Beyond the strong maximum principle for quasilinear equations

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Abstract

We study an extension of the strong maximum principle for quasilinear Dirichlet problems with sign-changing right-hand sides. For equations driven by the p-Laplacian, we establish sufficient conditions ensuring that supersolutions remain strictly positive in the domain even when the datum is indefinite. We also show that, under additional assumptions, positive solutions may fail to satisfy the Hopf-Oleinik boundary point lemma and instead exhibit flat boundary behavior, characterized by a vanishing weighted normal derivative. These results extend the previous linear theory to the quasilinear setting and are applied to sub-homogeneous indefinite equations and to a sign-indefinite version of Vázquez's strong maximum principle.

1 Introduction

The origins of the strong maximum principle go back to the classical work of S. Zaremba in 1910 [36], where one of the foundational results in the theory of second-order elliptic partial differential equations was established. In its simplest form, the principle states that if a smooth bounded domain $\Omega \subset \mathbb{R}^N$ is considered and a function u satisfies

$$\begin{cases} -\Delta u \geq f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

then one necessarily has

$$u(x) > 0 \qquad \text{for every } x \in \Omega, \tag{2}$$

whenever

$$f(x) \geq 0 \quad \text{in } \Omega, \qquad f \not\equiv 0. \tag{3}$$

The subsequent extension of this principle to general second-order uniformly elliptic operators is due to Hopf, whose 1927 paper [23] became a cornerstone of the subject. Later, in 1952, Hopf [24] and, independently, Oleinik [27], proved the celebrated boundary point lemma, which ensures that the normal derivative of any supersolution u of (1) satisfies

$$\frac{\partial u}{\partial n} < 0 \qquad \text{on } \partial\Omega. \tag{4}$$

Here, we focus on the case where the datum $f(x)$ is allowed to change sign. In a previous article [17], the second author, jointly with J. Hernández, proved that the positivity requirement (3) can be removed in the case of *linear operators*. Namely, under suitable assumptions, any supersolution u of (1) with a sign-changing right-hand side remains strictly positive in Ω. Furthemore, such a supersolution does not satisfy condition (4), provided that the sign-changing datum $f(x)$ fulfills appropriate hypotheses; this phenomenon leads to the notion of *flat solution*, which will be recalled below.

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The aim of the present paper is to show that the above phenomena, which go significantly beyond the classical form of the strong maximum principle, persist in the framework of degenerate quasilinear operators such as the p–Laplace operator. More precisely, we study supersolutions of

$$\begin{cases} -\Delta_p u \geq f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where

$$\Delta_p u = \operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big), \qquad p \in (1,\infty).$$

To the best of our knowledge, the adaptation of the strong maximum principle to sign–indefinite right–hand side in this quasilinear setting has not been previously developed (see, for instance, the monographs or surveys by Protter–Weinberger [29], Pucci–Serrin [30], Vázquez [32], Brezis–Ponce [10], or the extensive review by Apushkinskaya and Nazarov [1]). Important contributions concerning the p–Laplace operator itself—such as those by Vázquez [32], Tolksdorf [34], Ederson–Braga–Moreira [8], or Benedikt et al. [6]—do not address the case where $f(x)$ changes sign.

Following [17], we pay particular attention to situations in which $f(x)$ is negative in a neighborhood of the boundary and positive in an interior region. In the following, we write

$$f(x) = f^+(x) - f^-(x), \qquad f^+ = \max(f,0), \quad f^- = -\min(f,0),$$

and we assume, for simplicity, the existence of a subset $\Omega^+ \subset \Omega$ such that

$$\begin{cases} f(x) \geq 0 & \text{a.e. in } \Omega^+, \\ f(x) \leq 0 & \text{a.e. in } \Omega \setminus \Omega^+, \\ \partial\Omega \text{ satisfies the interior sphere condition.} \end{cases} \tag{6}$$

The new type of assumptions guaranteeing the positivity of supersolutions may be grouped into:

(H_1) a balance condition relating the "negative region" of f to its proximity to the boundary;

(H_2) a suitable decay of $f(x)$ as the boundary is approached.

Under these hypotheses we will prove that:
(A) the positivity property (2) of u still holds and $|\nabla u|^{p-2}\frac{\partial u}{\partial n} \leq 0$ on $\partial\Omega$,
(B) under additional conditions on $f(x)$, the positive *solution* of the quasilinear equation

$$\begin{cases} -\Delta_p u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

[i.e., now with the equality symbol =, instead ≥] does not satisfy the condition (4) but $|\nabla u|^{p-2}\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$.

Property (B) corresponds to the already known notion of *flat solutions*, which appears in several nonlinear contexts (see, e.g., Díaz [13], [17]). The existence of flat solutions shows that assumption (3) is necessary to derive (4). Notice also that a flat solution u on a problem (7) on the domain Ω can be extended by zero to get the unique solution $\widetilde{u}$ of a similar problem associated to an extended domain $\widetilde{\Omega} \supsetneq \Omega$ with the right hand side given by

$$\widetilde{f}(x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \widetilde{\Omega} \setminus \Omega. \end{cases}$$

In this way, we can construct *solutions with compact support* for data with compact support that becomes negative near the boundary of its support. This shows that the version of the strong maximum principle obtained in [10] – which ensures that the solution $u \geq 0$ of a linear problem (7) corresponding to a datum $f \geq 0$, cannot vanish on some positively measured subset of Ω except if $u \equiv 0$ on Ω) – has optimal conditions on $f(x)$. Analogous mechanisms occur in quasilinear equations with non-Lipschitz perturbations, a class of problems that has been extensively studied for decades. For instance, in the study of quasilinear problems of the type

$$\begin{cases} -\Delta_p u = g(x) - \beta(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{8}$$

with $g \geq 0$ and β a continuous function, for instance, such that $\beta(0) = 0$, it is well known the existence of a flat solution under suitable conditions on β and g. For the case where β is non-decreasing and is the subdifferential of the convex function j, $\beta = \partial j$, such that

$$\int_0 \frac{ds}{j(s)^{1/p}} < +\infty$$

and $g \neq 0$ we send the reader to Theorem 1.16 of [13] (the so called "non-diffusion of the support property" valid also for the case of β a multivalued maximal monotone graph). For the autonomous case $g \equiv 0$ and β non monotone see, e.g., [16]. This means that if we take

$$f(x) = g(x) - \beta(u(x)) \tag{9}$$

with u the flat solution of (8) then u is also a flat solution of the corresponding quasilinear problem (7). Note that, necessarily, such $f(x)$ becomes negative near the boundary $\partial\Omega$. In some sense, the solution has *a concave part in the center part* of the domain and *a convex part near the boundary* (where $f(x) \leq 0$).

The organization of this paper is the following: in Section 2 we recall some of the results of [17] for the linear problem (a version, as simple as possible, for the one-dimensional case in which assumptions (H_1) and (H_2) can be easily formulated in an optimal way). Section 3 deals with problems involving a quasilinear operator. We give sufficient conditions for the positivity of the solution (in subsection 3.1), and we study the flat solutions (in subsection 3.2), giving sufficient conditions to get globally flat solutions or merely locally flat solutions. An application to some *sub-homogeneous indefinite* quasilinear equations $-\Delta_p u = \lambda u^{p-1} + m(x)u^q$, $q \in (0, p-1)$, i.e., $m(x)$ a sign–changing function, generalizing some previous result for the case $p = 2$ (see, e.g., [22] and [17]) will be given in Section 4. Finally, in Section 5 we show an extension of the Vázquez's strong maximum principle proved in [32] when the right hand side is given by a sign–changing function $f(x)$.

2 Known results for the linear problem

It is useful to recall how the assumptions of type (H_1) and (H_2), mentioned in the Introduction, can be easily formulated and optimally formulated in the case of supersolutions $u(x)$ of the symmetric one-dimensional linear problem on the domain $\Omega = (-R, R)$

$$\begin{cases} -u''(x) \geq f(x) & \text{in } (-R, R), \\ u(\pm R) = 0. \end{cases} \tag{10}$$

We assume the symmetry condition

$$f(x) = f(-x), \ \ f = f^+ - f^-,$$

and we work in the framework of the space $L^1(\Omega : \delta)$, with $\delta(x) = d(x, \partial\Omega)$ (i.e., in this case, $\delta(r) = R - r$ if $r \in (0, R)$). We assume

$$f \in L^1(\Omega : \delta), \text{ i.e. } \int_0^R |f(s)|\,(R - s)ds < \infty, \tag{11}$$

and we consider *very weak supersolutions*, i.e., functions $u \in L^1(-R, R)$,, with $u'' \in L^1(\Omega : \delta)$, satisfying

$$-\int_\Omega u\psi'' \geq \int_\Omega f\psi \tag{12}$$

for any $\psi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ such that $\psi \geq 0$. Notice that since any function $\psi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$ satisfies the inequality $|\psi(x)| \leq C\delta(x)$ for any $x \in \overline{\Omega}$, for some $C > 0$, then the expressions in (12) make sense. The notion of *very weak solution* is defined in a similar way, with the symbol $\geq$ replaced by $=$. By well-known results (see, e.g. [9]), if u is a solution, we have that $u \in C([0,R]) \cap C^1[0,R)$, $u = u(r)$, $r = |x|$

$$\begin{cases} -u''(r) = f(r) & \text{in } (0,R), \\ u(R) = 0, \ u'(0) = 0. \end{cases} \tag{13}$$

The following result was proved in [17].

Theorem 1 *Assume that $f(x)$ becomes negative near the boundary in the following sense: there exists $r_0 \in (0,R)$, such that*

$$\begin{cases} f(x) = f^+(x) \geq 0 \text{ if } x \in (0,r_0), \ f^+ \not\equiv 0 \text{ on } (0,r_0), \\ \qquad f(x) = -f^-(x) \leq 0 \text{ if } x \in (r_0,R). \end{cases} \tag{14}$$

(A) Assume that the "balance condition"

$$\int_0^{r_0} f^+(s)(R-r_0)ds > \int_{r_0}^R f^-(s)(R-s)ds \tag{15}$$

and the "decay condition"

$$\int_r^R \left(\int_{r_0}^t f^-(s)ds\right) dt < (R-r)\int_0^{r_0} f^+(s)ds, \text{ for any } r \in (r_0,R) \tag{16}$$

hold.

Then any symmetric supersolution u satisfies that $u > 0$ in $(-R,R)$. Moreover, if $u \in C^1([0,R])$, then we have $u'(R) \leq 0$ and $u'(-R) \geq 0$. In addition, if hypothesis (15) is satisfied and u is a solution then $u > 0$ if and only if the decay condition (16) holds.

(B) Assume that (15), (16) and

$$f^- \in L^1(\Omega), \text{ i.e., } \int_{r_0}^R f^-(s)ds < \infty \tag{17}$$

hold. Let u be the unique solution u of (13). Then u is flat ($u'(\pm R) = 0$) if and only if the following condition holds

$$\int_0^{r_0} f^+(s)ds = \int_{r_0}^R f^-(s)ds, \text{ i.e. } \int_0^R f(s)ds = 0. \blacksquare \tag{18}$$

The study of the N-dimensional linear case in a general bounded regular domain Ω is much more complicated. The problem under consideration is now problem (1), referred to in the Introduction. We recall that, if $f \in L^1(\Omega : \delta)$, with $\delta(x) = d(x, \partial\Omega)$, by a *very weak supersolution* of (1) we mean a function $u \in L^1(\Omega)$ such that

$$-\int_\Omega u\Delta\psi \geq \int_\Omega f\psi, \tag{19}$$

for any $\psi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ such that $\psi \geq 0$. As in the one-dimensional case, since any function $\psi \in W^{2,\infty}(\Omega) \cap W_0^{1,\infty}(\Omega)$ satisfies $|\psi(x)| \leq C\delta(x)$ for any $x \in \overline{\Omega}$, for some $C > 0$, then the integrals involved in (19) are well defined. The notion of *very weak solution* is defined in a similar way, with the symbol $\geq$ replaced by $=$.

It is assumed that there exists a subset $\Omega^+ \subset \Omega$ such that

$$\begin{cases} f(x) \geq 0 & \text{a.e. } x \in \Omega^+, \\ f(x) \leq 0 & \text{a.e. } x \in \Omega \setminus \Omega^+, \\ \partial\Omega \text{ satisfies the interior sphere condition.} \end{cases} \tag{20}$$

In a first step, a result weaker than the one obtained in the first step of the proof of Theorem 1 is established. The positivity of u on Ω^+ is not obtained; instead, positivity is proved only on a compact set K contained in Ω^+. More precisely, it is shown that there exists a positive constant C^+ such that any supersolution u of (1) satisfies

$$u \geq C^+ \text{ on } K. \tag{21}$$

The second step consists in proving that, under suitable balance and decay conditions, the unique solution v of the linear problem on the ring $\Omega - K$

$$\begin{cases} -\Delta v = -f^-(x) & \text{in } \Omega - K, \\ v = 0 & \text{on } \partial\Omega, \\ v = C^+ & \text{on } \partial K, \end{cases} \tag{22}$$

is a positive subsolution, and thus

$$0 < v(x) \leq u(x) \text{ a.e. } x \in \Omega - K.$$

Note that this subsolution $v(x)$ is constructed in terms of a suitable power of φ_1, the normalized first eigenfunction of the Laplacian operator on Ω. In conclusion, the following result was proved in [17].

Theorem 2 *Let* $f \in L^1(\Omega : \delta)$.
(A) Assume that the following balance and decay near the boundary $\partial\Omega$ *conditions are satisfied:*
(H_1) Condition (20) holds, and there exists a compact set $K \subset \Omega^+$ *where* $f \neq 0$ *on* K *and*

$$c_K \int_\Omega f^+\delta - C_K \int_\Omega f^-\delta > 0, \tag{23}$$

with c_K *and* C_K *suitable positive constants.*[1]
(H_2) There exists $\alpha > 1$, $\varepsilon > 0$ *and* $M > 0$ *such that*

$$\begin{cases} \varepsilon \leq \min_{\Omega - K}((\alpha - 1)\left|\nabla\varphi_1\right|^2 - \lambda_1\varphi_1^2) & \text{and} \\ f(x) \geq -M\varphi_1(x)^{\alpha-2} \text{ a.e. } x \in \Omega - \Omega^+. \end{cases} \tag{24}$$

Then any supersolution u *of (1) satisfies that* $u(x) > 0$ *a.e.* $x \in \Omega$.
(B) If in addition $f \in L^1(\Omega)$ *and*

$$\int_\Omega f(x)dx = 0, \tag{25}$$

then the unique weak solution $u \in W_0^{1,1}(\Omega)$ *of the linear problem (7), with* $p = 2$, *is a flat solution.*■

Many remarks and comments on extensions to other linear second order elliptic operators can be found in [17].

It is clear that the mathematical study of this positivity property for quasilinear operators, such as the p-Laplacian, requires very different techniques. For instance, the notion of a very weak solution in the quasilinear framework is not well-defined, as it is not possible to integrate by parts twice, in contrast to the linear case. Consequently, new ideas are developed in the remaining sections of this paper.

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[1] Given a compact subset $K \subset \Omega^+$, let $\varrho > 0$ and $x_1, ..., x_m \in K$ such that $K \subset \bigcup_{i=1,\dots,m} B_\varrho(x_i) \subset \Omega^+$. If ς_i solves

$$\begin{cases} -\Delta\varsigma_i = \chi_{B_\varrho(x_i)} & \text{in } \Omega, \\ \varsigma_i = 0 & \text{on } \partial\Omega, \end{cases}$$

there exist two positive constants $c_K < C_K$ such that

$$c_K\delta(x) \leq \varsigma_i(x) \leq C_K\delta(x) \text{ a.e. } x \in \Omega.$$

3 The p-Laplace operator

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded regular domain and

$$f \in L^m(\Omega), \quad m \geq 1.$$

We assume that f is positive far the boundary of Ω and negative near the boundary of Ω; namely we assume that there exists $\Omega^+ \subset \Omega$ such that the following conditions hold

$$\begin{cases} f(x) \geq 0 \quad \text{a.e. } x \in \Omega^+ \\ \\ f(x) \leq 0 \quad \text{a.e. } x \in \Omega \setminus \Omega^+ \\ \\ \partial\Omega \text{ satisfies the interior sphere condition, dist } (\partial\Omega, \Omega^+) > 0. \end{cases} \tag{26}$$

As in the previous section, let us introduce the following decomposition

$$f(x) = f^+(x) - f^-(x)$$

with

$$f^+(x) = \max(f(x), 0), \quad f^-(x) = -\min(f(x), 0)$$

(notice that $f^-(x) \geq 0$).

Our results are new even when f is regular (for instance, $f \in L^{p'}(\Omega)$ with $p' = p/(p-1)$); thus, they are also valid for standard weak solutions. Nevertheless, for the sake of generality, we assume that $f \in L^1(\Omega)$. In this setting, as is well known, it is necessary to work with special classes of weak solutions to ensure uniqueness. In this paper, we adopt the notion of entropy solution introduced in [4]. However, our results can also be stated and proven using renormalized solutions, a framework first introduced in [7] and subsequently extended to various problems by several authors (see [21]) for the equivalence between these two notions).

To recall the notion of entropy solution we begin by introducing the truncation operator $T_k : \mathbb{R} \to \mathbb{R}$, defined, for arbitrary $k > 0$, by

$$T_k(s) = \begin{cases} s & \text{if } |s| \leq k \\ k \operatorname{sign}(s) & \text{if } |s| > k. \end{cases}$$

Then, we denote by $\mathcal{T}^{1,1}_{loc}(\Omega)$ the set of measurable functions $u : \Omega \to \mathbb{R}$ such that $T_k(u) \in W^{1,1}_{loc}(\Omega)$ for any $k > 0$. We also define, for $p \in (1, +\infty)$,

$$\mathcal{T}^{1,p}(\Omega) = \left\{ u \in \mathcal{T}^{1,1}_{loc}(\Omega) \text{ such that } T_k(u) \in W^{1,p}(\Omega) \ \forall\, k > 0 \right\}.$$

In [4] it has been proved that given $u \in \mathcal{T}^{1,p}(\Omega)$ there exists a unique measurable function $v : \Omega \to \mathbb{R}$ such that

$$\nabla T_k(u) = v \chi_{\{|v|<k\}} \ \forall\, k > 0.$$

This function v is also denoted by ∇u. It is clear that if $u \in W^{1,p}(\Omega)$ then $v \in L^p(\Omega)$ and $v = \nabla u$ in the usual weak sense. Moreover, as in [4], $\mathcal{T}^{1,p}_0(\Omega)$ is the subset of $\mathcal{T}^{1,p}(\Omega)$ formed by the functions which can be approximated by smooth functions with compact support in Ω in the following sense: $u \in \mathcal{T}^{1,p}_0(\Omega)$ if $u \in \mathcal{T}^{1,p}(\Omega)$ and $\forall\, k > 0$ there exists a sequence $\phi_n \in \mathcal{C}^\infty_0(\Omega)$ such that

$$\nabla \phi_n \to \nabla T_k(u) \text{ in } L^p(\Omega) \text{ and } \phi_n \to T_k(u) \text{ in } L^1_{loc}(\Omega).$$

Next, we introduce the appropriate notion of solution for the problem

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{27}$$

Definition. Given $f \in L^1(\Omega)$ a function $u \in \mathcal{T}_0^{1,p}(\Omega)$ is an "entropy solution" of (27) if

$$\int\limits_{\{|u-\phi|<k\}} \left\langle |\nabla u|^{p-2}\nabla u, \nabla u - \nabla\phi \right\rangle dx \leq \int\limits_{\Omega} T_k(u-\phi) f dx,\ \forall k>0,\ \forall \phi \in \mathcal{C}_0^\infty(\Omega).$$

By the results of [4], we know that for any $f \in L^1(\Omega)$ there exists a unique entropy solution of (27) and that the comparison principle holds.

Remark 3 *Originally, the notion of entropy solution was introduced for the study of nonlinear hyperbolic systems (works by P.Lax, S.N. Kruzhkov, and many others) in which the entropy (arising in Thermodynamics) served as a correct motivation (acceptable shocks along the characteristics, etc.). On the other hand, some problems involving the p-Laplacian operator came through an equivalent formulation of suitable systems of first-order hyperbolic systems. This was shown in [19] (see also [20]). The system*

$$\left\{ \begin{array}{c} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0 \\ \varepsilon\frac{\partial v}{\partial t} + \frac{\partial p}{\partial x} = -\lambda\rho\,|v|\,v \\ p/\rho = T = 1 \end{array} \right. \tag{28}$$

was derived in the study of a turbulent gas flowing in a very long pipeline with a frictional force (λ is known as the Darcy-Weissbach frictional coefficient). Then, after renormalizing to avoid constants they arrive to the equation (when $\varepsilon = 0$)

$$\frac{\partial \rho}{\partial t} - \Delta_{3/2}\rho^3 = 0. \tag{29}$$

For $\varepsilon > 0$ the characteristic lines intersect (especially for discontinuous data) and only entropy solutions have admissible shocks.

The positivity result for functions satisfying the quasilinear inequality (i.e. supersolutions of (27))

$$\left\{ \begin{array}{ll} -\Delta_p u \geq f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{30}$$

will be obtained under the following basic assumption:

$$f^- \in L^{\frac{N}{p-\epsilon_-}}(\Omega^-), \quad \text{with } \epsilon_- \in]0,p[\tag{H$_1$}$$

(it will allows to conclude that the (entropy) supersolution u has a "finite floor": see Lemma 5 below).

In addition, the "local positivity" of f , far from the boundary, will be assumed in the following way

$$\left\{ \begin{array}{c} \exists\delta>0, \quad \exists x^+ \in \Omega^+ \text{ with } B_\delta(x^+) \subset\subset \Omega^+ \text{ and there exists a radially symmetric function } \underline{f^+}\text{such that} \\ f^+(x) \geq \underline{f^+}(|x-x^+|) \text{ a.e. } x \in B_\delta(x^+) \text{ and} \\ \int_{B_\delta(x^+)} \underline{f^+}(x)dx > 0. \end{array} \right. \tag{H$_2$}$$

We will also need a suitable balance ensuring that $f^+(x)$ dominates over $f^-(x)$. This will be stated in terms of certain constants that will be specified explicitly later

$$C^+ := C(f^+) - C(f^-) > 0, \tag{H$_3$}$$

(with $C(f^-)$ and $C(f^+)$ defined by the conditions (36) and (37), below, respectively). This will allow to conclude that, in fact, $u(x) \geq C^+$ on the ball $B_\delta(x^+)$ (see Lemma 10 below).

Finally, an assumption on the growth of f^- will be needed:

$$\left\{ \begin{array}{c} f(x) \geq -M[\varphi_1(x)]^{(p-1)\alpha-p}, \quad \text{a.e. in } \Omega^-, \\ M := (\alpha C^+)^{p-1}[C(\Omega\setminus\overline{B_\delta(x^+)})^p(p-1)(\alpha-1)-\lambda_1], \\ \text{for some } \alpha > 1 + \frac{\lambda_1}{C(\Omega\setminus\overline{B_\delta(x^+)})^p(p-1)}, \end{array} \right. \tag{H$_4$}$$

for some constant $C(\Omega \setminus \overline{B_\delta(x^+)})$ which will be defined now, where φ_1 denotes the first eigenfunction of the p-Laplacian operator in Ω, given by

$$\left\{ \begin{array}{ll} -\Delta_p \varphi_1 = \lambda_1 \varphi_1^{p-1} & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{31}$$

Recall that $\phi_1 \in L^\infty(\Omega)$ is the unique positive first eigenfunction normalized such that $\|\phi_1\|_{L^\infty(\Omega)} = 1$. Moreover, the constant $C(\Omega \setminus \overline{B_\delta(x^+)})$ in (H$_4$) is given through the property

$$C(\Omega \setminus \overline{B_\delta(x^+)}) \leq |\nabla \varphi_1(x)| \quad \text{a.e.} \quad x \in \Omega \setminus \overline{B_\delta(x^+)}. \tag{32}$$

Of course, we will assume that

$$C(\Omega \setminus \overline{B_\delta(x^+)}) > 0. \tag{H$_5$}$$

This condition has a geometrical meaning: an easy argument shows that it requires that, if $x_M \in \Omega$ is such $\varphi_1(x_M) = 1$, then $x_M \notin \Omega^-$. Notice also that the value of $\alpha > 1$ which makes possible condition (H$_4$) must increase if $d(\Omega \setminus \Omega^+, 0)$ decreases.

Conditions (H$_4$) and (H$_5$) will allow to extend the positivity of u from $B_\delta(x^+)$ to the rest of Ω.

Theorem 4 *Let $f \in L^1(\Omega)$. Assume that the hypotheses (H$_1$), (H$_2$), (H$_3$), (H$_4$) and (H$_5$) are satisfied and let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be an entropy supersolution of problem (30). Then*

$$u(x) > 0 \ \textit{a.e.} \ x \in \Omega. \tag{33}$$

Before giving the proof, we will prove some Lemmas, proving the effects implied by each assumption.

Lemma 5 *Let $f \in L^1(\Omega)$. Assume that the hypothesis (H$_1$) is satisfied and let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be an entropy supersolution of the problem (30). Then, there exists a constant $C(\Omega) > 0$ such that*

$$u(x) \geq -C(\Omega) \|f^-\|_{L^{\frac{N}{p-\epsilon_-}}(\Omega^-)} \qquad \textit{a.e.} \ x \in \Omega.$$

Proof. Due to the assumption (H$_1$) we know that there exists a unique $U^- \in W_0^{1,p}(\Omega)$ weak solution of the Dirichlet problem

$$\left\{ \begin{array}{ll} -\Delta_p U^- = -f^- & \text{in } \Omega, \\ U^- = 0 & \text{on } \partial\Omega, \end{array} \right. \tag{34}$$

and, moreover, by a result due to Stampacchia [31] we know that U^- is bounded. More precisely, there exists a positive constant $C(\Omega)$ such that

$$\|U^-\|_{L^\infty(\Omega)} \leq C(\Omega) \|f^-\|_{L^{\frac{N}{p-\epsilon_-}}(\Omega^-)}^{\frac{1}{p-1}}. \tag{35}$$

In addition, by the comparison principle, since $-f^-(x) \leq 0$ a.e. $x \in \Omega$, we have

$$U^-(x) \leq 0 \quad \text{a.e. } x \in \overline{\Omega},$$

so that

$$-U^-(x) \leq C(\Omega) \|f^-\|_{L^{\frac{N}{p-\epsilon_-}}(\Omega^-)}, \quad \text{a.e. } x \in \Omega.$$

Finally, since $f(x) \geq -f^-(x)$ a.e. $x \in \Omega$, by the comparison principle for entropy solutions, we deduce

$$u(x) \geq U^-(x) \geq -C(\Omega) \|f^-\|_{L^{\frac{N}{p-\epsilon_-}}(\Omega^-)}, \quad \text{a.e. } x \in \Omega. \blacksquare$$

■

Remark 6 *According to the result of Talenti ([33] [pag. 176, formula (3.1)], for* $p \geq 2$*, the best constant in the* L^∞*- estimate (35) is*

$$C(\Omega) = \left(\frac{|\Omega|^{\frac{1}{p'}}}{N^{\frac{N}{p}} \omega_N} \right)^{\frac{p'}{N}}.$$

Let us prove now that the "local positivity" of f, i.e. assumption (H$_2$), and the balance condition (H$_3$) allow to derive the positivity of u in a part of Ω^+ (the ball $B_\delta(x^+)$). We start by defining the constants associated to each signed part of f. Concerning f^- we define

$$C(f^-) := C(\Omega) \|f^-\|_{L^{\frac{N}{p-\epsilon_-}}(\Omega^-)}. \tag{36}$$

The definition of $C(f^+)$ is more technical (once again, it depends on N and p)

$$C(f^+) := \int_0^\delta \left(\frac{1}{t^{N-1}} \int_0^\delta s^{N-1} \underline{f^+}(s) ds \right)^{1/(p-1)} dt, \tag{37}$$

where function $\underline{f^+}(s)$ was given in (H$_2$).

Remark 7 *In the special case* $\underline{f^+}(s) \equiv \epsilon > 0$ *we get that*

$$C(f^+) = \begin{cases} \frac{p-1}{N-p} \left(\frac{\epsilon}{N} \delta^N \right)^{\frac{1}{p-1}} \left(\delta^{-\frac{N-p}{p-1}} - R^{-\frac{N-p}{p-1}} \right) & \textit{if } p < N, \\ \frac{p-1}{p-N} \left(\frac{\epsilon}{N} \delta^N \right)^{\frac{1}{p-1}} \left(R^{\frac{p-N}{p-1}} - \delta^{\frac{p-N}{p-1}} \right) & \textit{if } \ p > N, \\ \left(\frac{\epsilon}{N} \delta^N \right)^{\frac{1}{p-1}} \log \frac{R}{\delta} & \textit{if } \ p = N. \end{cases}$$

Remark 8 *Notice that the balance condition can be equivalently stated in the following terms*

$$\int_0^\delta \left(\frac{1}{t^{N-1}} \int_0^\delta s^{N-1} \underline{f^+}(s) ds \right)^{1/(p-1)} dt > C(\Omega) \left(\int_{\Omega^-} |f^-(x)|^{\frac{N}{p-\epsilon_-}} \right)^{(p-\epsilon_-)/N}. \tag{38}$$

Remark 9 *Notice that if* $\alpha \geq p/(p-1)$ *then* $f^- \in L^\infty(\Omega)$ *if (H*$_4$*) holds and then assumption (H*$_1$*) is trivially satisfied. Moreover, if* $\alpha \in (1, p/(p-1))$ *then, if (H*$_4$*) holds,* f^- *may be unbounded and, in fact, it satisfies (H*$_1$*) when* $\alpha \in (p(N-1)/(p-1)N, p/(p-1))$*. This last case is a curious result in which we get the positivity of the solution even for* f^+ *very small and* f^- *unbounded near the boundary. Notice also that the value of* $\alpha > 1$ *which makes possible (H*$_4$*) must increase if* $d(\Omega^-, x^+)$ *decreases.*

Lemma 10 *Let* $f \in L^1(\Omega)$*. Assume that the hypotheses (H*$_1$*), (H*$_2$*) and (H*$_3$*) are satisfied and let* $u \in \mathcal{T}_0^{1,p}(\Omega)$ *be an entropy supersolution of the problem (30). Then, we have*

$$u(x) \geq C^+ \quad \textit{a.e. in } B_\delta(x^+), \tag{39}$$

where C^+ *is the constant defined in (H*$_3$*).*

Proof. By (H$_2$) $B_\delta(x^+) \subset\subset \Omega^+$, thus we know that there exists $R > \delta$ such that $B_\delta(x^+) \subset B_R(x^+) \subset \Omega^+$. The conclusion will be obtained by comparison with an auxiliary function $w \in W^{1,p}(B_R(x^+)) \cap L^\infty(B_R(x^+))$ defined trough a transmission problem. We define

$$w(x) = \begin{cases} w_1(|x - x^+|) & \text{if } x \in B_\delta(x^+), \\ w_2(|x - x^+|) & \text{if } x \in B_R(x^+) \setminus \overline{B_\delta(x^+)}, \end{cases} \tag{40}$$

where w_1, w_2 are radially symmetric (with respect the point x^+) functions such that

$$-\Delta_p w_1 = \underline{f^+} \tag{41}$$

where $\underline{f^+}$ was given in (H$_2$),

$$-\Delta_p w_2 = 0 \quad \text{in } B_R(x^+) \setminus \overline{B_\delta(x^+)} \tag{42}$$

$$w_1 = w_2 \quad \text{on } \partial B_\delta(x^+) \tag{43}$$

$$\nabla w_1 = \nabla w_2 \quad \text{on } \partial B_\delta(x^+) \tag{44}$$

and

$$w_2 = 0 \quad \text{on } \partial B_R(x^+). \tag{45}$$

We set $r = |x - x^+|$. Let us build such radially symmetric and decreasing functions w_1 and w_2. Firstly, we note that in $]0, \delta]$ the function w_1 solves the ordinary differential equation

$$\frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{N-1} \varphi \left(-\frac{\mathrm{d}w_1}{\mathrm{d}r} \right) \right) = \underline{f^+}(r), \tag{46}$$

where

$$\varphi(s) = |s|^{p-2} s. \tag{47}$$

Notice that if $s = -\tau \leq 0$ then $\varphi(s) = -\tau^{p-1}$. Thus, by integrating (46) from 0 to $r \in]0, \delta]$ it yields

$$\frac{\mathrm{d}w_1(r)}{\mathrm{d}r} = -\varphi^{-1} \left(\frac{1}{r^{N-1}} \int\limits_0^r s^{N-1} \underline{f^+}(s) ds \right), \quad \forall r \in]0, \delta]. \tag{48}$$

Writing the previous identity with $r = \delta$, we have.

$$\frac{\mathrm{d}w_1(\delta)}{\mathrm{d}r} = -\varphi^{-1} \left(\frac{1}{\delta^{N-1}} \int\limits_0^\delta s^{N-1} \underline{f^+}(s) ds \right). \tag{49}$$

Moreover, integrating (48) between r and δ we obtain

$$w_1(r) = w_1(\delta) + \int\limits_r^\delta \varphi^{-1} \left(\frac{1}{\delta^{N-1}} \int\limits_0^\delta s^{N-1} \underline{f^+}(s) ds \right). \tag{50}$$

Notice that $w_1(\delta)$ is still unknown but it will be determined by the transmission conditions. Concerning the function w_2, it satisfies

$$\frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{N-1} \varphi \left(\frac{\mathrm{d}w_2}{\mathrm{d}r} \right) \right) = 0, \quad \forall\, r \in [\delta, R].$$

Integrating between δ and $r \in [\delta, R]$ we get

$$r^{N-1} \varphi \left(\frac{\mathrm{d}w_2(r)}{\mathrm{d}r} \right) = \delta^{N-1} \varphi \left(\frac{\mathrm{d}w_2(\delta)}{\mathrm{d}r} \right).$$

Since, by (44), $\frac{\mathrm{d}w_2(\delta)}{\mathrm{d}r} = \frac{\mathrm{d}w_1(\delta)}{\mathrm{d}r}$, taking into account formula (49) we obtain

$$\frac{\mathrm{d}w_2(r)}{\mathrm{d}r} = -\varphi^{-1} \left(\frac{1}{r^{N-1}} \int\limits_0^\delta s^{N-1} \underline{f^+}(s) ds \right), \quad \forall\, r \in [\delta, R].$$

Integrating, again, and since $w_2(R) = 0$, we get that $\forall\, r \in [\delta, R]$

$$w_2(r) = \int\limits_0^r \varphi^{-1} \left(\frac{1}{t^{N-1}} \int\limits_0^\delta s^{N-1} \underline{f^+}(s) ds \right) dt. \tag{51}$$

In particular, using the transmission condition $w_1(\delta) = w_2(\delta)$ we get

$$w_1(\delta) = w_2(\delta) = \int_0^\delta \varphi^{-1}\left(\frac{1}{t^{N-1}}\int_0^\delta s^{N-1}\underline{f^+}(s)ds\right)dt. \tag{52}$$

Note that $w_2(\delta) = C(f^+)$, with $C(f^+)$ defined as in (37). Moreover, coming back to (50) we have

$$w_1(r) = C(f^+) + \int_r^\delta \varphi^{-1}\left(\frac{1}{\delta^{N-1}}\int_0^\delta s^{N-1}\underline{f^+}(s)ds\right). \tag{53}$$

Finally, the function w defined by (40) is such that $w \in W^{1,p}(B_R(x^+)) \cap L^\infty(B_R(x^+))$ and itsatisfies

$$-\Delta_p w(x) = \begin{cases} -\Delta_p w_1(|x-x^+|) \le f^+(x) = f(x) & \text{if } x \in B_\delta(x^+), \\ \\ -\Delta_p w_2(|x-x^+|) = 0 & \text{if } x \in B_R(x^+) \setminus \overline{B_\delta(x^+)}. \end{cases} \tag{54}$$

Then, since the $p-$Laplacian is invariant under translations by a constant, we get

$$-\Delta_p(u + C(f^-)) = -\Delta_p(u) \ge f \quad \text{in } \Omega.$$

and $u + C(f^-) \ge 0 = w$ on $\partial B_R(x^+)$. In consequence, we can apply the comparison principle in $B_R(x^+)$ and thus $w \le u + C(f^-)$ in $B_R(x^+)$. In particular, on $B_\delta(x^+)$ it holds $w_1 \le u + C(f^-)$. Consequently,

$$u(x) + C(f^-) \ge \inf_{]0,\delta]} w_1(r) = C(f^+) \quad \text{a.e. } x \in B_\delta(x^+),$$

and using (H$_3$) we get the result. ■

Now we are in a position to end the proof of Theorem 4.

Proof of Theorem 4. We note that by virtue of Lemma 10 $u(x) > C^+ > 0$ a.e. $x \in B_\delta(x^+)$. We are going to prove that

$$u(x) > 0 \qquad \text{a.e. } x \in \Omega \setminus B_\delta(x^+).$$

Let $v \in \mathcal{T}^{1,p}(\Omega \setminus \overline{B_\delta(x^+)})$ be the unique entropy solution of the problem

$$\begin{cases} -\Delta_p v = f & \text{in } \Omega \setminus \overline{B_\delta(x^+)}, \\ v = C^+ & \text{in } \partial B_\delta(x^+), \\ v = 0 & \text{in } \partial\Omega. \end{cases} \tag{55}$$

Note that by the comparison principle

$$v(x) \le u(x) \quad \text{a.e.} \quad x \in \Omega \setminus \overline{B_\delta(x^+)}.$$

Our conclusion will follow once we prove that

$$v(x) > 0 \quad \text{a.e. } x \in \Omega \setminus \overline{B_\delta(x^+)}.$$

To this aim, we will built a positive function w, which is a subsolution of the problem (55). Let

$$w(x) = C^+ \varphi_1^\alpha(x)$$

with α satisfying the condition in (H$_4$). Note that $w(x) > 0$ a.e. $x \in \Omega$. Moreover, we can easily show that w satisfies

$$\Delta_p w = [\alpha C^+]^{p-1} \varphi_1^{\alpha(p-1)-p}[(p-1)(\alpha-1)|\nabla\varphi_1|^p - \lambda_1 \varphi_1^p]$$

and using that $0 < \varphi_1 < 1$ we get

$$-\Delta_p w \le -[\alpha C^+]^{p-1} \varphi_1^{\alpha(p-1)-p}[(p-1)(\alpha-1)C_\Omega^p - \lambda_1]$$

$$< -M\varphi_1^{\alpha(p-1)-p}$$

with M defined in (H_4). Thanks to the assumption (H_4) it follows

$$-\Delta_p w \leq f = -\Delta_p v \quad \text{in} \quad \Omega \setminus \overline{B_\delta(x^+)},$$

(notice that it suffices to impose the growth condition (in Ω^- since on $\left(\Omega \setminus \overline{B_\delta(x^+)}\right) \cap \Omega^+$ it holds trivially). Moreover

$$w(x) \leq C^+ = v(x) \quad \text{on} \quad \partial B_\delta(x^+)$$

and

$$w(x) = v(x) = 0 \quad \text{on} \quad \partial\Omega.$$

Thus, by the comparison principle

$$w(x) \leq v(x) \quad \text{a.e.} \quad x \in \Omega \setminus \overline{B_\delta(x^+)}$$

which in turn implies that $v(x) > 0$ a.e. $x \in \Omega \setminus \overline{B_\delta(x^+)}$ due to the positivity of w. ■

4 Flat solutions

To express the conclusion of the above Theorem in terms of the "co-normal derivative" on the boundary $\partial\Omega$ we need to work with an appropriate notion of trace of a entropy solution. This was carried out in the papers [2], [3].

The difficulty comes from the fact that we are solving the problem in a framework different than the energy setting: note that neither $u \in \mathcal{T}_0^{1,p}(\Omega)$ nor the condition $u \in W_0^{1,p}(\Omega)$ are enough to conclude that the co-normal derivative of the entropy solution of

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{56}$$

is a function in $L^1(\partial\Omega)$. For the case of $p = 2$ that was proved in [5] since $u \in W_0^{1,1}(\Omega)$ and $\Delta u \in L^1(\Omega)$. For the case $p \neq 2$ the first deep result in this direction was obtained for the case $\Delta_p u \in L^\infty(\Omega)$ (see [25]). More in general, the notion of trace on the set $\mathcal{T}^{1,p}(\Omega)$ was introduced in [2], as follows. Firstly, we start by defining the set $\mathcal{T}_{tr}^{1,p}(\Omega)$ as the the set of functions u in $\mathcal{T}^{1,p}(\Omega)$ such that there exists a sequence $u_n \in W^{1,p}(\Omega)$ satisfying

(a) u_n converges to u a.e. in Ω,

(b) $\nabla T_k(u_n)$ converges to $\nabla T_k(u)$ in $L^1(\Omega)$ $\forall k > 0$,

(c) there exists a measurable function w on $\partial\Omega$ such that u_n converges to w a.e. in $\partial\Omega$.

Following [2], [3], we say that the function w is the trace of $u \in \mathcal{T}^{1,p}(\Omega)$ in the generalized sense and it is denoted as $w = tr(u)$ (and, sometimes, simply by u). Thus, if $u \in W^{1,p}(\Omega)$, $tr(u)$ coincides with the trace of u, $\tau(u)$, in the usual sense, and the space $\mathcal{T}_0^{1,p}(\Omega)$, introduced in [4] is equal to $Ker(tr)$.

In order to get a solution with an integrable co-normal on $\partial\Omega$ we need to assume some slight additional regularity to the left term $f \in L^1(\Omega)$ according the dimension N of the space. The following Banach space where introduced in [3]:

$$V^{1,p}(\Omega) := \left\{ f \in L^1(\Omega) : \exists M > 0 \text{ such that } \int_\Omega |fv| \leq M \, \|v\|_{W^{1,p}(\Omega)} \ \forall v \in W^{1,p}(\Omega) \right\},$$

(in some sense $V^{1,p}(\Omega) = (W^{1,p}(\Omega))' \cap L^1(\Omega)$). Using Sobolev embeddings we get that

$$L^{p'}(\Omega) \subset L^{(Np/(N-p))'}(\Omega) \subset V^{1,p}(\Omega) \text{ if } 1 \leq p < N,$$

$$L^q(\Omega) \subset V^{1,N}(\Omega) \text{ for any } q > 1,$$

$$V^{1,p}(\Omega) = L^1(\Omega) \text{ if } p < N.$$

The notion of “entropy solution” was extended, in the above two mentioned papers, to a general class of nonlinear Neummann type problems, including the following one

$$\begin{cases} -\Delta_p u = f & \text{in } \Omega, \\ |\nabla u|^{p-2} \nabla u \cdot n = g & \text{on } \partial\Omega, \end{cases} \tag{57}$$

assumed $f \in L^1(\Omega)$ and $g \in L^1(\partial\Omega)$: a function $u \in \mathcal{T}^{1,p}_{tr}(\Omega)$ is an entropy solution of (57) if

$$\int\limits_{\{|u-\phi|<k\}} \left\langle |\nabla u|^{p-2} \nabla u, \nabla u - \nabla\phi \right\rangle dx \leq \int\limits_{\Omega} T_k(u-\phi) f dx + \int\limits_{\partial\Omega} T_k(u-\phi) g d\sigma, \ \forall k > 0, \ \forall \phi \in L^\infty(\Omega) \cap W^{1,p}(\Omega). \tag{58}$$

Moreover, by applying Theorem 3.6 of [3], we conclude the following: assume $f \in V^{1,p}(\Omega)$ and let u be the unique entropy solution of the Dirichlet problem (56), then $|\nabla u|^{p-2} \nabla u \cdot n \in L^1(\partial\Omega)$ and it coincides with the unique entropy solution of the Neumman problem (57), with $g := |\nabla u|^{p-2} \nabla u \cdot n$.

Now we are in a position to state and prove a simple consequence of our Theorem 4 on the extension of the strong maximum principle:

Corollary 11 *Let $f \in V^{1,p}(\Omega)$. Assume that the hypotheses of Theorem 4 are satisfied and*

$$\int_\Omega f(x) dx = 0. \tag{59}$$

Then, the unique entropy solution $u \in \mathcal{T}^{1,p}_0(\Omega)$ of the Dirchlet problem (56) is a flat solution, in the sense that $u = 0$, and also $|\nabla u|^{p-2} \nabla u \cdot n = 0$, in $L^1(\partial\Omega)$.

Proof. Note that $u \in \mathcal{T}^{1,p}_0(\Omega)$ is also the entropy solution of the Neumman problem (57) with $g := |\nabla u|^{p-2} \nabla u \cdot n$. Let $j > 0$; taking $\phi \equiv T_j(u) \pm 1$ as test function in (58), we get

$$0 \leq \int\limits_{\Omega} T_k(u - T_j(u) \pm 1) f dx + \int\limits_{\partial\Omega} T_k(u - T_j(u) \pm 1) |\nabla u|^{p-2} \nabla u \cdot n d\sigma,$$

and letting $j \to +\infty$ we obtain

$$\int_{\partial\Omega} |\nabla u|^{p-2} \nabla u \cdot n d\sigma = \int_\Omega f(x) dx = 0.$$

Since by Theorem 4 we know that $|\nabla u|^{p-2} \nabla u \cdot n \leq 0$ on $\partial\Omega$, we conclude that $|\nabla u|^{p-2} \nabla u \cdot n = 0$ on $\partial\Omega$. ■

Corollary 12 *Let $\Omega^* \supset \Omega$ be an open extension of Ω, and let $f^* \in L^1(\Omega^*)$ be the extension, by zero, of a given function $f \in V^{1,p}(\Omega)$ as in Theorem 4, i.e.,*

$$f^*(x) = \begin{cases} f(x) & \textit{if } x \in \Omega, \\ 0 & \textit{if } x \in \Omega^* \setminus \Omega. \end{cases}$$

Let u be the unique solution of (56) and let u^* be the extension of u defined as

$$u^*(x) = \begin{cases} u(x) & \textit{if } x \in \Omega, \\ 0 & \textit{if } x \in \Omega^* \setminus \Omega. \end{cases}$$

Then u^* is the unique weak solution of the problem

$$\begin{cases} -\Delta_p u^* = f^*(x) & \textit{in } \Omega^*, \\ u^* = 0 & \textit{on } \partial\Omega^*. \end{cases} \tag{60}$$

Now we will give some sufficient local conditions on f in order to have a "locally flat solution", i.e. such that $u = 0$ and also $|\nabla u|^{p-2} \nabla u \cdot n = 0$ on some part $\Gamma \subset \partial\Omega$ (and then, lacking again the normal derivative conclusion of the classical strong maximum principle 4).

We will use some techniques developed in [13] to get a "local" condition on $f(x)$, on some neighborhood of a part $\Gamma_0 \subset \partial\Omega$, in order to construct suitable local supersolutions implying that $|\nabla u|^{p-2} \nabla u(x_0) \cdot n = 0$ on Γ_0. In addition to the assumptions of Theorem 4 we will assume that

$$f^+ \in L^{\frac{N}{p-\epsilon_+}}(\Omega^+), \quad \text{with } \epsilon_+ \in]0, p[. \tag{H$_6$}$$

Then, working as in the proof of Lemma 5 we know that there exists a positive constant $C(\Omega)$ such that

$$u(x) \leq C(\Omega) \|f^+\|^{\frac{1}{p-1}}_{L^{\frac{N}{p-\epsilon_+}}(\Omega^+)}. \tag{61}$$

We set

$$\hat{C}^+ = C(\Omega) \|f^+\|^{\frac{1}{p-1}}_{L^{\frac{N}{p-\epsilon_+}}}. \tag{62}$$

Theorem 13 *Assume that the hypotheses of Theorem 4 are satisfied and that (H$_6$) holds. Let* $\Gamma_0 \subset \partial\Omega$ *and let* $x_0 \in \Gamma_0$ *such that, there exist* $C_- > 0$ *and* $\theta > 1$ *for which*

$$f(x) \leq -C_- |x - x_0|^{(\theta-1)(p-1)-1} \quad a.e.\ x \in B_\tau(x_0) \cap \Omega, \tag{63}$$

where

$$\tau \geq \left(\frac{\hat{C}^+}{K} \right)^{1/\theta}, \tag{64}$$

with $K > 0$ *given by*

$$K = \frac{1}{\theta} \left(\frac{C_-}{[(\theta-1)(p-1) + N - 1]} \right)^{1/(p-1)}, \tag{65}$$

and $\hat{C}^+$ *defined in (62). Then*

$$0 \leq u(x) \leq K|x - x_0|^\theta, \quad a.e.\ x \in B_\tau(x_0) \cap \Omega. \tag{66}$$

In particular, if (63) holds for any $x_0 \in \Gamma_0$ *then* $|\nabla u|^{p-2} \nabla u \cdot n = 0$ *on* Γ_0.

Proof. Let U be the radial barrier function

$$U(x) =: K|x - x_0|^\theta.$$

As in Theorem 1.15 of [13], we know that $U(x)$ satisfies

$$-\Delta_p U = -(\theta K)^{p-1} [(\theta - 1)(p - 1) + N - 1] |x - x_0|^{(\theta-1)(p-1)-1}.$$

By the assumptions (65) and (64) it follows

$$-\Delta_p u \leq -\Delta_p U, \quad \text{a.e. } x \in B_\tau(x_0) \cap \Omega.$$

Moreover,

$$u = 0 \leq U \quad \text{on } \partial\Omega \cap \partial(B_\tau(x_0) \cap \Omega),$$

and, by (61) and assumption (64),

$$u \leq \hat{C}^+ \leq U, \quad \text{on } \partial(B_\tau(x_0) \cap \Omega) \setminus \partial\Omega.$$

Thus, by the comparison principle we get the estimate (66) on $B_\tau(x_0) \cap \Omega$. Since $\theta > 1$ we conclude that $|\nabla u|^{p-2} \nabla u \cdot n = 0$ on Γ_0 if (63) holds for any $x_0 \in \Gamma_0$. ■

Remark 14 *In a recent paper [28], the author studies whether the sign property of the solution of problems like (56), and its parabolic associated problem, is a local property or not.*

5 Application to some indefinite quasilinear problems

Let us give a short application of the results in previous sections to some *sub-homogeneous indefinite* quasilinear equations. We consider the question of the existence of nonnegative solutions of the problem

$$\left\{\begin{array}{ll} -\Delta_p u = \lambda u^{p-1} + m(x)u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{67}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $q \in (0, p-1)$, $m \in L^\infty(\Omega)$ changes sign on Ω and λ is a real parameter. Problems of this type arise in the study of problems in mathematical biology and porous media (see the papers [22] and [18] for $p = 2$).

For simplicity in the exposition we will assume

$$\left\{\begin{array}{c} m \in L^\infty(\Omega), \text{ and } |\Omega^+|, |\Omega^-| > 0, \text{ with} \\ \\ \Omega^+ = \{x \in \Omega \ \mid m(x) > 0\} \text{ and } \Omega^- = \{x \in \Omega \ \mid m(x) < 0\} \end{array}\right. \tag{68}$$

We have:

Theorem 15 *Assume that $f(x)$ satisfies the hypotheses of Theorem 4, (68) holds and let $0 \le \lambda < \lambda_1$. Then there is a solution $u_\lambda > 0$ to (67).*

Proof. We will follow an idea already used in Lemma 4.2 in [22] (see also [17]). Let U be the unique solution of the problem

$$\left\{\begin{array}{ll} -\Delta_p U = m(x) & \text{in } \Omega, \\ U = 0 & \text{on } \partial\Omega. \end{array}\right. \tag{69}$$

By Theorem 4 we know that

$$U > 0 \text{ in } \Omega. \tag{70}$$

Let $\alpha = q/(p-1) \in (0,1)$. Note that the function $u_0 = [(1-\alpha)U]^{1/(1-\alpha)}$ is a subsolution of problem (67). Indeed,

$$\nabla u_0 = [(1-\alpha)U]^{\alpha/(1-\alpha)} \nabla U,$$

$$|\nabla u_0|^{p-2} \nabla u_0 = [(1-\alpha)U]^{\alpha(p-1)/(1-\alpha)} |\nabla U|^{p-2} \nabla U,$$

and

$$-\Delta_p u_0 = -[(1-\alpha)U]^{\alpha(p-1)/(1-\alpha)} \Delta_p U - \frac{\alpha(p-1)}{(1-\alpha)} [(1-\alpha)U]^{\left(\frac{\alpha(p-1)}{(1-\alpha)}-1\right)} |\nabla U|^p .$$

Then

$$\begin{array}{rcl} -\Delta_p u_0 - \lambda u_0^{p-1} - m(x)(u_0)^{\alpha(p-1)} & = & -\dfrac{\alpha(p-1)}{(1-\alpha)} [(1-\alpha)U]^{\left(\frac{\alpha(p-1)}{(1-\alpha)}-1\right)} |\nabla U|^p \\ -\lambda [(1-\alpha)U]^{(p-1)/(1-\alpha)} & \le & 0, \end{array}$$

if $\lambda \ge 0$.
On the other hand, let $u^0 = C\psi$ where $\psi > 0$ is the unique solution of the problem

$$\left\{\begin{array}{ll} -\Delta_p \psi = \lambda \psi^{p-1} + 1 & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{71}$$

(recall that $0 \le \lambda < \lambda_1$). Easy calculations show that u^0 is a supersolution of problem (67) if

$$C > (\|m\|_{L^\infty} \|\psi\|_{L^\infty}^{\alpha(p-1)})^{1/(1-\alpha)(p-1)}.$$

Indeed,

$$-\Delta_p u^0 - \lambda \left(u^0\right)^{p-1} - m(x)(u^0)^{\alpha(p-1)} = C^{\alpha(p-1)}(C^{(1-\alpha)(p-1)} - m(x)\psi^{\alpha(p-1)}) > 0$$

Moreover, by the results of Lieberman [25], we know that U, $\psi \in C^1(\overline{\Omega})$, $\frac{\partial U}{\partial n} \leq 0$ and $\frac{\partial \psi}{\partial n} < 0$. Then, by taking $C > 0$ large enough we have that $u_0 \leq u^0$ on Ω. Then the existence of a solution $u_\lambda > 0$ to (67) follows from the super and subsolution method. ■

Remark 16 *Many other results on the problem (67) can be found in [22] and [18]. For instance, if in Theorem 15 we assume, in addition, that the unique weak solution U of problem (69) satisfies $\frac{\partial U}{\partial n} < 0$ on $\partial\Omega$ then there is uniqueness of positive solutions to problem (67). Indeed, the uniqueness of positive solutions can be obtained by means of a suitable change of unknowns (see Theorem 4.4 of [22] and the extension presented in [18]), or by means of some hidden convexity arguments (see, e.g., [14]).*

6 Beyond the Vázquez's strong maximum principle

In some sense the above result goes beyond the Vázquez's strong maximum principle [32]. Indeed, assume $\Omega = \Omega^- \cup \Omega^+$, and let

$$m(x) = \begin{cases} -\mu & \text{if } x \in \Omega^- \\ \gamma & \text{if } x \in \Omega^+ \end{cases}$$

with μ, γ such the assumptions of Theorem 4 hold. Then the solutions of (67) satisfy

$$-\Delta_p u + \mu \chi_{\Omega^-} u^q = \lambda u^{p-1} + \gamma \chi_{\Omega^+} u^q \geq 0 \text{ on } \Omega$$

and are positive ($u > 0$ in Ω) assumed $q \in (0, p-1)$, in contrast with the main result of [32] in which the positivity of the supersolutions

$$-\Delta_p u + \mu u^q \geq 0 \text{ on } \Omega,$$

(i.e.,when $\Omega = \Omega^-$) requires the opposite condition $q \geq (p-1)$.

Whe point out that the Vázquez's maximum principle can be extended to the case in which the right-hand side is a function $f(x)$ becoming negative near the boundary $\partial\Omega$.

Theorem 17 *Let β be a maximal monotone graph of $\mathbb{R}$ such that $\beta = \partial j$, with*

$$\int_0 \frac{ds}{j(s)^{1/p}} = +\infty.$$

Let $f \in L^1(\Omega)$ satisfying (H$_1$), (H$_2$), (H$_3$), (H$_4$) (with a constant M_β replacing M) and (H$_5$). Let $u \in \mathcal{T}_0^{1,p}(\Omega)$ be an entropy supersolution of the problem

$$\begin{cases} -\Delta_p u + \beta(u) \geq f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{72}$$

Then

$$u(x) > 0 \text{ a.e. } x \in \Omega. \tag{73}$$

Proof. We will give here only the idea of the proof for the case $\beta(u) \leq k\,|u|^{\gamma-1} u$ with $\gamma \geq (p-1)$ and $k > 0$. The extension of Lemma 10 is even easier than for the case $\beta \equiv 0$ (it suffices to apply the Vázquez's principle over $B_R(x^+) \subset \Omega^+$). Thus, we just have to adapt the proof of the second step of the proof of Theorem 4. Once again we take as subsolution on $\Omega \setminus \overline{B_\delta(x^+)}$ the function

$$w(x) = C^+ \varphi_1^\alpha(x)$$

with $\alpha > 1$ large enough. Then

$$-\Delta_p w + k\,|w|^{\gamma-1} w = [\alpha C^+]^{p-1} \varphi_1^{\alpha(p-1)-p} [(p-1)(\alpha-1)|\nabla\varphi_1|^p - \lambda_1 \varphi_1^p - k\varphi_1^{p+\alpha\gamma-\alpha(p-1)}]$$

and using the estimate $0 < \varphi_1 < 1$ and the assumption (H$_4$) we get

$$-\Delta_p w + k\,|w|^{\gamma-1} w \leq -M_\beta \varphi_1^{\alpha(p-1)-p}$$

with

$$M_\beta := (\alpha C^+)^{p-1}[C(\Omega \setminus \overline{B_\delta(x^+)})^p(p-1)(\alpha-1) - \max_{\Omega^-}\left(\lambda_1\varphi_1^p + k\varphi_1^{\alpha(\gamma-(p-1))}\right)],$$
$$\text{for some } \alpha > 1 + \frac{\max_{\Omega^-}\left(\lambda_1\varphi_1^p + k\varphi_1^{\alpha(\gamma-(p-1))}\right)}{C(\Omega\setminus\overline{B_\delta(x^+)})^p(p-1)}.$$

Thanks to the new assumption (H$_4$) it follows

$$-\Delta_p w + k\,|w|^{\gamma-1}\,w \le f \text{ in } \quad \Omega \setminus \overline{B_\delta(x^+)},$$

and thus the solution of

$$\begin{cases} -\Delta_p v + k\,|w|^{\gamma-1}\,w = f & \text{in } \Omega \setminus \overline{B_\delta(x^+)}, \\ v = C^+ & \text{in } \partial B_\delta(x^+), \\ v = 0 & \text{in } \partial\Omega, \end{cases} \tag{74}$$

satisfies

$$0 < w \le v \le u \text{ on } \Omega \setminus \overline{B_\delta(x^+)}.$$

■

Remark 18 *The methods of proof of this paper can be extended to several different equations, as, for instance, other degenerate operators ([26], [35]), the case of quasilinear equations with variable coefficients, two phase operators as in ([12],[11]) and the associated parabolic equations (as in [15], [17]) and ([28]).*

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References

[1] Apushkinskaya, D. E. and Nazarov, A. I.: The normal derivative lemma and surrounding issues, *Russian Math. Surveys* **77**(2), 189–249 (2022)

[2] Andreu,F., Mazón,J.M., Segura de León,S. and Toledo,J.: Quasi-linear elliptic and parabolic equations in L^1 with nonlinear boundary conditions, Adv. Math. Sci. Appl. 7 (1), 183-213 (1997)

[3] Andreu, F., Igbida, N., Mazón, J.M. and Toledo J.: L^1 existence and uniqueness results for quasi-linear elliptic equations with nonlinear boundary conditions. *Annales de l'IHP Analyse non linéaire* **24** (1), 61-89 (2007)

[4] Bénilan Ph., Boccardo, L., Gallouet, T., Gariepy, R., Pierre, M., and Vázquez, J. L.: An L^1-theory of existence and uniqueness of solutions of nonlinear elliptic equations. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze, 22 (2), 241-273 (1995)

[5] Bénilan, Ph., Crandall, M.G. and Sacks, P.: Some L^1 existence and dependence results for semilinear elliptic equations under nonlinear boundary conditions, Appl. Math. Optim. 17 (3), 203-224 (1988)

[6] Benedikt, J., Girg, P., Kotrla, L. and Takáč,P.: On the strong comparison principle for degenerate elliptic problems with convection, J. Math. Anal. Appl. 514, 126267 (2022)

[7] Boccardo, L., Díaz, J. I., Giacchetti, D., and Murat,F.: Existence of a solution for a weaker form of a nonlinear elliptic equation, in ”Recent Advances in Nonlinear Elliptic and Parabolic Problems (Proceedings. Nancy, 1988)” (P. Benilan, M. Chipot. L C. Evans, and M. Pierre, Eds.), pp. 229 -246, Pitman Research Notes in Mathematics Series, Vol. 208. Longman, Harlow (1989).

[8] Braga, J.E.M., and Moreira, D.: Inhomogeneous Hopf–Oleinik Lemma and regularity of semiconvex supersolutions via new barriers for the Pucci extremal operators, Advances in Mathematics, 334 184–242 (2018)

[9] Brezis, H., Cazenave T., Martel, Y. and Ramiandrisoa A.: Blow up for $u_t - \Delta u = g(u)$ revisited, *Adv. Differential Equations* **1**, 73–90 (1996)

[10] Brezis, H. and Ponce, A.C.: Remarks on the strong maximum principle, *Diff.Integ. Equ.* **16**, 1-12 (2003)

[11] Li,C., Papageorgiou,N.S. and Rădulescu, V.D.: Multiple and nodal solutions for parametric Dirichlet equations driven by the double phase differential operator. Complex Analysis and Operator Theory **17** 5 (2023): 62.

[12] Cirmi, G. R. and Díaz, J.I.: Qualitative properties of solutions of some quasilinear equations related to Bingham fluid *Communications in Optimization Theory (COT)* **5** `https://doi.org/10.23952/cot.2023.5` (a special issue on Numerical Analysis and Control dedicated to Roland Glowinski) (2025)

[13] Díaz, J.I.: *Nonlinear partial differential equations and free boundaries*, Research Notes in Mathematics **106**, Pitman, Boston, MA. (1985)

[14] Díaz, J.I.: New applications of monotonicity methods to a class of non-monotone parabolic quasilinear sub-homogeneous problems. *Journal Pure and Applied Functional Analysis,* **5** (4), 925-949 (2020)

[15] Díaz, J.I. and Fleckinger, J.: Positivité en temps grande des solutions de l’équation de la chaleur: le principe d’antimaximum parabolique, *Discrete and Continuous Dynamical Systems* **10**, 193-200 (2004)

[16] Díaz, J.I. and Hernández, J.: Global bifurcation and continua of nonnegative solutions for a quasilinear elliptic problem, *C.R. Acad. Sci. Paris* **329**, 587-592 (1999)

[17] Díaz, J.I. and Hernández, J.: Beyond the classical strong maximum principle: forcing changing sign near the boundary and at solutions. Advances in Nonlinear Analysis, vol. 13, no. 1, 2024, pp. 20230128 (2024)

[18] Díaz, J.I., Hernández, J. and Ilyasov, Y.: Non-negative and positive solutions for some indefinite sublinear elliptic problems. To appear in *Boundary Value Problems.*

[19] Díaz, J.I. and Liñán, A.: Gas discharge movement in long ducts.Modeling and study of a doubly nonlinear equation. In *Reunión Matemática en Honor de A. Dou* (J. I. Díaz and J. M. Vegas eds.) Universidad Complutense de Madrid, pp. 95-119 (1989)

[20] Díaz, J.I. and de Thelin F.: On a nonlinear parabolic problem arising in some models related to turbulent flows. SIAM Journal of Mathematical Analysis, 25 (4), 1085-1111 (1994)

[21] Droniou, J., and Prignet, A.: Equivalence between entropy and renormalized solutions for parabolic equations with smooth measure data. Nonlinear Differential Equations and Applications NoDEA, 14 181-205 (2007)

[22] Hernández,J., Mancebo, F.J., and Vega, J.M.: On the linearization of some singular nonlinear elliptic problems and applications. *Ann. Inst. H. Poincaré Anal.Non Lin.*,**19** 777-813 (2002)

[23] Hopf, E.: Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus, *Sitz Bet Preuss Akad Wissensch, Berlin. Math Phys kl*, 147-152 (1927)

[24] Hopf, E.: A remark on linear elliptic differential equations of second order, *Proc. Amer. Math. Soc.* **3** (5) 791–793 (1952)

[25] Lieberman G.M.: Boundary regularity for solutions of degenerate elliptic equations, Nonlinear Anal. 12 1203-1219 (1988)

[26] Moreira, D., dos Santos, J. A. and Soares, S. H. M.: A quantitative version of the Hopf-Oleinik lemma for a quasilinear non-uniformly elliptic operator. Annales Fennici Mathematici **49**.1 (2024), 337-348.

[27] Oleinik,O. A.: On properties of solutions of certain boundary value problens for equations of elliptic type, *Mat. Sb.* **30**(72):3 , 695–702 (Russian) (1952)

[28] Porzio, M. M.: Local or non-local properties of the solutions of some elliptic and parabolic pde's, to appear in *Differential and Integral Equations* (2026)

[29] Protter, M.H. and Weinberger,H. F.: *Maximum principles in differential equations*, Prentice Hall, Englewood Cliffs (1967)

[30] Pucci,P. and Serrin J.: The strong maximum principle revisited, *J. Differential Equations* **196**, 1–66 (2004)

[31] Stampacchia, G.: *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, Ann. Inst. Fourier, Grenoble, Vol. **15,** 1 (1965).

[32] Vázquez, J.L.: A strong maximum principle for some quasilinear elliptic equations, *Appl. Math. Optim.* **12**, 191–202 (1984)

[33] Talenti, G.: Nonlinear elliptic equations, rearrangements of functions and Orlicz spaces. Annali di Matematica Pura ed Applicata 120. 1 159-184.(1979)

[34] Tolksdorf,P.: On the Dirichlet problem for quasilinear equations in domains with conical boundary points, Comm. Partial Differential Equations 8:7, 773–817 (1983)

[35] Yongjun,H. Pinchover,Y. and Rasila, A.: Positive solutions of the A-Laplace equation with a potential. Potential Analysis **60**.2 (2024): 721-758

[36] Zaremba, S.: Sur un problème mixte relatif à l'équation de Laplace, *Bull. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Ser. A*, 313–344 (1910)