\alpha+1)} \left(\frac{\displaystyle\int_\Omega m(x)\, |v|^{\alpha+1}}{\displaystyle\int_\Omega \left(|\nabla v|^2 - \lambda v^2 \right)} \right)^{\frac{2}{1-\alpha}} \int_\Omega \left(|\nabla v|^2 - \lambda v^2 \right) \\
&= \frac{1-\alpha}{2(\alpha+1)} \frac{\left(\displaystyle\int_\Omega m(x)\, |v|^{\alpha+1} \right)^{\frac{2}{1-\alpha}}}{\left(\displaystyle\int_\Omega \left(|\nabla v|^2 - \lambda v^2 \right) \right)^{\frac{1+\alpha}{1-\alpha}}} \\
&\leq \frac{1-\alpha}{2(\alpha+1)} \|m\|_{L^\infty}^{\frac{2}{1-\alpha}} \frac{\left(\displaystyle\int_\Omega v^{\alpha+1} \right)^{\frac{2}{1-\alpha}}}{\left(\displaystyle\int_\Omega \left(|\nabla v|^2 - \lambda v^2 \right) \right)^{\frac{1+\alpha}{1-\alpha}}} \\
&\leq C_1 \frac{\|v\|_{L^{\alpha+1}}^{\frac{2(1+\alpha)}{1-\alpha}}}{\delta^{\frac{1+\alpha}{1-\alpha}} \|v\|^{\frac{2(1+\alpha)}{1-\alpha}}} \leq C_2
\end{aligned}$$

by (24) and the Sobolev embedding, where $C_i > 0$ are constants independent of v.

We have thus proved the

Lemma 6 *E_λ is bounded below on N^+.* □

Theorem 4 *E_λ attains its minimum on N^+.*

Proof. Let (u_n) be a minimizing sequence. We have

$$E_\lambda(u_n) = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x)\, |u_n|^{\alpha+1} \to \inf_{N^+} E_\lambda < 0,$$

and from (24) it follows that $\|u_n\| \leq C$. Then there exists a subsequence u_n of (u_n) (again denoted by (u_n)) such that $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$ (where 2^* is the critical Sobolev exponent. We have

$$\lim \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \lim \int_\Omega m(x)\, |u_n|^{\alpha+1} = \int_\Omega m(x)\, |u_0|^{\alpha+1} > 0$$

and then $u_0 \neq 0$, $\frac{u_0}{\|u_0\|} \in B^+$. From (24), $\frac{u_0}{\|u_0\|} \in L^+$, which implies $\frac{u_0}{\|u_0\|} \in L^+ \cap B^+$, $t(u_0)u_0 \in N^+$ and the corresponding fibering function ϕ_{u_0} is as in Figure 3.

We have also $u_n \to u_0$. If not, $\|u_0\| < \underline{\lim} \|u_n\|$ and this gives

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \lim \int_\Omega m(x)\, |u_n|^{\alpha+1} = \int_\Omega m(x)\, |u_0|^{\alpha+1}$$

and $t(u_0) > 1$ from (18). Moreover, we have

$$\inf_{N^+ N^*} \int E_\lambda = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x)\, |u_0|^{\alpha+1}.$$

On the other hand, since $t(u_0)u_0 \in N$

$$t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) = t(u_0)^{\alpha+1} \int_\Omega m(x)\, |u_0|^{\alpha+1}.$$

This gives

$$\begin{aligned} E_\lambda(t(u_0)u_0) &= \frac{t(u_0)^2}{2}\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2\right) - \frac{t(u_0)^{\alpha+1}}{\alpha+1}\int_\Omega m(x)\,|u_0|^{\alpha+1} \\ &= t(u_0)^{\alpha+1}\frac{(\alpha-1)}{2(\alpha+1)}\int_\Omega m(x)\,|u_0|^{\alpha+1} < \inf_{N^+} E_\lambda, \end{aligned}$$

since $t(u_0) > 1$, a contradiction with $t(u_0)u_0 \in N^+$.
We have thus $u_n \to u_0$, $E_\lambda(u_0) = \inf_{N^+} E_\lambda(u_n)$ where

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2\right) = \int_\Omega m(x)\,|u_0|^{\alpha+1} > 0,$$

and $u_0 \notin N^0$. □

Theorem 5 *There exists (at least) a non-negative solution $u_\lambda \geq 0$ to (4) for any $\lambda < \lambda_1$.*

Proof. Since $N = N^+ \cup \{0\}$, $E_\lambda(|u_0|) = E_\lambda(u_0)$, and $|u_0| \in N^+$, $u_0 \geq 0$, $u_0 \neq 0$ minimizer of E_λ on N^+, $u_0 \notin N^0$, then it is a critical point of the functional (see [15]). □

Case $\lambda < \lambda_1$, λ near λ_1: Now we consider the behaviour of the branch $u_\lambda \geq 0$ of solutions close to λ_1.

In the following we add the assumption

$$\int_\Omega m(x)\varphi_1^{\alpha+1} > 0 \tag{25}$$

which has not been used in the above arguments. See also [15]. Notice that if (25) holds, $\varphi_1 \in N^+ \neq \phi$.

Proposition 2 *Assume that (25) holds. Then we have that if $\lambda_n \nearrow \lambda_1$*

$$\lim_{\lambda_n \nearrow \lambda_1} \inf_{N^+} E_{\lambda_n} = \lim_{\lambda_n \nearrow \lambda_1} E_{\lambda_n}(u_n) = -\infty. \tag{26}$$

Proof. If (25) holds, $\varphi_1 \in L^+ \cap B^+$ and this gives $t(\varphi_1)\varphi_1 \in N^+$, where

$$t(\varphi_1) = \left(\frac{\int_\Omega m(x)\varphi_1^{\alpha+1}}{\int_\Omega \left(|\nabla\varphi_1|^2 - \lambda\varphi_1^2\right)}\right)^{\frac{1}{1-\alpha}} \tag{27}$$

and we obtain

$$\begin{aligned} E_{\lambda_n}(t(\varphi_1)\varphi_1) &= \frac{\alpha-1}{2(\alpha+1)} t(\varphi_1)^2 \int_\Omega \left(|\nabla\varphi_1|^2 - \lambda_n\varphi_1^2\right) \\ &= \frac{\alpha-1}{2(\alpha+1)} \frac{1}{(\lambda_1-\lambda_n)^{\frac{1+\alpha}{1-\alpha}}} \frac{\left(\int_\Omega m(x)\varphi_1^{\alpha+1}\right)^{\frac{2}{1-\alpha}}}{\delta^{\frac{1+\alpha}{1-\alpha}}\left(\int_\Omega \varphi_1^2\right)^{\frac{1+\alpha}{1-\alpha}}} \underset{\lambda_n \nearrow \lambda_1}{\rightarrow} -\infty. \end{aligned}$$

Finally,

$$\lim_{\lambda_n \nearrow \lambda_1} \inf_{N^+} E_{\lambda_n} \leq \lim_{\lambda_n \nearrow \lambda_1} E_{\lambda_n}(t(\varphi_1)\varphi_1) = -\infty.$$

□

Proposition 3 *Assume that (25) holds. If $E_{\lambda_n}(u_n) = \inf\limits_{N^+} E_{\lambda_n}$, where $\lambda_n \nearrow \lambda_1$ as $n \to +\infty$ then there exists a subsequence (n_j) such that $n_j \to +\infty$ as $j \to +\infty$, and*

i) $\left\|u_{n_j}\right\| \underset{\lambda_{n_j} \nearrow \lambda_1}{\rightarrow} +\infty$,

ii) $\dfrac{u_{n_j}}{\left\|u_{n_j}\right\|} \underset{\lambda_{n_j} \nearrow \lambda_1}{\rightarrow} \varphi_1$.

Proof. i) Assume that this is not the case. If $\|u_n\| \leq C$, $u_n \rightharpoonup u_0$ and $u_n \rightharpoonup u_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$. Then we have

$$\lim \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) \left|u_n\right|^{\alpha+1} = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) \left|u_0\right|^{\alpha+1} > -\infty,$$

against Proposition 2, a contradiction.
ii) From i) we have $\|u_n\| \underset{\lambda_n \nearrow \lambda_1}{\rightarrow} +\infty$, with

$$\int_\Omega \left(\left|\nabla u_n\right|^2 - \lambda_n u_n^2\right) = \int_\Omega m(x) \left|u_n\right|^{\alpha+1} > 0.$$

If we define $v_n = \frac{u_n}{\|u_n\|}$, $v_n \rightharpoonup v_0$ and $v_n \rightharpoonup v_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$, and

$$\int_\Omega \left(\left|\nabla v_n\right|^2 - \lambda_n v_n^2\right) = \|u_n\|^{\alpha-1} \int_\Omega m(x) \left|v_n\right|^{\alpha+1} \to 0,$$

and $1 - \lambda_1 \int_\Omega v_0^2 = 0$, $v_0 \neq 0$. If $v_n \nrightarrow v_0$, $\displaystyle\int_\Omega \left|\nabla v_0\right|^2 < \underline{\lim} \int_\Omega \left|\nabla v_n\right|^2$ giving

$$\int_\Omega \left(\left|\nabla v_0\right|^2 - \lambda_1 v_0^2\right) < \lim \int_\Omega \left(\left|\nabla v_n\right|^2 - \lambda_n v_n^2\right) = 0,$$

a contradiction. Thus $\|v_0\| = 1$ with $\displaystyle\int_\Omega \left(\left|\nabla v_0\right|^2 - \lambda_1 v_0^2\right) = 0$, giving $v_0 = \varphi_1$. □

We have obtained weak solutions in $H_0^1(\Omega)$. But we can show that they are actually more regular.

Lemma 7 *If $u \in H_0^1(\Omega)$, $u \geq 0$ is a solution to (4), $u \in C^{1,\gamma}(\overline{\Omega})$, for any $\gamma \in (0,1)$. Moreover, if if m(x) is Hölder continuous then $u \in C^2$.*

Proof. The bootstrap argument in [33] gives $u \in L^\infty(\Omega)$ and the regularity L^p and C^α in [1, 38] gives $u \in W^{2,p}(\Omega)$ for any $p > N$ and then $u \in C^{1,\gamma}(\overline{\Omega})$, for any $\gamma \in (0,1)$. It is also possible to reason as in Appendix B of [58] using that $|\lambda u + m(x)u^\alpha| \leq C(1+|u|)$ for some $C > 0$. Moreover, if $m(x)$ is Hölder continuous, then Δu is Hölder continuous, and by Schauder estimates $u \in C^2$. □

Remark 4 *The existence of non-negative solutions for $\lambda \in (0, \lambda_1)$ was already proved in Theorem 1.4 of [55] but the uniqueness of solution was not considered there.*

The case of λ near λ_1, but now for $\lambda > \lambda_1$, will be considered in Section 5.

We sketch an alternative approach to the existence of nonnegative solutions which consists in writing the problem as an equivalent nonlinear eigenvalue problem $Pv = \lambda v$ in some function space (e.g. $L^2(\Omega)$), where $P : L^2(\Omega) \to L^2(\Omega)$ is a compact nonlinear operator such that there is a "derivative at infinity", i.e., a compact linear operator $A : L^2(\Omega) \to L^2(\Omega)$ such that

$$\frac{\|Pu - Au\|_{L^2(\Omega)}}{\|u\|_{L^2(\Omega)}} \to 0, \text{ as } \|u\|_{L^2(\Omega)} \to +\infty,$$

(see [40], [20]). This allows to apply global asymptotic bifurcation results for non-negative solutions in [4] and [19] following the work in [56].

It is well known that nodal properties of solutions are preserved along the bifurcation branches in the case of ordinary bifurcation from the trivial solution $u \equiv 0$, but this is not the case any more in the case of bifurcation at the infinity. There is already an example in [56], some more examples can be found in [25] and [28].

It is possible to prove the following result

Theorem 6 *There exists an unbounded continuum of non negative solutions to (4) bifurcating at infinity from* $\lambda = \lambda_1$.

Corollary 1 *If (25) holds, then for any* $\lambda < \lambda_1$ *there exists (at least) a non-negative solution of (4).*

PROOF. It is enough to show that there exists a continuous function $\varphi(\lambda)$ such that if $u \geq 0$ is a solution for $\lambda < \lambda_1$, then $0 \leq \|u\|_{L^2(\Omega)} \leq \varphi(\lambda)$. Indeed, if we multiply equation (4) by u and integrate by parts we obtain

$$\lambda_1 \int_\Omega u^2 \leq \int_\Omega |\nabla u|^2 = \lambda \int_\Omega u^2 + \int_\Omega m(x)\,|u|^{\alpha+1} \leq \lambda \int_\Omega u^2 + \|m\|_{L^\infty(\Omega)} \int_\Omega |u|^{\alpha+1}$$

and then , for some $c > 0$

$$(\lambda_1 - \lambda) \int_\Omega u^2 \leq \|m\|_{L^\infty(\Omega)} \|u\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \leq c\,\|m\|_{L^\infty(\Omega)} \|u\|^{\alpha+1}_{L^2(\Omega)}$$

and we obtain

$$\|u\|_{L^2(\Omega)} \leq \left(\frac{c\,\|m\|_{L^\infty(\Omega)}}{(\lambda_1 - \lambda)} \right)^{1/(1-\alpha)} := \psi(\lambda).$$

Now, we obtain finally

$$\int_\Omega |\nabla u|^2 \leq \lambda \psi(\lambda)^2 + c\,\|m\|_{L^\infty(\Omega)}\,\psi(\lambda)^{\alpha+1} := \varphi(\lambda).$$

□

Remark 5 *The above result proves that* $\int_\Omega |\nabla u|^2 \to 0$ *as* $\lambda \searrow -\infty$ *(and thus* $\|u\|_{L^\infty(\Omega)} \to 0$ *as* $\lambda \searrow -\infty$*).*

Remark 6 *It is quite easy to adapt the results of [23] and [25] for the one-dimensional case* $N = 1$ *(see also [21], [28] for* $N > 1$*) to this problem to show that if* $\lambda \leq 0$ *then the solutions may have a compact support depending of the data (*Ω *and* $m(x)$*). Nevertheless, if* $\lambda \in (0, \lambda_1)$ *it can be proved that, necessarily,* $u > 0$ *in* Ω *(for suitable* $m(x)$*). On the other hand, using the results of [21] it can be shown that* $u > 0$ *for* $\lambda \leq 0$ *depending of the size of* Ω *and the values of* m^-.

Remark 7 *A similar result to our Theorem 6 was proved in [52] in the case* $m(x) \leq 0$, $m =$*constant. Actually, this is a particular case of Theorem 2.1 in [20] and the results in [40]. We notice that the approximation method used in the proof in [52], which is quite close to the one used in the non-monotone case in [18] is completely different from the direct bifurcation arguments in [20], [40] where a theorem of Brezis allows us to exploit the monotonicity of the nonlinearity. It is astonishing to find the result in [52] fifty years later than [20], [40] which are not quoted. These results were also mentioned in Section 2.2 in [28].*

4 Uniqueness and stability of positive solutions

In this Section we study the uniqueness of *positive* solutions (if they exist, notice that until now we have proved only existence of *non-negative* solutions except Theorem 3 in Section 2 under assumption (7) for $0 \leq \lambda < \lambda_1$).

We study a case somewhat more general than u^α $(0 < \alpha < 1)$. We consider the problem

$$\begin{cases} -\Delta u = \lambda u + m(x) f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{28}$$

where

$$f \;:\; [0,+\infty) \to [0,+\infty) \text{ is } C^1 \text{ on } (0,+\infty), \text{ continuous on } [0,+\infty) \text{ and} \tag{29}$$

$$f(0) \;=\; 0,\; f' > 0 \text{ and } f'' < 0 \text{ on } (0,+\infty). \tag{30}$$

In what follows, we consider solutions $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$. We use a change of unknown already used in [11], [57], [41] (see also [5]). We extend the argument in [41].

This change consists in introducing $u = h(v)$ (and then $\nabla u = h'(v)\nabla v$) and writing

$$v = \int_0^u \frac{dt}{f(t)} = \int_0^{h(v)} \frac{dt}{f(t)}.$$

By differentiating we obtain

$$1 = \frac{1}{f(h(v))} h'(v),$$

giving

$$h'(v) = f(h(v)) = f(u)$$

(notice that $f > 0$ implies $h' > 0$). We also get

$$\nabla v = \frac{\nabla u}{f(u)}$$

and then

$$\begin{aligned} \operatorname{div} \nabla v &= \Delta v = -\frac{f'(u)}{f(u)^2} |\nabla u|^2 + \frac{1}{f(u)} \Delta u \\ &= -\frac{f'(u)}{f(u)^2} |\nabla u|^2 + \frac{1}{f(u)}(-\lambda u - m(x) f(u)) \\ &= -\frac{f'(u)}{f(u)^2} |\nabla u|^2 - \frac{\lambda u}{f(u)} - m(x) \\ & -f'(h(v)) |\nabla v|^2 - \tfrac{\lambda h(v)}{f(h(v))} - m(x), \end{aligned}$$

otherwise stated

$$-\Delta v = f'(h(v)) |\nabla v|^2 + \frac{\lambda h(v)}{f(h(v))} + m(x) \text{ in } \Omega.$$

Assume there are two solutions $v_1, v_2 > 0$ of (28)

$$\begin{aligned} -\Delta v_1 &= f'(h(v_1)) |\nabla v_1|^2 + \frac{\lambda h(v_1)}{f(h(v_1))} + m(x) \quad \text{in} \quad \Omega, \\ -\Delta v_2 &= f'(h(v_2)) |\nabla v_2|^2 + \frac{\lambda h(v_2)}{f(h()v_2)} + m(x) \quad \text{in} \quad \Omega, \\ v_1 &= v_2 = 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

If we write $w = v_1 - v_2$ we have

$$\begin{aligned} -\Delta w &= f'(h(v_1)) |\nabla v_1|^2 - f'(h(v_2)) |\nabla v_2|^2 + \lambda \left(\frac{h(v_1)}{f(h(v_1))} - \frac{h(v_2)}{f(h()v_2)} \right) \\ &= f'(h(v_1))(|\nabla v_1|^2 - |\nabla v_2|^2) + (f'(h(v_1)) - f'(h(v_2))) |\nabla v_2|^2 + \lambda \left(\frac{h(v_1)}{f(h(v_1))} - \frac{h(v_2)}{f(h()v_2)} \right) \\ &= f'(h(v_1))(\nabla(v_1 + v_2)\nabla w + \frac{f'(h(v_1)) - f'(h(v_2))}{v_1 - v_2} |\nabla v_2|^2 w + \lambda \left(\frac{h(v_1)}{f(h(v_1))} - \frac{h(v_2)}{f(h()v_2)} \right), \end{aligned}$$

and then

$$\begin{cases} -\Delta w - \dfrac{f'(h(v_1)) - f'(h(v_2))}{v_1 - v_2} |\nabla v_2|^2 w - f'(h(v_1))(\nabla(v_1 + v_2)\nabla w \\ \qquad -\dfrac{\lambda}{v_1 - v_2} \left(\dfrac{h(v_1)}{f(h(v_1))} - \dfrac{h(v_2)}{f(h()v_2)} \right) w = 0, \text{ in } \Omega, \\ \\ \qquad\qquad w = 0, \text{ on } \partial\Omega. \end{cases}$$

In order to apply the Maximum Principle we need a nonnegative zero order term or at least greater than $-\lambda_1$. For this we need $(f'\circ h)(v)$ strictly decreasing in v, which follows from

$$(f'\circ h)' = (f''\circ h)h' < 0$$

since $f''<0$, $h>0$, $h' = f\circ h > 0$. For the last term we can write, introducing

$$F(v) = \frac{h(v)}{f(h(v))} = \frac{h(v)}{h'(v)}$$

and applying the Mean Value Theorem

$$\begin{aligned}|F(v_1)-F(v_2)| &\le \max_{[v_1,v_2]} |F'(v_1+\theta(x)(v_1-v_2))|\,|(v_1-v_2)| \\ &= \max_{[v_1,v_2]} \left|\left(\frac{h(v)}{h'(v)}\right)'\right| |v_1-v_2| = \max_{[v_1,v_2]} \left|\frac{h'(v)^2-h(v)h''(v)}{h'(v)^2}\right| |v_1-v_2| \\ &\le \max_{[v_1,v_2]} \left|1-\frac{h(v)h''(v)}{h'(v)^2}\right| |v_1-v_2| \le |v_1-v_2|\end{aligned}$$

since

$$\left|\frac{h(v)h''(v)}{h'(v)^2}\right| = \frac{h(v)h''(v)}{h'(v)^2} = \frac{(f\circ h)'(v)h(v)}{h'(v)} < 1$$

is equivalent to

$$(f'(h(v))h(v) < h'(v) = f(h(v))$$

or either

$$f'(h(v)) < \frac{f(h(v))}{h(v)}$$

which follows from $f''<0$.
We obtain, finally

$$-\frac{\lambda}{v_1-v_2}\left(\frac{h(v_1)}{f(h(v_1))} - \frac{h(v_2)}{f(h(v_2))}\right) \ge 0$$

for $\lambda \le 0$. For $0<\lambda<\lambda_1$ we can write

$$-\frac{\lambda}{v_1-v_2}\left(\frac{h(v_1)}{f(h(v_1))} - \frac{h(v_2)}{f(h(v_2))}\right)(v_1-v_2) \ge -\lambda w > -\lambda_1 w.$$

We have then proved the

Theorem 7 *Under the assumption (29) there is at most a unique positive solution to 4.* □

Corollary 2 *Assume $f(s)=s^\alpha$, $0<\alpha<1$ and $0<\lambda<\lambda_1$. Then there is at most a unique positive solution to problem (4).* □

Summarizing, recalling Theorem 3, we have thus proved the

Theorem 8 *Under assumption (7) there is a unique solution of problem (4) for any $0\le\lambda<\lambda_1$* □.

Remark 8 *In Theorem 1.1 of [55] it was proved that $u_\lambda \to 0$ if $\lambda\to-\infty$ and that the branch of unique solutions u_λ is increasing in λ when $\lambda\le 0$. It seems possible to extend their proof to the case $\lambda\in(0,\lambda_1)$. These authors show (in their Theorem 1.8.) that there are solutions with an internal dead core. Here we will follow a different strategy by analyzing the existence of flat or compact support solutions (see Section 6).*

The uniqueness of positive solutions to problem (4) for $\lambda\in(-\infty,0)$, and the monotone dependence of solutions with respect to the coefficient $m(x)$, can be proved by means of the *hidden convexity technique* (see, e.g., [32], [22] and its many references).

Proposition 4 *Let* $m, \widehat{m} \in L^\infty(\Omega)$ *satisfying (5) and let* $u, \widehat{u}$ *be positive solutions of the corresponding problems (4) for* $\lambda < 0$. *Then*

$$m \leq \widehat{m} \text{ on } \Omega \text{ implies that } u \leq \widehat{u} \text{ on } \Omega. \tag{31}$$

In particular, there is a unique positive solution of problem (4) under assumption (7) for any $\lambda < 0$.

Proof. We have

$$\frac{-\Delta u}{u^\alpha} + (-\lambda)u^{1-\alpha} = m(x) \text{in } \Omega. \tag{32}$$

Then the function $w = u^q$ with $q = 1 + \alpha$ satisfies

$$-\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + (-\lambda)w^{\frac{2-q}{q}} = m(x), \text{ in } \Omega. \tag{33}$$

From Lemma 2.4 of [22] we know that the operator $w \to -\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}}$ is the subdifferential of a convex T-monotone operator in $L^2(\Omega)$. Thus, multiplying the difference of equations (33) corresponding to $m(x)$ and $\widehat{m}(x)$, respectively, by $(w - \widehat{w})^+$, with $h^+ = \max(0, h)$

$$\int_\Omega \left[-\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + \frac{\Delta(\widehat{w}^{\frac{1}{q}})}{\widehat{w}^{\frac{q-1}{q}}} \right] (w - \widehat{w})^+ + (-\lambda) \int_\Omega [w^{\frac{2-q}{q}} - \widehat{w}^{\frac{2-q}{q}}](w - \widehat{w})^+ = \int_\Omega [m - \widehat{m}](w - \widehat{w})^+.$$

Then, since

$$\int_\Omega \left[-\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + \frac{\Delta(\widehat{w}^{\frac{1}{q}})}{\widehat{w}^{\frac{q-1}{q}}} \right] (w - \widehat{w})^+ \geq 0$$

we have

$$(-\lambda) \int_\Omega [w^{\frac{2-q}{q}} - \widehat{w}^{\frac{2-q}{q}}](w - \widehat{w})^+ \leq \int_\Omega [m - \widehat{m}]^+(w - \widehat{w})^+. \tag{34}$$

Now, if $m \leq \widehat{m}$ on Ω we have $[m - \widehat{m}]^+ \equiv 0$, and thus, from (34) we deduce that $(w - \widehat{w})^+ \equiv 0$. □

Proposition 5 *Let* $\lambda < \mu < 0$ *and let* w, $\widehat{w}$ *be the positive solutions for the corresponding problem (4). Then* $w \leq \widehat{w}$.

Proof. We can write, reasoning as above

$$-\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + (-\lambda)w^{\frac{2-q}{q}} = m(x), \text{ in } \Omega, \tag{35}$$

$$-\frac{\Delta(\widehat{w}^{\frac{1}{q}})}{\widehat{w}^{\frac{q-1}{q}}} + (-\mu)\widehat{w}^{\frac{2-q}{q}} = m(x), \text{ in } \Omega. \tag{36}$$

Then we obtain, multiplying the difference by $(w - \widehat{w})^+$

$$\int_\Omega \left[-\frac{\Delta(w^{\frac{1}{q}})}{w^{\frac{q-1}{q}}} + \frac{\Delta(\widehat{w}^{\frac{1}{q}})}{\widehat{w}^{\frac{q-1}{q}}} \right] (w - \widehat{w})^+ + (-\lambda) \int_\Omega [w^{\frac{2-q}{q}} - \widehat{w}^{\frac{2-q}{q}}](w - \widehat{w})^+$$
$$[(-\lambda) - (-\mu)] \textstyle\int_\Omega \widehat{w}^{\frac{2-q}{q}}(w - \widehat{w})^+ = 0.$$

Since all three terms are non-negative, we have $w \leq \widehat{w}$. □

We study now the linearized stability of positive solutions $u > 0$ with $u \in C^1(\overline{\Omega})$ satisfying $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$. We use the linearized stability results in [41], [42] (see also [8], [26]).

Consider again the more general problem (28) under the same assumptions on f. If $u > 0$ is a solution with the above smoothness, the linearized problem is

$$\begin{cases} -\Delta w - m(x)f'(u)w - \lambda w = \mu w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{37}$$

From [41], [26] we know that (37) possesses a first eigenvalue μ_1 with eigenfunction $\psi_1 > 0$ which satisfies $\frac{\partial \psi_1}{\partial n} < 0$ on $\partial\Omega$. We have

$$\begin{cases} -\Delta\psi_1 - m(x)f'(u)\psi_1 - \lambda\psi_1 = \mu_1\psi_1 & \text{in } \Omega, \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$

If we multiply (28) by ψ_1 and (38) by u and integrate on Ω, using Green's Formula we obtain

$$\begin{cases} \displaystyle\int_\Omega \nabla u \cdot \nabla\psi_1 - \lambda \int_\Omega u\psi_1 - \int_\Omega m(x)f(u)\psi_1 = 0 = \int_\Omega \nabla u \cdot \nabla\psi_1 \\ \displaystyle -\lambda \int_\Omega u\psi_1 - \int_\Omega m(x)f'(u)u\psi_1 - \mu_1 \int_\Omega u\psi_1 = 0, \end{cases},$$

which gives

$$\mu_1 = \frac{\displaystyle\int_\Omega m(x)[f(u) - uf'(u)]\psi_1}{\displaystyle\int_\Omega u\psi_1}.$$

If $f(s) = s^\alpha$, $0 < \alpha < 1$, we have

$$\mu_1 = \frac{\displaystyle(1-\alpha)\int_\Omega m(x)u^\alpha\psi_1}{\displaystyle\int_\Omega u\psi_1}, \tag{39}$$

and $\mu_1 > 0$ if $m(x) > 0$ on Ω. But we don't know how to prove that $\mu_1 > 0$ if $\int_\Omega m(x)\varphi_1^{1+\alpha} > 0$, the expected result.

Remark 9 *Notice that for $f(s) = s^\alpha$, $0 < \alpha < 1$, we have $f(u) - uf'(u) = (1-\alpha)u^\alpha > 0$ and $\mu_1 > 0$ if $m \equiv 1$. If $m \equiv -1$, $\mu_1 < 0$ if $0 < \alpha < 1$, and also $\mu_1 < 0$ if $m \equiv 1$ and $\alpha > 1$. If $m \equiv -1$ and $\alpha > 1$, the logistic equation, $\mu_1 > 0$. Notice also that $f(u) - uf'(u) > 0$ is precisely the condition giving uniqueness of the positive solution in the classical "concavity" argument.*

Another approach where $m(x)$ "disappears" was used by Brown and Hess in [14]. Multiplying (28) by $f'(u)\psi_1$ and (38) by $f(u)$ and integrating on Ω we obtain

$$\begin{cases} \displaystyle -\int_\Omega \Delta u \cdot f'(u)\psi_1 - \lambda \int_\Omega uf'(u)\psi_1 - \int_\Omega m(x)f(u)f'(u)\psi_1 = 0 \\ \displaystyle = -\int_\Omega \Delta\psi_1 \cdot f(u) - \lambda \int_\Omega f(u)\psi_1 - \int_\Omega m(x)f(u)f'(u)\psi_1 - \mu_1 \int_\Omega f(u)\psi_1 = 0, \end{cases}$$

giving

$$\mu_1 = \frac{\displaystyle\int_\Omega \Delta u \cdot f'(u)\psi_1 - \int_\Omega \Delta\psi_1 \cdot f(u) - \lambda \int_\Omega [f(u) - uf'(u)]\psi_1}{\displaystyle\int_\Omega f(u)\psi_1}.$$

We can calculate, using Green's Formula

$$\begin{aligned} &\int_\Omega \Delta u \cdot f'(u)\psi_1 - \int_\Omega \Delta\psi_1 \cdot f(u) - \lambda \int_\Omega [f(u) - uf'(u)]\psi_1 \\ &= \int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot f'(u)\psi_1 - \int_\Omega \nabla u \cdot \nabla(f'(u)\psi_1) + \int_\Omega \nabla\psi_1 \cdot \nabla f(u) - \int_{\partial\Omega} \frac{\partial\psi_1}{\partial n} \cdot f(u) \\ &\quad -\lambda \int_\Omega [f(u) - uf'(u)]\psi_1 \\ &= -\int_\Omega f''(u)\left|\nabla u\right|^2 \psi_1 - \lambda \int_\Omega [f(u) - uf'(u)]\psi_1 \end{aligned}$$

and we arrive to

$$\mu_1 = \frac{-\displaystyle\int_\Omega f''(u)\,|\nabla u|^2\,\psi_1 - \lambda \int_\Omega [f(u) - uf'(u)]\psi_1}{\displaystyle\int_\Omega f(u)\psi_1}. \tag{40}$$

Now we have $\mu_1 > 0$, for $f(s) = s^\alpha$, $0 < \alpha < 1$, if $\lambda \le 0$.

We have thus proved the

Theorem 9 *If $\lambda \le 0$, positive solutions to equation (1) are asymptotically stable.* □

The above argument gives the linearized stability for positive solutions if $\lambda \le 0$. From Theorem 8 follow the existence and uniqueness of a positive solution for $0 \le \lambda < \lambda_1$ obtained by the method of sub and supersolutions. But both methods used above are unable to provide stability. It is well-known (e.g., Brezis-Nirenberg Unpublished Notes [10]) that in this case $\mu_1 \ge 0$. But we are not able to exclude $\mu_1 = 0$, as for $m \equiv 1$ or $m(x) \ge m_0 > 0$. However, we can prove the following:

Theorem 10 *Under the above assumptions the unique solution to (4) for $0 \le \lambda < \lambda_1$ is globally asymptotically stable in $L^\infty(\Omega)$.*

Proof. Let $\overline{u} > 0$ be the unique solution to (4) for $0 \le \lambda < \lambda_1$ given. From the method of sub and supersolutions we know that there are sequences u_n of subsolutions (u^n of supersolutions) such that $u_n \to \overline{u}$ ($u^n \to \overline{u}$) in $L^\infty(\Omega)$ (actually in $C_0^1(\overline{\Omega})$) with $u_n \le \overline{u} \le u^n$. Then, for any $\epsilon > 0$, there exists n_0 such that for any $n \ge n_0$ we have

$$\|u^n - u_n\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}, \quad \|\overline{u} - u_n\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}, \quad \|u^n - \overline{u}\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.$$

Let now $v(t)$ be any solution of the associated parabolic problem

$$\begin{cases} v_t - \Delta v = \lambda v + m(x)\,|v|^{\alpha-1}\,v & \text{in } \Omega \times (0,+\infty), \\ v = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ v(x,0) = v_0(x) & \text{on } \Omega. \end{cases} \tag{41}$$

We recall that by the the method of sub and supersolutions we have that if

$$\underline{v}_0 \le v_0 \le \overline{v}_0$$

then we get that

$$\underline{v}_n(x,t) \le v(x,t) \le \overline{v}_n(x,t) \text{ in } \Omega \times (0,+\infty),$$

with

$$\begin{cases} \underline{v}_{n+1,t} - \Delta \underline{v}_{n+1} + m^-(x)\,|\underline{v}_{n+1}|^{\alpha-1}\,\underline{v}_{n+1} = \lambda \underline{v}_{n-} + m^+(x)\,|\underline{v}_n|^{\alpha-1}\,\underline{v}_n & \text{in } \Omega \times (0,+\infty), \\ \underline{v}_{n+1} = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ \underline{v}_{n+1}(x,0) = \underline{v}_0(x) & \text{on } \Omega. \end{cases} \tag{42}$$

A similar problem arises for the study of $\overline{v}_n(x,t)$.Let us show that for any $\epsilon > 0$ there exists a $\delta > 0$ such that if

$$\|\overline{u} - v_0\|_{L^\infty(\Omega)} < \delta$$

then

$$\|\overline{u} - v(.,t)\|_{L^\infty(\Omega)} < \epsilon, \text{ for any } t > 0.$$

Indeed: let $\delta = \frac{\epsilon}{3}$ and let v_0 such that

$$u_n \le v_0 \le u^n \text{ on } \Omega, \text{ for some } n \ge n_0. \tag{43}$$

Then, since

$$\begin{cases} u_{n+1,t}-\Delta u_{n+1}+m^{-}(x)\left|u_{n+1}\right|^{\alpha-1}u_{n+1}\leq \lambda u_n+m^{+}(x)\left|u_n\right|^{\alpha-1}u_n & \text{in } \Omega\times(0,+\infty) \\ u_n=0 & \text{on } \partial\Omega\times(0,+\infty), \\ u_n\leq v_0 & \text{on } \Omega, \end{cases} \tag{44}$$

from the comparison principle, by iteration, we get that

$$u_{n+1}(x)\leq \underline{v}_{n+1}(x,t) \text{ in } \Omega\times(0,+\infty).$$

Analogously, we get that $\overline{v}_{n+1}(x,t)(x,t)\leq u^{n+1}(x)$ in $\Omega\times(0,+\infty)$. Then we conclude that

$$\|v(.,t)-\overline{u}\|_{L^\infty}\leq \|v(.,t)-u_{n+1}\|_{L^\infty}+\left\|u_n-u^{n+1}\right\|_{L^\infty}+\left\|u^{n+1}-\overline{u}\right\|_{L^\infty}<\epsilon.$$

□

Remark 10 *Notice that in the above argument we can take also the choice of* $v_0(x)=u_0(x)$ *(respectively* $v_0(x)=u^0(x)$*) since in this case we can prove that* $\underline{v}_{n+1,t}\geq 0$ *(respectively* $\overline{v}_{n+1,t}(x,t)\leq 0$*) and then we get that* $\lim_{t\to\infty}v(x,t)=\overline{u}(x)$*. We conjecture that in fact the results holds for any initial datum in the interval* $[u_0,u^0]$ *(i.e.,* $u_0\leq v_0\leq u^0$*) but our proof requires a stronger condition.*

Sufficient conditions for existence of only positive solutions can be found in Theorem 1.6 and Propositions 4.5 and 4.6 of [51]. In particular this happens for $\alpha(\lambda)<\alpha<1$ for some $\alpha(\lambda)>0$ depending on λ.

Here we can prove that positive solutions form actually a smooth curve. More precisely we have the

Theorem 11 *Under the above assumptions for existence of only positive solutions for* $\lambda\leq 0$*, these solutions form an increasing smooth* C^∞ *map* $\lambda\to u(\lambda)$ *of* $(-\infty,0]$ *into* $C_0^1(\overline{\Omega})$*. Moreover, they are asymptotically stable.*

Proof. Proposition 3 gives the uniqueness and Proposition 4 that this branch is monotonically increasing. Then it follows from Theorem 10 that they are asymptotically stable and $\mu_1>0$ allows to apply the differentiability properties in [41] proving the C^∞ smoothnes of $u(\lambda)$. □

We have extended, in this way, the result for $m\equiv 1$ in Theorem 1 (Figure 1).

5 Existence results: case $\lambda\geq\lambda_1$

Now we study the existence of non-negative solutions of the problem

$$\begin{cases} -\Delta u=\lambda u+m(x)\left|u\right|^{\alpha-1}u & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{45}$$

where Ω is again a smooth bounded domain in $\mathbb{R}^N$, $0<\alpha<1$, $m\in L^\infty(\Omega)$ changes sign in Ω satisfying (5) and, in this case, $\lambda\geq\lambda_1$.
We start with the case $\lambda>\lambda_1$. If we multiply (45) by u and integrate by parts on Ω we obtain

$$\lambda_1\int_\Omega u^2\leq\int_\Omega|\nabla u|^2=\lambda\int_\Omega u^2+\int_\Omega m(x)\left|u\right|^{\alpha+1},$$

otherwise stated

$$\int_\Omega m(x)\left|u\right|^{\alpha+1}\geq(\lambda_1-\lambda)\int_\Omega u^2.$$

This means that, in principle, the integral $\displaystyle\int_\Omega m(x)\left|u\right|^{\alpha+1}$ may be ≥ 0 or ≤ 0 and then maybe we will have both $N^+\neq\phi$ and $N^-\neq\phi$, getting two (at least) non-negative solutions.

Now we have

$$\int_\Omega \left(|\nabla\varphi_1|^2 - \lambda\varphi_1^2\right) = (\lambda_1 - \lambda)\int_\Omega \varphi_1^2 < 0.$$

and if we also have

$$\int_\Omega m(x)\varphi_1^{\alpha+1} < 0. \tag{46}$$

$\varphi_1 \in L^- \cap B^-$ and $t(\varphi_1)\varphi_1 \in N^-$. We may wonder if $N^+ \neq \phi$ as well.

Reasoning as in [9] we prove the

Lemma 8 *If (46) holds, then $N^+ \neq \emptyset$ for any $\lambda > \lambda_1$.*

PROOF. Pick $x_0 \in \Omega$ such that $B_r(x_0) \subset \Omega^+$ and $\overline{u} \in C_0^\infty(\Omega)$, $\overline{u} > 0$ such that supp $\overline{u} \subset B_r(x_0)$ and define $u_n(x) = n^N\overline{u}(n(x - x_0))$. Then we have

$$\nabla u_n = n^{N+1}\nabla\overline{u}(n(x - x_0))$$

and

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = n^{2(N+1)}\int_\Omega |\nabla\overline{u}(n(x - x_0))|^2 - n^{2N}\lambda\int_\Omega (\overline{u}(n(x - x_0))^2$$

which tends to $+\infty$ with n for any $\lambda > \lambda_1$. Hence $u_n \in L^+ \cap B^+$ and $t(u_n)u_n \in N^+$. □

The following Lemma extends the results in [15] to $\lambda_1 \leq \lambda < \lambda^*$ with λ^* defined below.

Lemma 9 *Assume that (46) holds. Then $\overline{L^-} \cap \overline{B^+} = \emptyset$ if $\lambda_1 \leq \lambda < \lambda^*$, where*

$$\lambda^* = \inf\left\{\frac{\int_\Omega |\nabla u|^2}{\int_\Omega u^2} \mid \int_\Omega m(x)\,|u|^{\alpha+1} \geq 0\right\}. \tag{47}$$

PROOF. First we prove, as in [34], [15], that there exists $\delta > 0$ such that the result holds if $\lambda_1 \leq \lambda < \lambda_1 + \delta$. If not, there exists $\lambda_n \searrow \lambda_1$ and u_n with $\|u_n\| = 1$ such that

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2\right) \leq 0 \quad \int_\Omega m(x)\,|u_n|^{\alpha+1} \geq 0.$$

Then, there exists $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 \leq r < 2^*$. If $u_n \not\to u_0$ we have

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2\right) < \lim\int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2\right) \leq 0$$

which is impossible. Hence $\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2\right) = 0$, $u_0 = k\varphi_1$ with

$$\int_\Omega m(x)k^{\alpha+1}\varphi_1^{\alpha+1} \geq 0$$

and $k = 0$ by (46), a contradiction.

Now, it is enough to show that, for any $\lambda_1 + \delta \leq \overline{\lambda} < \lambda^*$, there exists $\eta > 0$ such that if $\overline{\lambda} \leq \lambda < \overline{\lambda} + \eta$ then $\overline{L^-} \cap \overline{B^+} = \emptyset$. Indeed, if not there exists $\lambda_n \searrow \overline{\lambda}$ and u_n with $\|u_n\| = 1$ such that

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2\right) \leq 0 \quad \int_\Omega m(x)\,|u_n|^{\alpha+1} \geq 0.$$

Then, there exists $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 \leq r < 2^*$ and we have by the l.s.c. of the norm

$$\int_\Omega \left(|\nabla u_0|^2 - \overline{\lambda} u_0^2\right) \leq \underline{\lim}\int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2\right) \leq 0$$

with

$$\int_\Omega m(x)\,|u_0|^{\alpha+1} = \lim \int_\Omega m(x)\,|u_n|^{\alpha+1} \geq 0.$$

Hence

$$\frac{\displaystyle\int_\Omega |\nabla u_0|^2}{\displaystyle\int_\Omega u_0^2} \leq \overline{\lambda} < \lambda^*,$$

with

$$\int_\Omega m(x)\,|u_0|^{\alpha+1} \geq 0.$$

a contradiction. This ends the proof. □

Lemma 10 *For any $\lambda \geq \lambda^*$, we have $\overline{L^-} \cap \overline{B^+} \neq \emptyset$.*

Proof. Let $\lambda_n \searrow \lambda^*$, then, there exists u_n with $\|u_n\| = 1$ such that

$$\frac{\int_\Omega |\nabla u_n|^2}{\int_\Omega u_n^2} \leq \lambda_n, \qquad \int_\Omega m(x)\,|u_n|^{\alpha+1} \geq 0.$$

Then, there exists $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$ for $1 \leq r < 2^*$. We have

$$\int_\Omega m(x)\,|u_0|^{\alpha+1} = \lim \int_\Omega m(x)\,|u_n|^{\alpha+1} \geq 0$$

and $\frac{u_0}{\|u_0\|} \in \overline{B^+}$. If $u_n \nrightarrow u_0$ we have

$$\int_\Omega |\nabla u_0|^2 < \underline{\lim} \int_\Omega |\nabla u_n|^2 \leq \lim \lambda_n \int_\Omega u_n^2 = \lambda^* \int_\Omega u_0^2$$

with $u_0 \neq 0$. Now

$$\frac{\displaystyle\int_\Omega |\nabla u_0|^2}{\displaystyle\int_\Omega u_0^2} < \lambda^*$$

with $\displaystyle\int_\Omega m(x)\,|u_0|^{\alpha+1} \geq 0$, a contradiction.
Hence, $u_n \to u_0$ and

$$\int_\Omega |\nabla u_0|^2 = \lambda^* \int_\Omega u_0^2$$

and $\frac{u_0}{\|u_0\|} \in \overline{L^-} \cap \overline{B^+}$. This shows that λ^* is attained with $u_0 \geq 0$.
If $\lambda > \lambda^*$ we have

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2\right) < \int_\Omega \left(|\nabla u_0|^2 - \lambda^* u_0^2\right) = 0$$

and we have $\frac{u_0}{\|u_0\|} \in \overline{L^-} \cap \overline{B^+}$. □

Corollary 3 *λ^* is attained at $u_0 \geq 0$ such that $\int_\Omega m(x)\,|u_0|^{\alpha+1} = 0$.*

Proof. The first part is contained in the proof of Lemma 10. For the second part, if we assume that $\displaystyle\int_\Omega m(x)\,|u_0|^{\alpha+1} > 0$ then the inf on this open subset is associated with $w \geq 0$ corresponding to the eigenvalue λ_1, and then $\lambda^* = \lambda_1$. But since $\int_\Omega m(x)\varphi_1^{\alpha+1} < 0$, $\lambda_1 < \lambda^*$, a contradiction. □

Lemma 11 *Assume* $\overline{L^-} \cap \overline{B^+} = \emptyset$*. Then we have*
i) $N^0 = \{0\}$,
ii) $0 \notin \overline{N^-}$ *and* N^- *is closed,*
iii) $\overline{N^-} \cap \overline{N^+} = \emptyset$,
iv) N^+ *is bounded.*

PROOF. The proof is similar to the proof of Theorem 4.2 in [15]. i) If $u \in N^0$, $u \neq 0$, then $u/\|u\| \in L^0 \cap B^0 \subset \overline{L^-} \cap \overline{B^+}$, which is impossible.
ii) If not, there exists $u_n \in N^-$ such that $u_n \to 0$. We have

$$0 > \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \int_\Omega m(x) |u_n|^{\alpha+1} \to 0,$$

and if $v_n = \frac{u_n}{\|u_n\|}$, we may suppose that $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$. Then

$$0 > \int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) = \|u_n\|^{\alpha-1} \int_\Omega m(x) |v_n|^{\alpha+1}, \tag{48}$$

and

$$0 \geq \lim \int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) = 1 - \lambda \lim \int_\Omega v_n^2 = 1 - \lambda \int_\Omega v_0^2$$

giving $v_0 \neq 0$. Since the left-hand side in (48) is bounded and $\|u_n\|^{\alpha-1}$ goes to $+\infty$, it should be

$$\lim \int_\Omega m(x) |v_n|^{\alpha+1} = \int_\Omega m(x) |v_0|^{\alpha+1} = 0$$

and $\frac{v_0}{\|v_0\|} \in \overline{B^+}$. We also have $v_n \to v_0$, if not

$$\int_\Omega \left(|\nabla v_0|^2 - \lambda v_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) \leq 0$$

and $\frac{v_0}{\|v_0\|} \in \overline{L^-} \cap \overline{B^+}$, a contradiction. Hence $v_n \to v_0$ and again $\frac{v_0}{\|v_0\|} \in \overline{L^-} \cap \overline{B^+}$, again a contradiction.
iii) We have $\overline{N^-} \cap \overline{N^+} \subset N^- \cap (N^+ \cup N^0) = N^- \cap (N^+ \cup \{0\}) \subset (N^- \cap N^+) \cup (N^- \cap \{0\}) = \emptyset$.
iv) Suppose that N^+ is unbounded. Then there exists $u_n \in N^+$ such that $\|u_n\| \to +\infty$, where

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \int_\Omega m(x) |u_n|^{\alpha+1} > 0.$$

If $v_n = \frac{u_n}{\|u_n\|}$, $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$. We get

$$\int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) = \|u_n\|^{\alpha-1} \int_\Omega m(x) |v_n|^{\alpha+1} \to 0$$

and $0 = 1 - \lambda \lim \int_\Omega v_n^2 = 1 - \lambda \int_\Omega v_0^2$, $v_0 \neq 0$ and $\frac{v_0}{\|v_0\|} \in \overline{B^+}$ since $\int_\Omega m(x) |v_n|^{\alpha+1} > 0$. If $v_n \nrightarrow v_0$ we have

$$\int_\Omega \left(|\nabla v_0|^2 - \lambda v_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) = 0$$

and $\frac{v_0}{\|v_0\|} \in \overline{L^-} \cap \overline{B^+}$, again a contradiction. Hence

$$\lim \int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2 \right) = \int_\Omega \left(|\nabla v_0|^2 - \lambda v_0^2 \right) = 0$$

and $\frac{v_0}{\|v_0\|} \in \overline{L^-} \cap \overline{B^+}$, a contradiction. □

Lemma 12 *The functional* E_λ *is bounded below on* N^+*, for any* $\lambda < \lambda^*$*.*

Proof. If $\|u\| \leq M$ (iv) in Lemma 11) we have

$$\begin{aligned} |E_\lambda(u)| &= \left| \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) |u|^{\alpha+1} \right| \\ &\leq \frac{1-\alpha}{2(\alpha+1)} \|m\|_{L^\infty(\Omega)} \|u\|^{\alpha+1}_{L^{\alpha+1}(\Omega)} \leq \frac{(1-\alpha)c}{2(\alpha+1)} \|m\|_{L^\infty(\Omega)} M^{\alpha+1} \end{aligned}$$

by Sobolev's embedding. □

Theorem 12 *The functional E_λ attains its minimum on N^+, for any $\lambda < \lambda^*$.*

Proof. Let (u_n) be a minimizing sequence. Then

$$E_\lambda(u_n) = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) |u_n|^{\alpha+1} \to \inf_{N^*} E_\lambda < 0,$$

with

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \int_\Omega m(x) |u_n|^{\alpha+1} > 0.$$

N^+ is bounded and then $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$, giving

$$\lim \int_\Omega m(x) |u_n|^{\alpha+1} = \int_\Omega m(x) |u_0|^{\alpha+1} > 0,$$

$u_0 \neq 0$, $\frac{u_0}{\|u_0\|} \in B^+$. But $\overline{L^-} \cap \overline{B^+} = \emptyset$ and then $B^+ \subset L^+$, $\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) > 0$ and $t(u_0)u_0 \in N^+$. Now, if $u_n \nrightarrow u_0$

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \lim \int_\Omega m(x) |u_n|^{\alpha+1} = \int_\Omega m(x) |u_0|^{\alpha+1}$$

and $t(u_0) > 1$ (both integrals are positive).
Now we have

$$\inf_{N^+} E_\lambda = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) |u_0|^{\alpha+1}.$$

On the other hand, since $t(u_0)u_0 \in N$

$$t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) = t(u_0)^{\alpha+1} \int_\Omega m(x) |u_0|^{\alpha+1}$$

and then

$$\begin{aligned} E_\lambda(t(u_0)u_0) &= \frac{1}{2} t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) - \frac{t(u_0)^{\alpha+1}}{\alpha+1} \int_\Omega m(x) |u_0|^{\alpha+1} \\ &= \frac{t(u_0)^{\alpha+1}}{2} \int_\Omega m(x) |u_0|^{\alpha+1} - \frac{t(u_0)^{\alpha+1}}{\alpha+1} \int_\Omega m(x) |u_0|^{\alpha+1} \\ &= \frac{(\alpha-1)t(u_0)^{\alpha+1}}{2(\alpha+1)} \int_\Omega m(x) |u_0|^{\alpha+1} < \inf_{N^+} E_\lambda, \end{aligned}$$

since $t(u_0) > 1$, a contradiction. If $u_n \to u_0$ we have

$$\begin{aligned} \lim E_\lambda(u_n) &= \frac{\alpha-1}{2(\alpha+1)} \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) \\ &= \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) |u_0|^{\alpha+1} = \inf_{N^+} E_\lambda, \end{aligned}$$

and $u_0 \in N^+$ is a minimizer since $u_0 \notin N^0$. □

Now we study the case of the component N^-. Again, we prove first some auxiliary results.

Lemma 13 *All minimizing sequences for N^- are bounded.*

PROOF. Let (u_n) be a minimizing sequence in N^-, we have

$$E_\lambda(u_n) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega m(x)\,|u_n|^{\alpha+1} \to \inf_{N^-} E_\lambda > 0.$$

If $\|u_n\| \to +\infty$, with $v_n = \frac{u_n}{\|u_n\|}$ we have $v_n \rightharpoonup v_0$ and $v_n \to v_0$ in $L^r(\Omega)$, $1 \le r < 2^*$. This implies

$$\int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2\right) = \|u_n\|^{\alpha-1}\int_\Omega m(x)\,|v_n|^{\alpha+1} \to 0$$

and $0 = 1 - \lambda \lim \int_\Omega v_n^2 = 1 - \lambda\int_\Omega v_0^2$, giving $v_0 \neq 0$. If $v_n \nrightarrow v_0$ we have

$$\int_\Omega \left(|\nabla v_0|^2 - \lambda v_0^2\right) < \underline{\lim}\int_\Omega \left(|\nabla v_n|^2 - \lambda v_n^2\right) = 0.$$

It follows that $\frac{v_0}{\|v_0\|} \in L^-$. On the other hand,

$$\lim\int_\Omega m(x)\,|u_n|^{\alpha+1} = \lim \|u_n\|^{\alpha+1}\int_\Omega m(x)\,|v_n|^{\alpha+1} = c < 0.$$

Now we should have

$$\lim\int_\Omega m(x)\,|v_n|^{\alpha+1} = \int_\Omega m(x)\,|v_0|^{\alpha+1} = 0,$$

and $\frac{v_0}{\|v_0\|} \in \overline{L^-} \cap \overline{B^+}$, a contradiction. □

Lemma 14 *We have* $\inf\limits_{N^-} E_{\lambda>0}$.

PROOF. If $u \in N^-$, $E_\lambda(u) \ge 0$. If $\inf\limits_{N^-} E_\lambda = 0$, there exists $u_n \in N^-$ such that $\lim E_\lambda(u_n) = 0$. Then

$$E_\lambda(u_n) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega m(x)\,|u_n|^{\alpha+1} \to 0.$$

Since u_n is bounded by Lemma 13, $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$, $1 \le r < 2^*$. Now, $\int_\Omega m(x)\,|u_0|^{\alpha+1} = 0$ and $\frac{u_0}{\|u_0\|} \in \overline{B^+}$. If $u_n \nrightarrow u_0$ we have

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2\right) < \underline{\lim}\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = 0,$$

and $\frac{u_0}{\|u_0\|} \in \overline{L^-} \cap \overline{B^+}$, again a contradiction. Hence $u_n \to u_0 \neq 0$ by ii) in Lemma 11. Then $\frac{u_0}{\|u_0\|} \in \overline{L^-} \cap \overline{B^+}$, another contradiction. □

Theorem 13 *The functional E_λ attains its minimum on N^-.*

PROOF. Let (u_n) be a minimizing sequence. From Lemmas 13 and 14 we have

$$E_\lambda(u_n) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = \frac{\alpha-1}{2(\alpha+1)}\int_\Omega m(x)\,|u_n|^{\alpha+1} \to \inf E_\lambda > 0,$$

and then

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2\right) = \int_\Omega m(x)\,|u_n|^{\alpha+1} \to c,$$

for some $c < 0$. Since u_n is bounded by Lemma 13, $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$, $1 \le r < 2^*$. We have

$$\lim\int_\Omega m(x)\,|u_n|^{\alpha+1} = \int_\Omega m(x)\,|u_0|^{\alpha+1} < 0,$$

which gives $u_0 \neq 0$ and $\frac{u_0}{\|u_0\|} \in B^-$. If $u_n \nrightarrow u_0$ we get

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla u_n|^2 - \lambda u_n^2 \right) = \lim \int_\Omega m(x) \, |u_n|^{\alpha+1} = \int_\Omega m(x) \, |u_0|^{\alpha+1}$$

and then $t(u_0) < 1$ (both integrals are negative). Moreover $\frac{u_0}{\|u_0\|} \in L^-$ and then $t(u_0)u_0 \in N^-$. Then we have

$$\inf_{N^-} E_\lambda = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x) \, |u_0|^{\alpha+1} .$$

On the other side, since $t(u_0)u_0 \in N$

$$t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) = t(u_0)^{\alpha+1} \int_\Omega m(x) \, |u_0|^{\alpha+1}$$

and we obtain

$$\begin{aligned} E_\lambda(t(u_0)u_0) &= \frac{t(u_0)^2}{2} \int_\Omega \left(|\nabla u_0|^2 - \lambda u_0^2 \right) - \frac{t(u_0)^{\alpha+1}}{\alpha+1} \int_\Omega m(x) \, |u_0|^{\alpha+1} \\ &= \frac{(\alpha-1) t(u_0)^{\alpha+1}}{2(\alpha+1)} \int_\Omega m(x) \, |u_0|^{\alpha+1} < \inf_{N^-} E_\lambda \end{aligned}$$

since $t(u_0) < 1$, a contradiction. The proof ends as in Theorem 12. □

Theorem 14 *Assume (46). For $\lambda_1 < \lambda < \lambda^*$, there are (at least) two non-negative solutions to (45). Both are in $C^{1,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$.*

Proof. Existence follows from Theorems 12 and 13 and $N_\lambda^0 = \{0\}$ insuring that both minimizers on N^+ and N^- are critical points of the functional ([15]). That they are different follows from iii) in Lemma 11. Since $E_\lambda(|u|) = E_\lambda(u)$ both solutions are nonnegative The last part is proved as in Lemma 7. □

Proposition 6 *Assume (46) and that $E_{\lambda_n}(u_n) = \inf_{N_{\lambda_n}^-} E_{\lambda_n}$ where $\lambda_n \downarrow \lambda_1$ (in fact, it refers to a subsequence, as in Proposition 3). We have,*
i) $\|u_n\| \to +\infty$,
ii) $\frac{u_n}{\|u_n\|} \to \varphi_1$,
iii) $\liminf_{\lambda_n \downarrow \lambda_1} E_{\lambda_n}(u_n) = \lim_{\lambda_n \downarrow \lambda_1} E_{\lambda_n}(u_n) = +\infty$.

Proof. i) If not $\|u_n\| \leq C$, $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$. If $u_n \nrightarrow u_0$ we have

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2 \right) = \lim \int_\Omega m(x) \, |u_n|^{\alpha+1} = \int_\Omega m(x) \, |u_0|^{\alpha+1} \leq 0.$$

which is impossible. Then $u_n \to u_0$ and $\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) \leq 0$ giving $v_0 = k\varphi_1$, for some k. Finally

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) = 0 = \int_\Omega m(x) k^{\alpha+1} \varphi_1^{\alpha+1}.$$

Hence $k = 0$ and $u_n \to 0$, a contradiction with $0 \notin \overline{N^-}$ in Lemma 11, ii).
ii) As in Proposition 3.
iii) Assume the result is false. Then

$$\left| \int_\Omega \left(|\nabla u_n|^2 - \lambda_n u_n^2 \right) \right| = \left| \int_\Omega m(x) \, |u_n|^{\alpha+1} \right| \leq C$$

with $\|u_n\| \to +\infty$ by i). If $v_n = \frac{u_n}{\|u_n\|}$ we have $v_n \to \varphi_1$ by ii) and then by i)

$$\int_\Omega \left(|\nabla\varphi_1|^2 - \lambda_1\varphi_1^2\right) = \lim\int_\Omega \left(|\nabla v_n|^2 - \lambda_n v_n^2\right) = \lim \|u_n\|^{\alpha-1}\int_\Omega m(x)\,|v_n|^{\alpha+1} = 0.$$

This means that, by ii),

$$\lim\int_\Omega m(x)\,|v_n|^{\alpha+1} = \int_\Omega m(x)\varphi_1^{\alpha+1} = 0,$$

a contradiction. □

We have proved existence of non-negative solutions for an interval “close” to λ_1. Moreover, we know from Proposition 6, parts i) and ii), that there is bifurcation at infinity at λ_1 and that “close” to $(\lambda_1, +\infty)$ solutions are not only positive but in the interior of the positive cone of $C_0^1(\overline{\Omega})$.

We recall that we already proved that there are not positive solutions for $\lambda > \lambda_1$ “large” without requiring condition (46).

We have just seen that condition (46) implies existence of solutions for $\lambda > \lambda_1$. There is a converse result for positive solutions (not for compact support solutions in general).

Proposition 7 *If there exists a positive solution* $u > 0$ *of*

$$\begin{cases} -\Delta u = \lambda u + m(x)\,|u|^{\alpha-1}\,u & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\end{cases} \tag{49}$$

with $0 < \alpha < 1$ *and* $\lambda > \lambda_1$*, then (46) holds.*

Proof. We follow the one in ([2]) for $\alpha > 1$. If we divide the equation by u^α, multiply by $\varphi_1^{\alpha+1}$ and integrate by parts twice, we get

$$\begin{aligned}\int_\Omega m(x)\varphi_1^{\alpha+1} &= -\int_\Omega \Delta u\frac{\varphi_1^{\alpha+1}}{u^\alpha} - \lambda_1\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}} - (\lambda-\lambda_1)\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}\\ &= \frac{(\alpha+1)\alpha}{\alpha-1}\int_\Omega |\nabla\varphi_1|^2\,\frac{\varphi_1^{\alpha-1}}{u^{\alpha-1}} - \frac{2\lambda_1\alpha}{\alpha-1}\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}} - \alpha\int_\Omega |\nabla u|^2\,\frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}\\ &\quad -(\lambda-\lambda_1)\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}.\end{aligned}$$

On the other hand we have

$$|\nabla\varphi_1|^2 = -\varphi_1^2\Delta\lg\varphi_1 - \lambda_1\varphi_1^2,$$

by using the identity

$$\int_\Omega w\Delta(\log w) = -4\int_\Omega \left|\nabla\sqrt{w}\right|^2$$

and

$$|\nabla u|^2 = -m(x)u^{\alpha+1} - u^2\Delta\lg u - \lambda u^2$$

and replacing above we obtain

$$\begin{aligned}-(\alpha-1)\int_\Omega m(x)\varphi_1^{\alpha+1} &= -\frac{(\alpha+1)\alpha}{\alpha-1}\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}\Delta\lg\varphi_1 + \alpha\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}\Delta\lg u\\ &\quad -\frac{4\lambda_1\alpha}{\alpha-1}\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}} + (\lambda-\lambda_1)(\alpha-1)\int_\Omega \frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}\\ &\le 4\frac{\alpha}{\alpha-1}\left[\int_\Omega \left|\nabla\sqrt{\frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}}\right|^2 - \lambda_1\left|\sqrt{\frac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}}\right|^2\right] < 0\end{aligned}$$

since $\sqrt{\dfrac{\varphi_1^{\alpha+1}}{u^{\alpha-1}}} \notin [\varphi_1]$ and hence (46) holds. □

Corollary 4 *If we assume* $\int_\Omega m(x)\varphi_1^{\alpha+1} \geq 0$*, then there is no positive solution for* $\lambda > \lambda_1$*.*

In particular, this means that bifurcation at infinity at λ_1 is necessarily from the left in this case.

5.1 Case $\lambda = \lambda_1$

The problem under consideration is now

$$\begin{cases} -\Delta u = \lambda_1 u + m(x)\,|u|^{\alpha-1}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{50}$$

The functional in this case becomes

$$J(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) - \frac{1}{\alpha+1}\int_\Omega m(x)\,|u|^{\alpha+1}$$

and the Nehari manifold is defined by

$$N = \left\{u \mid \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) = \int_\Omega m(x)\,|u|^{\alpha+1}\right\} \neq \{0\}.$$

The fibering functions are defined as

$$\phi_u(t) = \frac{t^2}{2}\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) - \frac{t^{\alpha+1}}{\alpha+1}\int_\Omega m(x)\,|u|^{\alpha+1}. \tag{51}$$

Calculating as above we obtain

$$N^+_{\lambda_1} \quad = \quad \left\{u \in N_{\lambda_1} : \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) > 0\right\} = \left\{u \in N : \int_\Omega m(x)\,|u|^{\alpha+1} > 0\right\} \tag{52}$$

$$\tag{53}$$

and similarly

$$N^- = \left\{u \in N : \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) < 0\right\} = \emptyset \tag{54}$$

and

$$N^0 = \left\{u \in N : \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) = 0\right\}, \tag{55}$$

by (46). Analogously

$$L^+ = \left\{u : \|u\| = 1,\ \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) > 0\right\} = \{u : \|u\| = 1,\, u \perp \varphi_1\},$$

$$L^- = \left\{u : \|u\| = 1,\ \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) < 0\right\} = \emptyset,$$

$$L^0 = \left\{u : \|u\| = 1,\ \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2\right) = 0\right\} = \{\pm\varphi_1\}$$

and in a similar way for B^+, B^- and B^0. Notice that we can show as in Section 3 that $N^+ \neq \emptyset$. □

We can prove easily

Lemma 15 *If* $u > 0$ *is a solution to (50) then*

$$\int_\Omega m(x)u^{\alpha+1} \geq 0. \tag{56}$$

□

Lemma 16 *If $u > 0$ is a solution to (50) then*

$$\int_\Omega m(x) u^\alpha \varphi_1 = 0.$$

□

The following result is a counterpart of Lemma 4.

Lemma 17 *There exists $\delta > 0$ such that for any $u \in N^+$ we have*

$$\int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2 \right) \geq \delta \left\| u \right\|^2 .$$

Proof. If not, there exists $u_n \in N^+$ and $\delta_n > 0$ with $\delta_n \to 0$ such that

$$\int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) < \delta_n \left\| u_n \right\|^2 \leq \delta_n C$$

where $\|u_n\| \leq C$ (Lemmas 9 and 11, iv). Then

$$\lim \int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) = 0.$$

Using again that N^+ is bounded, there exists a subsequence such that $u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^2(\Omega)$. If $u_n \nrightarrow u_0$ we have

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) < \lim \int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) = 0,$$

which is impossible. Hence If $u_n \to u_0$ and

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) = 0,$$

giving $u_0 = K\varphi_1$ and finally

$$\int_\Omega m(x) \left| u_0 \right|^{\alpha+1} = \lim \int_\Omega m(x) \left| u_n \right|^{\alpha+1} = \int_\Omega m(x) K^{\alpha+1} \varphi_1^{\alpha+1} \geq 0,$$

a contradiction. □

We see immediately that for any $u \in N^+$,

$$J(u) = \frac{\alpha - 1}{2(\alpha+1)} \int_\Omega \left(|\nabla u|^2 - \lambda_1 u^2 \right) \leq 0.$$

Lemma 18 *We have $\inf\limits_{N^+} J(u) > -\infty$.*

Proof. Indeed, let $u \in N^+$, if $v = \frac{u}{\|u\|}$ we have

$$\begin{aligned} |J(u)| = |J(t(v)v)| &= \frac{1-\alpha}{2(\alpha+1)} t(v)^2 \int_\Omega \left(|\nabla v|^2 - \lambda_1 v^2 \right) \\ &= \frac{1-\alpha}{2(\alpha+1)} \frac{\left(\int_\Omega m(x) \left| v \right|^{\alpha+1} \right)^{\frac{2}{1-\alpha}}}{\left(\int_\Omega \left(|\nabla v|^2 - \lambda_1 v^2 \right) \right)^{\frac{1+\alpha}{1-\alpha}}} \\ &\leq c_1 \frac{\left\| v \right\|_{L^{1+\alpha}}^{\frac{2(1+\alpha)}{1-\alpha}}}{\delta^{\frac{1+\alpha}{1-\alpha}} \left\| v \right\|^{\frac{2(1+\alpha)}{1-\alpha}}} \leq c_2 \end{aligned}$$

by Lemma 17 and Sobolev embedding. □

Lemma 19 *We have* $\inf\limits_{N^+} J(u) < 0$.

Proof. Assume that there exists $u_n \in N^+$ such that

$$\lim \int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) = \lim \int_\Omega m(x) |u_n|^{\alpha+1} = 0.$$

Since (u_n) is bounded by Lemma **??**, $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^r(\Omega)$, $1 \le r < 2^*$. If $u_n \nrightarrow u_0$

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) < \underline{\lim} \int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) = \lim \int_\Omega m(x) |u_n|^{\alpha+1} = \int_\Omega m(x) |u_0|^{\alpha+1} = 0,$$

with $u_0 \neq 0$, which is impossible. Now $u_n \to u_0$ and $\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) = 0$, $u_0 = k\varphi_1$ a contradiction since

$$\int_\Omega m(x) k^{\alpha+1} \varphi_1^{\alpha+1} = 0,$$

if $k \neq 0$. □

Theorem 15 *Assume (46), then* $\inf_{N^+} J$ *is attained and there is a solution to (50).*

Proof. By Lemma **??** the minimizing sequence (u_n) is bounded and then $u_n \rightharpoonup u_0$, $u_n \to u_0$ in $L^r(\Omega)$, $1 \le r < 2^*$. If $u_n \nrightarrow u_0$

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) < \lim \int_\Omega \left(|\nabla u_n|^2 - \lambda_1 u_n^2 \right) = \lim \int_\Omega m(x) |u_n|^{\alpha+1} = \int_\Omega m(x) |u_0|^{\alpha+1},$$

which gives $u_0 \neq 0$ and $t(u_0) > 1$. But in this case $\frac{u_0}{\|u_0\|} \in L^+ \cap B^+$, $t(u_0)u_0 \in N^+$ and

$$\inf_{N^+} J_\lambda = \frac{\alpha - 1}{2(\alpha+1)} \int_\Omega m(x) |u_0|^{\alpha+1}$$

and, on the other hand, since $t(u_0)u_0 \in N$

$$t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) = t(u_0)^{\alpha+1} \int_\Omega m(x) |u_0|^{\alpha+1},$$

giving

$$\begin{aligned} J_\lambda(t(u_0)u_0) &= \frac{1}{2} t(u_0)^2 \int_\Omega \left(|\nabla u_0|^2 - \lambda_1 u_0^2 \right) - \frac{t(u_0)^{\alpha+1}}{\alpha+1} \int_\Omega m(x) |u_0|^{\alpha+1} \\ &= \frac{(\alpha-1) t(u_0)^{\alpha+1}}{2(\alpha+1)} \int_\Omega m(x) |u_0|^{\alpha+1} < \inf_{N^+} J_\lambda, \end{aligned}$$

a contradiction following from $t(u_0) > 1$. Moreover $u_0 \notin N^0$, since $u_0 \neq 0$. □

5.2 Case $\lambda = \lambda^*$

Theorem 16 *There exists* $u_0 \in H_0^1(\Omega)$, $u_0 \ge 0$ *which is a minimizer of* E_{λ^*} *on* $N_{\lambda^*}^+$.

Proof. Let us pick a sequence $\lambda_n \nearrow \lambda^*$ and let $u_n \in N_{\lambda_n}^+$ such that $u_n \ge 0$ and

$$E_{\lambda_n}(u_n) = \inf_{N_{\lambda_n}^+} E_{\lambda_n}$$

with

$$\begin{cases} -\Delta u_n = \lambda_n u_n + m(x) |u_n|^{\alpha-1} u_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{57}$$

Then, the sequence (u_n) is bounded. Indeed, if not λ^* will be a bifurcation point for solutions $u_n \geq 0$ such that $\frac{u_n}{\|u_n\|} \to \varphi_1$ (use Proposition 3, ii) and Proposition 6. ii)) and then $\lambda^* = \lambda_1$, a contradiction.
Since (u_n) is bounded, $\exists\, u_n \rightharpoonup u_0$ and $u_n \to u_0$ in $L^r(\Omega)$, $1 \leq r < 2^*$. This implies that $u_0 \geq 0$ is a weak solution of

$$\begin{cases} -\Delta u = \lambda^* u + m(x)\,|u|^{\alpha-1}\,u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{58}$$

Indeed, for any function $\varphi \in H_0^1(\Omega)$ we have

$$\int_\Omega \nabla u_n \cdot \nabla\varphi = \lambda_n \int_\Omega u_n\varphi + \int_\Omega m(x)\,|u_n|^{\alpha-1}\,u_n\varphi.$$

and going to the limit we obtain

$$\int_\Omega \nabla u_0 \cdot \nabla\varphi = \lambda^* \int_\Omega u_0\varphi + \int_\Omega m(x)\,|u_0|^{\alpha-1}\,u_0\varphi$$

since $|u_n|^{\alpha-1}\,u_n \to |u_0|^{\alpha-1}\,u_0$ in $L^{\frac{2N}{\alpha(N-2)}}(\Omega)$. Next, we prove that

$$\overline{\lim} E_{\lambda_n}(u_n) < 0.$$

If $w \in N^+_{\lambda^*}$

$$\int_\Omega \left(|\nabla w|^2 - \lambda^* w^2\right) > 0$$

and by continuity

$$\int_\Omega \left(|\nabla w|^2 - \lambda_n w^2\right) > 0,$$

for n "large". Then $t_n(w)w \in N^+_{\lambda_n}$. Hence

$$E_{\lambda_n}(u_n) = \inf_{N^+_{\lambda_n}} E_{\lambda_n} \leq E_{\lambda_n}(t_n(w)w) = \inf_{t>0} E_{\lambda_n}(tw) \leq E_{\lambda_n}(w)$$

and passing to the limit we obtain

$$\overline{\lim_{n\to+\infty}} E_{\lambda_n}(u_n) = \overline{\lim_{n\to+\infty}} (\inf_{N_{\lambda_n}} E_{\lambda_n}) \leq E_{\lambda^*}(w) < 0$$

and then

$$E_{\lambda^*}(u_0) \leq E_{\lambda^*}(w)$$

and $u_0 \neq 0$. This means that

$$\int_\Omega \left(|\nabla u_0|^2 - \lambda^* u_0^2\right) > 0$$

and then

$$E_{\lambda^*}(u_0) = \frac{\alpha-1}{2(\alpha+1)} \int_\Omega m(x)\,|u_0|^{\alpha+1} < 0,$$

$u_0 \in N^+_{\lambda^*}$, $u_0 \notin N^0_{\lambda^*}$, and hence u_0 is a minimizer of E_{λ^*} on $N^+_{\lambda^*}$. □

6 Positive and compact support solutions

As it was pointed out in the Introduction, our main interest is to find, among all non-negative solutions, obtained by using the Nehari manifold method, those which are positive on Ω (including flat solutions such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$), or solutions $u \geq 0$ with supp $u \subset \Omega$.

In the one-dimensional case $N = 1$, classical energy methods in ordinary differential equations allow to provide a complete description of the solution set. We exhibit above the example of equation (1) with $\Omega = (-1, 1)$ and $m \equiv -1$, showing also the "transition" along the bifurcation branch (from λ_1) of non-negative solutions, going from solutions $u>0$ with $u'(\pm1)\neq 0$ to compact support solutions through a flat solution $u>0$ with $u'(\pm1)=0$ for a unique value λ^* of λ (see [23] and [25] for details).

For the general case $N>1$ only partial results seem to be available. In [28] we study the case $m(x)\leq 0$, and a necessary condition is given for the existence of solutions such that $\frac{\partial u}{\partial n}=0$ on $\partial\Omega$ if Ω is star-shaped by using an identity of Pohozaev type and, in this case, it is possible to obtain estimates on the parameter λ concerning the situation of the (possible) solutions with this property. We apply below these ideas to our problem. But we start by giving some sufficient conditions to get solutions with compact support by using the method of local supersolutions such as presented in [21]. In fact the following results extend Theorem 5.1 and other results presented in [28].

First of all we will construct some local supersolutions defined in some suitable balls $B_R(x_0)$ for $x_0\in\Omega^-$. To get more global conclusions we will need a certain uniformity on the coefficient $m^-(x)$. Thus, given $q_0\in(0,\|m^-\|_{L^\infty(\Omega)}]$ we introduce the subset of Ω^- given by

$$\Omega^-_{q_0}=\left\{x\in\Omega^- : m^-(x)\geq q_0\right\}.$$

On this subset, $\Omega^-_{q_0}$, $m^+(x)=0$, and then any u_λ solution of (1) satisfies

$$-\Delta u_\lambda+q_0\left|u_\lambda\right|^{\alpha-1}u_\lambda-\lambda u_\lambda\leq-\Delta u_\lambda+m^-(x)\left|u_\lambda\right|^{\alpha-1}u_\lambda-\lambda u_\lambda=0.$$

The behavior on $\Omega^-_{q_0}$, of u_λ solution of (1), depends of the sign of parameter λ. This will be proved with the help of the following auxiliary result.

Proposition 8 *Let $x_0\in\Omega^-$ such that*

$$B_R(x_0)\subset\Omega^-_{q_0},\ \textit{for some } R>0. \tag{59}$$

i) Assume $\lambda\leq 0$. Then, for any $R>0$ satisfying (59) the function

$$U(x:x_0)=C\left|x-x_0\right|^{2/(1-\alpha)} \tag{60}$$

is a local supersolution on $B_R(x_0)$, in the sense that

$$\begin{aligned} -\Delta U+q_0\left|U\right|^{\alpha-1}U-\lambda U &> 0,\ \textit{in } B_R(x_0)-\{x_0\}\,,\ \textit{respectively,}\\ -\Delta U+q_0\left|U\right|^{\alpha-1}U-\lambda U &= 0,\ \textit{in } B_R(x_0), \end{aligned}$$

depending on whether C is such that

$$C < \left[\frac{q_0(1-\alpha)^2}{2(2\alpha+N(1-\alpha))}\right]^{1/(1-\alpha)},\ \textit{or} \tag{61}$$

$$C = \left[\frac{q_0(1-\alpha)^2}{2(2\alpha+N(1-\alpha))}\right]^{1/(1-\alpha)},\ \textit{respectively.} \tag{62}$$

ii) Assume $\lambda>0$. Define

$$\begin{aligned} &f(u):=q_0u^\alpha-\lambda u, \qquad F(u)=\int_0^u f(t)dt=\alpha q_0\frac{u^{\alpha+1}}{\alpha+1}-\frac{\lambda}{2}u^2,\\ &\tau_M=\left[\frac{q_0\alpha}{\lambda}\right]^{1/(1-\alpha)}, \qquad \tau_f=\left[\frac{q_0}{\lambda}\right]^{1/(1-\alpha)}, \qquad \tau_F=\left[\frac{2q_0}{(\alpha+1)\lambda}\right]^{1/(1-\alpha)}. \end{aligned} \tag{63}$$

Then $f(u)\geq 0$ iff $u\in[0,\tau_f]$, $f'(u)\geq 0$ iff $u\in[0,\tau_M]$ and $F(u)\geq 0$ iff $u\in[0,\tau_F]$. Moreover, if for $\mu>0$ we define $\psi_\mu:[0,\tau_F)\to[0,+\infty)$ by

$$\psi_\mu(\tau):=\frac{1}{\sqrt{2\mu}}\int_0^\tau\frac{ds}{\sqrt{F(\tau)}}, \tag{64}$$

then $\psi_\mu(\tau)$ *is strictly increasing and we have:*

$$\begin{aligned}
\psi_\mu(\tau) &= \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\ \arcsin\left(\sqrt{\frac{\lambda(\alpha+1)}{2q_0}\tau^{1-\alpha}}\right), \quad 0 \le \tau < \tau_F, \\
\psi_\mu([0,\tau_F)) &= \Big[0,\ \frac{\pi}{(1-\alpha)\sqrt{\mu\lambda}}\Big) := \Big[0, R_F\Big), \\
\psi_\mu([0,\tau_f)) &= \Big[0,\ \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\arcsin\Big(\sqrt{\frac{\alpha+1}{2}}\Big)\Big) := \Big[0, R_f\Big), \\
\psi_\mu([0,\tau_M)) &= \Big[0,\ \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\arcsin\Big(\sqrt{\frac{(\alpha+1)\alpha}{2}}\Big)\Big) := \Big[0, R_M\Big).
\end{aligned}$$

Let now $\eta(.,\mu) : [0, R_F) \to [0, \tau_F)$ *be the inverse function,.i.e. defined by*

$$\sqrt{2\mu}\tau = \int_0^{\eta(\tau,\mu)} \frac{ds}{\sqrt{F(\tau)}}. \tag{65}$$

Then,

$$\eta(r,\mu) = \left[\frac{2q_0}{\lambda(\alpha+1)}\ \sin^2\left(\frac{(1-\alpha)\sqrt{\mu\lambda}}{2}\,r\right)\right]^{\frac{1}{1-\alpha}}, \quad 0 \le r < R_F. \tag{66}$$

In particular, the function

$$U(x : x_0) = \eta(|x - x_0|\,, \mu)$$

satisfies that

$$-\Delta U + q_0|U|^{\alpha-1}U - \lambda U \ge 0 \quad \textit{in } B_R(x_0), \tag{67}$$

if

$$0 < \mu \le \tfrac{1}{N} \ \textit{and } R \in [0, R_M). \tag{68}$$

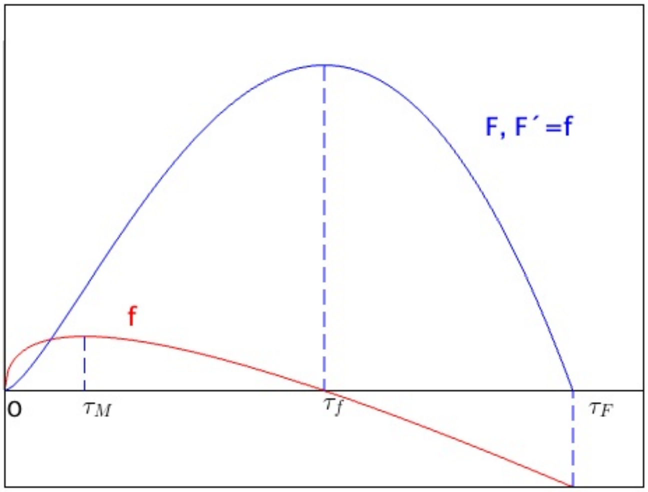

Figure 4: Functions f and F for $\alpha = 1/2, q_0 = 30, \lambda = 10$.

Proof. Part i) was shown, for $\lambda = 0$, in Lemma 1.6 of [21], and it applies also to $\lambda < 0$ since $U(x : x_0) \ge 0$.

To prove ii) we observe that the positive zeros of f and F verify $\tau_f < \tau_F$. In order to get an explicit form of $\psi_\mu(\tau)$, let $A := \dfrac{q_0}{\alpha+1}$, $B := \dfrac{\lambda}{2}$, so that $F(s) = As^{\alpha+1} - Bs^2$.
We introduce the change of variables $U(s) := \dfrac{B}{A}s^{1-\alpha} = \dfrac{\lambda(\alpha+1)}{2q_0}\, s^{1-\alpha}$. Then $As^{\alpha+1} - Bs^2 = As^{\alpha+1}(1-U)$, and $dU = (1-\alpha)\dfrac{B}{A}s^{-\alpha}ds$. Hence

$$\frac{ds}{\sqrt{F(s)}} = \frac{1}{(1-\alpha)\sqrt{B}}\, U^{-1/2}(1-U)^{-1/2}\, dU.$$

Substituting this we get

$$\begin{aligned}\psi_\mu(\tau) &= \frac{1}{\sqrt{2\mu}}\cdot\frac{1}{(1-\alpha)\sqrt{B}}\int_0^{U(\tau)} U^{-1/2}(1-U)^{-1/2}\, dU \\ &= \frac{1}{(1-\alpha)\sqrt{\mu\lambda}}\int_0^{U(\tau)} U^{-1/2}(1-U)^{-1/2}\, dU.\end{aligned}$$

On the other hand, by well-known properties of the Euler Beta functions (see, e.g. [43])

$$B_z\left(\frac{1}{2},\frac{1}{2}\right) = \int_0^z t^{-1/2}(1-t)^{-1/2}\, dt = 2\arcsin(\sqrt{z}).$$

Then, we obtain the closed form (8), i.e.,

$$\psi_\mu(\tau) = \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\ \arcsin\Big(\sqrt{U(\tau)}\Big)\,, \qquad U(\tau) = \frac{\lambda(\alpha+1)}{2q_0}\tau^{\,1-\alpha}.$$

This formula is valid for $0 \le \tau < \tau_F$ (where $U(\tau) \le 1$). Then, as $\tau \to \tau_F$, we have $U(\tau) \to 1^-$. Therefore

$$\psi_\mu([0,\tau_F)) = \left[0,\ \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\arcsin(1)\right) = \left[0,\ \frac{\pi}{(1-\alpha)\sqrt{\mu\lambda}}\right).$$

If one includes the endpoint $\tau = \tau_F$ as a limit, $\psi_\mu(\tau_F)$ reaches the finite value $\dfrac{\pi}{(1-\alpha)\sqrt{\mu\lambda}}$.
Analogously, at $\tau = \tau_f$,

$$U(\tau_f) = \frac{\lambda(\alpha+1)}{2q_0}\tau_f = \frac{\alpha+1}{2},$$

since $0 < \alpha < 1$, this value lies in $(0,1)$ and thus

$$\psi_\mu([0,\tau_f)) = \left[0,\ \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\ \arcsin\Big(\sqrt{\tfrac{\alpha+1}{2}}\Big)\right).$$

Notice that ψ_μ is strictly increasing on $[0,\tau_F)$ and that $\psi_\mu'(\tau) = \dfrac{1}{\sqrt{2\mu}}\dfrac{1}{\sqrt{F(\tau)}}$ diverges as $\tau \to \tau_F$.
Consequently, the image $\psi_\mu([0,\tau_F))$ is a finite open interval, with the derivative blowing up at its right endpoint.
Let us compute now of the inverse function $\eta(r,\mu)$. Let $r = \psi_\mu(\tau)$ and solve for τ as a function of r. We have

$$r = \frac{2}{(1-\alpha)\sqrt{\mu\lambda}}\arcsin\left(\sqrt{\frac{\lambda(\alpha+1)}{2q_0}\,\tau^{\,1-\alpha}}\right).$$

Then

$$\tau^{\,1-\alpha} = \frac{2q_0}{\lambda(\alpha+1)}\ \sin^2\!\left(\frac{(1-\alpha)\sqrt{\mu\lambda}}{2}\, r\right).$$

Hence, we get the explicit expression of function η given in (66). Its domain is clear. The endpoint R_F corresponds to $\tau \to \tau_F$, where the argument of the arcsine tends to 1. This happens when

$$R_F = \frac{\pi}{(1-\alpha)\sqrt{\mu\lambda}}.$$

Therefore

$$\eta(\cdot,\mu) : [0, R_F) \longrightarrow [0, \tau_F), \qquad R_F = \frac{\pi}{(1-\alpha)\sqrt{\mu\lambda}}, \quad \tau_F = \left(\frac{2q_0}{(\alpha+1)\lambda}\right)^{\frac{1}{1-\alpha}}.$$

Moreover, as $r \to R_F^-$, the sine term tends to 1 and hence $\eta(r,\mu) \to \tau_F$, and the derivative

$$\frac{\partial\eta}{\partial r}(r,\mu) = \frac{\sqrt{\mu\lambda}}{\sqrt{2}} \sin((1-\alpha)\sqrt{\mu\lambda}\, r) \cos((1-\alpha)\sqrt{\mu\lambda}\, r) \left[\frac{2q_0}{\lambda(\alpha+1)} \sin^2\!\left(\frac{(1-\alpha)\sqrt{\mu\lambda}}{2}\, r\right)\right]^{\frac{\alpha}{1-\alpha}}$$

converges to 0 as $r \to R_F^-$, since $\cos((1-\alpha)\sqrt{\mu\lambda}\, r) \to 0^+$.

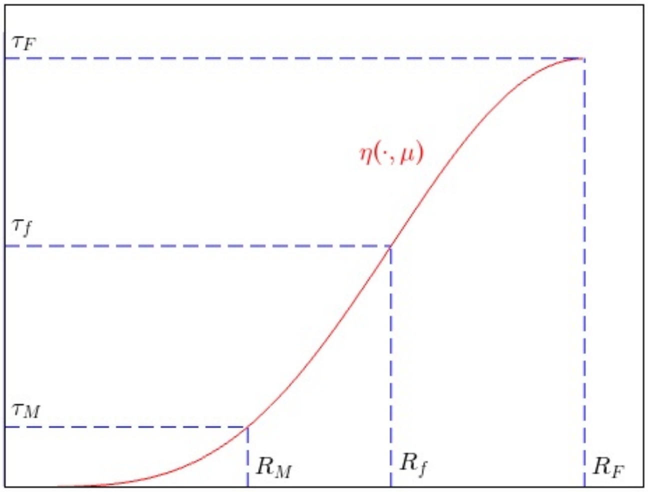

Figure 5: Profile of the barrier function η

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Finally, let us show the supersolution property of the radial function $U(x) = \eta(|x-x_0|,\mu)$. Differentiating $\psi_\mu(\eta(r,\mu)) = r$ yields

$$\psi'_\mu(\eta)\,\eta'(r) = 1,$$

$$\psi'_\mu(\tau) = \frac{1}{\sqrt{2\mu F(\tau)}} \quad \text{and} \quad \text{then } \eta'(r) = \sqrt{2\mu F(\eta(r))}.$$

Differentiating again we get

$$\eta''(r) = \mu\, f(\eta(r)). \tag{69}$$

Let $r = |x-x_0|$. The Laplacian of a radial function is

$$\Delta U = \eta''(r) + \frac{N-1}{r}\eta'(r).$$

Hence

$$-\Delta U + q_0 U^\alpha - \lambda U = -\eta''(r) - \frac{N-1}{r}\eta'(r) + f(\eta(r))$$

$$= (1-\mu)\, f(\eta(r)) - \frac{N-1}{r}\eta'(r), \tag{70}$$

where we used (69). Now we will use now an argument in the proof of Theorem 1.5 of [21]. Let us consider the case $0 < \mu \leq 1/N$ and prove that then we get a supersolution if $R \in [0, R_M)$. For general $\mu > 0$, equation (70) can be rewritten as

$$-\Delta U + q_0 U^\alpha - \lambda U = (1-\mu)\, f(\eta) - \frac{N-1}{r}\eta'$$

$$= (1-\mu)\, f(\eta) - \frac{N-1}{r}\sqrt{2\mu F(\eta(r))}.$$

Now, if $R \in [0, R_M)$ we know that $\eta''(r) = \mu\, f(\eta(r)) \geq 0$, so $\eta(r)$ is a convex function. Consider now the auxiliary function

$$\Phi(r) = \sqrt{2\mu F(\eta(r))} \text{ for } 0 \leq r \leq R_M.$$

Then $\Phi(0) = 0$, $\Phi(r) > 0$ and moreover $\Phi(r)$ is a convex function since $\Phi'(r) = \mu f(\eta(r))$, which is an increasing function if $0 \leq r \leq R_M$ (since in this range of values f is increasing). Then, by elementary results

$$\Phi(r) \leq \Phi'(r) r \text{ for all } 0 < r < R_M,$$

and then

$$-\Delta U + q_0 U^\alpha - \lambda U \geq (1-\mu)\, f(\eta) - (N-1)\mu f(\eta(r)) = (1-\mu N)\, f(\eta) \geq 0 \quad \text{for all } 0 < r < R_M.$$

□

What Proposition 8 indicates us is that for $\lambda > 0$ the useful barrier function $\eta(r, .)$ has a limited height and a limited set of definition $r \in (0, R_M]$, $R_M = \dfrac{2}{(1-\alpha)\sqrt{\mu\lambda}} \arcsin\Big(\sqrt{\dfrac{(\alpha+1)\alpha}{2}}\Big))$, given by $\tau_M = \left[\dfrac{q_0\alpha}{\lambda}\right]^{1/(1-\alpha)}$, which is inversely proportional to λ, and $\tau_M \to +\infty$ when $\lambda \searrow 0$. This coincides with the fact that the barrier function (60) is well defined for any $r > 0$, when $\lambda \leq 0$, and does not have any height limitation.

Since the function $f(u) := q_0 u^\alpha - \lambda u$ is monotone increasing if $u \in [0, \tau_M]$, when $\lambda > 0$, in order to have an estimate on the location of the support of the solution u_λ it is enough to know where is located the level set $[u_\lambda \leq \tau_M] = \big\{x \in \Omega_{q_0}^- : u_\lambda(x) \leq \tau_M\big\}$.

Theorem 17 *Let $\lambda > 0$ and let u_λ be a solution of (1). Assume $0 < \mu \leq \frac{1}{N}$ and that*

$$\Omega_{q_0}^- \cap [u_\lambda \leq \tau_M] \textit{ is not empty.}$$

Let $x_0 \in \Omega_{q_0}^- \cap [u_\lambda \leq \tau_M]$ be such that

$$B_{R_M}(x_0) \cap \Omega \subset \Omega_{q_0}^- \cap [u_\lambda \leq \tau_M], \tag{71}$$

with

$$R_M = \frac{2}{(1-\alpha)\sqrt{\mu\lambda}} \arcsin\Big(\sqrt{\frac{(\alpha+1)\alpha}{2}}\Big). \tag{72}$$

Then

$$u_\lambda(x_0) = 0.$$

In particular, if

$$[u_\lambda \leq \tau_M] \subset \Omega_{q_0}^-$$

then

$$\textit{supp } u_\lambda \subset \Omega \setminus \{x \in \Omega_{q_0}^- \textit{ such that } d(x, [u_\lambda = \tau_M]) \geq R_M\}.$$

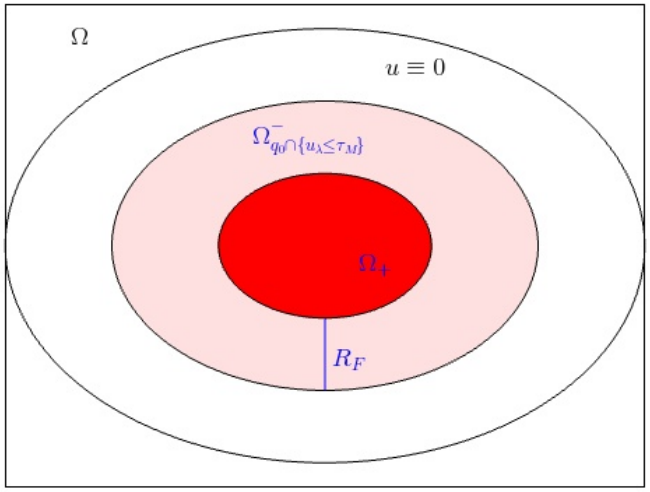

Figure 6: Estimate on the compact support.

Proof. We can apply the method of local super and subsolution, such as presented in the monograph [21] since $f(u) := q_0 u^\alpha - \lambda u$ is monotone increasing if $u \in [0, \tau_M]$. By Proposition 8

$$-\Delta U + q_0 \left|U\right|^{\alpha-1} U - \lambda U \geq 0 \text{ on } B_{R_M}(x_0).$$

Thus, from (71), given any u_λ solution of (1) on Ω^- we know that $m^+(x) = 0$ and then

$$-\Delta u_\lambda + m^-(x) \left|u_\lambda\right|^{\alpha-1} u_\lambda - \lambda u_\lambda = 0 \leq -\Delta U + q_0 \left|U\right|^{\alpha-1} U - \lambda U \text{ in } B_{R_M}(x_0) \cap \Omega^-.$$

Moreover, we have

$$u_\lambda(x) = 0 \leq U(x; x_0) \text{ on } \partial\Omega \cap \partial(B_{R_M}(x_0) \cap \Omega)$$

and, from (72) we get

$$u_\lambda(x) = \tau_M = U(x; x_0) \text{ on } (B_{R_M}(x_0) \cap \Omega) \setminus \partial\Omega.$$

Then, since the comparison principle applies for the operator $v \to -\Delta v + q_0 \left|v\right|^{\alpha-1} v - \lambda v$ with Dirichlet boundary conditions, when $v(x) \in [0, \tau_M]$, we deduce that

$$0 \leq u_\lambda(x) \leq U(x; x_0) \text{ on } B_{R_M}(x_0) \ \cap \Omega^-,$$

and, in particular, $u_\lambda(x_0) = 0$ and the proof is complete. □

Remark 11 *The similar statement when $\lambda \leq 0$ is more standard since there is no constraints for the barrier function, the level τ_M is replaced by any level τ (for instance $\tau = \left\|u_\lambda\right\|_{L^\infty(\Omega)}$) and the radius R_M is replaced by $\psi_\mu(\tau)$ (see the exposition made in [21]). We also point out that when $\lambda > 0$ it is possible to use the barrier function (66) at a level lower than τ_M, but then the location estimate is more limited.*

The proof that under suitable conditions the non-negative solutions u_λ have a compact support (strictly contained on Ω) is a consequence of Theorem 17, nevertheless we can make explicit a more global supersolution which illustrate this fact. For simplicity we will assume now that Ω is a ball.

Theorem 18 *Let $\lambda > 0$, $\Omega = B_{R_\Omega}(0)$, and assume that there exists $R_{q_0} < R_\Omega$ such that $R_{q_0}+R_M \leq R_\Omega$ and*

$$[u_\lambda \leq \tau_M] \subset \Omega_{q_0}^{-} := B_{R_\Omega}(0) \setminus B_{R_{q_0}}(0). \tag{73}$$

Then, if $u_\lambda(x)$ is a radially symmetric solution of (1),

$$\begin{cases} 0 \leq u_\lambda(x) \leq \eta(R_{q_0} - |x|, 1) \text{ on the ring } B_{R_{q_0}+R_M}(0) \setminus B_{R_{q_0}}(0) \\ \text{and} \\ u_\lambda(x) = 0 \text{ on the ring } B_{R_\Omega}(0) \setminus B_{R_{q_0}+R_M}(0), \end{cases}$$

where $\eta(.,1)$ is the function given in (66) corresponding to $\mu = 1$, i.e.,

$$\eta(r,1) = \left[\frac{2q_0}{\lambda(\alpha+1)} \sin^2\left(\frac{(1-\alpha)\sqrt{\lambda}}{2} r \right) \right]^{\frac{1}{1-\alpha}}, \quad 0 \leq r < R_M \tag{74}$$

and

$$R_M = \frac{2}{(1-\alpha)\sqrt{\lambda}} \arcsin\left(\sqrt{\frac{(\alpha+1)\alpha}{2}} \right).$$

Proof. It suffices to show that the function

$$U(x) = \begin{cases} \eta(R_{q_0} - |x|, 1) & \text{if } x \in B_{R_{q_0}+R_M}(0) \setminus B_{R_{q_0}}(0), \\ 0 & \text{if } x \in B_{R_\Omega}(0) \setminus B_{R_{q_0}+R_M}(0), \end{cases}$$

is a supersolution of the problem

$$\begin{cases} -\Delta w + q_0 |w|^{\alpha-1} w - \lambda w = 0, & \text{in } B_{R_\Omega}(0) \setminus B_{R_{q_0}}(0) \\ w = \tau_M = \left[\dfrac{q_0 \alpha}{\lambda} \right]^{1/(1-\alpha)}, & \text{on } \partial B_{R_{q_0}}(0) \\ w = 0 & \text{on } B_{R_\Omega}(0). \end{cases} \tag{75}$$

Indeed, as in the precedent result, we know that if u_λ is a radially symmetric solution of (1) (since on $B_{R_{q_0}}(0) \subset \Omega^-$ $m^+(x) = 0$), then

$$-\Delta u_\lambda + q_0 |u_\lambda|^{\alpha-1} u_\lambda - \lambda u_\lambda \leq -\Delta u_\lambda + m^-(x) |u_\lambda|^{\alpha-1} u_\lambda - \lambda u_\lambda = 0 \text{ in } B_{R_M}(x_0) \cap \Omega^-.$$

Moreover, from the assumption (73) it follows necessarily,

$$\frac{\partial u_\lambda}{\partial r}(r) \leq 0, \; r = |x|,$$

and

$$u_\lambda(R_{q_0}) \leq \tau_M,$$

and thus u_λ is a subsolution of the problem (75). Finally, since functions $U(x)$ and $u_\lambda(x)$ take values in $[0, \tau_M]$ and there the function $f(u) := q_0 u^\alpha - \lambda u$ is monotone increasing, by the comparison principle, we get that

$$0 \leq u_\lambda(x) \leq U(x) \text{ on } B_{R_\Omega}(0) \setminus B_{R_{q_0}}(0),$$

which is the wanted conclusion (74). Now, to check that $U(x)$ is a supersolution of (75) it suffices to remark that now since $\mu = 1$

$$-\Delta U + q_0 U^\alpha - \lambda U = \frac{N-1}{r} \eta'(R_{q_0} - r, 1) > 0 \text{ if } r \in (R_{q_0}, R_{q_0} + R_M).$$

On the other hand,

$$U = 0 \text{ and } \nabla U = 0 \text{ on } \partial B_{R_{q_0}+R_M}(0)$$

and thus the prolongation by zero verifies that $U \in H^1(B_{R_\Omega}(0) \setminus B_{R_{q_0}}(0))$, it is a supersolution of (75), and the proof is complete. □

It is then interesting to know when assumption (73) holds, i.e. when the level set $[u_\lambda \le \tau_M]$ is localized near the boundary $\partial\Omega$, since our main interest now is the study of compact support solutions, arising when $\Omega^+ \subset\subset \Omega$. This can be done in several ways. One possibility is to construct a global supersolution. We already know that if $u_\lambda(x)$ is a solution of (1), with Ω bounded non-necessarily symmetric, then $\|u_\lambda\|_{L^\infty(\Omega)} \le M(m,\alpha,\lambda)$ for some $M(m,\alpha,\lambda) > 0$. Then $u_\lambda(x)$ is also a subsolution of the problem

$$\begin{cases} -\Delta v = H(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{76}$$

with

$$H(x) = \lambda M(m,\alpha,\lambda) + m^+(x) M(m,\alpha,\lambda)^\alpha,$$

and then if v is the unique solution of (76), by the comparison principle for the Laplacian operator, we have that

$$0 \le u_\lambda(x) \le v(x) \text{ on } \Omega.$$

A better estimate can be obtained for small values of λ. The case $\lambda \le 0$ is more standard and we send the reader to the presentation made in [21]. A different case corresponds to when

$$0 < \lambda < \lambda_1. \tag{77}$$

Then we can construct a sharper auxiliary problem in order to estimate u_λ, since u_λ coincides with the unique solution of the problem

$$\begin{cases} -\Delta v + m^-(x) |v|^{\alpha-1} v - \lambda \chi_{\Omega^-}(x) v = G(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{78}$$

with

$$G(x) = \lambda \chi_{\Omega^+}(x) u_\lambda(x) + m^+(x) |u_\lambda(x)|^{\alpha-1} u_\lambda(x).$$

Notice that $G \in L^\infty(\Omega)$, $G = 0$ on Ω^-. Since the comparison principle holds for the problem (78), thanks to the assumption (77), it suffices now to construct a global supersolution for problem (78). It is useful to simplify its formulation by considering the problem

$$\begin{cases} -\Delta w + q_0 \chi_{\Omega^-} |w|^{\alpha-1} w - \lambda \chi_{\Omega^-}(x) w = G(x) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{79}$$

with the assumption

$$0 \le q_0 \chi_{\Omega^-} \le m^-(x) \quad \text{on } \Omega^- \tag{80}$$

Corollary 5 *Assume (77) and (80). Let w be the unique solution of problem (79). Then $0 \le u_\lambda \le w$ in Ω.*

It suffices now to construct an explicit supersolution for the semilinear problem (79). We will do that for the case of a ball, but it can adapted to the a general open bounded set Ω.

Proposition 9 *Let $\Omega = B_{R_\Omega}(0)$ and assume $q_0 > 0$, $\Omega^- = \Omega^-_{q_0} := B_{R_\Omega}(0) \setminus B_{R_+}(0)$, $\Omega^+ = B_{R_+}(0)$, for some $R_+ \in (0, R_\Omega)$, $R_+ + R_M \le R_\Omega$ with R_M given in (72) for $\mu = 1$. Assume the data balance condition*

$$(q_0\alpha)^{1/(1-\alpha)} \lambda^{-(1+\alpha)/(2(1-\alpha))} \sqrt{\frac{(1-\alpha)(2+\alpha)}{\alpha(\alpha+1)}} > \frac{\|G\|_{L^\infty(\Omega)}}{2N} R_+. \tag{81}$$

Then, the function

$$U(x) = \begin{cases} K_0 - K_1 |x| - K_2 |x|^2 & \text{if } x \in B_{R_+}(0), \\ \eta(R_+ + R_M - |x|, 1) & \text{if } x \in B_{R_+ + R_M}(0) \setminus B_{R_+}(0), \\ 0 & \text{if } x \in B_{R_\Omega}(0) \setminus B_{R_+ + R_M}(0), \end{cases}$$

is a supersolution to problem (79) for some suitable positive constants K_i, $i = 0, 1, 2$.

Proof. Once $K_1 > 0$, on $B_{R_+}(0)$ we have

$$-\Delta U + q_0 \chi_{\Omega^-} |U|^{\alpha-1} U - \lambda \chi_{\Omega^-}(x) = -\Delta U \geq 2NK_2 \geq \|G\|_{L^\infty(\Omega)}$$

if

$$K_2 \geq \frac{\|G\|_{L^\infty(\Omega)}}{2N}.$$

On the other hand, to get that $U \in H^1(B_{R_\Omega}(0))$ we must have

$$\begin{cases} K_0 - K_1 R_+ - K_2 R_+^2 = \eta(R_M, 1), \\ K_1 + 2K_2 R_+ = \eta'(R_M, 1). \end{cases}$$

From the second equation we can take, for instance

$$K_1 = \eta'(R_M, 1) - \frac{\|G\|_{L^\infty(\Omega)}}{2N} R_+.$$

Thanks to the information on the function η given in Proposition 8 we know that $K_1 > 0$ if the balance inequality (81) holds. Indeed, we recall that $u_M = \left(\frac{q_0 \alpha}{\lambda}\right)^{1/(1-\alpha)}$, $\eta(R_M, 1) = u_M$, and then

$$\eta'(R_M, 1) = \sqrt{2\,F(u_M)} = u_m \sqrt{\lambda \frac{(1-\alpha)(2+\alpha)}{\alpha(\alpha+1)}} = (q_0 \alpha)^{1/(1-\alpha)} \lambda^{-(1+\alpha)/(2(1-\alpha))} \sqrt{\frac{(1-\alpha)(2+\alpha)}{\alpha(\alpha+1)}},$$

which proves that $K_1 > 0$. Finally, we take

$$K_0 = \eta(R_M, 1) + \left(\eta'(R_M, 1) - \frac{\|G\|_{L^\infty(\Omega)}}{2N} R_+\right) R_+ + \frac{\|G\|_{L^\infty(\Omega)}}{2N} R_+^2,$$

and the proof is complete. □

Remark 12 *Assumption (81) is of the same nature than the so called "balance among the data" made when $\lambda = 0$, $\Omega^- = \Omega$, and in the presence of a non-zero right hand side forcing term $G(x)$ (see Section 1.2b in [21]).*

Remark 13 *In a future paper (in preparation) we will obtain some a priori estimate on $\|u_\lambda\|_{L^\infty(\Omega)}$ when u_λ is solution of a degenerate problem in which $\lambda > \lambda_1$, $\Omega^+ = \phi$ and $\Omega^- \subsetneq \Omega$, as in problem (79).*

Finally, we will use the super and subsolution method to get a solution with compact support, under suitable conditions, but now for the case

$$\lambda \geq \lambda_1. \tag{82}$$

Notice that this solution does not have to coincide with a possible variational solution to the problem.

Theorem 19 *Let $\Omega = B_{R_\Omega}(0)$ and assume $q_0 > 0$, $\Omega^- = \Omega^-_{q_0} := B_{R_\Omega}(0) \setminus B_{R_+}(0)$, $\Omega^+ = B_{R_+}(0)$. Assume (82), (80). There exists a $R \in (0, \min(1, R_M))$ such that, if we assume*

$$R_+ \text{ small enough, and } \lambda \leq \lambda(R_+, R, \left\|m^+\right\|_{L^\infty(\Omega^+)}, \alpha), \tag{83}$$

for some $\lambda(R_+, R, \|m^+\|_{L^\infty(\Omega^+)}, \alpha)) \in [\lambda_1, \lambda_1(B_{R_+}(0)))$, with $\lambda_1(B_{R_+}(0))$ the first eigenvalue of the Laplacian operator on the set $B_{R_+}(0)$ with homogeneous Dirichlet boundary conditions, then there exists at least a solution $u \geq 0$ with compact support to problem (1).

PROOF. The function $\Lambda(u) = \lambda u + m^+(x) |u|^{\alpha-1} u$ is monotone increasing and the problem

$$\left\{ \begin{array}{ll} -\Delta w + q_0 \chi_{\Omega^-} |w|^{\alpha-1} w = G & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{84}$$

satisfies the maximum principle when $G \in L^2(\Omega)$. Then, we can apply the super and subsolution method: if we construct a supersolution u^0 and a subsolution u_0, of problem (1) such that

$$0 \le u_0(x) \le u^0(x) \text{ a.e. } x \in \Omega, \tag{85}$$

then, problem (1) has a maximal solution u^* and a minimal solution u_* on the "interval" $[u_0, u^0]$ of $H_0^1(\Omega)$, i.e., any other solution $u \in [u_0, u^0]$ is such that

$$u_0 \le u_* \le u \le u^* \le u^0 \text{ a.e. } x \in \Omega. \tag{86}$$

Here we are using the iterative algorithm

$$\left\{ \begin{array}{ll} -\Delta u^1 + q_0 \chi_{\Omega^-} \left|u^1\right|^{\alpha-1} u^1 \le \lambda u^0 + m^+(x) \left|u^0\right|^{\alpha-1} u^0 & \text{in } \Omega, \\ u^1 = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{87}$$

with

$$\left\{ \begin{array}{ll} -\Delta u^0 + q_0 \chi_{\Omega^-} \left|u^0\right|^{\alpha-1} u^0 \ge \lambda u^0 + m^+(x) \left|u^0\right|^{\alpha-1} u^0 & \text{in } \Omega, \\ u^0 = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{88}$$

and similarly for the case of the subsolution u_0. We will construct u^0 by adapting the function $U(x)$ used in Proposition 9. For $R \in (0, R_M]$, we consider the function

$$U(x) = \left\{ \begin{array}{ll} K_0 - K_1 |x| - K_2 |x|^2 & \text{if } x \in B_{R_+}(0), \\ C(R_+ + R - |x|)^{2/(1-\alpha)} & \text{if } x \in B_{R_+ + R}(0) \setminus B_{R_+}(0), \\ 0 & \text{if } x \in B_{R_\Omega}(0) \setminus B_{R_+ + R}(0), \end{array} \right.$$

for some suitable positive constants, C and K_i, $i = 0, 1, 2$, to be chosen now. So, in $B_{R_+ + R}(0) \setminus B_{R_+}(0)$ we must have

$$-\Delta U + q_0 \chi_{\Omega^-} |U|^{\alpha-1} U \ge \lambda \chi_{\Omega^-} U. \tag{89}$$

It is easy to see that

$$-\Delta U + q_0 \chi_{\Omega^-} |U|^{\alpha-1} U \ge (q_0 C^\alpha - \frac{2(1+\alpha)}{(1-\alpha)^2} C)(R_+ + R - |x|)^{2\alpha/(1-\alpha)}$$

and , since $\alpha < 1$, condition (89) holds if

$$0 < R \le 1 \tag{90}$$

(since $0 \le s \le s^\alpha$, when $\alpha \in (0,1)$ and $s \in [0,1]$), and

$$q_0 C^\alpha - \frac{2(1+\alpha)}{(1-\alpha)^2} C \ge \lambda C,$$

which holds if

$$C \le \left[\frac{q_0}{\frac{2(1+\alpha)}{(1-\alpha)^2} + \lambda} \right]^{1/(1-\alpha)}. \tag{91}$$

Now, let us chose K_i, $i = 0, 1, 2$ taking into account the rest of conditions in order to get $U \in H^1$, U supersolution. If $K_1 \ge 0$, in $B_{R_+}(0)$, we have

$$-\Delta U + q_0 \chi_{\Omega^-} |U|^{\alpha-1} U \ge 2N K_2 \ge \lambda K_0 + \left\|m^+\right\|_{L^\infty(\Omega^+)} K_0^\alpha \ge \lambda \chi_{\Omega^-} U + m^+(x) |U|^{\alpha-1} U, \tag{92}$$

by taking, e.g.,

$$K_2 = \frac{\lambda K_0 + \|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{2N}. \tag{93}$$

Moreover, in order to get $U \in H^1(\Omega)$ we must verify the two following conditions: continuity on $|x| = R_+$

$$K_0 - K_1 R_+ - K_2 R_+^2 = C R^{\frac{2}{(1-\alpha)}}, \tag{94}$$

and continuity of the gradient on $|x| = R_+$

$$K_1 + 2K_2 R_+ = \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}}. \tag{95}$$

Substituting (93) in (95) we get

$$K_1 = \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}} - \frac{\lambda K_0 + \|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{N} R_+^2.$$

Then condition (94) is equivalent to

$$K_0 \left(1 + \frac{\lambda K_0}{2N} R_+^2 + \frac{\|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{2N} R_+^2 \right) = C R^{\frac{2}{(1-\alpha)}} + \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}}, \tag{96}$$

which determines K_0 and then K_2. We must ensure now that $K_1 \geq 0$. Since

$$K_0 \geq C R^{\frac{2}{(1-\alpha)}} + \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}},$$

we see that $K_1 \geq 0$ once we have

$$\frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}} \geq \frac{\lambda K_0 + \|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{N} R_+^2. \tag{97}$$

But, from (96) we can write

$$K_0 = \Psi_{R_+}^{-1}\left(C R^{\frac{2}{(1-\alpha)}} + \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}}\right),$$

where $\Psi_{R_+} : [0. + \infty) \to [0. + \infty)$ is the strictly increasing function defined by

$$\Psi_{R_+}(s) = s\left(1 + \frac{\lambda}{2N} R_+^2 s + \frac{\|m^+\|_{L^\infty(\Omega^+)}}{2N} R_+^2 s^\alpha\right).$$

Notice that

$$\lim_{R_+ \to 0} \Psi_{R_+}(s) = s.$$

Then,

$$\frac{\lambda K_0 + \|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{N} R_+^2 = \frac{C R^{\frac{2}{(1-\alpha)}} + \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}}}{\Psi_{R_+}^{-1}\left(C R^{\frac{2}{(1-\alpha)}} + \frac{2C}{(1-\alpha)} R^{\frac{(1+\alpha)}{(1-\alpha)}}\right)} - 1.$$

Notice that

$$\frac{s}{\Psi_{R_+}^{-1}(s)} > 1 \text{ since } s < \Psi(s),$$

and that

$$\lim_{R_+ \to 0} \left[\frac{\lambda K_0 + \|m^+\|_{L^\infty(\Omega^+)} K_0^\alpha}{N} R_+^2 \right] = 0.$$

Thus, if we prescribe $R \leq 1$, condition (97) is verified when R_+ is small enough (which also implies that λ is bounded by some $\lambda(R_+, R, \|m^+\|_{L^\infty(\Omega^+)}, \alpha))$. We know that $\lambda(R_+, R, \|m^+\|_{L^\infty(\Omega^+)}, \alpha) \geq \lambda_1$ since condition (97) also holds, in this case, for R_+ small enough.
Once we have constructed a supersolution $u^0 = U$ let us construct now a subsolution u_0, of problem (1), such that

$$0 \leq u_0(x) \leq u^0(x) \text{ a.e. } x \in \Omega. \tag{98}$$

We consider the function

$$\underline{U}(x) = \begin{cases} v(x) & \text{if } x \in B_{R_+}(0), \\ \underline{C}(R_+ + \underline{R} - |x|)^{2/(1-\alpha)} & \text{if } x \in B_{R_+ + \underline{R}}(0) \setminus B_{R_+}(0), \\ 0 & \text{if } x \in B_{R_\Omega}(0) \setminus B_{R_+ + \underline{R}}(0) \end{cases}$$

for some suitable positive constants $\underline{R} < R$ and $\underline{C} > 0$ to be chosen. The subsolution condition over $B_{R_+ + \underline{R}}(0) \setminus B_{R_+}(0)$ holds if we have

$$-\Delta \underline{U} + q_0 \chi_{\Omega^-} |\underline{U}|^{\alpha - 1} \underline{U} \leq 0 \text{ in } B_{R_+ + R}(0) \setminus B_{R_+}(0),$$

and it is satisfied if

$$\underline{C} \geq \left[\frac{q_0 (1-\alpha)^2}{2(2\alpha + N(1-\alpha)} \right]^{1/(1-\alpha)}$$

(see [21]). On $B_{R_+}(0)$ we have

$$-\Delta \underline{U} + q_0 \chi_{\Omega^-} |\underline{U}|^{\alpha - 1} \underline{U} = -\Delta v.$$

Thus we will take $v \geq 0$ such that

$$\begin{cases} -\Delta v = \lambda v & \text{in } B_{R_+}(0), \\ v = \tau & \text{on } B_{R_+}(0), \end{cases}$$

where $\tau \in (0, \tau_M]$ related with R by the expression

$$\tau = \underline{C}\underline{R}^{2/(1-\alpha)}. \tag{99}$$

By well-known results, we know that $\lambda_1 < \lambda_1(B_{R_+}(0)))$. Let $\lambda \in [\lambda_1, \lambda_1(B_{R_+}(0)))$.Then we know (see, e.g, [59]) that defining $w = v - \tau$, we obtain

$$(-\Delta - \lambda) w = \lambda \tau, \qquad w|_{\partial B_R} = 0. \tag{100}$$

This is an inhomogeneous Dirichlet problem which admits a unique solution and then $v = w + \tau$. For a radial solution $v(x) = W(r)$, the equation becomes

$$W''(r) + \frac{n-1}{r} W'(r) + \lambda W(r) = 0, \quad 0 < r < R, \quad W(R) = \tau. \tag{101}$$

It follows that

$$W''(r) = -\frac{n-1}{r} W'(r) - \lambda W(r). \tag{102}$$

Since $W'(r) < 0$ and $W(r) > 0$, the sign of W'' is not fixed. The function is concave ($W'' \leq 0$) if and only if $\frac{n-1}{r}(-W'(r)) \leq \lambda W(r) \quad \forall r \in (0, R]$. This is the case for $N = 1$ (for $N \geq 2$: W'' can change sign because the geometric term $\frac{N-1}{r} W'$ may dominate near the boundary). In any case, we can get a bound of the boundary derivative $W'(R)$. Indeed, let $\nu = \frac{n}{2} - 1$ and $\beta = \sqrt{\lambda} R$. The regular radial solution is

$$W(r) = A\, r^{-\nu} J_\nu(\sqrt{\lambda}\, r), \qquad A = \frac{\tau R^\nu}{J_\nu(\beta)}. \tag{103}$$

Hence

$$W'(R) = \tau\left(-\frac{\nu}{R} + \sqrt{\lambda}\frac{J_\nu'(\beta)}{J_\nu(\beta)}\right). \tag{104}$$

For $0 < \lambda < \lambda_1(B_{R_+}(0)))$, $J_\nu(\beta) > 0$ and $J_{\nu+1}(\beta) > 0$, which implies

$$W'(R) = \tau\left(-\frac{\nu}{R} + \sqrt{\lambda}\frac{J_\nu'(\beta)}{J_\nu(\beta)}\right) < 0. \tag{105}$$

As $\lambda \nearrow \lambda_1(B_{R_+}(0)))$, $J_\nu(\beta) \to 0^+$, $J_\nu'(\beta)$ remains finite (negative), and hence $W'(R) \to -\infty$.
In addition, we can estimate the maximum of $v(x)$. Because $W'(r) < 0$ for $r > 0$, the maximum of W occurs at the center:

$$\max_{B_R} W = W(0). \tag{106}$$

Using the series expansion (see [59]) $J_\nu(z) \sim \dfrac{z^\nu}{2^\nu \Gamma(\nu+1)}$ as $z \to 0$, we find

$$W(0) = \tau\,\frac{\beta^\nu}{2^\nu \Gamma(\nu+1)\,J_\nu(\beta)} \tag{107}$$

or equivalently,

$$\max_{B_R} v = W(0) = \tau\,\frac{(\sqrt{\lambda}\,R)^\nu}{2^\nu \Gamma(\nu+1)\,J_\nu(\sqrt{\lambda}\,R)}\,. \tag{108}$$

(for $N = 1, W(0) = \dfrac{\tau}{\cos(\beta)}$). Then we get that, as $\lambda \nearrow \lambda_1(B_{R_+}(0)))$, since $J_\nu(\beta) \to 0^+$ while $J_\nu'(\beta)$ remains finite we conclude that $W(0) \to +\infty$.
In conclusion, to get a comparison with U we will not ask $\underline{U} \in H^1(\Omega)$. This time, we can apply some well-known results ([44], [6]) saying that if a measure is generated in the transmission curve, it is enough to verify that the measure have the correct sign. So, as said before, the continuity on $|x| = R_+$ requires the condition (99), $\tau = \underline{CR}^{\frac{2}{(1-\alpha)}}$, and the negative sign of the measure generated by the gradient, on $|x| = R_+$, is automatic if we take $|W'(r)|$ large enough (i.e., λ near $\lambda_1(B_{R_+}(0))$ since $|W'(R)| \to +\infty$). Finally, to have $u_0(x) \leq u^0(x)$ a.e. $x \in \Omega$, it is enough to take $\underline{R}$ small (i.e. τ small), so that, for instance,

$$\tau\,\frac{(\sqrt{\lambda}\,R)^\nu}{2^\nu \Gamma(\nu+1)\,J_\nu(\sqrt{\lambda}\,R)} \leq \underline{CR}^{\frac{2}{(1-\alpha)}}.$$

This completes the proof. □

Remark 14 *Results of this kind for (1) or related problems giving information on the existence (or not) of internal "dead cores" can be found, e.g., in [5], where positive solutions are not considered. Necessary and / or sufficient conditions for these "dead cores" can be found in several papers dealing with the version of problem (1) for the* $p-$*Laplacian. A sufficient condition for the existence of "dead cores" for* $\lambda \leq \lambda_1$ *is given in [51], Theorem 1.8 and also in Proposition 5.2 (see Remark 1.9 for details). Another sufficient condition is given in Proposition 2.19 in [9]. On the other side, sufficient conditions for the existence of positive solutions are given in Theorem 1.6 and Propositions 4.5 and 4.6 of [51].*

In the next Section we will make some comments and study the necessary conditions for the formation of the free boundary for the case $N > 1$.

7 Application of Pohozaev's Identity

Next, in order to study the existence (or not) of flat ($u > 0$ in Ω, with $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$) or compact support solutions ($u \geq 0$ in Ω, with supp $u \subset \Omega$), we make the following additional assumption

$$m \in C^1(\overline{\Omega}). \tag{109}$$

The following result was proved in [28]

Lemma 20 *Assume that Ω is starshaped, $\partial\Omega$ is C^2-manifold and (109) holds. Let $u \in C^1(\overline{\Omega})$ be a weak solution of (1). Then the following Pohozaev Identity holds*

$$\frac{(N-2)}{2N}\int_\Omega |\nabla u|^2 - \lambda\frac{1}{2}\int_\Omega |u|^2 - \frac{1}{\alpha+1}\int_\Omega m(x)|u|^{\alpha+1} \tag{110}$$

$$-\frac{1}{N(\alpha+1)}\int_\Omega (x\cdot\nabla m(x))|u|^{\alpha+1} = -\frac{1}{2N}\int_{\partial\Omega}\left|\frac{\partial u}{\partial \nu}\right|^2 (x\cdot\nu(x))\mathrm{d}\sigma(x). \tag{111}$$

If u is a solution such that $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ we have

$$\frac{(N-2)}{2N}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{\alpha+1}\int_\Omega m(x)|u|^{\alpha+1} - \frac{1}{N(\alpha+1)}\int_\Omega (x\cdot\nabla m(x))|u|^{\alpha+1} = 0.$$

In order to study the existence of compactly support solution at λ, we introduce the Pohozaev functional $P_\lambda : H^1_0(\Omega) \to \mathbb{R}$

$$P_\lambda(u) := \frac{(N-2)}{2N}\int_\Omega |\nabla u|^2 - \lambda\frac{1}{2}\int_\Omega |u|^2 - \frac{1}{\alpha+1}\int_\Omega m(x)|u|^{\alpha+1} - \frac{1}{N(\alpha+1)}\int_\Omega (x\cdot\nabla m(x))|u|^{\alpha+1}.$$

Thus, we have

Corollary 6

- *If $u \in C^1(\overline{\Omega})$ is a weak solution of (4) then $P_\lambda(u) \le 0$.*
- *If $u \in C^1(\overline{\Omega})$ is a compactly supported weak solution of (4) then $P_\lambda(u) = 0$.*
- *Under an additional assumption that Ω is strictly star-shaped the converse is also true: if $P_\lambda(u) = 0$ and $u \in C^1(\overline{\Omega})$ is a weak solution of (4), then it has a compact support or it is a flat solution.*

With the notations

$$T(u) = \int_\Omega |\nabla u|^2, \quad G(u) = \int_\Omega |u|^2,$$
$$M(u) = \int_\Omega m(x)|u|^{\alpha+1}, \quad M_\nabla(u) = \int_\Omega (x\cdot\nabla m(x))|u|^{\alpha+1}$$

we have

$$\begin{cases} T(u) - \lambda G(u) - M(u) = 0, \\ \dfrac{N-2}{2N}T(u) - \dfrac{\lambda}{2}G(u) - \dfrac{1}{\alpha+1}M(u) - \dfrac{1}{N(\alpha+1)}M_\nabla(u) = 0. \end{cases} \tag{112}$$

Notice that if u is a compactly support solution,

$$P_\lambda(u) = E_\lambda(u) - \frac{1}{N}T(u) - \frac{1}{N(\alpha+1)}M_\nabla(u).$$

Hence, we obtain

Corollary 7 *If $(x\cdot\nabla m(x)) \ge 0$, $x \in \Omega$, any compactly supported weak solution $u \in C^1(\overline{\Omega})$ of (4) belongs to the set $N^-_\lambda = \{u \in H^1_0(\Omega) \cap C^1(\overline{\Omega}) \mid E'_\lambda(u) = 0,\ E''_\lambda(u) < 0\}$, and thus, $E_\lambda(u) > 0$.* □

Recall that if $E'_\lambda(u) = 0$ and $E''_\lambda(u) < 0$, then $M(u) < 0$ and $T(u) - \lambda G(u) < 0$, and therefore, $\lambda > \lambda_1$ and $\Omega^- \neq 0$.

By multiplying the first equation in (112) by 1/2 and then subtracting the second equation from it, we derive:

$$T(u)=-\frac{N(1-\alpha)M(u)+2M_{\nabla}(u)}{2(1+\alpha)} \tag{113}$$

Hence $N(1-\alpha)M(u)+2M_{\nabla}(u)<0$. Furthermore, under this assumption we obtain

$$\lambda=\Lambda(u):=\frac{T(u)}{G(u)}+\frac{2(1+\alpha)T(u)M(u)}{G(u)[N(1-\alpha)M(u)+2M_{\nabla}(u)]}. \tag{114}$$

Therefore, if $u\in C^1(\overline{\Omega})$ is a compactly supported weak solution of (4), and $M(u)<0$, then

$$\begin{aligned}\Lambda(u)\quad &=\tfrac{T(u)}{G(u)}+\frac{2(1+\alpha)T(u)M(u)}{G(u)[N(1-\alpha)M(u)+2M_{\nabla}(u)]}\\ &\geq \lambda_1+\lambda_1\frac{2(1+\alpha)M(u)}{N(1-\alpha)M(u)+2M_{\nabla}(u)}>\lambda_1.\end{aligned}$$

Thus by Corollary 6 we have

Corollary 8 *Assume that Ω is starshaped, $\partial\Omega$ is C^2-manifold and (109) holds. Let $\lambda\in\mathbb{R}$, and $u\in C^1(\overline{\Omega})$ be a compactly supported weak solution of (4). Then*

- $\displaystyle\int_\Omega [N(1-\alpha)m(x)+2(x,\nabla m(x))]|u|^{\alpha+1}\,\mathrm{d}x<0,$
- $\lambda=\Lambda(u),$
- *if* $M(u)<0$, $\lambda>\lambda_1$.

Introduce

$$\mathcal{E}_m:=\{u\in N_\lambda^-:\ \int_\Omega [N(1-\alpha)m(x)+2(x,\nabla m(x))]|u|^{\alpha+1}\,\mathrm{d}x<0\}$$

Consider

$$\lambda^c=\inf_{u\in\mathcal{E}_m}\Lambda(u), \tag{115}$$

where $\lambda^c=+\infty$ if $\mathcal{E}_m=\emptyset$. From [28] we have

Lemma 21 *Assume that Ω is starshaped, $\partial\Omega$ is C^2-manifold and (109) holds. Then $\lambda^c>\lambda_1$, and there exists a nonnegative minimizer $w_c\in\mathcal{E}_m$ of (115), i.e., $\lambda^c=\Lambda(w_c)$ and $w_c\geq 0$ in Ω.*

Furthermore, for any $\lambda<\lambda^c$, equation (4) has no compactly supported weak solution $u\in C^1(\overline{\Omega})$ with $E_\lambda(u)\geq 0$.

In the case $M(u)=\displaystyle\int_\Omega m(x)|u|^{\alpha+1}<0$, the inequality $N(1-\alpha)M(u)+2M_{\nabla}(u)<0$ holds if $M_{\nabla}(u)<0$. This condition is particularly holds if $(x\cdot\nabla m(x))\leq 0$ and $(x\cdot\nabla m(x))<0$ on some domain $D\subset\Omega$ with $|D|>0$. Note that while this is a sufficient condition, it is not necessary.

Remark 15 *Consider*

$$m(x)=\beta-k\,|x|^{\omega}\,,\ \beta,k,>0,\ \omega>1.$$

Then $m\in C^1(\overline{\Omega})$, and

$$x\cdot\nabla m(x)=-k\omega\,|x|^{\omega}<0,\ \ x\neq 0.$$

Hence, the necessary condition $N(1-\alpha)M(u)+2M_{\nabla}(u)<0$ can be rewritten as follows

$$\{x\in\Omega\mid\ N(1-\alpha)\beta<k(N(1-\alpha)+2\omega)\,|x|^{\omega}\}\neq\emptyset.$$

Thus, we have the following sufficient condition on the nonexistence of compactly supported solutions for (4). Indeed, let $\Omega=\{x:|x|<R\}$. Then the assumptions of Corollary 8 are satisfied, and therefore, equation (4) has no solutions with compact support if

$$N(1-\alpha)\beta>k(N(1-\alpha)+2\omega)R^{\omega}. \tag{116}$$

In the case $M(u) = \int_\Omega m(x)|u|^{\alpha+1}\,dx \geq 0$, the inequality $N(1-\alpha)M(u) + 2M_\nabla(u) < 0$ holds only if $M_\nabla(u) < 0$. Consequently, if $m(x) \geq 0$ and $(x \cdot \nabla m(x)) \geq 0$ for all $x \in \Omega$, then all solutions to equation (4) satisfy $\frac{\partial u}{\partial n} \neq 0$ on a subset $D \subset \partial\Omega$ with $|D| > 0$. In particular, this includes solutions $u > 0$ for which $\frac{\partial u}{\partial n} < 0$ on $\partial\Omega$.

Remark 16 *It is clear that condition* (116) *on the nonexistence of compactly supported solutions of* (4) *does not conflict with the result of Theorem 19 on the existence of such solution.*

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