Homogenization of Dynamic Signorini-Type Problems on Critically Oscillating Boundaries under Time-Periodic Forcing

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We study the homogenization, in the critical scaling regime, of a boundary value problem with a nonlinear dynamic Signorini-type condition posed on a rapidly oscillating portion of the boundary. The source term is time-periodic and we look for time-periodic solutions. Using the method of oscillating test functions (Tartar), compactness, and monotonicity arguments, we identify the homogenized problem and the effective nonlinear boundary operator. In contrast with the evolutionary (initial value) setting, the periodic framework eliminates memory effects and yields an instantaneous time-periodic operator defined through a periodic-in-time cell problem.

Keywords: homogenization, Signorini-type condition, critical case, rapidly alternating boundary condition.

1. Introduction

The main goal of this paper is to study a dynamic Signorini-type problem with unilateral boundary conditions posed on a part of the boundary which is rapidly oscillating. The source term is time-periodic and we look for time-periodic solutions. There are multiple physical problems leading to these type of formulations. For instance, the model under consideration in this paper arises in some simplifications related to nano-composite membranes and nano-osmosis (in the spirit of the paper [7]). In this framework, u represents pressure, concentration, or chemical potential, the oscillating boundary models nanopores, Signorini-type conditions model selective permeability.

The problem consists of the homogenization of a time-periodic Poisson equation with a time-periodic source term, where the boundary condition rapidly alternates between a dynamic Signori-type boundary condition and a pure Neumann boundary condition. This type of equation models systems where macroscopic, stable phenomena interact with microstructures that oscillate rapidly in space (on the boundary of the geometrical domain). The systems are subjected to coupled dynamics and sources that are periodic in time. Some other prominent physical and engineering contexts for this problem include: a) *Fuel Cells and Lithium-Ion Batteries.* Porous electrode frontiers usually present a rough or oscillating geometry at the microscopic scale. Chemical reactions at the interface are neither instantaneous nor static, they are modeled via dynamic Signorini-type boundary conditions (which involve time derivatives at the boundary due to charge or mass accumulation). If the device operates under alternating current or charge/discharge cycles, both the source and the boundary conditions acquire a time-periodic nature; b) *Heat*

Transfer with Boundary Thermal Capacitance. Consider a solid with a rough surface exposed to a cycling thermal flux (e.g., day/night cycles or pistons in an engine). If the thin surface layer significantly absorbs/stores heat, the traditional Neumann boundary condition transforms into a dynamic condition (Robin-dynamic type); c) *Microfluidic Devices and Chemical Sensors*. Channels where the walls have active patches (catalyzed periodic chemical reactions) interleaved with insulating patches (Neumann conditions).

Let us mention now some reasons to consider the time-periodic formulation in some of those applications. *Electrochemical Impedance Spectroscopy (EIS) and Batteries:* This is currently the most direct field of application, where the homogenization techniques provide significant industrial value. The phenomenon is to determine the state of health of a lithium-ion battery or a fuel cell, a low-amplitude alternating current with a fixed period $T = 2\pi/\omega$ is injected into the system. As a simplified model, we can consider the Poisson equation governing the electric potential or ion concentration. The source (the injected current) is T-periodic. The reason to consider some dynamic oscillating boundary is that the electrodes are porous and rough at the microscopic scale ε. At the electrode surface, the double capacitive layer occurs (which stores charge, generating the dynamic condition $\partial_t u$) alongside the charge transfer reaction in active patches, while the insulating zones act as a Neumann condition. Finally, the relevance of the homogenization limit is that the appearance of some "strange term" non-local in time predicted by mathematical theory translates physically into the effective impedance of the electrode. This explicitly explains how microscopic roughness alters the global resistance and capacitance of the battery (see, e.g. [22], and [6]).

In the case of *Heat Conduction in Engines and Bioclimatic Architecture,* some natural or mechanical thermal cycles impose a strict temporal periodicity. In *daily cycles* (Climatization and Geothermal Energy), the outdoor temperature or solar radiation on the rough facade of a building (or the ground) varies with a fixed period of $T = 24$ hours and the heat source is essentially periodic. In the case of Internal Combustion Engine Cycles, the walls of the cylinders in an engine receive intermittent heat pulses due to injection and combustion strokes (T depends on the engine's RPM). The dynamic oscillating boundary is justified because the outer surface has microscopic cooling fins or rough textures to dissipate heat, and these textures have their own thermal capacity (storing heat before transmitting it), the boundary condition is dynamic. Upon applying spatial homogenization, the 24-hour period (or the RPM cycle) remains fixed, but the strange term reveals the effective thermal inertia that the rough material will exhibit macroscopically (see [13]).

Finally, the case of *Microfluidic Devices and Organs-on-a-Chip (Lab-on-a-Chip)* is very relevant in modern bioengineering where microscopic channels are used to transport biological fluids or chemical reagents via piezoelectric or peristaltic pumping. The interesting phenomenon is that microfluidic pumps operate in a pulsatile manner, generating pressure fields or solute concentrations with a fixed temporal periodicity T (the pump cycle). The dynamic oscillating boundary is justified since the walls of these channels are often decorated with patches of chemical catalysts or biological receptors (active patches with dynamic adsorption/reaction conditions) interleaved with inert plastic (Neumann condition). The relevance of the limit (the spatial homogenization) allows chip designers to determine the total percentage of reagent that will be absorbed per pumping cycle T, eliminating the need to computationally simulate millions of microscopic patches inside the channel (see, e.g., [2]).

Thus, time periodicity corresponds to cyclic operational regimes: oscillating pressure, alternating electric fields, periodic concentration gradients. Although, we will present the technical details on the domain in the next section, we outline now that the problem under consideration can be simply formulated in the following terms. Given a time period $T > 0$, in the cylinder $Q^\infty = \Omega \times \mathbb{R}$, and assuming a source term $f \in H^1(\mathbb{R}, L^2(\Omega))$ is T-time-periodic, i.e.

$$f(x,t) = f(x, t+T) \text{ for any } t \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Notice that we can also use a different natural notation within the time torus $\mathbb{T} = \mathbb{R}/T\mathbb{Z}$, where T is the fixed period, and then, replacing the above condition, $f \in H^1(\mathbb{R}, L^2(\Omega))$ T-time-periodic, by using the Bochner spaces integrated over time, such as $L^2(\mathbb{T}; H^1(\Omega))$ (and as it will appear later $H^1(\mathbb{T}; L^2(\partial\Omega))$). We look for time-periodic solutions $u_\varepsilon(x,t)$ of the Poisson problem with a dynamic Signorini-type boundary condition on the rapidly oscillating portion of the boundary $l_\varepsilon^\infty = l_\varepsilon \times \mathbb{R}$

$$(PP)\begin{cases} -\Delta_x u_\varepsilon(x,t) = f(x,t), & (x,t) \in Q^\infty, \\ u_\varepsilon \geq 0, \beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) \geq 0, & \\ u_\varepsilon(\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon)) = 0, & (x,t) \in l_\varepsilon^\infty = l_\varepsilon \times \mathbb{R}, \\ \partial_\nu u_\varepsilon = 0, & (x,t) \in \gamma_\varepsilon^\infty = \gamma_\varepsilon \times \mathbb{R}, \\ u_\varepsilon(x,t) = u_\varepsilon(x, t+T) & \text{for any } t \in \mathbb{R},\ x \in l_\varepsilon, \\ u_\varepsilon(x,t) = 0, & (x,t) \in \Gamma_1^\infty = \Gamma_1 \times \mathbb{R}, \end{cases} \tag{1}$$

where $l_\varepsilon \cup \gamma_\varepsilon = \Gamma_2$, $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\beta(\varepsilon) = \exp(\alpha^2/\varepsilon)$, $\alpha > 0$, ν is the unit outward normal vector to l_ε. The function $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function, such that $\sigma(0) = 0$ and $0 < k_1 \leq \sigma'(u) \leq k_2$ for an arbitrary $u \in \mathbb{R}$, and some constants $k_1, k_2 > 0$. We remark that on the *microscopic boundary* $\Gamma_2 = \partial\Omega \setminus \Gamma_1$, we are assuming a structure starting with a periodic reference cell $Y = (0,1)^{d-1}$ with a subset $S \subset Y$ representing the solid part. The oscillatory boundary Γ_2 is generated by scaling and repetition of S with characteristic size ε and critical thickness so that surface effects persist in the limit. We recall that, as explained in the monograph [9], this choice of the parameter $\beta(\varepsilon)$ corresponds to the more interesting case: the so called *Critical Case*. It can be shown that there are two other different cases. The *Subcritical Case* corresponds to when the dynamic zone is small in measure, and in the limit the entire boundary behaves as a pure Neumann boundary. The *Supercritical Case* corresponds to when the dynamic zone dominates, the homogenized boundary fully absorbs the dynamic condition. Finally, the *Critical Case* arises when both zones are perfectly balanced, and passing to the limit $\varepsilon \to 0$ in space generates a *modified effective boundary condition*. It implies the emergence of *"strange* *terms" (*in our case *strange operators)* arising from dynamic and non-linear boundary conditions on critical microstructures. Due to the time derivative at the original boundary, $\partial_t u$, the final homogenized boundary acquires a "reactive capacitance" or a damped memory term. Since the source is time-periodic and the operator is linear, the limit solution $u_0(x,t)$ preserves exactly the same period T, but its spatial profile is dictated by the homogenized coefficients that capture the microstructural geometry of the boundary.

In order to study the periodical problem (PP) we will start by considering the associate initial value problem in the cylinder $Q^T = \Omega \times (0,T)$,

$$(IVP)\begin{cases} -\Delta_x u_\varepsilon(x,t) = f(x,t), & (x,t) \in Q^T, \\ u_\varepsilon \geq 0, \beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) \geq 0, & \\ u_\varepsilon(\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon)) = 0, & (x,t) \in l_\varepsilon^T = l_\varepsilon \times (0,T), \\ \partial_\nu u_\varepsilon = 0, & (x,t) \in \gamma_\varepsilon^T = \gamma_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u^0(x), & x \in l_\varepsilon, \\ u_\varepsilon(x,t) = 0, & (x,t) \in \Gamma_1^T = \Gamma_1 \times (0,T), \end{cases} \tag{2}$$

where $u^0 \in H_0^1(-l,l)$, $u^0(x) \geq 0$. We will show the existence of solutions to the periodic problem (PP) by getting a fixed point function u^0 of the Poincaré map

$$F : u^0(\cdot) \to u_\varepsilon(\cdot, T) \text{ from } L^2(l_\varepsilon) \text{ into } L^2(l_\varepsilon).$$

We will show the existence and uniqueness of $u_\varepsilon^T(x,t)$, T-time-periodic solution of (PP) by two different methods. First, we check the assumptions of some abstract results (see [15]) to the case of our formulation. Moreover, if we assume

$$f \in H^1(\mathbb{R}, L^2(\Omega)),\ T - \text{time-periodic},$$

we can get some useful estimates on the T-time-periodic solution $u_\varepsilon^T(x,t)$ of (PP). Indeed, we can define

$$f_1(x) := \operatorname*{ess\,min}_{t\in[0,T]} f(x,t) \leq f_2(x) := \operatorname*{ess\,max}_{t\in[0,T]} f(x,t),$$

(see Lemma 5) and since $f_i \in L^2(\Omega)$, for $i = 1, 2$, we can define the (unique) solutions of the stationary problems

$$(SP)\begin{cases} -\Delta_x u_\varepsilon^i(x) = f_i(x), & x, \in \Omega, \\ u_\varepsilon^i \geq 0, \partial_\nu u_\varepsilon^i + \beta(\varepsilon)\sigma(u_\varepsilon^i) \geq 0, & \\ u_\varepsilon^i(\partial_\nu u_\varepsilon^i + \beta(\varepsilon)\sigma(u_\varepsilon^i)) = 0, & x \in l_\varepsilon, \\ \partial_\nu u_\varepsilon^i = 0, & x \in \gamma_\varepsilon, \\ u_\varepsilon^i(x) = 0, & x \in \Gamma_1. \end{cases} \tag{3}$$

Then, we will prove that

$$u_\varepsilon^1(x) \leq u_\varepsilon^T(x,t) \leq u_\varepsilon^2(x) \text{ for any } t \in \mathbb{R},\ \text{ on } l_\varepsilon.$$

The main result of this paper is to prove that $u_\varepsilon^T(x,t) \rightharpoonup u_0^T(x,t)$ as $\varepsilon \to 0$ with $u_0^T(x,t)$ a $T-$periodic solution of the homogenized problem of (PP), given by the stationary problem (depending $T-$periodically on t, as a parameter). When passing to the limit, as $\varepsilon \to 0$, only in the spatial heterogeneity and not in the temporal periodicity T, the temporal frequency remains fixed at a macroscopic scale. In this scenario, time essentially acts as a continuous parameter during the spatial limit process. We will prove that $u_\varepsilon^T(x,t) \rightharpoonup u_0^T(x,t)$ with $u_0^T(x,t)$ $T-$periodic solution of the homogenized problem of (PP), given by the stationary problem (dependinhg on t as a parameter)

$$(PP)_{Hom}\begin{cases} -\Delta_x u_0(x,t) = f(x,t), & (x,t)\in Q^\infty, \\ -\partial_{x_2}u_0 + \mathcal{M}u_0 = \mathcal{M}H_{u_0}, & (x,t)\in\Gamma_2^\infty, \\ H_{u_0}\geq 0,\ \partial_t H_{u_0} + \mathcal{L}H_{u_0} + \sigma(H_{u_0}) \geq \mathcal{L}u_0, & (x,t)\in\Gamma_2^\infty, \\ H_{u_0}(\partial_t H_{u_0} + \mathcal{L}H_{u_0} + \sigma(H_{u_0}) - \mathcal{L}u_0) = 0, & (x,t)\in\Gamma_2^\infty, \\ H_{u_0}(x,t) = H_{u_0}(x,t+T), & \text{for any } t\in\mathbb{R},\ x\in\Gamma_2, \\ u_0(x,t) = 0, & (x,t)\in\Gamma_1^\infty, \end{cases} \tag{4}$$

where $\mathcal{M} = \frac{\pi}{\alpha^2}$, $\mathcal{L}= \frac{\pi}{2C_0 l_0\alpha^2}$. While keeping time as a continuous parameter, the time derivative at the boundary, i.e. $\partial_t u$, propagates into the homogenized domain or boundary. Crucially, it does not appear as a simple derivative, but rather as a *memory operator or a non-local term in time* H_{u_0}. This memory operator simplifies into a *compact periodic operator* (or an impedance matrix) that couples the harmonics of the source with the geometric response of the boundary.

To carry out such a program, previously, we will prove that $u_\varepsilon(x,t) \rightharpoonup u_0(x,t)$ as $\varepsilon\to 0$ with $u_0(x,t)$ a weak solution of the homogenized problem of the (IVP), given by the stationary problem (dependinhg on t as a parameter)

$$(IVP)_{Hom}\begin{cases} -\Delta_x u_0(x,t) = f(x,t), & (x,t)\in Q^T, \\ -\partial_{x_2}u_0 + \mathcal{M}u_0 = \mathcal{M}H_{u_0}, & (x,t)\in\Gamma_2^T, \\ H_{u_0}\geq 0,\ \partial_t H_{u_0} + \mathcal{L}H_{u_0} + \sigma(H_{u_0}) \geq \mathcal{L}u_0, & (x,t)\in\Gamma_2^T, \\ H_{u_0}(\partial_t H_{u_0} + \mathcal{L}H_{u_0} + \sigma(H_{u_0}) - \mathcal{L}u_0) = 0, & (x,t)\in\Gamma_2^T, \\ H_{u_0}(x,0) = u^0(x), & x\in\Gamma_2, \\ u_0(x,t) = 0, & (x,t)\in\Gamma_1^T, \end{cases} \tag{5}$$

with $\mathcal{M}$ and $\mathcal{L}$ as before. Thus the homogenized problem $(PP)_{Hom}$ reaches a stabilized cyclic regime. The periodic-in-time homogenization of dynamic Signorini-type conditions $(PP)_{Hom}$ produces an effective nonlinear boundary operator encoding microscopic *cyclic dynamics,* in contrast with the homogenized initial value problem $(IVP)_{Hom}$, initial memory effects disappear and are replaced by a stabilized periodic response.

The organization of the paper is as follows: Section 2 deals with a detailed study of the initial problem which is divided into subsections: the statement of the problem, the main theorem about the homogenization for the initial problem with its detailed proof. Section 3 deals with the T-periodic formulation and its also divided in subsections: the existence of time periodic solutions of the microscopic problem and, finally, the homogenization result.

2. On the initial problem

2.1. **Statement of the problem.** Let Ω be a bounded domain in $\mathbb{R}^2\cap\{x_2>0\}$ with a smooth boundary, consisting of two parts $\partial\Omega = \Gamma_1\cup\Gamma_2$, where $\Gamma_1 = \partial\Omega\cap\{x_2>0\}$ and

$\Gamma_2 = \partial\Omega \cap \{x_2 = 0\} = [-l, l]$ for some $l > 0$. We define

$$Y_1 = \{(y_1, 0) : -1/2 < y_1 < 1/2\},$$
$$\hat{l}_0 = \{(y_1, 0) : -l_0 < y_1 < l_0\} \subset Y_1,\ l_0 \in (0, 1/2).$$

For a small parameter $\varepsilon > 0$ and a parameter $0 < a_\varepsilon \ll \varepsilon$ whose value is "critical", that is $a_\varepsilon = C_0\varepsilon\exp(-\alpha^2/\varepsilon)$, C_0 and α are positive constants, we introduce the sets

$$\widetilde{G_\varepsilon} = \bigcup_{j\in\mathbb{Z}'} (a_\varepsilon \hat{l}_0 + \varepsilon j) = \bigcup_{j\in\mathbb{Z}'} l_\varepsilon^j,$$

where $\mathbb{Z}' = \mathbb{Z} \times \{0\}$ is the set of vectors of the form $i = (j_1, 0)$, $j_1 \in \mathbb{Z}$. We set

$$\Upsilon_\varepsilon = \{j \in \mathbb{Z}' : \overline{l_\varepsilon^j} \subset [-l + 2\varepsilon, l - 2\varepsilon] \times \{0\}\}.$$

Next, we define the sets

$$Y_\varepsilon^j = \varepsilon Y_1 + \varepsilon j,\ j \in \mathbb{Z}',\ l_\varepsilon = \bigcup_{j\in\Upsilon_\varepsilon} l_\varepsilon^j.$$

It is easy to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$.

We introduce the set $\gamma_\varepsilon = \Gamma_2 \setminus \overline{l_\varepsilon}$. Note that $|l_\varepsilon^j| = 2a_\varepsilon l_0$ for an arbitrary $j \in \mathbb{Z}'$ and $|l_\varepsilon| \cong d a_\varepsilon \varepsilon^{-1}$, $d =$*const* > 0. In the cylinder $Q^T = \Omega \times (0, T)$, we consider the problem

$$\begin{cases} -\Delta_x u_\varepsilon(x,t) = f(x,t), & (x,t) \in Q^T, \\ u_\varepsilon \geq 0, \beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon) \geq 0, & \\ u_\varepsilon(\beta(\varepsilon)\partial_t u_\varepsilon + \partial_\nu u_\varepsilon + \beta(\varepsilon)\sigma(u_\varepsilon)) = 0, & (x,t) \in l_\varepsilon^T = l_\varepsilon \times (0,T), \\ \partial_\nu u_\varepsilon = 0, & (x,t) \in \gamma_\varepsilon^T = \gamma_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u^0(x), & x \in l_\varepsilon, \\ u_\varepsilon(x,t) = 0, & (x,t) \in \Gamma_1^T = \Gamma_1 \times (0,T), \end{cases} \tag{6}$$

where $\beta(\varepsilon) = \exp(\alpha^2/\varepsilon)$, $\alpha > 0$, $f \in H^1(0,T; L^2(\Omega))$, $u^0 \in H_0^1(-l,l)$, $u^0(x) \geq 0$, ν is the unit outward normal vector to l_ε^T. The function $\sigma : \mathbb{R}^1 \to \mathbb{R}^1$ *is a differentiable* continuous function, $\sigma(0) = 0$, and there exist constants $k_1, k_2 > 0$ such that $0 < k_1 \leq \sigma'(u) \leq k_2$ for an arbitrary $u \in \mathbb{R}^1$.

We define the convex closure sets

$$\mathcal{K}_\varepsilon = \{\phi \in H^1(\Omega, \Gamma_1) \,|\, \phi \geq 0 \text{ on } l_\varepsilon\},$$
$$\mathbb{K}_\varepsilon = \{v \in L^2(0,T; H^1(\Omega,\Gamma_1)) : v(\cdot,t) \in \mathcal{K}_\varepsilon,\ a.e.\ t \in (0,T)\}.$$

Here, $H^1(\Omega, \Gamma_1)$ *is the closure in* H^1*-norm of the set of infinitely differentiable functions vanishing near the boundary* Γ_1.

Given $f \in H^1(0,T;L^2(\Omega))$, *we say that* $u_\varepsilon \in \mathbb{K}_\varepsilon$ *is a strong solution to* (6), *if* $\partial_t u_\varepsilon \in L^2(0,T;L^2(l_\varepsilon))$, $u_\varepsilon(x,0) = u^0(x)$ *for* $x \in l_\varepsilon$, *and it satisfies the variational inequality*

$$\beta(\varepsilon)\int\limits_{l_\varepsilon^T} \partial_t u_\varepsilon(\phi - u_\varepsilon)dx_1dt + \int\limits_{Q^T} \nabla u_\varepsilon \nabla(\phi - u_\varepsilon)dxdt + \beta(\varepsilon)\int\limits_{l_\varepsilon^T} \sigma(u_\varepsilon)(\phi - u_\varepsilon)dx_1dt \geq \\ \geq \int\limits_{Q^T} f(\phi - u_\varepsilon)dxdt, \tag{7}$$

for an arbitrary function $\phi \in \mathbb{K}_\varepsilon$.

Theorem 1. *For any* $\varepsilon > 0$ *the problem* (6) *has a unique strong solution* u_ε. *Moreover,* u_ε *satisfies the following estimates*

$$\beta(\varepsilon)\max_{[0,T]}\|u_\varepsilon\|^2_{L^2(l_\varepsilon)} + \|u_\varepsilon\|^2_{L^2(0,T;H^1(\Omega,\Gamma_1))} \leq K(\|f\|^2_{L^2(Q^T)} + \|u^0\|^2_{L^2(-l,l)}), \\ \beta(\varepsilon)\|\partial_t u_\varepsilon\|^2_{L^2(l_\varepsilon^T)} + \max_{[0,T]}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} \\ \leq K\Big(\max_{[0,T]}\|f(\cdot,t)\|^2_{L^2(\Omega)} + \|f\|^2_{H^1(0,T;L^2(\Omega))} + \|u^0\|^2_{H_0^1(-l,l)}\Big). \tag{8}$$

Proof. For $\delta > 0$, we consider the problem

$$\begin{cases} -\Delta u_\varepsilon^\delta = f(x,t), & (x,t) \in Q^T, \\ \partial_\nu u_\varepsilon^\delta + \beta(\varepsilon)\partial_t u_\varepsilon^\delta + \beta(\varepsilon)\sigma(u_\varepsilon^\delta) = -\beta(\varepsilon)\delta^{-1}(u_\varepsilon^\delta)^-, & (x,t) \in l_\varepsilon^T, \\ u_\varepsilon^\delta(x,0) = u^0(x), & x \in l_\varepsilon, \\ u_\varepsilon^\delta(x,t) = 0, & (x,t) \in \Gamma_1^T, \end{cases} \tag{9}$$

where u^- is a negative part of the function u, i.e. $u^- = \inf(0,u)$.

By the strong solution of the problem (9), we call a function $u_\varepsilon^\delta \in L^2(0,T;H^1(\Omega,\Gamma_1))$ such that $\partial_t u_\varepsilon^\delta \in L^2(0,T;L^2(l_\varepsilon))$, $u_\varepsilon^\delta(x,0) = u^0(x)$ for $x \in l_\varepsilon$, and satisfying the integral identity

$$\beta(\varepsilon)\int\limits_{l_\varepsilon^T} \partial_t u_\varepsilon^\delta \phi dx_1dt + \int\limits_{Q^T} \nabla u_\varepsilon^\delta \nabla\phi dxdt + \beta(\varepsilon)\int\limits_{l_\varepsilon^T} \sigma(u_\varepsilon^\delta)\phi dx_1dt + \\ +\beta(\varepsilon)\delta^{-1}\int\limits_{l_\varepsilon^T} (u_\varepsilon^\delta)^-\phi dx_1dt = \int\limits_{Q^T} f\phi dxdt, \tag{10}$$

where ϕ is an arbitrary function $\phi \in L^2(0,T;H^1(\Omega,\Gamma_1))$.

Taking into account that $\sigma(w) + \delta^{-1}w^-$ is a monotone function of $w \in \mathbb{R}^1$, we conclude that problem (9) has a unique strong solution. In addition, we have the estimate

$$\beta(\varepsilon)\max_{[0,T]}\|u_\varepsilon^\delta\|^2_{L^2(l_\varepsilon)} + \|\nabla_x u_\varepsilon^\delta\|^2_{L^2(Q^T)} + \beta(\varepsilon)\delta^{-1}\|(u_\varepsilon^\delta)^-\|^2_{L^2(l_\varepsilon^T)} \leq \\ \leq K(\|f\|^2_{L^2(Q^T)} + \|u^0\|^2_{L^2(-l,l)}), \tag{11}$$

where $K = const > 0$ does not depend on ε and δ.

Using Galerkin's approximations, we get

$$\beta(\varepsilon)\|\partial_t u_\varepsilon^\delta\|^2_{L^2(l_\varepsilon^T)} + \max_{[0,T]}\|\nabla_x u_\varepsilon^\delta\|^2_{L^2(\Omega)} + \beta(\varepsilon)\delta^{-1}\max_{[0,T]}\|(u_\varepsilon^\delta)^-\|^2_{L^2(l_\varepsilon)} \leq$$

$$\leq K(\|f\|^2_{H^1(0,T;L^2(\Omega))} + \|u^0\|^2_{H^1_0(-l,l)}). \tag{12}$$

From estimates (11), (12), *we derive that there exists a subsequence (we preserve the notation of the original sequence) such that, as* $\delta \to 0$, *we have*

$$\begin{aligned} &u^\delta_\varepsilon \rightharpoonup u_\varepsilon \text{ weakly in } L^2(0,T;H^1(\Omega,\Gamma_1)),\\ &\partial_t u^\delta_\varepsilon \rightharpoonup \partial_t u_\varepsilon \text{ weakly in } L^2(0,T;L^2(l_\varepsilon)),\\ &u^\delta_\varepsilon \to u_\varepsilon \text{ strongly in } C([0,T];L^2(l_\varepsilon)),\\ &\left(u^\delta_\varepsilon\right)^- \to 0 \text{ strongly in } L^2(0,T;L^2(l_\varepsilon)). \end{aligned} \tag{13}$$

Let us prove that $u_\varepsilon \in \mathbb{K}_\varepsilon$ *is a strong solution of the problem* (6). *From the integral identity* (10), *we conclude*

$$\beta(\varepsilon)\int\limits_{l^T_\varepsilon} \partial_t u^\delta_\varepsilon(\phi - u^\delta_\varepsilon)dx_1dt + \int\limits_{Q^T} \nabla u^\delta_\varepsilon \nabla(\phi - u^\delta_\varepsilon)dxdt + \beta(\varepsilon)\int\limits_{l^T_\varepsilon} \sigma(u^\delta_\varepsilon)(\phi - u^\delta_\varepsilon)dx_1dt +$$

$$+\beta(\varepsilon)\delta^{-1}\int\limits_{l^T_\varepsilon}\left(u^\delta_\varepsilon\right)^-(\phi - u^\delta_\varepsilon)dx_1dt = \int\limits_{Q^T} f(\phi - u^\delta_\varepsilon)dx_1dt, \tag{14}$$

where ϕ *is an arbitrary function from* $\mathbb{K}_\varepsilon$. *Taking into account that* $\phi \in \mathbb{K}_\varepsilon$, *we have*

$$\beta(\varepsilon)\delta^{-1}\Big(\int\limits_{l^T_\varepsilon}(u^\delta_\varepsilon)^-\phi dx_1dt - \int\limits_{l^T_\varepsilon}|(u^\delta_\varepsilon)^-|^2dx_1dt\Big) \leq 0, \tag{15}$$

and, hence, we get

$$\beta(\varepsilon)\int\limits_{l^T_\varepsilon} \partial_t u^\delta_\varepsilon(\phi - u^\delta_\varepsilon)dx_1dt + \int\limits_{Q^T} \nabla u^\delta_\varepsilon \nabla(\phi - u^\delta_\varepsilon)dxdt +$$

$$+\beta(\varepsilon)\int\limits_{l^T_\varepsilon} \sigma(u^\delta_\varepsilon)(\phi - u^\delta_\varepsilon)dx_1dt \geq \int\limits_{Q^T} f(\phi - u^\delta_\varepsilon)dxdt. \tag{16}$$

Using the estimate

$$\|u_\varepsilon\|_{L^2(0,T;H^1(\Omega,\Gamma_1))} \leq \lim_{\delta\to 0}\|u^\delta_\varepsilon\|_{L^2(0,T;H^1(\Omega,\Gamma_1))},$$

and applying

$$\beta(\varepsilon)\int\limits_{l^T_\varepsilon}(\sigma(u^\delta_\varepsilon) - \sigma(u_\varepsilon))(\phi - u^\delta_\varepsilon)dx_1dt =$$

$$= \beta(\varepsilon)\int\limits_{l^T_\varepsilon}(\sigma(u^\delta_\varepsilon) - \sigma(u_\varepsilon))(\phi - u_\varepsilon)dx_1dt - \beta(\varepsilon)\int\limits_{l^T_\varepsilon}(\sigma(u^\delta_\varepsilon) - \sigma(u_\varepsilon))(u^\delta_\varepsilon - u_\varepsilon)dx_1dt \leq$$

$$\leq \beta(\varepsilon)\int\limits_{l^T_\varepsilon}(\sigma(u^\delta_\varepsilon) - \sigma(u_\varepsilon))(\phi - u_\varepsilon)dx_1dt \to 0,\ \delta \to 0,$$

we conclude that u^δ_ε *converges to the strong solution* $u_\varepsilon \in \mathbb{K}_\varepsilon$ *of the problem* (6). *From the estimates* (11), (12), *we obtain the estimates* (8). *The uniqueness of the strong solution of the problem* (6) *immediately follows from the inequality* (7). ∎

From the estimates (8), *we have as* $\varepsilon \to 0$

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } L^2(0,T;H^1(\Omega,\Gamma_1)). \tag{17}$$

2.2. **Main theorem for the initial problem.**

Theorem 2. *Let* u_ε *be a strong solution of the problem* (6)*. Then the function* u_0*, defined by* (17) *is a weak solution of the system*

$$\begin{cases} -\Delta_x u_0(x,t) = f(x,t), & (x,t) \in Q^T, \\ -\partial_{x_2} u_0 + \mathcal{M} u_0 = \mathcal{M} H_{u_0}, & (x,t) \in \Gamma_2^T, \\ H_{u_0} \ge 0,\ \partial_t H_{u_0} + \mathcal{L} H_{u_0} + \sigma(H_{u_0}) \ge \mathcal{L} u_0, & (x,t) \in \Gamma_2^T, \\ H_{u_0}(\partial_t H_{u_0} + \mathcal{L} H_{u_0} + \sigma(H_{u_0}) - \mathcal{L} u_0) = 0, & (x,t) \in \Gamma_2^T, \\ H_{u_0}(x,0) = u^0(x), & x \in \Gamma_2, \\ u_0(x,t) = 0, & (x,t) \in \Gamma_1^T, \end{cases} \tag{18}$$

where $\mathcal{M} = \frac{\pi}{\alpha^2}$, $\mathcal{L} = \frac{\pi}{2C_0 l_0 \alpha^2}$.

Remark 1. *In addition, if* $u \in H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$*, then,* $u(x,0)$ *is given as the unique solution of the stationary problem*

$$\begin{cases} -\Delta u(x,0) = f(x,0), & x \in \Omega, \\ u(x,0) = 0, & x \in \Gamma_1, \\ \partial_{x_2} u(x,0) = \mathcal{M}(u(x,0) - u^0(x)), & x \in \Gamma_2. \end{cases}$$

Remark 2. *As we can see, a new "strange term" has appeared in the boundary condition of the homogenized problem. This new term is a non-local nonlinear operator that requires solving an obstacle problem for the ordinary differential operator to determine it*

$$\begin{cases} \frac{d}{dt} H_\phi + \mathcal{L} H_\phi + \sigma(H_\phi) \ge \mathcal{L}\phi, & \\ H_\phi \ge 0, & t \in (0,T), \\ H_\phi(\frac{d}{dt} H_\phi + \mathcal{L} H_\phi + \sigma(H_\phi) - \mathcal{L}\phi) = 0, & \\ H_\phi(0) = u^0. & \end{cases} \tag{19}$$

The function H_ϕ *is a weak solution of the problem* (19)*, if* $H_\phi \in H^1(0,T)$, $H_\phi(0) = u^0 \ge 0$, $H_\phi(t) \ge 0$ *for any* $t \in [0,T]$ *and for an arbitrary function* $\psi \in L^2(0,T)$, $\psi(t) \ge 0$, $t \in (0,T)$*, it satisfies the variational inequality*

$$\int_0^T \frac{d}{dt} H_\phi (\psi - H_\phi) dt + \mathcal{L} \int_0^T H_\phi (\psi - H_\phi) dt + \int_0^T \sigma(H_\phi)(\psi - H_\phi) dt \ge$$
$$\ge \mathcal{L} \int_0^T \phi(\psi - H_\phi) dt. \tag{20}$$

It is easy to show that problem (19) *has the unique weak solution. Indeed, if we denote by* $H_{1,\phi}$ *and* $H_{2,\phi}$ *two weak solutions of the problem* (19)*, then taking* $\psi = \frac{1}{2}(H_{1,\phi} + H_{2,\phi})$

as a test-function in (20), *we get*

$$\int_0^T \frac{d}{dt}(H_{2,\phi}-H_{1,\phi})(H_{1,\phi}-H_{2,\phi})dt+\mathcal{L}\int_0^T (H_{2,\phi}-H_{1,\phi})(H_{1,\phi}-H_{2,\phi})dt+$$

$$+\int_0^T (\sigma(H_{2,\phi})-\sigma(H_{1,\phi}))(H_{1,\phi}-H_{2,\phi})dt \geq 0. \tag{21}$$

Evidently, the left-hand side has a non-positive value. So, we conclude that $H_{1,\phi}=H_{2,\phi}$. *To prove the existence of a weak solution of the problem* (19), *we use the penalized method. Consider the problem*

$$\begin{cases} \frac{d}{dt}H_{\phi,\delta}+\mathcal{L}H_{\phi,\delta}+\sigma(H_{\phi,\delta})+\delta^{-1}H^-_{\phi,\delta}=\mathcal{L}\phi, \quad t\in(0,T), \\ H_{\phi,\delta}(0)=u^0, \end{cases} \tag{22}$$

where $\delta>0$, $g^+(t)=\sup\{0,g(t)\}$, $g^-=g-g^+$. *By the solution of the problem* (20), *we consider a function* $H_{\phi,\delta}\in H^1(0,T)$, $H_{\phi,\delta}(0)=u^0$, *that satisfies the following identity*

$$\int_o^T \frac{d}{dt}H_{\phi,\delta}\psi dt+\mathcal{L}\int_0^T H_{\phi,\delta}\psi dt+\int_0^T \sigma(H_{\phi,\delta})\psi dt+\delta^{-1}\int_0^T H^-_{\phi,\delta}\psi dt=\mathcal{L}\int_0^T \phi\psi dt, \tag{23}$$

for an arbitrary function $\psi\in L^2(0,T)$.

Taking $\psi=H_{\phi,\delta}$ *as a test function in* (23), *we obtain*

$$\max_{[0,T]} H^2_{\phi,\delta}+\|H_{\phi,\delta}\|^2_{L^2(0,T)}+\delta^{-1}\|H_{\phi,\delta}{}^-\|^2_{L^2(0,T)}\leq C(\|\phi\|^2_{L^2(0,T)}+(u^0)^2), \tag{24}$$

From the equation of the problem (22) *and the estimate* (24), *we conclude*

$$\|\partial_t H_{\phi,\delta}\|^2_{L^2(0,T)}\leq K(\|\phi\|^2_{L^2(0,T)}+(u^0)^2), \tag{25}$$

where K *and* C *do not depend on* ε *and* δ. *Indeed, we have*

$$\int_0^T (\frac{d}{dt}H_{\phi,\delta})^2dt+\mathcal{L}^2\int_0^T H^2_{\phi,\delta}dt+\int_0^T \sigma^2(H_{\phi,\delta})dt+\delta^{-2}\int_0^T (H^-_{\phi,\delta})^2dt+$$

$$+2\mathcal{L}\int_0^T H'_{\phi,\delta}H_{\phi,\delta}dt+2\int_0^T H'_{\phi,\delta}\sigma(H_{\phi,\delta})dt+2\delta^{-1}\int_0^T H'_{\phi,\delta}H^-_{\phi,\delta}dt+2\mathcal{L}\delta^{-1}\int_0^T H_{\phi,\delta}H^-_{\phi\delta}dt+$$

$$+2\mathcal{L}\int_0^T H_{\phi,\delta}\sigma(H_{\phi,\delta})dt+2\delta^{-1}\int_0^T \sigma(H_{\phi,\delta})H^-_{\phi,\delta}dt=\mathcal{L}^2\int_0^T \phi^2dt.$$

From here, taking into account that

$$2\delta^{-1}\int_0^T H'_{\phi,\delta}H^-_{\phi,\delta}dt=\delta^{-1}(H^-_{\phi,\delta}(T))^2\geq 0,$$

$$2\delta^{-1}\int_0^T \sigma(H_{\phi,\delta})H^-_{\phi,\delta}=2\delta^{-1}\int_0^T \sigma(H^-_{\phi,\delta})H^-_{\phi,\delta}dt\geq 0,$$

and using Cauchy inequality $ab \le \alpha a^2 + C_\alpha b^2$ *for any* $\alpha > 0$ *and* (24), *we get* (25).

Then, using the estimates (24) *and* (25), *we derive, as* $\delta \to 0$, *convergences*

$$H_{\phi,\delta} \rightharpoonup H_\phi \text{ weakly in } H^1(0,T),\ H_{\phi,\delta} \to H_\phi \text{ uniformly with respect to } t \in [0,T],$$

$$H^-_{\phi,\delta} \to 0 = H^-_\phi \text{ in } L^2(0,T),\ \frac{d}{dt}H_{\phi,\delta} \rightharpoonup \frac{d}{dt}H_\phi \text{ weakly in } L^2(0,T). \tag{26}$$

From (24)–(26), *we obtain the estimates*

$$\max_{[0,T]} |H_\phi| \le K(\|\phi\|_{L^2(0,T)} + u^0), \quad \|H_\phi\|_{H^1(0,T)} \le K(\|\phi\|_{L^2(0,T)} + u^0). \tag{27}$$

Using $\psi = v - H_{\phi,\delta}$, *where* $v \ge 0$ *for* $t \in [0,T]$, *we get*

$$\int_0^T \frac{d}{dt}H_{\phi,\delta}(v - H_{\phi,\delta})dt + \mathcal{L}\int_0^T H_{\phi,\delta}(v - H_{\phi,\delta})dt + \int_0^T \sigma(H_{\phi,\delta})(v - H_{\phi,\delta})dt +$$

$$+\delta^{-1}\int_0^T H^-_{\phi,\delta}(v - H_{\phi,\delta})dt = \mathcal{L}\int_0^T \phi(v - H_{\phi,\delta})dt. \tag{28}$$

Taking into account that

$$\delta^{-1}\int_0^T H^-_{\phi,\delta}(v - H_{\phi,\delta})dt \le 0,$$

and using convergences (26), *we derive the inequality* (20).

We denote by $\mathbf{H}$ *the operator from* $L^2(0,T) \times \mathbb{R}^+$ *to* $L^2(0,T)$ *that maps a function* ϕ *and nonnegative number* u^0 *to a solution* $H_{\phi,u^0}(t)$ *of the problem* (19). *Then, we have*

Theorem 3. *The operator* $\mathbf{H}$, *defined by the equality* $\mathbf{H}(\phi, u^0) = H_{\phi,u^0}(t)$ *satisfies the following inequalities: let* ϕ_1, $\phi_2 \in L^2(0,T)$, *and* $u_1^0, u_2^0 \ge 0$, *then*

$$\max_{[0,T]} |H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0}| \le K(|u_1^0 - u_2^0| + \|\phi_1 - \phi_2\|_{L^2(0,T)}), \tag{29}$$

$$\mathcal{L}\int_0^T (\phi_1 - \phi_2)(H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})dt + \frac{1}{2}(u_1^0 - u_2^0)^2 \ge$$

$$\ge \frac{1}{2}(H_{\phi_1,u_1^0}(T) - H_{\phi_2,u_2^0}(T))^2 + \mathcal{L}\int_0^T (H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})^2 dt. \tag{30}$$

Proof. We take $v = \frac{1}{2}(H_{\phi_1,u_1^0} + H_{\phi_2,u_2^0})$ *as a test function in variational inequalities for functions* $H_{\phi_1,u_1^0}(t)$, $H_{\phi_2,u_2^0}(t)$. *Then, we sum the obtained expressions and get*

$$\int_0^T \frac{d}{dt}(H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})(H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})dt + \mathcal{L}\int_0^T (H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})^2 dt +$$

$$+\int_0^T (\sigma(H_{\phi_1,u_1^0}) - \sigma(H_{\phi_2,u_2^0}))(H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})dt \le \mathcal{L}\int_0^T (\phi_1 - \phi_2)(H_{\phi_1,u_1^0} - H_{\phi_2,u_2^0})dt.$$

From these inequalities, we deduce (29), (30). ∎

2.3. ***Proof of the Main Theorem for the initial problem.*** *We define a set of obstacle problems*

$$\begin{cases} \frac{d}{dt}H^j_{\phi,\varepsilon} + \mathcal{L}H^j_{\phi,\varepsilon} + \sigma(H^j_{\phi,\varepsilon}) \geq \mathcal{L}\phi(P^j_\varepsilon, t), \ H^j_{\phi,\varepsilon}(t) \geq 0, & t \in (0,T), \\ (\frac{d}{dt}H^j_{\phi,\varepsilon} + \mathcal{L}H^j_{\phi,\varepsilon} + \sigma(H^j_{\phi,\varepsilon}) - \mathcal{L}\phi(P^j_\varepsilon, t))H^j_{\phi,\varepsilon} = 0, & t \in (0,T), \\ H^j_{\phi,\varepsilon}(0) = u^0(P^j_\varepsilon), \end{cases} \tag{31}$$

where $j \in \Upsilon_\varepsilon$, $P^j_\varepsilon = \varepsilon j$, $\phi(x,t) = \psi(x)\eta(t)$, $\psi \in C^\infty(\overline{\Omega}, \Gamma_1)$, $\eta \in C^1[0,T]$.

We denote by T^j_r *the ball of radius* r *with the center in* $P^j_\varepsilon = \varepsilon j$ *and* $(T^j_r)^+ = T^j_r \cap \{x_2 > 0\}$. *Consider auxiliary functions* w^j_ε *and* q^j_ε *which are solutions of the problems*

$$\Delta w^j_\varepsilon = 0,\ x \in T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}},\ \ w^j_\varepsilon = 1,\ x \in \partial T^j_{a_\varepsilon},\ w^j_\varepsilon = 0,\ x \in \partial T^j_{\varepsilon/4}, \tag{32}$$

and

$$\Delta q^j_\varepsilon = 0,\ \ x \in T^j_{\varepsilon/4} \setminus \overline{l^j_\varepsilon},\ \ q^j_\varepsilon = 1,\ x \in l^j_\varepsilon,\ \ q^j_\varepsilon = 0,\ \ x \in \partial T^j_{\varepsilon/4}. \tag{33}$$

Note that w^j_ε *and* q^j_ε *are solutions of the problems*

$$\begin{cases} \Delta w^j_\varepsilon = 0, & x \in (T^j_{\varepsilon/4})^+ \setminus \overline{T^j_{a_\varepsilon}}, \\ w^j_\varepsilon = 0, & x \in (\partial T^j_{\varepsilon/4})^+, \\ w^j_\varepsilon = 1, & x \in (\partial T^j_{a_\varepsilon})^+, \\ \partial_{x_2} w^j_\varepsilon = 0, & x \in \{x_2 = 0\} \cap (T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}}), \end{cases} \tag{34}$$

and

$$\begin{cases} \Delta q^j_\varepsilon = 0, & x \in (T^j_{\varepsilon/4})^+, \\ q^j_\varepsilon = 0, & x \in (\partial T^j_{\varepsilon/4})^+, \\ q^j_\varepsilon = 1, & x \in l^j_\varepsilon, \\ \partial_{x_2} q^j_\varepsilon = 0, & x \in (T^j_{\varepsilon/4} \cap \{x_2 = 0\}) \setminus \overline{l^j_\varepsilon}, \end{cases} \tag{35}$$

where $j \in \Upsilon_\varepsilon$. *It is easy to see that*

$$w^j_\varepsilon = \frac{\ln\left(\frac{4r}{\varepsilon}\right)}{\ln\left(\frac{4a_\varepsilon}{\varepsilon}\right)}.$$

We define the functions $W_\varepsilon(x)$ *and* $Q_\varepsilon(x)$ *by setting*

$$W_\varepsilon(x) = \begin{cases} w^j_\varepsilon(x), & x \in \left(T^j_{\varepsilon/4} \setminus \overline{T^j_{a_\varepsilon}}\right)^+,\ j \in \Upsilon_\varepsilon, \\ 1, & x \in \overline{(T^j_{a_\varepsilon})^+},\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \setminus \overline{\cup_{j \in \Upsilon_\varepsilon}(T^j_{\varepsilon/4})^+}, \end{cases} \tag{36}$$

and

$$Q_\varepsilon(x) = \begin{cases} q^j_\varepsilon(x), & x \in \left(T^j_{\varepsilon/4}\right)^+,\ j \in \Upsilon_\varepsilon, \\ 0, & x \in \Omega \setminus \overline{\cup_{j \in \Upsilon_\varepsilon}(T^j_{\varepsilon/4})^+}. \end{cases} \tag{37}$$

It is easy to see that W_ε, $Q_\varepsilon \in H^1(\Omega, \Gamma_1)$ *and* $W_\varepsilon \rightharpoonup 0$ *weakly in* $H^1(\Omega, \Gamma_1)$ *as* $\varepsilon \to 0$. *To compare these two functions, we will use the following lemma, proved in* [8].

Lemma 1. *Let W_ε and Q_ε are the functions defined by (36) and (37) correspondingly. Then, we have the estimate*

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega,\Gamma_1)} \le K\sqrt{\varepsilon}. \tag{38}$$

Let's construct two more auxiliary functions

$$Q_{\varepsilon,\phi}(x,t) = \begin{cases} q_\varepsilon^j(x)(\phi(x,t) - H^j_{\phi,\varepsilon}(t)), & x \in (T^j_{\varepsilon/4})^+,\ j \in \Upsilon_\varepsilon,\ t \in (0,T), \\ 0, & x \in \Omega \setminus \overline{\cup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}}, \end{cases} \tag{39}$$

and

$$W_{\varepsilon,\phi}(x,t) = \begin{cases} w_\varepsilon^j(x)(\phi(x,t) - H^j_{\phi,\varepsilon}(t)), & x \in \left(T^j_{\varepsilon/4}\right)^+ \setminus \overline{T^j_{a_\varepsilon}},\ j \in \Upsilon_\varepsilon,\ t \in (0,T), \\ 1, & x \in \overline{(T^j_{a_\varepsilon})^+},\ j \in \Upsilon_\varepsilon,\ t \in (0,T), \\ 0, & x \in \Omega \setminus \overline{\bigcup_{j\in\Upsilon_\varepsilon} T^j_{\varepsilon/4}},\ t \in (0,T). \end{cases} \tag{40}$$

Using the properties of W_ε, we have that $W_{\varepsilon,\phi} \rightharpoonup 0$ weakly in $L^2(0,T;H^1(\Omega,\Gamma_1))$ as $\varepsilon \to 0$.

From Lemma 1, we deduce

Lemma 2. *The following estimate is valid*

$$\|W_{\varepsilon,\phi} - Q_{\varepsilon,\phi}\|_{L^2(0,T;H^1(\Omega,\Gamma_1))} \le K\sqrt{\varepsilon}. \tag{41}$$

From the variational inequality (7), using the monotonicity of σ, we have

$$\beta(\varepsilon)\int_{l_\varepsilon^T} \partial_t\psi(\psi - u_\varepsilon)dx_1dt + \int_{Q^T} \nabla\psi\nabla(\psi - u_\varepsilon)dxdt + \beta(\varepsilon)\int_{l_\varepsilon^T} \sigma(\psi)(\psi - u_\varepsilon)dx_1dt \ge$$
$$\ge \int_{Q^T} f(\psi - u_\varepsilon)dxdt - \frac{\beta(\varepsilon)}{2}\|\psi(x,0) - u^0(x)\|^2_{L^2(l_\varepsilon)}, \tag{42}$$

where $\psi \in \mathbb{K}_\varepsilon$.

Let us take in (42) as a test function $\psi = \phi(x,t) - Q_{\varepsilon,\phi}(x,t)$, where ϕ is an arbitrary function from $L^2(0,T;H^1(\Omega,\Gamma_1))$. Note, that $(\phi(x,t) - Q_{\varepsilon,\phi}(x,t))\Big|_{x\in l^j_\varepsilon, t\in(0,T)} = \phi(x,t) - \phi(x,t) + H^j_{\phi,\varepsilon} = H^j_{\phi,\varepsilon}(t) \ge 0$, hence, $\psi \in \mathbb{K}_\varepsilon$. Also, we have $\psi(x,0) - u^0(x) = \phi(x,0) - \phi(x,0) + H^j_{\phi,\varepsilon}(0) - u^0(x) = u^0(P^j_\varepsilon) - u^0(x)$, $x \in l^j_\varepsilon$. Thus, from the inequality (42), we derive

$$\beta(\varepsilon)\int_{l_\varepsilon^T} \partial_t(\phi - Q_{\varepsilon,\phi})(\phi - Q_{\varepsilon,\phi} - u_\varepsilon)dx_1dt + \int_{Q^T} \nabla(\phi - Q_{\varepsilon,\phi})\nabla(\phi - Q_{\varepsilon,\phi} - u_\varepsilon)dxdt +$$
$$+\beta(\varepsilon)\int_{l_\varepsilon^T} \sigma(\phi - Q_{\varepsilon,\phi})(\phi - Q_{\varepsilon,\phi} - u_\varepsilon)dx_1dt \ge \int_{Q^T} f(\phi - Q_{\varepsilon,\phi} - u_\varepsilon)dxdt + \alpha_\varepsilon, \tag{43}$$

where $\alpha_\varepsilon \to 0$ as $\varepsilon \to 0$. Again, using that $(\phi(x,t) - Q_{\varepsilon,\phi}(x,t))\Big|_{x\in l^j_\varepsilon, t\in(0,T)} = H^j_{\phi,\varepsilon}$, from (43), we conclude

$$\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int_0^T\int_{l^j_\varepsilon} \partial_t H^j_{\varepsilon,\phi}(H^j_{\varepsilon,\phi} - u_\varepsilon)dx_1dt + \int_{Q^T} \nabla\phi\nabla(\phi - Q_{\varepsilon,\phi} - u_\varepsilon)dxdt -$$

$$-\int\limits_{Q^T}\nabla Q_{\varepsilon,\phi}\nabla(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt+\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{l_\varepsilon^j}\sigma(H^j_{\varepsilon,\phi})(H^j_{\varepsilon,\phi}-u_\varepsilon)dx_1dt\geq$$

$$\geq\int\limits_{Q^T}f(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt+\alpha\varepsilon. \qquad (44)$$

Using Lemma 2, we have

$$\lim_{\varepsilon\to0}\int\limits_{Q^T}\nabla(Q_{\varepsilon,\phi}-W_{\varepsilon,\phi})\nabla(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt=0. \qquad (45)$$

Let us consider the following integral

$$J_\varepsilon\equiv\int\limits_{Q^T}\nabla W_{\varepsilon,\phi}\nabla(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt.$$

We have

$$J_\varepsilon=\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(T^j_{\varepsilon/4})^+\setminus\overline{(T^j_{a_\varepsilon})^+}}\nabla(w^j_\varepsilon(\phi(x,t)-H^j_\varepsilon(t)))\nabla(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt=$$

$$=\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(T^j_{\varepsilon/4})^+\setminus\overline{(T^j_{a_\varepsilon})^+}}\nabla w^j_\varepsilon\nabla((\phi-H^j_\varepsilon)(\phi-Q_{\varepsilon,\phi}-u_\varepsilon))dxdt+\alpha_\varepsilon=$$

$$=\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}\partial_\nu w^j_\varepsilon(\phi(x,t)-H^j_\varepsilon(t))(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dsdt+$$

$$+\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{a_\varepsilon})^+}\partial_\nu w^j_\varepsilon(\phi(x,t)-H^j_\varepsilon)(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dsdt+\alpha_\varepsilon. \qquad (46)$$

Taking into account that $\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{\varepsilon/4}}=\frac{4}{-\alpha^2+\varepsilon\ln(4C_0)}$ *and* $\partial_\nu w^j_\varepsilon\Big|_{\partial T^j_{a_\varepsilon}}=\frac{\exp(\alpha^2/\varepsilon)}{C_0\alpha^2-C_0\varepsilon\ln(4C_0)}$, *we transform relation* (46) *into*

$$J_\varepsilon=-\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}(\phi(x,t)-H^j_\varepsilon(t))(\phi-u_\varepsilon)dsdt+$$

$$+\frac{\beta(\varepsilon)}{C_0\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{a_\varepsilon})^+}(\phi(x,t)-H^j_{\varepsilon,\phi}(t))(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dsdt+\alpha_\varepsilon, \qquad (47)$$

where $\alpha_\varepsilon\to0$ *as* $\varepsilon\to0$.

In what follows, we will need the following lemma of [8].

Lemma 3. *Let* $h\in H^1(\Omega,\Gamma_1)$. *Then*

$$\Big|\frac{\beta(\varepsilon)\pi}{2l_0}\int\limits_{l_\varepsilon}hdx_1-\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_{(\partial T^j_{a_\varepsilon})^+}hds\Big|\leq K\sqrt{\varepsilon}\|h\|_{H^1(\Omega,\Gamma_1)}. \qquad (48)$$

By virtue of Lemma 3, we have

$$\frac{1}{C_0\alpha^2}\Big|\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{a_\varepsilon})^+}(\phi(x,t)-H^j_{\varepsilon,\phi}(t))(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dsdt-$$

$$-\frac{\pi\beta(\varepsilon)}{2l_0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{l^j_\varepsilon}(\phi(x,t)-H^j_{\varepsilon,\phi}(t))(H^j_{\varepsilon,\phi}-u_\varepsilon)dx_1dt\Big|\le K\sqrt{\varepsilon}. \tag{49}$$

From (47) *and* (49), *we obtain*

$$J_\varepsilon=-\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}(\phi(x,t)-H^j_{\varepsilon,\phi}(t))(\phi-u_\varepsilon)dsdt+$$

$$+\frac{\pi\beta(\varepsilon)}{2\alpha^2C_0l_0}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{l^j_\varepsilon}(\phi(x,t)-H^j_{\varepsilon,\phi})(H^j_{\varepsilon,\phi}-u_\varepsilon)dx_1dt+\alpha_\varepsilon. \tag{50}$$

Now, we consider all integrals over l^T_ε included into variational inequality

$$\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{l^j_\varepsilon}(\partial_tH^j_{\varepsilon,\phi}+\mathcal{L}H^j_{\varepsilon,\phi}+\sigma(H^j_{\varepsilon,\phi})-\mathcal{L}\phi(x,t))(H^j_{\varepsilon,\phi}-u_\varepsilon)dx_1dt\equiv\mathcal{P}_\varepsilon, \tag{51}$$

where $\mathcal{L}=\frac{\pi}{2\alpha^2C_0l_0}$.

As $|\phi(x,t)-\phi(P^j_\varepsilon,t)|\le Ka_\varepsilon$ for $x\in l^j_\varepsilon$, we have

$$\mathcal{P}_\varepsilon=\beta(\varepsilon)\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{l^j_\varepsilon}(\partial_tH^j_{\varepsilon,\phi}+\mathcal{L}H^j_{\varepsilon,\phi}+\sigma(H^j_{\varepsilon,\phi})-\mathcal{L}\phi(P^j_\varepsilon,t))(H^j_\varepsilon-u_\varepsilon)dx_1dt+\alpha_\varepsilon, \tag{52}$$

where $\alpha_\varepsilon\to 0$ as $\varepsilon\to 0$. Taking into account that $H^j_{\varepsilon,\phi}(t)$ is a solution of the problem (31), *and using that $u_\varepsilon\Big|_{l^T_\varepsilon}\ge 0$, we conclude that $\mathcal{P}_\varepsilon\le 0$.*

Using the last conclusion, from (44), *we obtain the inequality*

$$\int\limits_{Q^T}\nabla\phi\nabla(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt+\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}(\phi-H^j_{\varepsilon,\phi}(t))(\phi-u_\varepsilon)dsdt\ge$$

$$\ge\int\limits_{Q^T}f(\phi-Q_{\varepsilon,\phi}-u_\varepsilon)dxdt+\alpha_\varepsilon. \tag{53}$$

The assertion proved in [8],[24] *implies that*

$$\lim_{\varepsilon\to 0}\frac{4}{\alpha^2}\sum_{j\in\Upsilon_\varepsilon}\int\limits_0^T\int\limits_{(\partial T^j_{\varepsilon/4})^+}(\phi(P^j_\varepsilon,t)-H^j_{\varepsilon,\phi}(t))(\phi-u_\varepsilon)dsdt=$$

$$=\frac{\pi}{\alpha^2}\int\limits_0^T\int\limits_{\Gamma_2}(\phi(x,t)-H_\phi(x,t))(\phi-u_\varepsilon)dx_1dt, \tag{54}$$

where $H_\phi(x,t)$ *is defined by* (19).

Using the properties of the function $Q_{\varepsilon,\phi}$*, from* (53), (54)*, we conclude, as* $\varepsilon \to 0$*, that* u_0 *satisfies the following variational inequality*

$$\int\limits_{Q^T} \nabla\phi\nabla(\phi - u_0)dxdt + \frac{\pi}{\alpha^2}\int\limits_{\Gamma_2^T}(\phi - H_\phi)(\phi - u_0)dx_1dt \geq \int\limits_{Q^T} f(\phi - u_0)dxdt, \tag{55}$$

where ϕ *is an arbitrary function from* $L^2(0,T;H^1(\Omega,\Gamma_1))$*, and* $H_\phi(x,t) \in H^1(0,T;L^2(\Gamma_2))$ *is a solution of the obstacle problem for a.e.* $x \in \Gamma_2$

$$\begin{cases} \frac{d}{dt}H_\phi + \mathcal{L}H_\phi + \sigma(H_\phi) \geq \mathcal{L}\phi,\ H_\phi \geq 0, & t \in (0,T), \\ H_\phi(\frac{d}{dt}H_\phi + \mathcal{L}H_\phi + \sigma(H_\phi) - \mathcal{L}\phi) = 0, & t \in (0,T), \\ H_\phi(x,0) = u^0(x). \end{cases}$$

3. The time periodic problem

3.1. **Time periodic solutions of the microscopic problem.** In order to show the existence and uniqueness of time-periodic solutions of (PP) it is useful to reformulate the problem in terms of the abstract subdifferentials of convex functions on Hilbert space H

$$\begin{cases} \frac{du}{dt}(t) + \partial\phi^t(u(t)) \ni 0 \quad \text{on } H \\ u(0) = u_0. \end{cases} \tag{56}$$

Obviously, to apply such abstract formulation, since the term $\beta(\varepsilon)\partial_t u_\varepsilon$ appears in the dynamic boundary condition, we must introduce a change of time-variable

$$t = \widetilde{t}_\varepsilon\beta(\varepsilon) \text{ and } \widetilde{u}_\varepsilon(x,\widetilde{t}_\varepsilon) = u_\varepsilon(x,\widetilde{t}_\varepsilon\beta(\varepsilon)) = u_\varepsilon(x,t). \tag{57}$$

In this way,

$$\beta(\varepsilon)\frac{\partial u_\varepsilon}{\partial t}(x,t) = \frac{\partial\widetilde{u}_\varepsilon}{\partial\widetilde{t}_\varepsilon}(x,\widetilde{t}_\varepsilon),$$

and thus we can apply the abstract theory to the function $\widetilde{u}_\varepsilon(x,\widetilde{t}_\varepsilon)$. To simplify the notation we dispense with that distinction and identify u_ε with $\widetilde{u}_\varepsilon$.

Our key point of view is based on the main theorem of [10] (see also Remark 3.1 of [5]). When applied to our formulation of (IVP), we get the following: let $H = L^2(l_\varepsilon)$, and $\phi^t : H \to (-\infty,+\infty]$, with $\phi^t \not\equiv +\infty$, $D(\phi^t) = \{u \in H, \phi^t(u) < +\infty\}$ is the function given by

$$\phi^t(u) = \begin{cases} \frac{1}{2}\int\limits_\Omega \|\nabla U(x)\|^2dx - \int\limits_\Omega f(x,t)U(x)dx + \beta(\varepsilon)\int\limits_{l_\varepsilon} j_\sigma(u(\sigma))d\sigma + \int\limits_{l_\varepsilon} j_S(u(\sigma))d\sigma, \\ \quad \text{if } U \in H^1(\Omega), U|_{l_\varepsilon} = u, \partial_\nu U|_{\gamma_\varepsilon} = 0, U|_{\Gamma_1} = 0, j_\sigma(u), j_S(u) \in L^1(l_\varepsilon), \\ \quad +\infty \ \text{ in the rest,} \end{cases}$$

where the subdifferentials of the convex functions j_σ and j_S are given by $\partial j_\sigma(r) = \sigma(r)$ and $\partial j_S(r) = \theta(r)$, the Signorini maximal monotone graph in $\mathbb{R}^2$ given by

$$\theta(r) = \begin{cases} \{0\}, & r > 0, \\ [0,+\infty), & r = 0, \\ \emptyset, & r < 0, \end{cases}$$

i.e. the functions are

$$j_\sigma(r) = \int_0^r \sigma(s)ds \text{ and } j_S(r) = 0 \text{ if } r \geq 0, j_S(r) = +\infty \text{ if } r < 0.$$

ϕ^t is convex lower semi-continuous and $\phi^t \not\equiv +\infty$ on $H = L^2(l_\varepsilon)$. As in [10], it is easy to prove that $\overline{D(\phi^t)} = L^2_+(l_\varepsilon) = \{w \in L^2(l_\varepsilon) : w \geq 0 \text{ on } l_\varepsilon\}$. Moreover, after the change of variables (57) the initial value problem (IVP) can be formulated in terms of the abstract theory for subdifferentials of convex functions (56). In addition, we have:

Lemma 4. *Assuming $f \in H^1(0, T; L^2(\Omega))$, $\phi^t(u)$ satisfies the Kenmochi assumption (see Section 1.5 of* [15]*): for any $r > 0$ $\exists \alpha_r \in H^1(0,T)$ and $\beta_r \in W^{1,1}(0,T)$ such that for each $s, t \in [0,T]$ with $s \leq t$ and $z \in D(\phi^s)$ with $|z|_H \leq r$ there exists $\widehat{z} \in D(\phi^t)$ such that*

$$|z - \widehat{z}|_H \leq |\alpha_r(t) - \alpha_r(s)|\,(1 + |\phi^s(z)|_H^{1/2}) \tag{58}$$

and

$$\phi^t(\widehat{z}) - \phi^s(z) \leq |\beta_r(t) - \beta_r(s)|\,(1 + |\phi^s(z)|_H). \tag{59}$$

Proof. Let $r > 0$, $s, t \in [0,T]$ with $s \leq t$, and let $z \in D(\phi^s)$, $|z|_H \leq r$. Since the only time dependence of ϕ^t is contained in the linear term $\int_\Omega f(x,t)U(x)\,dx$, the effective domain is independent of t:

$$D(\phi^t) = D(\phi^s), \qquad \forall\, s, t \in [0,T].$$

Therefore, to check both conditions in the statement, we simply choose $\widehat{z} = z$. For the first condition (58), we have $|z - \widehat{z}|_H = 0$. Hence, it is enough to take $\alpha_r(t) \equiv 0$ and (58) holds. To verify (59), let $U \in H^1(\Omega)$ be the extension associated with z. Since $\widehat{z} = z$,

$$\phi^t(\widehat{z}) - \phi^s(z) = \phi^t(z) - \phi^s(z) = -\int_\Omega (f(x,t) - f(x,s))U(x)\,dx.$$

Therefore, we get the estimate

$$|\phi^t(z) - \phi^s(z)| \leq \|f(\cdot,t) - f(\cdot,s)\|_{L^2(\Omega)} \|U\|_{L^2(\Omega)}.$$

Since $U = 0$ on Γ_1, Poincaré's inequality yields

$$\|U\|_{L^2(\Omega)} \leq C_P \|\nabla U\|_{L^2(\Omega)}.$$

Moreover,

$$\phi^s(z) = \frac{1}{2}\|\nabla U\|^2_{L^2(\Omega)} - \int_\Omega f(x,s)U(x)\,dx$$
$$+ \beta(\varepsilon)\int_{l_\varepsilon} j_\sigma(z)\,d\sigma + \int_{l_\varepsilon} j_S(z)\,d\sigma.$$

Since the last two terms are nonnegative,

$$\phi^s(z) \geq \frac{1}{2}\|\nabla U\|^2_{L^2(\Omega)} - \|f(\cdot,s)\|_{L^2(\Omega)}\|U\|_{L^2(\Omega)}.$$

Using Poincaré's inequality and Young's inequality,

$$\phi^s(z) \geq \frac{1}{4}\|\nabla U\|^2_{L^2(\Omega)} - C,$$

where $C > 0$ depends only on

$$\sup_{\tau\in[0,T]} \|f(\cdot,\tau)\|_{L^2(\Omega)}.$$

Hence

$$\|\nabla U\|_{L^2(\Omega)}^2 \le C\big(1+\phi^s(z)\big),$$

and consequently

$$\|U\|_{L^2(\Omega)} \le C\big(1+\phi^s(z)\big)^{1/2}.$$

Substituting into the previous estimate,

$$|\phi^t(z)-\phi^s(z)| \le C\,\|f(\cdot,t)-f(\cdot,s)\|_{L^2(\Omega)}\big(1+\phi^s(z)\big)^{1/2}.$$

Since

$$(1+a)^{1/2} \le 1+a, \text{ for any } a \ge 0,$$

we obtain

$$|\phi^t(z)-\phi^s(z)| \le C\,\|f(\cdot,t)-f(\cdot,s)\|_{L^2(\Omega)}\big(1+\phi^s(z)\big).$$

Since $f \in H^1(0,T;L^2(\Omega))$, we have

$$\|f(\cdot,t)-f(\cdot,s)\|_{L^2(\Omega)} \le \int_s^t \|f'(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau.$$

Define

$$\beta_r(t) = C\int_0^t \|f'(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau.$$

Then $\beta_r \in W^{1,1}(0,T)$ and

$$\phi^t(\widehat{z})-\phi^s(z) \le |\beta_r(t)-\beta_r(s)|\big(1+\phi^s(z)\big),$$

and (59) holds.■

Then we have

Theorem 4. *Assume $f \in H^1(\mathbb{R}, L^2(\Omega))$ T-periodic. Then, there exists a unique strong T-periodic solution of (PP).*

Proof. The existence of a T-periodic solution of (PP) is a consequence of Theorem 2.3.1 of [15] (thanks to the conditions (58) and (59)). The uniqueness is consequence from the fact that $\phi^t(u)$ is strictly convex and Theorem 2.3.2 of [15].■

Remark. Let $\Omega \subset \mathbb{R}^n$ be an open set (bounded or not) and $T > 0$. We consider the Bochner-Sobolev space

$$H^1(0,T;L^2(\Omega)) := \big\{f : (0,T)\times\Omega \to \mathbb{R} \mid f \in L^2(0,T;L^2(\Omega)),\ \ \partial_t f \in L^2(0,T;L^2(\Omega))\big\}.$$

Here, $\partial_t f$ is taken in the distributional sense. The norm is

$$\|f\|_{H^1(0,T;L^2(\Omega))}^2 = \int_0^T\int_\Omega |f(x,t)|^2\,dx\,dt + \int_0^T\int_\Omega |\partial_t f(x,t)|^2\,dx\,dt.$$

For almost every $x \in \Omega$, the map $t \mapsto f(x,t)$ belongs to $H^1(0,T)$, and hence to $C([0,T])$ by the Sobolev embedding in one dimension. Therefore, we can define

$$M(x) := \sup_{t\in[0,T]} |f(x,t)|.$$

Lemma 5. *If $f \in H^1(0,T;L^2(\Omega))$ then $\sup_{t\in[0,T]} |f(\cdot,t)| \in L^2(\Omega)$.*

For the proof, it is useful to recall a well-known result (here it is given with an explicit estimate)

Lemma 6 (Morrey inequality in 1D)**.** *For any $g \in H^1(0,T)$, we have*

$$\|g\|_{L^\infty(0,T)} \le C_T \|g\|_{H^1(0,T)}, \tag{60}$$

with $C_T = \sqrt{\frac{2}{T} + 2T}$.

Proof of the estimate (60). For any $s,t \in [0,T]$,

$$g(t) = g(s) + \int_s^t g'(\tau)\,d\tau,$$

hence $|g(t)| \le |g(s)| + \sqrt{T}\|g'\|_{L^2(0,T)}$. Averaging over $s \in [0,T]$ gives

$$|g(t)| \le \frac{1}{T}\int_0^T |g(s)|\,ds + \sqrt{T}\|g'\|_{L^2(0,T)}.$$

By Cauchy-Schwarz,

$$\frac{1}{T}\int_0^T |g(s)|\,ds \le \frac{1}{\sqrt{T}}\|g\|_{L^2(0,T)}.$$

Thus for every t,

$$|g(t)| \le \frac{1}{\sqrt{T}}\|g\|_{L^2(0,T)} + \sqrt{T}\|g'\|_{L^2(0,T)}.$$

Taking the supremum over t and using $(a+b)^2 \le 2(a^2+b^2)$ yields

$$\|g\|^2_{L^\infty(0,T)} \le 2\left(\frac{1}{T}\|g\|^2_{L^2(0,T)} + T\|g'\|^2_{L^2(0,T)}\right) \le 2\max\left\{\frac{1}{T}, T\right\}\|g\|^2_{H^1(0,T)}.$$

So $C_T = \sqrt{2\max\{1/T, T\}}$ works, but the simpler constant $\sqrt{2/T + 2T}$ is also valid (since $\max\{1/T, T\} \le \frac{1}{T} + T$ when $T \ge 1$, but the inequality $(a+b)^2 \le 2a^2 + 2b^2$ is uniform). Actually from the pointwise bound:

$$\sup_t |g(t)| \le \frac{1}{\sqrt{T}}\|g\|_{L^2} + \sqrt{T}\|g'\|_{L^2}$$

and by Cauchy-Schwarz in $\mathbb{R}^2$,

$$\frac{1}{\sqrt{T}}\|g\|_{L^2} + \sqrt{T}\|g'\|_{L^2} \le \sqrt{\frac{1}{T} + T}\,\sqrt{\|g\|^2_{L^2} + \|g'\|^2_{L^2}} = \sqrt{\frac{1}{T} + T}\,\|g\|_{H^1}.$$

Hence $C_T = \sqrt{T + \frac{1}{T}}$. ∎

Proof of Lemma 5. Fix $x \in \Omega$ such that $f(x,\cdot) \in H^1(0,T)$ (this holds for almost every x). Applying the Morrey Lemma with $g(t) = f(x,t)$, we obtain, for every $t \in [0,T]$,

$$|f(x,t)| \leq \sqrt{T + \frac{1}{T}} \left(\int_0^T |f(x,s)|^2 ds \right)^{1/2} + \sqrt{T + \frac{1}{T}} \left(\int_0^T |\partial_s f(x,s)|^2 ds \right)^{1/2}.$$

Taking the supremum in t gives

$$M(x) \leq C_T \left(\int_0^T |f(x,s)|^2 ds \right)^{1/2} + C_T \left(\int_0^T |\partial_s f(x,s)|^2 ds \right)^{1/2},$$

with $C_T = \sqrt{T + \frac{1}{T}}$.
Now, let us find an estimate in $L^2(\Omega)$ of M. Square the inequality and use $(a+b)^2 \leq 2(a^2+b^2)$:

$$M(x)^2 \leq 2C_T^2 \int_0^T |f(x,s)|^2 ds \ + \ 2C_T^2 \int_0^T |\partial_s f(x,s)|^2 ds.$$

Now integrate over Ω:

$$\int_\Omega M(x)^2 dx \leq 2C_T^2 \int_\Omega \int_0^T |f(x,s)|^2 ds\, dx + 2C_T^2 \int_\Omega \int_0^T |\partial_s f(x,s)|^2 ds\, dx.$$

By Tonelli's theorem (all integrands are nonnegative), we may swap the order:

$$\int_\Omega M(x)^2 dx \leq 2C_T^2 \int_0^T \int_\Omega |f(x,s)|^2 dx\, ds + 2C_T^2 \int_0^T \int_\Omega |\partial_s f(x,s)|^2 dx\, ds.$$

But these two terms are exactly $2C_T^2$ times the two summands of the H^1 norm:

$$\int_\Omega M(x)^2 dx \leq 2C_T^2 \|f\|^2_{L^2(0,T;L^2(\Omega))} + 2C_T^2 \|\partial_t f\|^2_{L^2(0,T;L^2(\Omega))}.$$

Thus

$$\|M\|^2_{L^2(\Omega)} \leq 2C_T^2 \|f\|^2_{H^1(0,T;L^2(\Omega))}.$$

This completes the proof.$\blacksquare$

Theorem 5. *Assume*

$$f \in H^1(\mathbb{R}; L^2(\Omega)),\ T-\textit{time-periodic}.$$

For $i = 1,2$, define $f_i \in L^2(\Omega)$, by

$$f_1(x) := \operatorname*{ess\,min}_{t\in[0,T]} f(x,t) \leq f_2(x) := \operatorname*{ess\,max}_{t\in[0,T]} f(x,t).$$

Let the $u_\varepsilon^i(x)$ be (unique) solutions of the stationary problems

$$(SP)\begin{cases} -\Delta_x u_\varepsilon^i(x) = f_i(x), & x, \in \Omega, \\ u_\varepsilon^i \geq 0, \partial_\nu u_\varepsilon^i + \beta(\varepsilon)\sigma(u_\varepsilon^i) \geq 0, & \\ u_\varepsilon^i(\partial_\nu u_\varepsilon^i + \beta(\varepsilon)\sigma(u_\varepsilon^i)) = 0, & x \in l_\varepsilon, \\ \partial_\nu u_\varepsilon^i = 0, & x \in \gamma_\varepsilon, \\ u_\varepsilon^i(x) = 0, & x \in \Gamma_1. \end{cases} \tag{61}$$

Then, we have

$$u_\varepsilon^1(x) \leq u_\varepsilon^T(x,t) \leq u_\varepsilon^2(x) \text{ for any } t \in \mathbb{R}, \text{ on } l_\varepsilon. \tag{62}$$

Proof. We point out that, thanks to Lemma 4 we can apply Theorem 1.1.2 of [15] and then, for any $u^0 \in L_+^2(l_\varepsilon)$ there exists a unique weak solution $u \in C([0,T]; L^2(l_\varepsilon))$ of (IVP) (i.e. of (56) associated to the convex function $\phi^t(u)$ defined above). Moreover, for any $t \in (0,T]$, such a solution satisfies (the regularizing effect) that $u(\cdot,t) \in D(\phi^t)$. We will get the estimate (62) by finding a fixed point function u^0 of the Poincaré map $F: K \to K$, with $K \subset L^2(l_\varepsilon)$ a closed and convex set,

$$F(u^0(\cdot)) = u_\varepsilon(\cdot,T).$$

As in ([3]), we will apply the Schauder fixed point theorem on the space $L^2(l_\varepsilon)$. We need to select a closed and convex set $K \subset L^2(l_\varepsilon)$ such that i) $F(K) \subset K$; ii) $F|_K$ is continuous, iii) $F(K)$ is relatively compact in $L^2(l_\varepsilon)$. As the set K, we will take the *interval* $[u_\varepsilon^1(\cdot), u_\varepsilon^2(\cdot)] = \{w \in L^2(l_\varepsilon),\ u_\varepsilon^1(x) \leq w(x) \leq u_\varepsilon^2(x) \text{ a.e. } x \in l_\varepsilon\}$. Obviously K is a closed, convex and non empty set of $L^2(l_\varepsilon)$. It is useful to prove property i) as a separate auxiliary result.

Lemma 7. *Let $f(x,t)$ and $f_i(x)$, $i = 1,2$, as in Theorem 5. Let $u_\varepsilon(x,t)$ be the unique solution of (IVP) corresponding to an initial datum $u^0 \in K$. Then, $u_\varepsilon(\cdot,t) \in K$ for any $t \in [0,T]$.*

Proof. By the continuous dependence of solutions of (56) with respect to the initial datum (see, e.g. Theorem 1.1.1 of [15]) in order to get the comparisons included in the information $u_\varepsilon(\cdot,t) \in K$ it is enough to assume that $u^0 \in D(\phi^t)$. Let us prove that $u_\varepsilon(x,t) \leq u_\varepsilon^2(x)$ (the proof of the other comparison $u_\varepsilon(x,t) \geq u_\varepsilon^1(x)$ is similar). We subtract the differential equations for $u_\varepsilon(x,t)$ and for $u_\varepsilon^2(x)$, multiply by $[u_\varepsilon(x,t) - u_\varepsilon^2(x)]_+ = \max(u_\varepsilon(x,t) - u_\varepsilon^2(x), 0)$ and integrate by parts. After using the boundary conditions and the monotonicity of the nonlinear terms $\partial j_\sigma(r) = \sigma(r)$ and $\partial j_S(r) = \theta(r)$, we arrive to the inequality

$$\begin{aligned} &\int_\Omega \left|\nabla \left[u_\varepsilon(x,t) - u_\varepsilon^2(x)\right]_+\right|^2 dx + \frac{d}{dt}\int_{l_\varepsilon} \left|\left[u_\varepsilon(s,t) - u_\varepsilon^2(s)\right]_+\right|^2 ds \\ &\leq \int_{\{x\in\Omega | u_\varepsilon(x,t) > u_\varepsilon^2(x)\}} (f(x,t) - f^2(x))(u_\varepsilon(x,t) - u_\varepsilon^2(x))_+ dx. \end{aligned}$$

Then, since $f(x,t) \leq f^2(x)$ on Ω, we get that

$$\int_{l_\varepsilon} \left|\left[u_\varepsilon(s,t) - u_\varepsilon^2(s)\right]_+\right|^2 ds \leq \int_{l_\varepsilon} \left|\left[u^0(s) - u_\varepsilon^2(s)\right]_+\right|^2 ds,$$

and since $u^0 \in K$, we get that $u_\varepsilon(\cdot,t) \le u_\varepsilon^2(\cdot)$ on l_ε.■

Proof of Theorem 5 (continuation). The above Lemma shows that $F([u_\varepsilon^1, u_\varepsilon^2]) \subset [u_\varepsilon^1, u_\varepsilon^2]$. Let us check that $F|_K$ is continuous (property ii)). It is a trivial consequence of the $L^2(l_\varepsilon)$ estimate

$$\int_{l_\varepsilon} |u_\varepsilon(s,t) - u_\delta(s,t)|^2 \, ds \le \int_{l_\varepsilon} \left|u^0(s) - u_\delta^0(s)\right|^2 ds,$$

which holds, thanks to Theorem1.1.1 of [15] if $u_\delta(x,t)$ denotes the solution of

$$\begin{cases} \frac{du}{dt}(t) + \partial\phi^t(u(t)) \ni 0 \quad \text{on } H \\ u(0) = u_\delta^0, \end{cases} \tag{63}$$

and $u_\delta^0 \to u^0$ in $L^2(l_\varepsilon)$. Finally, to prove that $F(K)$ is relatively compact in $L^2(l_\varepsilon)$ (property (iii)) it is enough to observe that by the regularization effect (see Theorem 2.1.1 of [15]) we have $u(T) \in D(\phi^T) \subset H^{1/2}(l_\varepsilon)$, and since the inclusion $H^{1/2}(l_\varepsilon) \subset L^2(l_\varepsilon)$ is compact, we get that $F(K)$ is relatively compact in $L^2(l_\varepsilon)$. Then, by the Schauder fixed point theorem, there exists a fixed point $u_\varepsilon^T \in K$ of the Poincaré map F and the proof is complete. ■

3.2. **Time periodic solutions of the homogenized problem.** Applying the techniques of the homogenization result for the initial value problem (IVP), we get

Theorem 6. *Assume $f \in H^1(\mathbb{R}, L^2(\Omega))$, $T-$time-periodic, then, we have that $u_\varepsilon^T(x,t) \rightharpoonup u_0^T(x,t)$ with $u_0^T(x,t)$ $T-$periodic solution of the homogenized problem of (PP), given by the stationary problem (dependinhg on t as a parameter)*

$$(PP)_{Hom} \begin{cases} -\Delta_x u_0(x,t) = f(x,t), & (x,t) \in Q^\infty, \\ -\partial_{x_2} u_0 + \mathcal{M} u_0 = \mathcal{M} H_{u_0}, & (x,t) \in \Gamma_2^\infty, \\ H_{u_0} \ge 0,\, \partial_t H_{u_0} + \mathcal{L} H_{u_0} + \sigma(H_{u_0}) \ge \mathcal{L} u_0, & (x,t) \in \Gamma_2^\infty, \\ H_{u_0}(\partial_t H_{u_0} + \mathcal{L} H_{u_0} + \sigma(H_{u_0}) - \mathcal{L} u_0) = 0, & (x,t) \in \Gamma_2^\infty, \\ H_{u_0}(x,t) = H_{u_0}(x,t+T), & \textit{for any } t \in \mathbb{R},\, x \in \Gamma_2, \\ u_0(x,t) = 0, & (x,t) \in \Gamma_1^\infty, \end{cases} \tag{64}$$

where $\mathcal{M} = \frac{\pi}{\alpha^2}$, $\mathcal{L} = \frac{\pi}{2C_0 l_0 \alpha^2}$. Moreover, if $u_0^i(x)$ is the homogenized stationary problem associated to the data f^i (see [9]) we have

$$u_0^1(x) \le u_0^T(x,t) \le u_0^2(x) \ \ a.e.\ x \in \Omega,\ for\ t \in (0,T). \tag{65}$$

Proof. It is an obvious adaptation of the proof of Theorem 2 to this setting and we escape the details. Moreover, if ϕ is an arbitrary function from $L^2(\mathbb{R}; H^1(\Omega, \Gamma_1))$, $T-$periodic, $\phi \ge 0$ on Ω for a.e. t, then, $u_\varepsilon^1(x)\phi(x,t) \le u_\varepsilon^T(x,t)\phi(x,t) \le u_\varepsilon^2(x)\phi(x,t)$ for any $t \in \mathbb{R}$, and by the weak convergence we get that

$$u_0^1(x)\phi(x,t) \le u_0^T(x,t)\phi(x,t) \le u_0^2(x)\phi(x,t)$$

for any arbitrary test function, which implies (65). ■

Remark 3. *Finally, by using the techniques of* [11] *it seems possible to prove that if the forcing $f(x,t)$ is strongly negative, then, the transport becomes impossible: the trace vanishes on Γ_2^T, the membrane becomes inactive. Thus, one obtains two regimes: active transport regime, or blocked membrane regime according to the values of $f(x,t)$.*

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