Nonlinear evolution equations with a non-Lipschitz perturbation: convergence of successive approximations and uniqueness of solutions

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Dedicated to Roger Temam, always admired, on the occasion of his 85th birthday

Abstract

This paper investigates the existence and uniqueness of solutions for a nonlinear evolution equation governed by an m-accretive operator $\mathcal{A}$ in a Banach space, presenting a perturbation term $\mathrm{F}(t, \cdot)$ that does not satisfy the Lipschitz condition. Motivated by nonlinear diffusion models in climatology, we first establish the validity of the Variation of Constants Formula in this nonlinear framework, thereby reformulating the problem as a fixed-point problem for an integral operator. Under structural, boundedness, and unique continuation conditions on an associated scalar integral equation, we prove the convergence of the method of successive approximations towards a unique mild solution. This constructive approach extends previous uniqueness results for semilinear partial differential equations with non-Lipschitz perturbations to the more general setting of nonlinear operators in Banach spaces, which need not be reflexive.

1 Introduction

One of the main motivations of the present work is the intention of the authors to extend to the case of a quasilinear diffusion operator whose results ([19]) dealing with a stochastic diffusive energy balance climate model with a multiplicative noise modeling the Solar variability. In that paper the discontinuous (or multivalued) co-albedo function was replaced by a non-Lipschitz function $\beta(u)$ (since the considered noise was of multiplicative type) which also allows us to identify the location of the polar caps with a precision similar to that achieved by the discontinuous function proposed by Budyko in 1969, but presents fewer difficulties in the treatment of the stochastic framework. The reason for such non-linear diffusion of the type $-((1-x^2)|u'|^{p-2}u')' + \epsilon e(u) = \lambda f(u)$, with $p = 3$, was supported by the paper of Stone [33] (see the mathematical study in [21], and [7])). In fact, a more general model in which the Earth is modelled by a Riemannian manifold without boundary and a diffusion operator similar to the usual p-Laplace operator $\Delta_p u$, with $p > 1$, was considered in ([23] and [25]). Here we will not present the study in a stochastic framework (it could not be too difficult with the help of the treatment of the stochastic p–Laplace operator made, for instance, in [29]). We will consider here a pure deterministic nonlinear evolution problem stated in an abstract setting for a non-linear operator $\mathcal{A}\mathrm{u}(t)$

$$(P)\begin{cases} \dfrac{d\mathrm{u}(t)}{dt} + \mathcal{A}\mathrm{u}(t) = \mathrm{F}\big(t, \mathrm{u}(t)\big), \quad 0 < t < \mathrm{T} < \infty, \\ \mathrm{u}(0) = \mathrm{u}_0. \end{cases} \tag{1}$$

More precisely, we consider a Banach space $\mathcal{X}$, not necessarily reflexive, and assume that

$$\mathcal{A} \text{ is a m-accretive operator on } \mathcal{X}$$

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(to simplify the notation, we write the problem as for $\mathcal{A}$ univalued, but all the results of this paper are valid for the general case of $\mathcal{A}$ multivalued). We also assume that

$$\overline{\mathrm{D}(\mathcal{A})} = \mathcal{X} \text{ and } \mathrm{u}_0 \in \mathcal{X}.$$

On the perturbation term $\mathrm{F}(t, \mathrm{u})$ we will assume the following conditions:

a1) Structural condition. There exists a function $\mathrm{K}(t, \mathrm{U}) \geq 0,\ t \in [0, \mathrm{T}],\ \mathrm{U} \geq 0$, locally integrable function in t for each fixed $\mathrm{U} \geq 0$, continuous and nondecreasing in U for fixed *a.e.* $t \in (0, \mathrm{T})$ and such $\mathrm{K}(t, 0) \equiv 0$ and

$$\|\mathrm{F}(t, \mathrm{u}) - \mathrm{F}(t, \mathrm{v})\|_{\mathcal{X}} \leq \mathrm{K}\big(t, \|\mathrm{u} - \mathrm{v}\|\big)_{\mathcal{X}}\big) \tag{2}$$

for *a.e.* $t \in (0, \mathrm{T})$ and all $\mathrm{u}, \mathrm{v} \in \mathcal{X}$,

a2) Boundedness condition. For a function $\mathrm{K}(t, \mathrm{U})$ as in **a1)** the integral equation

$$\mathrm{U}(t) = \mathrm{U}_0 + \int_0^t \mathrm{K}\big(s, \mathrm{U}(s)\big) ds, \quad 0 \leq t \leq \mathrm{T}, \tag{3}$$

admits a scalar global solution $\mathrm{U}(t)$ on $[0, \mathrm{T}]$, where $\mathrm{U}_0 \geq 0$.

a3) Unique continuation condition. The function $\mathrm{U}(t) \equiv 0$ is the only non negative solution of

$$\mathrm{U}(t) \leq \int_0^t \mathrm{K}\big(s, \mathrm{U}(s)\big) ds, \quad 0 \leq t \leq \mathrm{T}. \tag{4}$$

(see [34] or [19]). We will use the following notation of solution: a function $\mathrm{u} \in \mathcal{C}\big([0, \mathrm{T}] : \mathcal{X}\big)$ is said a *mild solution* of problem (P) if $\mathrm{F}\big(\cdot, \mathrm{u}(\cdot)\big) \in \mathrm{L}^1(0, \mathrm{T}; \mathcal{X})$ (the usual Bochner integral) and

$$\mathrm{u}(t) = \mathrm{S}(t)\mathrm{u}_0 + \int_0^t \mathrm{S}(t - s)\mathrm{F}\big(s, \mathrm{u}(s)\big) ds, \quad 0 \leq t < \mathrm{T}, \tag{5}$$

where $\{\mathrm{S}(t)\}_{t \geq 0}$ is the semigroup of contractions generated by $\mathcal{A}$.

We point out that expression (5) comes from the usual Constants Variations Formula (sometimes called the Variation of parameters or Duhamel's principle) in the case in which operator $\mathcal{A}$ is linear (see *e.g.* [6, Section 5.4]). Nevertheless, we will prove here, it seems that by first time in the literature, that such formula also holds when operator $\mathcal{A}$ is a nonlinear m-accretive operator with $\mathrm{D}(\mathcal{A}) = \mathcal{X}$ (see Section 2). It is clear that the abstract framework allows to apply our results for many other applications different than the mentioned diffusive energy balance climate models (see, *e.g.*, the monographs [9], [5], [6], [3], [20] and [35], to mention only some few of them).

Let us mention that the existence of solutions for perturbed nonlinear Cauchy problems as (P) was obtained in the literature even for more general assumptions than the assumed here: for instance for the case in which $\mathrm{F}\big(\cdot, \mathrm{U}\big)$ is a multivalued operator (see, *e.g.*, [27], [36], [21], [23], [22]). One of the main tools used in those studies is the Kakutani fixed point theorem for multivalued operators. Moreover, it is also well known that when $\mathrm{F}\big(\cdot, \mathrm{u}(\cdot)\big)$ is a multivalued operator the uniqueness of solution only holds in the class of non-degenerate solutions (see [16], [21], [23]). In fact the uniqueness of solutions for quasilinear partial differential equations with some non-Lipschitz perturbations were already studied in the very important paper [17] (see also [11], [24]). The uniqueness of the solution in the semilinear PDEs under the above mentioned conditions were already considered in [37], [34], [15], [4] and [19] in the stochastic framework for semilinear equations.

We want to go further than the paper [17] where criteria for the uniqueness and non-uniqueness of solutions are presented because we want to get a constructive convergent successive approximations scheme for the solutions, in this abstract framework, implying the uniqueness of solutions in the absence of any Lipschitz condition on the perturbation term. We recall that for ordinary differential equations, the uniqueness of solutions can still be obtained under weaker conditions, such as the classical Osgood criterion [31]. In the context of partial differential equations, the situation is significantly more subtle. The pioneering work of Fujita and Watanabe [17] showed that quasi-linear parabolic equations may exhibit both uniqueness and non-uniqueness phenomena depending on the growth behavior of the nonlinear term. Their results highlight important differences between parabolic and elliptic problems. In particular, conditions ensuring uniqueness in elliptic equations, often based on sub- and supersolution techniques (see, *e.g.*, Amann [2] and Pao [32] for linear operators and Díaz-Saa [26] and [22] for the case of the p-Laplace operator), do not directly extend to the parabolic framework.

Our main argument is to search for mild solutions through fixed points u of the operator $\mathcal{G}^{\mathrm{u}_0}$

$$\mathrm{u} = \mathcal{G}^{\mathrm{u}_0}\mathrm{u},$$

defined as

$$\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t) \doteq \mathrm{S}(t)\mathrm{u}_0 + \int_0^t \mathrm{S}(t-s)\mathrm{F}\big(s,\mathrm{u}(s)\big)ds, \quad 0 \le t < \mathrm{T}.$$

In order to prove the existence of the fixed point u a key point is to find some adequate topology in which the relative Picard type approximations

$$\mathrm{u}_{n+1} = \mathcal{G}^{\mathrm{u}_0}\mathrm{u}_n,\ n \ge 0, \tag{6}$$

starting at u_0, converge to some function u.

Theorem 1 *Let $\mathcal{A}$ be a m-accretive operator on the Banach space $\mathcal{X}$, such that $\overline{\mathrm{D}(\mathcal{A})} = \mathcal{X}$, and let $\mathrm{u}_0 \in \mathcal{X}$. Assume* **a1)**, **a2)** *and* **a3)**. *Then the problem (P) has a unique mild solution $\mathrm{u} \in \mathcal{C}\big([0,\mathrm{T}] : \mathcal{X}\big)$. Moreover, u is the fixed point of operator $\mathcal{G}^{\mathrm{u}_0}$ and it can be constructed as the limit of the Picard type approximation $\{\mathrm{u}_n\}_{n\ge 0} \subset \mathcal{C}\big([0,\mathrm{T}] : \mathcal{X}\big)$ given by (6). Moreover, we have the estimate*

$$\|\mathrm{u}\|_{\mathcal{C}\left([0,t]:\mathcal{X}\right)} \le \mathrm{U}(t), \qquad \textit{for all } t \in [0,\mathrm{T}], \tag{7}$$

where $\mathrm{U}(t)$ is the scalar function given as the solution of the integral equation

$$\mathrm{U}(t) = \mathrm{U}_0 + \int_0^t \mathrm{K}\big(s,\mathrm{U}(s)\big)ds, \quad t \in [0,\mathrm{T}], \tag{8}$$

with $\mathrm{U}_0 = \|\mathrm{u}_0\|_{\mathcal{X}}$. In addition, the solution u can be also identified as the unique limit of the implicit Euler scheme associated to the nonhomogeneous equation with the forcing term $g(t) = \mathrm{F}(t,\mathrm{u}(\mathrm{t}))$ a.e. $t \in (0,\mathrm{T})$.

In fact, in the proof of Theorem 1 we will prove the inequality

$$\big\|\mathrm{u} - \widehat{\mathrm{u}}\big\|_{\mathbb{B}_t} \le \|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} + \int_0^t \mathrm{K}\big(s, \|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_s}\big)ds, \quad t \in [0,\mathrm{T}],$$

where u and $\widehat{\mathrm{u}}$ are two eventual fixed points of $\mathcal{G}^{\mathrm{u}_0}$ and $\mathcal{G}^{\widehat{\mathrm{u}}_0}$, respectively, in $\mathcal{C}\big([0,\mathrm{T}] : \mathcal{X}\big)$. For the particular case

$$\mathrm{K}(t,\mathrm{U}) = \phi(t)\vartheta(\mathrm{U}), \quad t \in [0,\mathrm{T}],\ \mathrm{U} \ge 0$$

we have the estimate

$$\int_{\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}}}^{\|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathcal{C}\left([0,t]:\mathcal{X}\right)}} \frac{ds}{\vartheta(s)} \le \int_0^t \phi(s)ds, \quad 0 \le t \le \mathrm{T}$$

provided $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} > 0$ (see Corollay 2 below). Moreover, the Osgood condition

$$\int_{0^+} \frac{d\mathrm{U}}{\vartheta(\mathrm{U})} = \infty$$

implies **a3)**.

The organization of the rest of this paper is as follows. In Section 2 we prove that the classical Constant Variations Formula remains valid beyond the linear framework: we will prove it for a possible nonlinear operator m-accretive operator $\mathcal{A}$ with dense domain on $\mathcal{X}$. Section 3 is devoted to the proof of Theorem 1. Finally, in Section 4 we study the integral equation (3) and give several examples of the kernel $\mathrm{K}(s, \mathrm{U})$ which satisfy the unique continuation condition **a3)**. In particular, we apply the above Theorem 1 to a relevant illustration (see Examples 1 and 2 below) introduced in [19] in studying Stochastic Energy Balance Climate models. Here (3) becomes

$$\mathrm{U}(t) = \mathrm{U}_0 + \mathrm{S}_0 \int_0^t \beta(\mathrm{U}(s))ds, \quad 0 \leq t \leq \mathrm{T}, \tag{9}$$

where $\mathbf{S}_0$ is positive insulation constant and $\beta(s)$ is a co-albedo profile in the hybrid model of [19].

2 Extension of the Constant Variation Formula to nonlinear operators on Banach spaces

Let $\mathcal{X}$ be a Banach space. Let $\mathcal{A} : \mathrm{D}(\mathcal{A}) \subset \mathcal{X} \to 2^{\mathcal{X}}$ be an m-accretive operator, *i.e.* such that

$$\begin{cases} \|(x - \widehat{x}) + \lambda(y - \widehat{y})\| \geq \|x - \widehat{x}\|, \quad (x, y),\ (\widehat{x}, \widehat{y}) \in \mathcal{A}),\ \lambda \geq 0 \\ \text{Rank } (\mathrm{I} + \mathcal{A}) = \mathbb{X}, \end{cases}$$

hold. We assume throughout that

$$\overline{\mathrm{D}(\mathcal{A})} = \mathcal{X}.$$

Let $f \in L^1(0, T; X)$. We consider the evolution problem

$$\begin{cases} u'(t) + \mathcal{A}u(t) \ni f(t), \\ u(0) = u_0. \end{cases}$$

For $\lambda > 0$ define the resolvent

$$\mathrm{J}_\lambda = (\mathrm{I} + \lambda\mathcal{A})^{-1}.$$

Since $\mathcal{A}$ is m-accretive,

$$\mathrm{J}_\lambda : \mathcal{X} \to \mathrm{D}(\mathcal{A})$$

is well defined and nonexpansive:

$$\|\mathrm{J}_\lambda x - \mathrm{J}_\lambda y\| \leq \|x - y\|.$$

The nonlinear semigroup generated by $\mathcal{A}$ is defined by

$$\mathrm{S}(t)x = \lim_{n\to\infty} \left(\mathrm{I} + \frac{t}{n}\mathcal{A}\right)^{-n} x.$$

The Crandall–Liggett Theorem ([14]) guarantees that the limit exists for every $x \in \mathcal{X}$ and defines a contraction semigroup. Moreover, for the non-homogeneous problem we can use the implicit Euler scheme to solve the problem. Let $\lambda = \frac{T}{n}$, $t_k = k\lambda$. From u_0, we consider the approximation

$$u_{k+1} = \mathrm{J}_\lambda(u_k + \lambda f(t_k))$$

Define the piecewise constant interpolation

$$u_\lambda(t) = u_k, \quad t \in [t_k, t_{k+1}).$$

By the Crandall–Liggett theorem we know the convergence of $u_\lambda(t)$ to the unique solution $u(t)$ of the problem. Our goal is to extend the representation formula

$$u(t) = \mathrm{S}(t)u_0 + \int_0^t \mathrm{S}(t - s)f(s)\, ds$$

which is well-known for linear operators, to the case of nonlinear operators $\mathcal{A}$, where now $\mathrm{S}(t)$ is the nonlinear semigroup generated by $\mathcal{A}$.

Theorem 2 *Let $\mathcal{A}$ be an m-accretive operator on a Banach space $\mathcal{X}$ with dense domain. Let $f \in \mathrm{L}^1(0,T;\mathcal{X})$. Then the associated implicit Euler scheme converges to a limit function $u(t)$ which satisfies*

$$u(t) = \mathrm{S}(t)u_0 + \int_0^t \mathrm{S}(t-s)f(s)\,ds.$$

The proof will be a consequence of several general properties. We start with the comparison with the homogeneous scheme. Let

$$v_{k+1} = \mathrm{J}_\lambda v_k, \qquad v_0 = u_0.$$

Then $v_k = \mathrm{J}_\lambda^k u_0$. Recall that by the Crandall–Liggett Theorem ([14]), $v_k \to \mathrm{S}(t_k)u_0$.

Lemma 1 *The following estimate holds*

$$\|u_{k+1} - v_{k+1}\| \leq \|u_k - v_k\| + \lambda\|f(t_k)\|.$$

Proof. Using the nonexpansiveness of J_λ,

$$\begin{aligned} \|u_{k+1} - v_{k+1}\| &= \|\mathrm{J}_\lambda(u_k + \lambda f(t_k)) - \mathrm{J}_\lambda v_k\| \\ &\leq \|u_k + \lambda f(t_k) - v_k\| \\ &\leq \|u_k - v_k\| + \lambda\|f(t_k)\|. \end{aligned}$$

Iterating this inequality gives □

$$\|u_k\| - \|v_k\| \leq \|u_k - v_k\| \leq \sum_{i=0}^{k-1} \lambda\|f(x_i)\|.$$

It will be useful to prove the boundedness of u_k.

Lemma 2 *The sequence $\{u_k\}_k$ is bounded.*

Proof. Using the previous estimate,

$$\|u_k\| \leq \|v_k\| + \sum_{\mathrm{J}=0}^{k-1} \lambda\|f(t_\mathrm{J})\|.$$

Since $\mathrm{S}(t)$ is a contraction,

$$\|v_k\| \leq \|u_0\|.$$

Thus

$$\|u_k\| \leq \|u_0\| + \int_0^T \|f(s)\|ds.$$

□

Let us prove now a consequence of the a priori bounds of the implicit Euler scheme and the nonexpansiveness of the resolvent: a property that is consistent with the classical proof of the Crandall–Liggett Theorem (see ([14])).

Lemma 3 *There exists a constant $\mathrm{C} > 0$, independent of λ, such that the following estimate holds*

$$\|u - \mathrm{J}_\lambda u\| \leq \lambda \mathrm{C}, \tag{10}$$

PROOF. From the definition of the resolvent we have

$$u - \mathrm{J}_\lambda u \in \lambda \mathcal{A}(\mathrm{J}_\lambda u).$$

Thus, there exists $w_\lambda \in \mathcal{A}(\mathrm{J}_\lambda u)$ such that

$$u - \mathrm{J}_\lambda u = \lambda w_\lambda \quad \text{and hence} \quad \|u - \mathrm{J}_\lambda u\| = \lambda \|w_\lambda\|.$$

Therefore, it suffices to obtain a bound on $\|w_\lambda\|$ independent of λ. To this end, we use the implicit Euler scheme

$$u_{k+1} = \mathrm{J}_\lambda (u_k + \lambda f(t_k)).$$

From Lemma 2, the sequence $\{u_k\}_k$ is bounded:

$$\|u_k\| \leq \mathrm{M} \doteq \|u_0\| + \int_0^T \|f(s)\| \, ds.$$

Since J_λ is nonexpansive, it follows that

$$\|\mathrm{J}_\lambda (u_k + \lambda f(t_k))\| \leq \|u_k + \lambda f(t_k)\| \leq \mathrm{M} + \lambda \|f(t_k)\|,$$

so the sequence $\{u_{k+1}\}_k$ also remains in a bounded subset of $\mathcal{X}$, uniformly with respect to λ. Moreover, from the scheme we have

$$u_k + \lambda f(t_k) - u_{k+1} \in \lambda \mathcal{A}(u_{k+1}),$$

so there exists $z_k \in \mathcal{A}(u_{k+1})$ such that

$$u_k + \lambda f(t_k) - u_{k+1} = \lambda z_k.$$

Hence,

$$\|z_k\| \leq \frac{\|u_k - u_{k+1}\|}{\lambda} + \|f(t_k)\|.$$

Using again, the contractivity of J_λ,

$$\|u_{k+1} - u_k\| = \|\mathrm{J}_\lambda (u_k + \lambda f(t_k)) - \mathrm{J}_\lambda u_k\| \leq \lambda \|f(t_k)\|,$$

and therefore

$$\|z_k\| \leq 2\|f(t_k)\|.$$

This shows that the elements of $\mathcal{A}(u_{k+1})$ selected by the scheme are uniformly bounded in terms of $\|f\|_{L^1}$, independently on λ. Since $\mathrm{J}_\lambda u$ belongs to the same bounded region (by nonexpansiveness and the boundedness of the data), we conclude that there exists a constant $\mathrm{C} > 0$, independent of λ, such that

$$\|w_\lambda\| \leq \mathrm{C}.$$

This proves (10). □

Now, let us prove a discrete and nonlinear version of the so-called Variation of Constants Formula (in the framework of linear operators)

Proposition 1 *We have*

$$u_k = \mathrm{J}_\lambda^k u_0 + \sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1} \lambda f(t_i) + r_k$$

with $\|r_k\| \to 0$.

Proof. Let

$$u_{k+1} = \mathrm{J}_\lambda(u_k + \lambda f(t_k)), \qquad u_0 \in X.$$

Let the homogeneous scheme be

$$v_{k+1} = \mathrm{J}_\lambda v_k, \qquad v_0 = u_0.$$

Then $v_k = \mathrm{J}_\lambda^k u_0$. Define

$$w_k = u_k - v_k.$$

Using the recursion we have

$$w_{k+1} = \mathrm{J}_\lambda(u_k + \lambda f(t_k)) - \mathrm{J}_\lambda v_k.$$

Introduce

$$z_{k,\mathrm{J}} \doteq \mathrm{J}_\lambda^{k-i-1}\left(u_{i+1} - \mathrm{J}_\lambda u_i\right)$$

and observe that

$$u_k - v_k = \sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1}\left(u_{i+1} - \mathrm{J}_\lambda u_i\right).$$

Indeed this follows from the telescoping identity

$$u_k = \mathrm{J}_\lambda^k u_0 + \sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1}\big(u_{i+1} - \mathrm{J}_\lambda u_i\big).$$

From the scheme

$$u_{\mathrm{J}+1} = \mathrm{J}_\lambda(u_i + \lambda f(t_i))$$

we define the local error

$$e_i = u_{i+1} - \mathrm{J}_\lambda u_i.$$

Thus

$$u_k = \mathrm{J}_\lambda^k u_0 + \sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1} e_i.$$

The key point here is that

$$e_i = \mathrm{J}_\lambda(u_i + \lambda f(t_i)) - \mathrm{J}_\lambda u_i.$$

Since J_λ is nonexpansive,

$$\|e_i\| \le \lambda \|f(t_i)\|.$$

Denote $g_i \doteq \lambda f(t_i)$, and consider the approximation $\sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1} g_i$. We compare this with the true discrete contribution

$$\sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1} e_i.$$

Define the difference

$$r_k = \sum_{i=0}^{k-1} \mathrm{J}_\lambda^{k-i-1}(e_i - g_i).$$

This is the nonlinear error term. Using the nonexpansiveness of J_λ we obtain

$$\|r_k\| \le \sum_{i=0}^{k-1} \|e_i - g_i\|.$$

Now

$$e_i - g_i = \mathrm{J}_\lambda(u_i + \lambda f(t_i)) - \mathrm{J}_\lambda u_i - \lambda f(t_i).$$

Adding and subtracting $\mathrm{J}_\lambda(u_i) + \lambda f(t_i)$ we get

$$e_i - g_i = \big(\mathrm{J}_\lambda(u_i + \lambda f(t_i)) - \mathrm{J}_\lambda u_i\big) - \lambda f(t_i).$$

Using the Lipschitz property of J_λ

$$\|\mathrm{J}_\lambda(u_i + \lambda f) - \mathrm{J}_\lambda u_i\| \le \lambda \|f\|.$$

Therefore

$$\|e_i - g_i\| \le \|\mathrm{J}_\lambda(u_i + \lambda f(t_i)) - (u_i + \lambda f(t_i))\| + \|\mathrm{J}_\lambda u_i - u_i\|.$$

By the resolvent identity, we know that

$$u - \mathrm{J}_\lambda u \in \lambda \mathcal{A}(\mathrm{J}_\lambda u).$$

By Lemma 3, one knows that

$$\|u - \mathrm{J}_\lambda u\| \le \lambda \mathrm{C},$$

for some constant $\mathrm{C} > 0$ independent of λ. Since the sequence u_i is bounded, we obtain

$$\|e_i - g_i\| \le \mathrm{C}\lambda^2.$$

Thus

$$\|r_k\| \le \mathrm{C} \sum_{i=0}^{k-1} \lambda^2 = \mathrm{C} k \lambda^2.$$

But $k\lambda = t_k \le T$, hence

$$\|r_k\| \le \mathrm{C} T \lambda,$$

and therefore $r_k \to 0$ $(\lambda \to 0)$. □

Proof of the Theorem 2). We proceed in two steps. *Step 1: The case* $f \in \mathcal{C}([0,\mathrm{T}];\mathcal{X})$. Let $t_k \to t$. By the Crandall–Liggett Theorem ([14]), $\mathrm{J}_\lambda^k u_0 \to \mathrm{S}(t)u_0$. Moreover,

$$\mathrm{J}_\lambda^{k-i-1} x \to \mathrm{S}(t - t_i)x.$$

Therefore

$$\sum_{i=0}^{k-1} \lambda \mathrm{J}_\lambda^{k-i-1} f(t_i) \to \int_0^t \mathrm{S}(t-s) f(s)\, ds.$$

This is a Riemann sum for the Bochner integral. Combining the convergence of the homogeneous part and the Riemann convergence of the forcing term gives the desired result.
Step 2: Extension to $f \in \mathrm{L}^1(0,\mathrm{T};\mathcal{X})$. Choose a sequence $\{f_n\}_n \subset \mathcal{C}([0,\mathrm{T}];\mathcal{X})$ such that $f_n \to f$ in L^1. For each n, let $u_k^{(n)}$ be the scheme with forcing f_n, and let $u^{(n)}(t)$ be the limit from Step 1

$$u^{(n)}(t) = \mathrm{S}(t)u_0 + \int_0^t \mathrm{S}(t-s) f_n(s)\, ds.$$

By the discrete Duhamel estimate,

$$\|u_k - u_k^{(n)}\| \le \sum_{i=0}^{k-1} \lambda \|f(t_i) - f_n(t_i)\|.$$

As $\lambda \to 0$, the right-hand side converges to $\displaystyle\int_0^t \|f(s) - f_n(s)\|\, ds$. Hence,

$$\limsup_{\lambda\to 0} \|u_k - u^{(n)}(t)\| \le \limsup_{\lambda\to 0} \big\|u_k - u_k^{(n)}\big\| + \limsup_{\lambda\to 0} \big\|u_k^{(n)} - u^{(n)}(t)\big\| \le \int_0^t \|f(s) - f_n(s)\|\, ds.$$

Taking $n \to \infty$, the right-hand side tends to 0, so

$$u_k \to u(t) \doteq \lim_{n\to\infty} u^{(n)}(t).$$

Finally, we identify $u(t)$. By the continuous dependence estimate for mild solutions,

$$\left\| u^{(n)}(t) - \left(\mathrm{S}(t)u_0 + \int_0^t \mathrm{S}(t-s)f(s)\,ds \right) \right\| \le \int_0^t \|f_n(s) - f(s)\|\,ds \to 0.$$

Thus,

$$u(t) = \mathrm{S}(t)u_0 + \int_0^t \mathrm{S}(t-s)f(s)\,ds.$$

□

Remark 1 Notice that this nonlinear version of the Constant Variations Formula is different to the version given in [1] for nonlinear ordinary differential equations (see also [10]).

3 Existence and uniqueness of solutions of problem (P)

As it was pointed out above, we study the general Cauchy problem governed by a nonlinear equation

$$\begin{cases} \dfrac{d\mathrm{u}(t)}{dt} + \mathcal{A}\mathrm{u}(t) = \mathrm{F}\big(t, \mathrm{u}(t)\big), \quad 0 < t < \mathrm{T} < \infty, \\ \mathrm{u}(0) = \mathrm{u}_0, \end{cases}$$

(see (1) above) on a Banach space $\mathcal{X}$ where we are considering function $\mathrm{u} : [0, \mathrm{T}] \to \mathcal{X}$. In fact, we want to solve (1) in the functional space $\mathcal{C}\big([0, \mathrm{T}] : \mathcal{X}\big)$.

Here for term $\mathrm{F}\big(t, \mathrm{u}\big) \in \mathcal{X},\ (t, \mathrm{u}) \in [0, \mathrm{T}] \times \mathcal{X}$ we assume that the Bochner integral

$$\int_0^{\mathrm{T}} \mathrm{F}(s, \mathrm{u})ds$$

is posed in $\mathcal{X}$ as well as the condition **a1)**, **a2)** and **a3)** (see Introduction). Finally, we assume that $\mathcal{A}$ is a m-accretive operator on $\mathcal{X}$ generating a contraction semigroup $\{\mathrm{S}(t)\}_{t\ge 0}$ in $\overline{\mathrm{D}(\mathcal{A})} \subset \mathcal{X}$. Assume $\overline{\mathrm{D}(\mathcal{A})} = \mathcal{X}$. From the Constant Variations Method we want to solve (1) by *mild solutions*, thus by functions $\mathrm{u} \in \mathcal{C}\big([0, \mathrm{T}] : \mathcal{X}\big)$ satisfying

$$\mathrm{u}(t) = \mathrm{S}(t)\mathrm{u}_0 + \int_0^t \mathrm{S}(t-s)\mathrm{F}\big(s, \mathrm{u}(s)\big)ds, \quad 0 \le t < \mathrm{T}, \tag{11}$$

where $\mathrm{u}_0 \in \mathcal{X}$. So that, we search mild solutions by means of the existence of fixed points u of $\mathcal{G}^{\mathrm{u}_0}$

$$\mathrm{u} = \mathcal{G}^{\mathrm{u}_0}\mathrm{u}, \tag{12}$$

for the integral operator

$$\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t) \doteq \mathrm{S}(t)\mathrm{u}_0 + \int_0^t \mathrm{S}(t-s)\mathrm{F}\big(s, \mathrm{u}(s)\big)ds, \quad 0 \le t < \mathrm{T}. \tag{13}$$

The goal is to solve the fixed-point problem in the space $\mathcal{C}\big([0, \mathrm{T}] : \mathcal{X}\big)$. By simplicity we denote $\mathbb{B}_{\mathrm{T}} \doteq \mathcal{C}\big([0, \mathrm{T}] : \mathcal{X}\big)$ equipped with the norm

$$\|\mathrm{u}\|_{\mathbb{B}_{\mathrm{T}}} \doteq \sup_{0\le s\le \mathrm{T}} \|\mathrm{u}(s)\|_{\mathcal{X}}.$$

As it follows from the notations, the function $\mathrm{u} \in \mathbb{B}_{\mathrm{T}}$ is evaluated in the Banach space $\mathcal{X}$ and $\mathrm{U} \in \mathcal{C}([0, \mathrm{T}] : \mathbb{R})$ is an scalar function.

Remark 2 In considering assumptions on the term $\mathrm{F}(s,\mathrm{U})$ we may pick up the choice

$$\mathrm{K}(t,\mathrm{U})=\phi(t)\vartheta(\mathrm{U}),\quad t\in[0,\mathrm{T}],\ \mathrm{U}\geq 0,$$

where $\vartheta:]0,\infty[\to]0,\infty[$ is a continuous and increasing function verifying $\vartheta(0^+)=0$ and ϕ is an integrable and non negative function on $[0,\mathrm{T}]$. In this case, the assumption **a2**) is immedate (see Remark 6). We emphasize that equality in (3) cannot be substituted by an inequality (see the proof of Theorem 1 and Remark 5).
The condition **a3**) is studied in Section 4). As it is well known, in the Lipschitz case

$$\vartheta(\mathrm{U})=\mathrm{U},\quad t\in[0,\mathrm{T}],\ \mathrm{U}\geq 0,$$

the condition **a3**) follows from the Gronwall Inequality (see Corollary 3). □

In Section 4 below, we pick up some general comments in which the boundedness condition **a2**) and the unique continuation criterion **a3**) hold.

Remark 3 In order to provide inequalities as (2) we may consider real functions $\vartheta:[0,\infty[\to[0,\infty[$ satisfying $\vartheta(0)\geq 0$ with the subadditive property

$$\vartheta(\mathrm{U})+\vartheta(\mathrm{W})\geq\vartheta(\mathrm{U}+\mathrm{W})\quad\Leftrightarrow\quad\vartheta(\mathrm{U}+\mathrm{W})-\vartheta(\mathrm{U})\leq\vartheta(\mathrm{W}),\quad \mathrm{U},\mathrm{W}\geq 0.\tag{14}$$

So that, let $\mathrm{U},\mathrm{V}\geq 0$ such that $\mathrm{V}\geq\mathrm{U}$ form $\mathrm{W}=\mathrm{V}-\mathrm{U}\geq 0$ for which

$$0\leq\vartheta(\mathrm{V})-\vartheta(\mathrm{U})=\vartheta(\mathrm{W}+\mathrm{U})-\vartheta(\mathrm{U})\leq\vartheta(\mathrm{W})=\vartheta(\mathrm{V}-\mathrm{U}),$$

whenever ϑ is nondecreasing. By means of a similar reasoning, we conclude

$$|\vartheta(\mathrm{U})-\vartheta(\mathrm{V})|\leq\vartheta(|\mathrm{U}-\mathrm{V}|),\quad \mathrm{U},\mathrm{V}\geq 0,\tag{15}$$

provided ϑ is nondecreasing. In this case, under (14), the condition **a1)** holds for the choice $\mathrm{F}(t,\mathrm{U})=\mathrm{K}(t,\mathrm{U})=\phi(t)\vartheta(\mathrm{U}),\ t\in[0,\mathrm{T}],\mathrm{U}\geq 0$. □

Remark 4 The subaditive property (14) is satisfied by real concave functions $\vartheta:[0,\infty[\to[0,\infty[$ satisfying $\vartheta(0)\geq 0$. Indeed, it follows

$$\vartheta(\lambda\mathrm{Z})=\vartheta\big(\lambda\mathrm{Z}+(1-\lambda)0\big)\geq\lambda\vartheta(\mathrm{Z})+(1-\lambda)\vartheta(0)\geq\lambda\vartheta(\mathrm{Z}),\quad z\geq 0,\ 0<\lambda<1.$$

In particular, given $\mathrm{U},\mathrm{W}>0$ by choosing $\lambda_{\mathrm{U,W}}=\dfrac{\mathrm{U}}{\mathrm{U}+\mathrm{W}}\in]0,1[$ it follows

$$\begin{cases}\vartheta(\mathrm{U})=\vartheta\big(\lambda_{\mathrm{U,W}}(\mathrm{U}+\mathrm{W})\big)\geq\lambda_{\mathrm{U,W}}\vartheta(\mathrm{U}+\mathrm{W}),\\ \vartheta(\mathrm{W})=\vartheta\big((1-\lambda_{\mathrm{U,W}})(\mathrm{U}+\mathrm{W})\big)\geq(1-\lambda_{\mathrm{U,W}})\vartheta(\mathrm{U}+\mathrm{W}),\end{cases}$$

whence one concludes the subadditive property

$$\vartheta(\mathrm{U})+\vartheta(\mathrm{W})\geq\vartheta(\mathrm{U}+\mathrm{W}).$$

Arguing as in [18, Lemma 4.3], we may obtain the subadditive property (14) by transfer without concavity settings. Indeed, assume

$$q(\mathrm{U})+q(\mathrm{V})\geq q(\mathrm{U}+\mathrm{V}),\quad \mathrm{U},\mathrm{V}\geq 0$$

and

$$\frac{\vartheta(\mathrm{U})}{q(\mathrm{U})}\quad\text{is non increasing,}\tag{16}$$

provided $q(\mathrm{U})>0,\ \mathrm{U}>0$. Then

$$\vartheta(\mathrm{U})+\vartheta(\mathrm{V})=q(\mathrm{U})\frac{\vartheta(\mathrm{U})}{q(\mathrm{U})}+q(\mathrm{V})\frac{\vartheta(\mathrm{V})}{q(\mathrm{V})}\geq(q(\mathrm{U})+q(\mathrm{V}))\frac{\vartheta(\mathrm{U}+\mathrm{V})}{q(\mathrm{U}+\mathrm{V})}\geq\vartheta(\mathrm{U}+\mathrm{V}),$$

thus, the subadditivity of the function $q(\mathrm{U})$ is transferred to the function $\vartheta(\mathrm{U})$ provided (16). In particular, any function ϑ such that

$$\frac{\vartheta(\mathrm{U})}{\mathrm{U}^m} \quad \text{is non increasing}$$

for some $0 < m \leq 1$ is subadditive and the inequality (15) holds whenever ϑ is nondecreasing. $\square$

Example 1 Let us consider the decreasind function

$$\vartheta(\mathrm{U}) = \mathrm{U} \ln \mathrm{U}, \quad 0 \leq \mathrm{U} \leq e^{-1}. \tag{17}$$

From Remark 4 one proves that ϑ satisfies the inequality

$$|\vartheta(\mathrm{U}) - \vartheta(\mathrm{V})| \leq \vartheta(|\mathrm{U} - \mathrm{V}|), \quad \mathrm{U}, \mathrm{V} \geq 0.$$

Moreover, from (39) below the function ϑ satisfies (32). Then we come back to the co-albedo profile

$$\beta(\mathrm{U}) = \begin{cases} \beta_i, & \text{if } \mathrm{U} < 0, \\ \dfrac{\beta_w - \beta_i}{\delta \ln \delta} \vartheta(\mathrm{U}) + \beta_i, & \text{if } 0 \leq \mathrm{U} \leq \delta, \\ \beta_w, & \text{if } \mathrm{U} > \delta, \end{cases} \tag{18}$$

with $0 < \delta < e^{-1}$ that governs the co-albedo function introduced in [19] (see Figure 1). We note

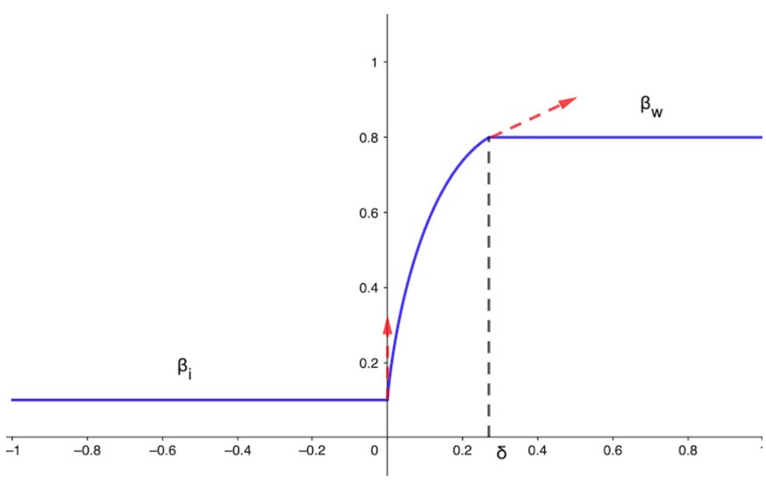

Figure 1: Co-albedo profile (see (18))

that $\beta \in \mathcal{C}(\mathbb{R}) \cap \mathcal{C}^1\big(\mathbb{R} \setminus \{0, \delta\}\big)$, with

$$\begin{cases} \beta'(0^-) = 0 \quad \text{and} \quad \beta'(0^+) = +\infty, \\ \beta'\big(\delta^-\big) > 0 \quad \text{and} \quad \beta'(\delta^+) = 0. \end{cases}$$

We claim that the profile β verifies

$$|\beta(\mathrm{U}) - \beta(\mathrm{V})| \leq \vartheta(|\mathrm{U} - \mathrm{V}|), \quad \mathrm{U}, \mathrm{V} \in \mathbb{R}.$$

Indeed, when $\mathrm{V} \leq 0$ one has

$$\beta(\mathrm{U}) - \beta(\mathrm{V}) = \vartheta(\mathrm{U}) - \vartheta(0) = \vartheta(\mathrm{U}) \leq \vartheta(\mathrm{U} - \mathrm{V}), \quad \mathrm{V} \leq 0 \leq \mathrm{U} \leq \delta.$$

Analogously, if $\mathrm{U}\geq\delta$ one has

$$\beta(\mathrm{U})-\beta(\mathrm{V})=\beta_w-\frac{\beta_w-\beta_i}{\delta\ln\delta}\vartheta(\mathrm{V})-\beta_i\leq 0\leq\vartheta(\mathrm{U}-\mathrm{V}),\quad \mathrm{U}\geq\delta\geq\mathrm{V}\geq 0.$$

Finally, the claim follows from

$$\begin{cases}\beta(\mathrm{U})-\beta(\mathrm{V})=\vartheta(\mathrm{U})-\vartheta(\mathrm{V})\leq\vartheta(\mathrm{U}-\mathrm{V}), & \delta\geq\mathrm{U}\geq\mathrm{V}\geq 0,\\ \beta(\mathrm{U})-\beta(\mathrm{V})=\vartheta(\delta)-\vartheta(0)=\vartheta(\delta)\leq\vartheta(\mathrm{U}-\mathrm{V}), & \mathrm{U}\geq\delta\geq 0\geq\mathrm{V}.\end{cases}$$

Thus, for the choice $\mathrm{F}(t,\mathrm{U})=\mathbf{S}_0\beta(\mathrm{U}),\ t\in[0,\mathrm{T}],\ \mathrm{U}\in\mathbb{R}$, the condition **a1)** holds for the function $\mathrm{K}(t,\mathrm{U})=\mathbf{S}_0\vartheta(\mathrm{U}),\ t\in[0,\mathrm{T}],\ \mathrm{U}\geq 0$. Here $\mathbf{S}_0$ is a positive constant. We come back to this choice in the Example 2 below. □

Proposition 2 *Assuming* **a1**), *the operator* $\mathcal{G}^{\mathrm{u}_0}:\mathbb{B}_\mathrm{T}\to\mathbb{B}_\mathrm{T}$ *given by* (13) *is well defined. Moreover, one has the estimate*

$$\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}-\mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{v}\right\|_{\mathbb{B}_t}\leq\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|_{\mathcal{X}}+\int_0^t\mathrm{K}\big(s,\|\mathrm{u}-\mathrm{v}\|_{\mathbb{B}_s}\big)ds,\quad t\in[0,\mathrm{T}]. \tag{19}$$

that implies the continuity of $\mathcal{G}^{\mathrm{u}_0}$.

Proof. Let $\mathrm{u}\in\mathbb{B}_\mathrm{T}$. Then

$$\begin{aligned}\sup_{0\leq t\leq\mathrm{T}}\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t)\right\|_{\mathcal{X}}&\leq\sup_{0\leq t\leq\mathrm{T}}\left\|\mathrm{S}(t)\mathrm{u}_0\right\|_{\mathcal{X}}+\sup_{0\leq t\leq\mathrm{T}}\left\|\int_0^t\mathrm{S}(t-s)\mathrm{F}\big(s,\mathrm{u}(s)\big)\right\|_{\mathcal{X}}ds\\&\leq\|\mathrm{u}_0\|_{\mathcal{X}}+\sup_{0\leq t\leq\mathrm{T}}\int_0^t\|\mathrm{F}\big(s,\mathrm{u}(s)\big)\|_{\mathcal{X}}ds\\&\leq\|\mathrm{u}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\mathrm{K}\big(s,\|\mathrm{u}(s)\|_{\mathcal{X}}\big)ds\\&\leq\|\mathrm{u}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\mathrm{K}\big(s,\|\mathrm{u}\|_{\mathbb{B}_\mathrm{T}}\big)ds<\infty,\end{aligned}$$

thus

$$\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}\right\|_{\mathbb{B}_\mathrm{T}}\leq\|\mathrm{u}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\mathrm{K}\big(s,\|\mathrm{u}\|_{\mathbb{B}_\mathrm{T}}\big)ds<\infty, \tag{20}$$

whence $\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}\right\|_{\mathbb{B}_\mathrm{T}}<\infty$ and $\mathcal{G}^{\mathrm{u}_0}\mathrm{u}\in\mathbb{B}_\mathrm{T}$. On the other hand, let $\mathrm{u},\mathrm{v}\in\mathbb{B}_\mathrm{T}$. Then, by reasoning as above, and using the fact that the semigroup is of contractions, we obtain

$$\begin{aligned}\sup_{0\leq t\leq\mathrm{T}}\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t)-\mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{v}(t)\right\|_{\mathcal{X}}&\leq\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\|\mathrm{F}\big(s,\mathrm{u}(s)\big)-\mathrm{F}\big(s,\mathrm{v}(s)\big)\|_{\mathcal{X}}ds\\&\leq\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\mathrm{K}\big(s,\|\mathrm{u}-\mathrm{v}\|_{\mathbb{B}_\mathrm{T}}\big)ds,\end{aligned} \tag{21}$$

briefly

$$\left\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}-\mathcal{G}^{\widehat{\mathrm{u}}_0}\mathrm{v}\right\|_{\mathbb{B}_\mathrm{T}}\leq\|\mathrm{u}_0-\widehat{\mathrm{u}}_0\|_{\mathcal{X}}+\int_0^\mathrm{T}\mathrm{K}\big(s,\|\mathrm{u}-\mathrm{v}\|_{\mathbb{B}_\mathrm{T}}\big)ds. \tag{22}$$

In particular the continuity of $\mathcal{G}^{\mathrm{u}_0}$ on $\mathbb{B}_\mathrm{T}$ follows. □

Corollary 1 *Under the growth assumption*

$$\mathrm{K}(t,\mathrm{U})\leq\phi(t)\mathrm{U}\quad\textit{for all } t\in[0,\mathrm{T}]\textit{ and }\mathrm{U}\geq 0, \tag{23}$$

for a function $\phi\in\mathrm{L}^p(0,\mathrm{T}),\ p\in(1,\infty]$ *the operator* $\mathcal{G}^{\mathrm{u}_0}$ *is a contraction on* $\mathbb{B}_\mathrm{T}$ *equipped with a kind of Bielecki (see [8]) norm*

$$|||\mathrm{u}|||_{\mathbb{B}_\mathrm{T}}\doteq\sup_{0\leq t\leq\mathrm{T}}e^{-\gamma t}\|\mathrm{u}(t)\|_{\mathcal{X}},\quad\mathrm{u}\in\mathbb{B}_\mathrm{T}, \tag{24}$$

provided $\gamma > \frac{p-1}{p}\|\phi\|_{\mathrm{L}^p(0,\mathrm{T})}^{\frac{p}{p-1}}$ *and* $\phi \in \mathrm{L}^p(0,\mathrm{T})$ *for some* $1 < p < \infty$ *or* $\gamma > \|\phi\|_{\mathrm{L}^\infty(0,\mathrm{T})}$ *if* $\|\phi\|_{\mathrm{L}^\infty(0,\mathrm{T})} < \infty$. *Therefore, the operator* $\mathcal{G}^{\mathrm{u}_0}$ *has a unique fixed point in both cases.*

PROOF. Applying the property (23) to (21) we obtain

$$\begin{aligned} e^{-\gamma t}\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t) - \mathcal{G}^{\mathrm{u}_0}\mathrm{v}(t)\|_{\mathcal{X}} &\le \int_0^t e^{-\gamma t}\mathrm{K}\big(s, \|\mathrm{u}(s) - \mathrm{v}(s)\|_{\mathcal{X}}\big)ds \\ &\le \int_0^t e^{-\gamma s}\|\mathrm{u}(s) - \mathrm{v}(s)\|_{\mathcal{X}}\phi(s)e^{-\gamma(t-s)}ds \\ &\le |||\mathrm{u} - \mathrm{v}|||_{\mathbb{B}_\mathrm{T}} \int_0^t \phi(s)e^{-\gamma(t-s)}ds. \end{aligned}$$

For $\frac{1}{p} + \frac{1}{p'} = 1$ Hölder's inequality leads to

$$e^{-\gamma t}\big\|\mathcal{G}^{\mathrm{u}_0}\mathrm{u}(t) - \mathcal{G}^{\mathrm{u}_0}\mathrm{v}(t)\big\|_{\mathcal{X}} \le |||\mathrm{u} - \mathrm{v}|||_{\mathbb{B}_\mathrm{T}}\|\phi\|_{\mathrm{L}^p(0,\mathrm{T})}\left(\int_0^t e^{-p'\gamma(t-s)}ds\right)^{\frac{1}{p'}}$$

and

$$\big|\big|\big|\mathcal{G}^{\mathrm{u}_0}\mathrm{u} - \mathcal{G}^{\mathrm{u}_0}\mathrm{v}\big|\big|\big|_{\mathbb{B}_\mathrm{T}} \le \frac{1}{(p'\gamma)^{\frac{1}{p'}}}\|\phi\|_{\mathrm{L}^p(0,\mathrm{T})}|||\mathrm{u} - \mathrm{v}|||_{\mathbb{B}_\mathrm{T}}$$

follows. □

PROOF OF THEOREM 1 Adapting the reasoning of the first part of the proof of Proposition 2 we obtain

$$\big\|\mathrm{u}_{n+1}\big\|_{\mathbb{B}_t} \le \mathrm{U}_0 + \int_0^t \mathrm{K}\big(s, \|\mathrm{u}_n\|_{\mathbb{B}_\mathrm{T}}\big)ds, \quad n \ge 1. \tag{25}$$

Next we consider the global nonnegative solution U on $[0,\mathrm{T}]$ of the integral equation

$$\mathrm{U}(t) = \mathrm{U}_0 + \int_0^t \mathrm{K}\big(s, \mathrm{U}(s)\big)ds, \quad t \in [0,\mathrm{T}], \tag{26}$$

(see (3) in condition **a2**)) for which one has

$$\mathrm{U}(t) - \big\|\mathrm{u}_{n+1}\big\|_{\mathbb{B}_t} \ge \int_0^t \Big(\mathrm{K}\big(s,\mathrm{U}(s)\big) - \mathrm{K}\big(s, \big\|\mathrm{u}_n\big\|_{\mathbb{B}_s}\big)\Big)\, ds.$$

Since $\mathrm{U}(t) \ge \mathrm{U}_0 \doteq \|\mathrm{u}_0\|_{\mathbb{B}_\mathrm{T}}$, the monotonicity of the function $\mathrm{K}(s,\mathrm{U})$ on the second variable (see **a1**)) implies, by induction, the inequality

$$\big\|\mathrm{u}_n\big\|_{\mathbb{B}_t} \le \mathrm{U}(t) \quad \text{for } t \in [0,\mathrm{T}],$$

where the function $\mathrm{U}(t)$ is independent on n. Thus, $\{\mathrm{u}_n\}_{n\ge 0}$ is a bounded sequence in $\mathbb{B}_\mathrm{T}$. Analogously, for each $n \ge 0$

$$\mathrm{R}_n(t) \doteq \sup_{m \ge n} \big\|\mathrm{u}_m - \mathrm{u}_n\big\|_{\mathbb{B}_t}$$

is a nonnegative, uniformly bounded, and nondecreasing function on $t \in [0,\mathrm{T}]$. By construction, we may assume that for each $t \in [0,\mathrm{T}]$ the real sequence $\{\mathrm{R}_n(t)\}_{n\ge 0}$ is nonincreasing. It implies the existence of a nonnegative and nondecreasing function given by

$$\mathrm{R}(t) = \lim_{n\to\infty} \mathrm{R}_n(t), \quad t \in [0,\mathrm{T}].$$

On the other hand, a similar reasoning as in the second part of the proof of Proposition 2, and using that the semigroup is of contractions, leads to

$$\big\|\mathrm{u}_m - \mathrm{u}_n\big\|_{\mathbb{B}_t} \le \int_0^t \mathrm{K}\big(s, \big\|\mathrm{u}_{m-1}(s) - \mathrm{u}_{n-1}(s)\big\|_{\mathbb{B}_s}\big)ds$$

(see (22)). Therefore, we obtain

$$\mathrm{R}(t) \leq \mathrm{R}_n(t) \leq \int_0^r \mathrm{K}\big(s, \mathrm{R}_{n-1}(s)\big)ds, \quad t \in [0, \mathrm{T}],$$

whence the Lebesgue Convergence Theorem implies

$$\mathrm{R}(t) \leq \int_0^t \mathrm{K}\big(s, \mathrm{R}(s)\big)ds, \quad t \in [0, \mathrm{T}].$$

So that we deduce $\mathrm{R}(t) \equiv 0$ for $t \in [0, \mathrm{T}]$ from the assumption **a3**). Since

$$\big\|\mathrm{u}_m - \mathrm{u}_n\big\|_{\mathbb{B}_\mathrm{T}} \leq \mathrm{R}_n(\mathrm{T})$$

we conclude

$$\big\|\mathrm{u}_m - \mathrm{u}_n\big\|_{\mathbb{B}_\mathrm{T}} \to 0 \quad \text{as } m, n \to \infty.$$

Therfore, we have proved the existence of a fixed point , u, of the operator $\mathcal{G}^{\mathrm{u}_0}$. Since

$$\big\|\mathrm{u}\big\|_{\mathbb{B}_\mathrm{T}} \leq \big\|\mathrm{u} - \mathrm{u}_n\big\|_{\mathbb{B}_\mathrm{T}} + \big\|\mathrm{u}_n\big\|_{\mathbb{B}_\mathrm{T}}$$

the estimate (7) follows by repeating the reasonings to changing the global horizont T by each $t \in [0, \mathrm{T}]$.

Let us prove now that the operator $\mathcal{G}^{\mathrm{u}_0}$ only admits at most a fixed point in $\mathbb{B}_\mathrm{T}$. Indeed, let u and $\widehat{\mathrm{u}}$ be two eventual fixed points of $\mathcal{G}^{\mathrm{u}_0}$ and $\mathcal{G}^{\widehat{\mathrm{u}}_0}$ in $\mathbb{B}_\mathrm{T}$. Then

$$\big\|\mathrm{u} - \widehat{\mathrm{u}}\big\|_{\mathbb{B}_t} \leq \|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} + \int_0^t \mathrm{K}\big(s, \|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_s}\big)ds, \quad t \in [0, \mathrm{T}] \tag{27}$$

(see (19)). Then from the property (4), it follows $\big\|\mathrm{u} - \widehat{\mathrm{u}}\big\|_{\mathbb{B}_t} \equiv 0$ for all $t \in [0, \mathrm{T}]$ whence $\mathrm{u} = \widehat{\mathrm{u}}_0$ in the space $\mathbb{B}_\mathrm{T}$, provided $\mathrm{u}_0 = \widehat{\mathrm{u}}_0$.

Finally, to prove that u can also be identified as the unique limit of the implicit Euler scheme associated with the nonhomogeneous equation with the forcing term $g(t) = \mathrm{F}(t, \mathrm{u}(t))$ a.e. $t \in (0, \mathrm{T})$ it suffices to use the uniqueness of solution of Crandall-Liggett's Theorem and the fact that, as already shown, the mild solution is unique. □

Remark 5 We emphasize that the equality is required in (26) (see (3) in condition **a2**)) in the above proof. An inequality does not apply in the reasoning. □

4 The integral equation. Some uniqueness criteria revisited

Certainly, as it follows from Theorem 1, the properties **a2**) and **a3**) are key stones in the proofs. So, we dedicate this Section to study these main properties. Theorems 3, 4 are revisited version of relative results in [31, 30, 28, 12, 13].

In order to do it, in what follows we assume

$$\mathrm{K}(t, \mathrm{U}) = \phi(t)\vartheta(\mathrm{U}), \quad t \in [0, \mathrm{T}],\ \mathrm{U} \geq 0, \tag{28}$$

where $\vartheta :]0, \infty[\to]0, \infty[$ is a continuous and increasing function verifying $\vartheta(0^+) = 0$ and ϕ is an integrable and non negative function on $[0, \mathrm{T}]$. In this case, we repeat the proof of Theorem 1 by using directly the function $\phi(t)\vartheta(\mathrm{U})$. Then we replace the equation (26) by

$$\mathrm{U}(t) = \mathrm{U}_0 + \int_0^t \phi(s)\vartheta\big(\mathrm{U}(s)\big)ds, \quad 0 \leq s \leq \mathrm{T}, \tag{29}$$

in which we focus in order to prove whenever the assumption **a2**) and **a3**) are fulfilled.

Theorem 3 (Osgood's criterion) *When* $\mathrm{U}_0 > 0$ *a positive function* $\mathrm{U}(t)$ *satisfying*

$$\mathrm{U}(t) \leq \mathrm{U}_0 + \int_0^t \phi(s)\vartheta\big(\mathrm{U}(s)\big)ds, \quad 0 \leq s \leq \mathrm{T} \leq +\infty \tag{30}$$

is given implicitly, in the whole interval $[0,\mathrm{T}]$*, by the property*

$$\int_{\mathrm{U}_0}^{\mathrm{U}(t)} \frac{ds}{\vartheta(s)} \leq \int_0^t \phi(s)ds, \quad 0 \leq t \leq \mathrm{T}. \tag{31}$$

Therefore the condition **a2***) holds. Moreover, under the Osgood condition*

$$\int_{0^+} \frac{d\mathrm{U}}{\vartheta(\mathrm{U})} = \infty, \tag{32}$$

the unique nonnegative solution $\mathrm{U}(t)$ *of*

$$\mathrm{U}(t) \leq \int_0^t \phi(s)\vartheta\big(\mathrm{U}(s)\big)ds, \quad 0 \leq s \leq \mathrm{T}$$

is the null function, i.e. we have **a3***).*

Proof. As it is well known, integral equation as (29) is equivalent to the Cauchy problem

$$\begin{cases} \mathrm{U}'(t) = \phi(t)\vartheta\big(\mathrm{U}(t)\big), \\ \mathrm{U}(0) = u_0 > 0 \end{cases} \tag{33}$$

whose positive and continuous global solution is represented by

$$\int_{\mathrm{U}_0}^{\mathrm{U}(t)} \frac{ds}{\vartheta(s)} = \int_0^t \phi(s)ds, \quad 0 \leq t \leq \mathrm{T}.$$

When $\mathrm{U}(t)$ solves the inequality (30) we require a sharp refinement because an equivalence as (29) and (33) does not hold in general. So, we introduce the positive and non decreasing function $\mathrm{V}(t) = \max\limits_{0\leq\tau\leq t} \mathrm{U}(\tau) = \mathrm{U}(\tau_t)$, for some $\tau_t \in [0,t]$. Next, we define

$$\widehat{\mathrm{V}}(t) = \mathrm{U}_0 + \int_0^t \phi(s)\vartheta\big(\mathrm{V}(s)\big)ds > 0, \quad t > 0$$

that satisfies $\widehat{\mathrm{V}}(0) = u_0$ as well as

$$\mathrm{V}(t) = \mathrm{U}(\tau_t) \leq \mathrm{U}_0 + \int_0^{\tau_t} \phi(s)\vartheta\big(\mathrm{U}(s)\big) \leq \mathrm{U}_0 + \int_0^t \phi(s)\vartheta\big(\mathrm{V}(s)\big) = \widehat{\mathrm{V}}(t)$$

and

$$\begin{cases} \widehat{\mathrm{V}}'(t) = \phi(t)\vartheta\big(\mathrm{V}(t)\big) \leq \phi(t)\vartheta\big(\widehat{\mathrm{V}}(t)\big), \\ \widehat{\mathrm{V}}(0) = \mathrm{U}_0 > 0 \end{cases}$$

close to (33). Hence, a kind of Leibnitz inquality

$$\int_{\mathrm{U}_0}^{\mathrm{U}(t)} \frac{dr}{\vartheta(r)} \leq \int_{\mathrm{U}_0}^{\widehat{\mathrm{V}}(t)} \frac{dr}{\vartheta(r)} = \int_0^t \frac{\widehat{\mathrm{V}}'(t)dt}{\vartheta\big(\widehat{\mathrm{V}}(t)\big)} \leq \int_0^{t_1} \phi(s)ds < +\infty$$

holds, whence (31) follows.

On the other hand, when $\mathrm{U}_0 = 0$, if we suppose $\mathrm{U}(t) > 0$ in some interval $t \in]0,t_1] \subset [0,\mathrm{T}]$. the above reasoning shows that one satisfies $\widehat{\mathrm{V}}(0) = 0$, $0 < \mathrm{V}(t) \leq \widehat{\mathrm{V}}(t)$ and $\widehat{\mathrm{V}}'(t) \leq \phi(t)\vartheta\big(\widehat{\mathrm{V}}(t)\big)$. Then we deduce $\widehat{\mathrm{V}}(t) > 0$ in $t \in]0,t_1]$ and

$$\int_0^{\widehat{\mathrm{V}}(t_1)} \frac{dr}{\vartheta(r)} = \int_0^{t_1} \frac{\widehat{\mathrm{V}}'(t)dt}{\vartheta\big(\widehat{\mathrm{V}}(t)\big)} \leq \int_0^{t_1} \phi(s)ds < +\infty$$

contrary to the condition (32). □

Remark 6 Since ϑ is a continuous and increasing function the relative equation (3) becomes

$$\int_{\mathrm{U}_0}^{\mathrm{U}(t)} \frac{ds}{\vartheta(s)} = \int_0^t \phi(s)ds, \quad t \geq 0. \tag{34}$$

□

Let us introduce the increasing function

$$\Psi_{\mathrm{U}_0}(\mathrm{U}) \doteq \int_{\mathrm{U}_0}^{\mathrm{U}} \frac{ds}{\vartheta(s)}, \quad \mathrm{U} \geq \mathrm{U}_0, \tag{35}$$

provided $\mathrm{U}_0 > 0$. Then we have the estimate (30) by the version

$$\mathrm{U}(t) = \Psi_{\mathrm{U}_0}^{-1}\left(\int_0^t \vartheta(s)ds\right), \quad 0 \leq t \leq \mathrm{T}, \tag{36}$$

of (31), where $\mathrm{U}(0) = \Psi^{-1}(0) = \mathrm{U}_0 > 0$. This function is defined by horizont T such that

$$\int_{\mathrm{U}_0}^{+\infty} \frac{ds}{\vartheta(s)} \geq \int_0^{\mathrm{T}} \phi(s)ds. \tag{37}$$

Remark 7 We emphasize that in the Osgood assumption (32) only the behaviour of the function ϑ near the origin is involved. □

Corollary 2 *Under* (28) *implies*

$$\int_{\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}}}^{\|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_t}} \frac{ds}{\vartheta(s)} \leq \int_0^t \phi(s)ds, \quad 0 \leq t \leq \mathrm{T} \tag{38}$$

provided $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} > 0$. *Moreover, under the Osgood condition* (32) *one has the property*

$$\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0 \quad \Rightarrow \quad \|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_t} = 0.$$

□

Proof. Since $\displaystyle\int_{0^+}^{\|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_t}} \frac{ds}{\vartheta(s)} = +\infty$ the substitution $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0$ in (38) only holds if the upper limit is $\|\mathrm{u} - \widehat{\mathrm{u}}\|_{\mathbb{B}_t} = 0$. □

Remark 8 We note that no convex function $\vartheta(\mathrm{U})$ satisfies (32). Certainly, if (32) holds then the function ϑ is not integrable near 0. For instance, the functions satisfying

$$\frac{\vartheta(\mathrm{U})}{\mathrm{U}} \leq \ln\frac{1}{\mathrm{U}}, \quad \ln\frac{1}{\mathrm{U}} \ln \overset{n}{\cdots} \ln\frac{1}{\mathrm{U}}, \quad n \geq 0, \quad \text{near } \mathrm{U} = 0 \tag{39}$$

provide examples for which (32) holds near the origin. The conditions (32) and (39) coincide with the classical Osgood's criterion (see [31]).
On the other hand, we also note that the examples given by (39) verify

$$\vartheta(\mathrm{U})|\ln \mathrm{U}| \leq \mathrm{U}|\ln \mathrm{U}| \ln\frac{1}{\mathrm{U}} \ln \overset{n}{\cdots} \ln\frac{1}{\mathrm{U}} = \mathrm{U}\big(\ln \mathrm{U}\big)^{n+2}, \quad n \geq 0, \quad \text{near } \mathrm{U} = 0,$$

thus they satisfy the Dini condition

$$\lim_{u \searrow 0} \vartheta(\mathrm{U})|\ln \mathrm{U}| = 0. \tag{40}$$

We emphasize that, as in (32) only the behaviour of ϑ near the origin is involved in the Dini condition. □

Next we give some simple power like cases $\vartheta_m(\mathrm{U}) = \mathrm{U}^m,\ m > 0$, for which the assumptions **a2**) and **a3**) hold. Certainly, these criteria must satisfy (32), thus

$$\int_{0^+} \frac{d\mathrm{U}}{\mathrm{U}^m} = +\infty.$$

We note that the assumption (32) holds if and only if $m \geq 1$.
Certainly, the most famous uniqueness criteria is related to Gronwall's Lemma whenever one considers the Lipschitz case

$$\vartheta_1(\mathrm{U}) = \mathrm{U}, \quad \mathrm{U} \geq 0. \tag{41}$$

Corollary 3 (Gronwall's inequality) *Let* $\mathrm{U}(t)$ *verifying*

$$0 \leq \mathrm{U}(t) \leq \mathrm{U}_0 + \int_0^t \phi(s)\mathrm{U}(s)ds < +\infty, \quad 0 \leq t,$$

provided $\mathrm{U}_0 \geq 0$ *and* $\phi \in \mathrm{L}^1_{loc}(0, +\infty),\ \phi \geq 0$, *then*

$$0 \leq \mathrm{U}(t) \leq \mathrm{U}_0 \exp\left(\int_0^t \phi(s)ds\right), \quad 0 \leq t. \tag{42}$$

In particular, $\mathrm{U}_0 = 0$ *implies* $\mathrm{U}(t) \equiv 0$. □

Remark 9 It follows from the representation (42) in which (35) is given by

$$\Psi_{\mathrm{U}_0}(\mathrm{U}) \doteq \int_{\mathrm{U}_0}^{\mathrm{U}} \frac{ds}{s} = \ln \frac{\mathrm{U}}{\mathrm{U}_0}, \quad \mathrm{U} \geq \mathrm{U}_0 > 0.$$

Here the maximal horizon $\mathrm{T}_\infty = +\infty$ is independent on U_0 (see (37)) and (38) becomes

$$\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} \leq \|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} \exp\left(\int_0^t \phi(s)ds\right), \quad 0 \leq t \leq \mathrm{T} < \infty. \tag{43}$$

Clearly,

$$\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0 \quad \Rightarrow \quad \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} = 0.$$

□

The other power like criterion is governed by $\vartheta_m(\mathrm{U}) = \mathrm{U}^m,\ m > 1$.

Corollary 4 (Power like criterion) *Let* $\mathrm{U}(t)$ *verifying*

$$0 \leq \mathrm{U}(t) = \mathrm{U}_0 + \int_0^t \phi(s)\big(\mathrm{U}(s)\big)^m ds < +\infty, \quad 0 \leq t < \mathrm{T}_\infty < \infty,$$

provided $u_0 \geq 0,\ m > 1$ *and* $\phi \in \mathrm{L}^1(0, \mathrm{T}),\ \phi \geq 0$, *then*

$$\big(\mathrm{U}(t)\big)^{1-m} + (m-1)\int_0^t \phi(s)ds = \mathrm{U}_0^{1-m}, \quad 0 \leq t < \mathrm{T}_\infty < \infty, \tag{44}$$

provided $\mathrm{U}_0 > 0$. *Here the maximal horizon* $\mathrm{T}_\infty < \infty$ *is dependent on* U_0 *given by*

$$\mathrm{U}_0^{1-m} = (m-1)\int_0^{\mathrm{T}_\infty(\mathrm{U}_0)} \phi(s)ds \tag{45}$$

(see (37)*). Moreover,* $\mathrm{U}_0 = 0$ *implies* $\mathrm{U}(t) \equiv 0$. □

Remark 10 Now the representation (44) is given by the version of (35)

$$\Psi_{\mathrm{U}_0}(\mathrm{U}) \doteq \int_{\mathrm{U}_0}^{\mathrm{U}} \frac{ds}{s^m} = \frac{1}{m-1}\left(\frac{1}{\mathrm{U}_0^{m-1}} - \frac{1}{\mathrm{U}^{m-1}}\right), \quad \mathrm{U} \geq \mathrm{U}_0 > 0.$$

Then (38) becomes

$$\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}}^{1-m} \leq \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}^{1-m} + (m-1)\int_0^t \phi(s)ds, \quad 0 \leq t \leq \mathrm{T}_\infty(\|\mathrm{u}_0\|_{\mathcal{X}}) < +\infty, \tag{46}$$

provided $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} > 0$. Once more,

$$\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0 \quad \Rightarrow \quad \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} = 0.$$

Indeed, the substitution $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0$ in (46) implies

$$+\infty \leq \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}^{1-m} + (m-1)\int_0^t \phi(s)ds, \quad 0 \leq t \leq \mathrm{T}_\infty(\|\mathrm{u}_0\|_{\mathcal{X}}) < +\infty,$$

whence

$$\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}^{1-m} = +\infty \quad \Rightarrow \quad \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} = 0.$$

□

Example 2 For the choice in Example 1 the inequality (30) becomes

$$\mathrm{U}(t) = \mathrm{U}_0 + \mathbf{S}_0 \int_0^t \vartheta\big(\mathrm{U}(s)\big)ds, \quad 0 \leq s \leq \mathrm{T} \leq +\infty. \tag{47}$$

where $\mathbf{S}_0$ is a positive constant and $\vartheta(\mathrm{U}) = \mathrm{U}\ln\mathrm{U},\ \mathrm{U} \geq 0$ (see (17)). In view of (36) the version of (35) is

$$\int_{\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}}}^{\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}} \frac{ds}{\vartheta(s)} \leq \mathbf{S}_0 t, \quad 0 \leq t, \tag{48}$$

provided $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} > 0$. In particular

$$\int_{\left\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\right\|_{\mathcal{X}}}^{\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}} \frac{ds}{s\ln s} \leq \mathbf{S}_0 t, \quad 0 \leq t \tag{49}$$

provided $0 < \left\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\right\|_{\mathcal{X}} \leq \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}$. As the Osgood condition (32) holds, we have

$$\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0 \quad \Rightarrow \quad \left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} = 0.$$

Indeed, as in Coriollary 2, since $\displaystyle\int_{0^+}^{\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t}} \frac{ds}{s\ln s} = +\infty$ the substitution $\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\|_{\mathcal{X}} = 0$ in (49) only holds if the upper limit is $\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} = 0$.
More precisely, in this case, one has

$$\int_{\mathrm{U}_0}^{\mathrm{U}} \frac{ds}{\vartheta(s)} = \ln\left(\frac{\ln\mathrm{U}}{\ln\mathrm{U}_0}\right) \quad \mathrm{U} \geq \mathrm{U}_0, \tag{50}$$

provided $\mathrm{U}_0 > 1$. Here the maximal horizon $\mathrm{T}_\infty = +\infty$ is independent on U_0 (see (37)). Then (49) becomes

$$\left\|\mathrm{u} - \widehat{\mathrm{u}}\right\|_{\mathbb{B}_t} \leq \left\|\mathrm{u}_0 - \widehat{\mathrm{u}}_0\right\|_{\mathcal{X}}^{e^{\mathrm{S}_0 t}}, \quad 0 \leq t. \tag{51}$$

□

Related to the study of the property **a3**) we may replace the inequality (28) by Lévy's extension (see [28, Theoréme 10.I]) of the well-known Nagumo‘s criterion (see [30]). Here we incorporate an improved generalization based on [12]. This corresponds with the case

$$\mathrm{K}(t,\mathrm{U}) \le \frac{\psi'(t)}{\psi(t)}\vartheta(\mathrm{U}), \tag{52}$$

where $\psi :]0,\infty[\to]0,\infty[$ is a differentiable function satisfying

$$\psi(0^+) = 0,\ \psi'(0^+) > 0 \quad \text{and} \quad \psi'(t) > 0$$

and $\vartheta :]0,\infty[\to]0,\infty[$ is a continuous and increasing function verifying $\vartheta(0^+) = 0$ and the Nagumo condition

$$\int_0^r \frac{\vartheta(s)}{s} ds \le r, \quad r \ge 0. \tag{53}$$

Then we may prove **a3**). The reasonings involve the functional space

$$\mathcal{Z} = \left\{ \mathrm{U} \in \mathcal{C}\big([0,\mathrm{T}] : [0,\infty[\big) \text{ such that } \lim_{t \searrow 0} \frac{\mathrm{U}(t)}{\psi(t)} = 0 \right\}$$

endowed with the norm

$$\|\mathrm{U}\| \doteq \sup_{0\le t\le \mathrm{T}} \frac{\mathrm{U}(t)}{\psi(t)}.$$

In fact, under (52), (53) and (55) we claim that a generalized version of Nagumo's criterion holds, *i.e.* the null function is the unique nonnegative solution of (4) in the space $\mathcal{Z}$ verifying (4). Indeed, assume that there exists $u \in \mathcal{Z}$ such that

$$0 \ne \|\mathrm{U}\| = \sup_{0\le t\le \mathrm{T}} \frac{\mathrm{U}(t)}{\psi(t)} = \frac{\mathrm{U}(t^*)}{\psi(t^*)}.$$

Since $\mathrm{U} \in \mathcal{Z} \setminus \{0\}$ and $\mathrm{U}(0) = 0$ by construction we have $\mathrm{U}(t) \not\equiv \|\mathrm{U}\|$ if $t \in [0,t^*]$, for some t^* that without loss of generality we may assume small. From (56) we deduce the inequality

$$\begin{aligned} \mathrm{U}(t^*) &\le \int_0^{t^*} \mathrm{K}\big(s,\mathrm{U}(s)\big)ds \le \int_0^{t^*} \frac{\psi'(s)}{\psi(s)}\vartheta(\mathrm{U}(s))ds < \int_0^{t^*} \frac{\psi'(s)}{\psi(s)}\vartheta\big(\psi(s)\|\mathrm{U}\|\big)ds \\ &= \int_{\tau^*}^{\infty} \vartheta\big(e^{-\tau}\|\mathrm{U}\|\big)d\tau = \int_0^{\psi(\tau^*)\|\mathrm{U}\|} \frac{\vartheta(r)}{r}dr \le \psi(t^*)\|\mathrm{U}\|, \end{aligned} \tag{54}$$

where first we have used the change of variable $\tau = -\ln\psi(s)$ and then $r = \|\mathrm{U}\|e^{-\tau}$. Therefore, we have obtained a contradiction

$$\|\mathrm{U}\| < \|\mathrm{U}\|$$

and the claim holds.

Remark 11 It follows that under the assumption

$$\lim_{t\searrow 0} \frac{\mathrm{K}(t,\mathrm{U})}{\psi'(t)} = 0 \quad \text{uniformly to small values of U}, \tag{55}$$

any nonnegative continuous function, U, verifying the integral inequation

$$\mathrm{U}(t) \le \int_0^t \mathrm{K}\big(s,\mathrm{U}(s)\big)ds, \quad t \in [0,\mathrm{T}]$$

(see (4)) belongs to $\mathcal{Z}$. Indeed, if U solves (4) one has

$$\frac{\mathrm{U}(t)}{\psi(t)} \le \frac{\displaystyle\int_0^t \mathrm{K}\big(s,\mathrm{U}(s)\big)ds}{\psi(t)} \le \frac{\displaystyle\int_0^t \mathrm{K}\big(s,\mathrm{M}_{t_0}\big)ds}{\psi(t)}, \quad 0 \le t \le t_0$$

where $\mathrm{M}_{t_0} = \sup_{0\le t\le t_0} \mathrm{U}(t)$. Hence by l'Hopital's rule, one obtains

$$\lim_{t\searrow 0} \frac{\mathrm{U}(t)}{\psi(t)} \le \lim_{t\searrow 0} \frac{\mathrm{K}(t,\mathrm{U})}{\psi'(t)} \quad \text{uniformly in small values of U.}$$

□

Remark 12 In proving (54) it is enough to use

$$\int_0^r \frac{\vartheta(s)}{s} ds \le r, \quad 0 \le r \le r^* \tag{56}$$

where r^* can be a small positive value. In this case, it is enough to choose t^* appropriately small. □

We note that $\vartheta(s) = s$ is the simplest case satisfying (56). It coincides with Nagumo's classical criterion. In [12], it was given another example verifying (56). On the other hand, we also note the inequality

$$\vartheta\left(\frac{r}{2}\right)\ln 2 = \vartheta\left(\frac{r}{2}\right)\int_{\frac{r}{2}}^r \frac{1}{s}ds \le \int_{\frac{r}{2}}^r \frac{\vartheta(s)}{s}ds \le \int_0^r \frac{\vartheta(s)}{s}ds,$$

whence (56) leads to

$$\vartheta\left(\frac{r}{2}\right)\left|\ln\frac{r}{2}\right| \le \frac{1}{\ln 2} r,$$

thus the integral condition (56) also implies the Dini condition (40).

Next, we deal with suitable combinations, non-necessarily convex, of Osgood and Nagumo uniqueness criteria. It extends [13, Theorem 2.1]. More precisely, here we consider

$$\mathrm{K}(t,\mathrm{U}) \le \mu\vartheta(t)\vartheta_{\mathrm{O}}(\mathrm{U}) + \lambda\frac{\psi'(t)}{\psi(t)}\vartheta_{\mathrm{N}}(\mathrm{U}), \quad t\in[0,\mathrm{T}],\ \mathrm{U}\ge 0,\ 0\le\mu, 0\le\lambda\le 1 \tag{57}$$

where ϑ is integrable on $[0,\mathrm{T}]$ as well as $\vartheta_{\mathrm{O}},\ \vartheta_{\mathrm{N}} : [0,\infty[\to [0,\infty[$ are continuos and nondecreasing functions such that $\vartheta_{\mathrm{O}}(0) = \vartheta_{\mathrm{N}}(0) = 0$ and (32) and (56) respectively hold. Moreover, we assume that $\psi :]0,\infty[\to]0,\infty[$ is a differentiable function satisfying

$$\psi(0^+) = 0,\ \psi'(0^+) > 0 \quad \text{and} \quad \psi'(t) > 0.$$

We note $\psi(t) = t$ is the simplest case of the function ψ. Really, it is sufficient that ψ let be defined in a bounded interval $]0,\mathrm{L}[$, L small. Clearly, the inequality (57) includes the Lipschitz case (41).

Theorem 4 *Let us assume* (57) *as before. Let* u *be a nonnegative continuous in the space* $\mathcal{Z}$ *veryfying the inequality*

$$\mathrm{U}(t) \le \int_0^t \mathrm{K}\big(s,\mathrm{U}(s)\big)ds, \quad t\in[0,\mathrm{T}].$$

Then if $\dfrac{\vartheta}{\psi}$ *is integrable on* $[0,\mathrm{T}]$ *the function* $\mathrm{U}(t)$ *is the null function.*

Proof. We emphasize that the cases $(\mu,\lambda) = (1,0)$ and $(\mu,\lambda) = (0,1)$ are proved in Osgood's criterion and Nagumo's criterion (see (Theorem 3 and the above arguments) respectively. So for $\mu > 0$ and $0 < \lambda < 1$ we form

$$\frac{\mathrm{U}(t)}{\psi(t)} \le \frac{\mu}{\psi(t)}\int_0^t \vartheta(s)\vartheta_{\mathrm{O}}\big(\mathrm{U}(s)\big)ds + \frac{\lambda}{\psi(t)}\int_0^t \frac{\psi'(s)}{\psi(s)}\vartheta_{\mathrm{N}}\big(\mathrm{U}(s)\big)ds, \quad t\in[0,\mathrm{T}].$$

In particular, the function $\widehat{\mathrm{U}}(t) = \dfrac{\mathrm{U}(t)}{\psi(t)}$ satisfies

$$\begin{aligned}\widehat{\mathrm{U}}(t) &\le \frac{\mu}{\psi(t)}\int_0^t \vartheta(s)\vartheta_{\mathrm{O}}\big(\psi(s)\widehat{\mathrm{U}}(s)\big)ds + \frac{\lambda}{\psi(t)}\int_0^t \frac{\psi'(s)}{\psi(s)}\vartheta_{\mathrm{N}}\big(\psi(s)\widehat{\mathrm{U}}(s)\big)ds \\ &\le \mu\int_0^t \frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\widehat{\mathrm{U}}(s)\big)ds + \frac{\lambda}{\psi(t)}\int_0^t \frac{\psi'(s)}{\psi(s)}\vartheta_{\mathrm{N}}\big(\psi(s)\widehat{\mathrm{U}}(s)\big)ds, \quad t\in[0,\mathrm{T}].\end{aligned}$$

We note that by

$$\lim_{t\searrow 0}\frac{\mathrm{U}(t)}{\psi(t)}=\lim_{t\searrow 0}\frac{\mathrm{K}(t,\mathrm{U})}{\psi'(t)}=0,$$

the function $\widehat{\mathrm{U}}$ is continuous in $[0,\mathrm{T}]$. Next, introducing the nondecreasing function $\mathrm{V}(t)\doteq\max_{0\le\tau\le t}\widehat{\mathrm{U}}(\tau)=\widehat{\mathrm{U}}(\tau_t)$, for some $\tau_t\in[0,t]$, we get

$$\widehat{\mathrm{U}}(\tau)\le\mu\int_0^\tau\frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\mathrm{V}(s)\big)ds+\frac{\lambda}{\psi(\tau)}\int_0^\tau\frac{\psi'(s)}{\psi(s)}\vartheta_{\mathrm{N}}\big(\psi(s)\mathrm{V}(\tau)\big)ds,\quad 0<\tau<t\le\mathrm{T}.$$

As in the reasoning of Nagumo's criterion (see (54))

$$\int_0^\tau\frac{\psi'(s)}{\psi(s)}\vartheta_{\mathrm{N}}\big(\psi(s)\mathrm{V}(\tau)\big)ds=\int_{-\ln\psi(\tau)}^{\infty}\vartheta_{\mathrm{N}}\big(e^{-\widehat{\tau}}\mathrm{V}(\tau)\big)d\widehat{\tau}=\int_0^{\psi(\tau)\mathrm{V}(\tau)}\frac{\vartheta_{\mathrm{N}}(r)}{r}dr\le\psi(\tau)\mathrm{V}(\tau),$$

where first we have used the change of variable $\widehat{\tau}=-\ln\psi(s)$ and then $r=\mathrm{V}(\tau)e^{-\widehat{\tau}}$. So that

$$\begin{aligned}\widehat{\mathrm{U}}(\tau)&\le\mu\int_0^\tau\frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\mathrm{V}(s)\big)ds+\lambda\mathrm{V}(\tau)\\&\le\mu\int_0^t\frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\mathrm{V}(s)\big)ds+\lambda\mathrm{V}(t),\quad 0<\tau<t\le\mathrm{T},\end{aligned}$$

and by construction, we obtain

$$\mathrm{V}(t)\le\mu\int_0^t\frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\mathrm{V}(s)\big)ds+\lambda\mathrm{V}(t),\quad 0<t\le\mathrm{T}.$$

Thus

$$0\le\psi(\mathrm{T})\mathrm{V}(t)\le\frac{\mu}{1-\lambda}\int_0^t\psi(\mathrm{T})\frac{\vartheta(s)}{\psi(s)}\vartheta_{\mathrm{O}}\big(\psi(\mathrm{T})\mathrm{V}(s)\big)ds,\quad 0<t\le\mathrm{T}.$$

We need a simple change of notation to use the reasoning of Theorem 3. In particular, since the function $\widehat{\vartheta}(t)=\psi(\mathrm{T})\dfrac{\vartheta(t)}{\psi(t)}$ is integrable on $[0,\mathrm{T}]$ and $v(0)=0$, by the Osgood's condition (32) one concludes

$$\psi(\mathrm{T})\mathrm{V}(t)\equiv 0\quad\text{in }[0,\mathrm{T}]\quad\Rightarrow\quad\mathrm{U}(t)\equiv 0\quad\text{in }[0,\mathrm{T}],$$

as in Theorem 3. □

Remark 13 Once more we emphasize that in assumption (57) only the behaviour of the function ϑ near the origin is involved (see Remarks 7 and 12). □

So that we may extend the Theorem 3 when

$$\vartheta(t)\le\frac{\psi'(t)}{\psi(t)}\tag{58}$$

where $\psi:]0,\infty[\to]0,\infty[$ is a differentiable function satisfying

$$\psi(0^+)=0,\ \psi'(0^+)>0\quad\text{and}\quad\psi'(t)>0$$

Corollary 5 *When* $\mathrm{U}_0>0$ *any positive function* $\mathrm{U}(t)$ *satisfying*

$$\mathrm{U}(t)\le\mathrm{U}_0+\int_0^t\frac{\psi'(s)}{\psi(s)}\vartheta\big(\mathrm{U}(s)\big)ds,\quad 0\le s\le\mathrm{T}\le+\infty\tag{59}$$

is given implicitly in the whole interval $[0,\mathrm{T}]$ *by the property*

$$\int_{\mathrm{U}_0}^{\mathrm{U}(t)}\frac{ds}{\vartheta(s)}\le\psi(t),\quad 0\le t\le\mathrm{T}.\tag{60}$$

Therefore, the condition **a2***) holds. Moreover, under the assumptions of Theorem 4, the property* **a3***) holds.* □

Remark 14 The Corollaries 3 and 4 and the inequalities (43) and (46) are immediately adapted to the case (58). □

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