Compactly Supported Solutions near a Nehari Critical Value

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Abstract

We study non-negative solutions of the indefinite sublinear Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u + m(x)|u|^{\alpha-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $0 < \alpha < 1$, $m \in L^\infty(\Omega)$ is a sign-changing or non-positive weight, and $\lambda \in \mathbb{R}$.

Using variational and fibering methods on the Nehari manifold, we construct two branches of non-negative solutions distinguished by the sign of the second derivative of the energy along rays. The branch u_λ^+, for which $E_\lambda''(u_\lambda^+) > 0$, has negative energy, whereas the branch u_λ^-, for which $E_\lambda''(u_\lambda^-) < 0$, has positive energy. Their existence and asymptotic behavior are governed by the critical value

$$\lambda^* = \inf\left\{ \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega |u|^2\,dx} : u \in H_0^1(\Omega) \setminus \{0\}, \quad \int_\Omega m(x)|u|^{\alpha+1}\,dx \geq 0 \right\}.$$

In the degenerate case $m \leq 0$, when the zero region $\Omega^0 = \operatorname{int}\{x \in \Omega : m(x) = 0\}$ has positive measure, we identify λ^* with the first Dirichlet eigenvalue λ_{1,Ω^0}. We show that the positive-energy branch u_λ^- tends to zero as $\lambda \uparrow \lambda_{1,\Omega^0}$, while its normalized profile converges to the first eigenfunction in Ω^0.

We then combine uniform L^∞-estimates with a local supersolution argument to obtain dead-core formation. In particular, if the weight is uniformly negative in a neighborhood of $\partial\Omega$, then solutions on the positive-energy branch u_λ^- have compact support for λ sufficiently close to λ_{1,Ω^0} from below. After obtaining explicit solutions to a one-dimensional degenerate problem, we prove the convergence of the free boundaries (the boundary of the support of u_λ^- to the boundary of Ω^0) as $\lambda \uparrow \lambda_{1,\Omega^0}$. For sign-changing weights, solutions on the negative-energy branch u_λ^+ also have compact support when λ is sufficiently negative.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded connected domain with sufficiently smooth boundary. In this paper, we study the existence and qualitative properties of non-negative solutions of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u + m(x)|u|^{\alpha-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $0 < \alpha < 1, \lambda \in \mathbb{R}, m \in L^\infty(\Omega)$.

The weight m may change sign and may vanish on a nonempty open subset of Ω. Problem (1) combines a linear spectral term with a sublinear indefinite reaction. The non-Lipschitz character of the nonlinearity at the origin allows non-negative solutions to develop a dead core, that is, a nonempty open set on which the solution vanishes identically. Consequently, besides strictly positive solutions, problem (1) may possess flat solutions satisfying $u > 0$ in Ω and $u = \partial u/\partial n = 0$ on $\partial\Omega$, as well as solutions whose support is compactly contained in Ω. Such phenomena arise in models of nonlinear diffusion, population dynamics, reaction–diffusion processes, and porous media. We refer, for example, to [25, 4, 8] and the references therein. The literature for $\alpha \geq 1$ is extensive, but will not be discussed here, since our interest is restricted to $0 < \alpha < 1$.

The variational analysis of elliptic problems with indefinite nonlinearities has a long history; see, among others, [2, 3]. The fibering method provides a natural framework for detecting and distinguishing variational branches; see [23]. Related parameter-dependent variational ideas and nonlinear Rayleigh quotient methods for the analysis of nonlinear bifurcations and extreme parameter values of Nehari-type manifolds were developed in[29, 30, 32, 33, 31].

For the more general quasilinear problem

$$-\Delta_p u = \lambda |u|^{p-2}u + a(x)|u|^{q-2}u, \qquad 1 < q < p,$$

Kaufmann and Ramos [35] established, among other results, the existence of a negative-energy ground state below a constrained spectral threshold and, under the condition $\int_\Omega a\varphi_p^q\,dx < 0$, the existence of a second positive-energy solution between the first eigenvalue and that threshold. They also studied uniqueness, positivity, and dead-core formation. Subsequently, Bobkov and Tanaka [5] gave a fine analysis of the critical and supercritical spectral intervals for the same p-Laplacian problem. In particular, they showed that, for genuinely nonlinear principal parts and under the condition $p > 2q$, non-negative solutions may persist to the right of the constrained critical value. This phenomenon cannot occur in the present semilinear sublinear case $p = 2$, $q = \alpha + 1 \in (1, 2)$.

The semilinear problem (1) was recently considered by Díaz, Hernández and Il'yasov [20]. That work pursued a broad qualitative program involving non-negative and positive solutions, bifurcation from infinity, uniqueness and stability questions, local barriers, compactly supported solutions, and Pohozaev-type restrictions. It also used the Nehari manifold to obtain two variationally distinguished non-negative solutions in the interval between the first eigenvalue and a constrained spectral threshold. Thus, neither the definition of the threshold nor the basic existence of the two variational levels is claimed here as a new result.

The purpose of the present paper is more focused. We isolate the critical mechanism specific to weights having a nonempty open zero region and give a self-contained treatment of the corresponding limiting branch. Our main emphasis is placed on the degenerate case

$$m \leq 0, \qquad \Omega^0 = \mathrm{int}\{m = 0\}, \qquad |\Omega^0| > 0.$$

We identify the constrained critical value with the first Dirichlet eigenvalue of Ω^0, prove that the positive-energy branch vanishes as the parameter approaches this value from below, and identify its normalized limit with the first eigenfunction of the zero region, extended by zero outside Ω^0. This spectral asymptotic information is then combined with a quantitative local supersolution argument to prove the formation of a dead core in a boundary ring whose width is uniform along the branch. Consequently, the solutions have compact support for parameters sufficiently close to the finite critical value.

We also analyze the opposite asymptotic regime $\lambda \to -\infty$ for the negative-energy branch. A direct maximum estimate yields uniform smallness of the solutions, and the same local dead-core mechanism then produces a fixed zero ring near $\partial\Omega$. These results describe two distinct routes to compact support: one is governed by the spectral geometry of the zero set of the weight at a finite endpoint, whereas the other is driven by uniform amplitude decay as the parameter tends to $-\infty$.

Accordingly, the contribution of this paper is not a further broad classification of all positive and non-negative solutions of (1). Rather, it is the rigorous identification and analysis of the chain

$$\lambda^* = \lambda_{1,\Omega^0}, \qquad u_\lambda^- \to 0, \qquad \frac{u_\lambda^-}{\|u_\lambda^-\|_{H_0^1(\Omega)}} \to \widetilde{\varphi}_{1,\Omega^0},$$

where $\widetilde{\varphi}_{1,\Omega^0}$ denotes the zero extension of φ_{1,Ω^0} to Ω, together with its consequence: a uniform boundary dead core and compact support of the corresponding variational solutions. We use the term "dead core" even though, in our setting, the zero region need not be located at the center of the domain.

We use the notation

$$\Omega^+ := \{x \in \Omega : m(x) > 0\}, \qquad \Omega^- := \{x \in \Omega : m(x) < 0\},$$

and

$$Z_m := \{x \in \Omega : m(x) = 0\}, \qquad \Omega^0 := \operatorname{int} Z_m.$$

Here and throughout the paper, $|A|$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$. Whenever one of the sets Ω^+, Ω^-, or Ω^0 is required to be nontrivial in the measure-theoretic sense, this will be stated explicitly.

Whenever spectral quantities associated with Ω^0 or $\Omega^0 \cup \Omega^+$ are used, we assume that the corresponding set is a nonempty connected Lipschitz domain. Whenever spectral quantities associated with Ω^0 are involved, we also assume that $\Omega^0 \Subset \Omega$.

Whenever an argument identifies a function vanishing a.e. on $\{m \neq 0\}$ with an element of $H_0^1(\Omega^0)$, we further assume that

$$Z_m = \overline{\Omega^0} \quad \text{up to a set of Lebesgue measure zero.}$$

In particular, in the degenerate case $m \leq 0$, this amounts to assuming that $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$. We denote by $\lambda_{1,\Omega}$ and $\varphi_{1,\Omega}$ the first Dirichlet eigenvalue and a corresponding positive eigenfunction of $-\Delta$ in Ω. Similarly, λ_{1,Ω^0} and φ_{1,Ω^0} denote the first Dirichlet eigenpair in Ω^0. When regarded as a function on Ω, φ_{1,Ω^0} is extended by zero outside Ω^0, and this extension is denoted by $\widetilde{\varphi}_{1,\Omega^0}$.

The variational functional associated with (1) is

$$E_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx - \frac{1}{\alpha+1}\int_\Omega m(x)|u|^{\alpha+1}\, dx, \qquad u \in H_0^1(\Omega).$$

We also write

$$H_\lambda(u) := \int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) dx, \qquad M(u) := \int_\Omega m(x)|u|^{\alpha+1}\, dx.$$

The corresponding Nehari manifold is

$$\mathcal{N}_\lambda := \{u \in H_0^1(\Omega) \setminus \{0\} : H_\lambda(u) = M(u)\}.$$

Its decomposition according to the sign of the second derivative of the fibering map leads to two different families of solutions, denoted by u_λ^+ and u_λ^-.

A central role is played by the Nehari critical value

$$\lambda^* := \inf\left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2\, dx}{\displaystyle\int_\Omega |u|^2\, dx} : u \in H_0^1(\Omega) \setminus \{0\}, \quad M(u) \geq 0 \right\}. \tag{2}$$

The value λ^* is an extreme parameter value associated with the Nehari constraint. Extreme values of this type and the behavior of solution branches near them are naturally described within the nonlinear Rayleigh quotient framework; see [30, 32, 33, 31] and the references therein. We adopt the convention $\lambda^* = +\infty$ if $|\Omega^+ \cup \Omega^0| = 0$. By the variational characterization of the first Dirichlet eigenvalue, $0 < \lambda_{1,\Omega} \leq \lambda^*$, and $\lambda^* < +\infty$ whenever $|\Omega^+ \cup \Omega^0| > 0$. The position of λ^* relative to $\lambda_{1,\Omega}$ is determined by the sign of the weight along the first eigenfunction:

$$\lambda^* > \lambda_{1,\Omega} \quad \Leftrightarrow \quad \int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\, dx < 0, \tag{3}$$

whereas

$$\lambda^* = \lambda_{1,\Omega} \quad \Leftrightarrow \quad \int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx \geq 0. \tag{4}$$

For completeness, and in order to fix the particular variational selections used later, we first recall the two solution levels below λ^*, together with the properties needed in the sequel. The role of the next theorem is preparatory: it identifies the precise branches whose asymptotic profiles and associated supports are analyzed in the subsequent sections.

Theorem 1. *Assume that $0 < \alpha < 1$ and $m \in L^\infty(\Omega)$.*

i) If $|\Omega^+| > 0$, then, for every $\lambda < \lambda^*$, problem (1) *possesses a non-negative weak solution $u_\lambda^+ \in H_0^1(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$, such that*

$$E_\lambda(u_\lambda^+) < 0, \qquad E''_\lambda(u_\lambda^+) > 0.$$

Moreover, the minimum energy level $(-\infty,\lambda^*) \ni \lambda \longmapsto E_\lambda(u_\lambda^+)$ is continuous and strictly decreasing, and

$$E_\lambda(u_\lambda^+) \longrightarrow 0 \qquad \text{as } \lambda \to -\infty.$$

ii) Suppose that

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0, \qquad |\Omega^+ \cup \Omega^0| > 0.$$

Then $0 < \lambda_{1,\Omega} < \lambda^* < +\infty$. For every $\lambda \in (\lambda_{1,\Omega},\lambda^*)$, problem (1) *possesses a non-negative weak solution $u_\lambda^- \in H_0^1(\Omega) \cap C^{1,\gamma}(\overline{\Omega})$ for any $\gamma \in (0,1)$, such that*

$$E_\lambda(u_\lambda^-) > 0, \qquad E''_\lambda(u_\lambda^-) < 0.$$

Moreover, the corresponding minimum energy level $(\lambda_{1,\Omega},\lambda^*) \ni \lambda \longmapsto E_\lambda(u_\lambda^-)$ is continuous and strictly decreasing, and

$$E_\lambda(u_\lambda^-) \longrightarrow +\infty \qquad \text{as } \lambda \downarrow \lambda_{1,\Omega}.$$

If, in addition, condition

$$\lambda_{1,\Omega^0} > \lambda_{1,\Omega^0\cup\Omega^+} \tag{M}$$

holds, then

$$E_\lambda(u_\lambda^-) \longrightarrow 0 \qquad \text{as } \lambda \uparrow \lambda^*.$$

In the degenerate case $m \leq 0$ a.e. in Ω, assume that $\Omega^0 = \operatorname{int}\{m = 0\}$ is a nonempty connected Lipschitz domain compactly contained in Ω and that $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$. Then

$$\lambda^* = \lambda_{1,\Omega^0}, \qquad E_\lambda(u_\lambda^-) \longrightarrow 0 \quad \text{as } \lambda \uparrow \lambda_{1,\Omega^0},$$

without assuming condition (M).

The proof of Theorem 1 will be given later. We point out that the basic existence assertions in parts i) and ii) belong to the variational framework developed for subhomogeneous indefinite problems in [35, 5, 20]. The continuity and strict monotonicity of the selected minimum energy levels are proved in the present manuscript. In the nondegenerate case, the endpoint relation $E_\lambda(u_\lambda^-) \to 0$ as $\lambda \uparrow \lambda^*$ follows under condition (M). In the degenerate case $m \leq 0$, however, the same endpoint limit remains valid without (M), as proved below.

The branches u_λ^+ and u_λ^- have different variational origins. The first corresponds to a stable critical point, in the sense of [24], of the fibering map and has negative energy. The second corresponds to an unstable critical point and has positive energy. Note that the stable energy branch is decreasing. This does not contradict the well-known result [9] that a decreasing bifurcation branch is unstable when represented in terms of the L^p norm, because here the branch represents the energy level rather than the L^p norm.

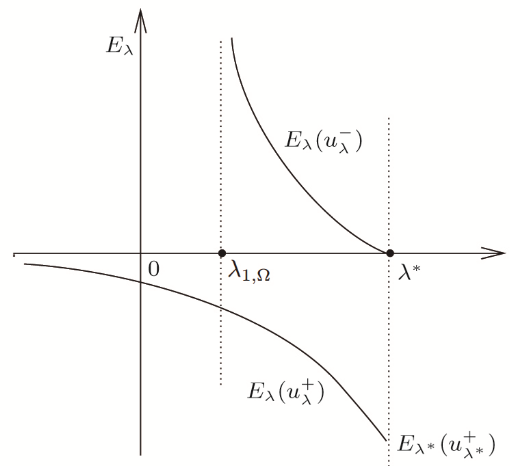

Figure 1: Qualitative behavior of the energy levels $E_\lambda(u_\lambda^+)$ and $E_\lambda(u_\lambda^-)$ when $\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0$.

The corresponding two-level structure is known for the general subhomogeneous indefinite problem; see [35] and [5, Theorem 1.7]. In the present semilinear case, λ^* is the terminal value of the positive-energy branch. Our subsequent analysis concerns what happens to this branch as the endpoint is approached and how its smallness affects the geometry of its support.

The degenerate case $m \le 0$ and $|\Omega^0| > 0$ has a particularly transparent spectral structure. Under the standing assumptions,

$$\lambda^* = \lambda_{1,\Omega^0}.$$

Moreover, since $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$,

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0, \qquad \lambda_{1,\Omega} < \lambda^* = \lambda_{1,\Omega^0}.$$

The solution u_λ^- then converges to zero as $\lambda \uparrow \lambda_{1,\Omega^0}$, whereas its normalized profile converges to $\widetilde{\varphi}_{1,\Omega^0}$. This vanishing property is the starting point for our compact-support analysis.

To obtain a dead core near $\partial\Omega$, negativity of the weight merely at boundary points is not sufficient. We require a full boundary ring on which the weight is uniformly negative. For $\rho > 0$, set

$$\mathcal{C}_\rho := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}.$$

Our first compact-support theorem concerns the degenerate case $m \le 0$.

Theorem 2. *Assume that $0 < \alpha < 1$, $m \in L^\infty(\Omega)$, $m \le 0$ a.e. in Ω, and that*

$$\Omega^0 := \operatorname{int}\{x \in \Omega : m(x) = 0\}$$

is a nonempty connected $C^{1,1}$ domain compactly contained in Ω, $|\Omega^0| > 0$. Assume also that $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$. Suppose that there exist $\rho_0 > 0$ and $m_{\rho_0}^- > 0$ such that

$$\mathcal{C}_{\rho_0} \subset \Omega^- \tag{5}$$

and

$$-m(x) \ge m_{\rho_0}^- \qquad \text{for a.e. } x \in \mathcal{C}_{\rho_0}. \tag{6}$$

Then there exists $\varepsilon > 0$ such that, for every $\lambda \in (\lambda_{1,\Omega^0} - \varepsilon, \lambda_{1,\Omega^0})$, the non-negative weak solution u_λ^- given by Theorem 5 has compact support in Ω. More precisely, there exists $\rho \in (0, \rho_0)$, independent of λ sufficiently close to λ_{1,Ω^0}, such that $u_\lambda^- = 0$ in $\mathcal{C}_\rho$. Consequently,

$$\operatorname{supp} u_\lambda^- \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \rho\} \Subset \Omega.$$

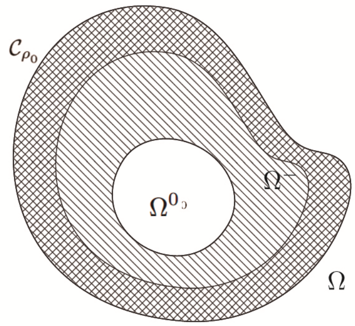

Figure 2: A uniformly negative boundary ring $\mathcal{C}_{\rho_0} \subset \Omega^-$ and the zero region $\Omega^0 = \mathrm{int}\{x \in \Omega : m(x) = 0\}$.

The preceding result concerns the upper endpoint of the branch u_λ^-. A complementary phenomenon occurs on the branch u_λ^+ as $\lambda \to -\infty$. In this regime, a direct maximum estimate shows that

$$\|u_\lambda^+\|_{L^\infty(\Omega)} \le \left(\frac{\|m^+\|_{L^\infty(\Omega)}}{|\lambda|} \right)^{\frac{1}{1-\alpha}},$$

and therefore

$$\|u_\lambda^+\|_{L^\infty(\Omega)} \longrightarrow 0 \qquad \text{as } \lambda \to -\infty.$$

Combined with the local dead-core criterion, this yields compactly supported solutions for sufficiently negative values of the parameter.

Theorem 3. *Assume that $0 < \alpha < 1$, $|\Omega^+| > 0$, $|\Omega^-| > 0$, and $m \in L^\infty(\Omega)$. Suppose that there exist $\rho_0 > 0$ and $m_{\rho_0}^-$ such that*

$$\mathcal{C}_{\rho_0} \subset \Omega^- \tag{7}$$

and

$$-m(x) \ge m_{\rho_0}^- \qquad \text{for a.e. } x \in \mathcal{C}_{\rho_0}. \tag{8}$$

Let u_λ^+ be the non-negative weak solution of (1) given by Theorem 1. Then there exists $\sigma < 0$ such that, for every $\lambda < \sigma$, the solution u_λ^+ has compact support in Ω.

More precisely, there exists $\rho \in (0, \rho_0)$, independent of all sufficiently negative λ, such that $u_\lambda^+ = 0$ in $\mathcal{C}_\rho$. Consequently,

$$\operatorname{supp} u_\lambda^+ \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \rho\} \Subset \Omega.$$

Theorems 2 and 3 describe two different mechanisms leading to compactly supported solutions. In Theorem 2, the amplitude tends to zero as the parameter approaches the finite spectral threshold λ_{1,Ω^0} from below. In Theorem 3, the amplitude tends to zero as $\lambda \to -\infty$. In both cases, uniform negativity of the weight in a boundary ring allows a local barrier argument to transform smallness of the solution into the formation of a dead core close to $\partial\Omega$.

The distinction from the results of [35] is that the weight is kept fixed: the dead core is produced by the vanishing of a distinguished variational branch rather than by increasing the negative part of the weight. The distinction from [5] is equally clear. Their principal new phenomenon concerns continuation into the supercritical interval for $p > 2q$, whereas our problem belongs to the semilinear regime $p = 2$, where such continuation does not occur. Our results instead identify the internal spectral object governing the terminal behavior and use it to derive uniform localization of the support.

The proofs combine nonlinear Rayleigh quotients, minimization on the Nehari manifold, compactness arguments, spectral properties of the zero region Ω^0, uniform L^∞ estimates, and local supersolutions of compact-support type. The paper is organized as follows. We first introduce the variational setting and analyze the Nehari critical value λ^*. We then establish the existence and asymptotic behavior of the two selected solution branches. Next, we identify λ^* with λ_{1,Ω^0} in the degenerate non-positive case and determine the normalized limiting profile. We develop a local dead-core criterion and apply it to obtain compactly supported solutions both near the finite spectral endpoint and for sufficiently negative values of λ. We then prove convergence of the free boundaries, i.e. the boundaries of the supports of u_λ^-, to $\partial\Omega^0$ as $\lambda \uparrow \lambda_{1,\Omega^0}$. For sign-changing weights, solutions on the negative-energy branch u_λ^+ also have compact support when λ is sufficiently negative. Finally, we establish free-boundary convergence in several additional cases.

2 Preliminaries

We write $H_0^1 := H_0^1(\Omega)$, endowed with the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2},$$

which, by the Poincaré inequality, is equivalent to the standard H^1-norm on $H_0^1(\Omega)$. Throughout the paper, integrals without an explicitly indicated domain are understood to be taken over Ω. We denote by $L^p(\Omega)$, $1 < p < \infty$, the usual Lebesgue spaces, with norm $\|\cdot\|_p$. For $u \in H_0^1(\Omega)$, set

$$H_\lambda(u) := \int_\Omega \left(|\nabla u|^2 - \lambda |u|^2\right) dx, \qquad M(u) := \int_\Omega m(x)|u|^{\alpha+1}\,dx,$$

and define the associated energy functional by

$$E_\lambda(u) := \frac{1}{2}H_\lambda(u) - \frac{1}{\alpha+1}M(u).$$

A weak solution of (1) is a critical point of E_λ in $H_0^1(\Omega)$. We construct solutions by minimizing E_λ on the two components of the Nehari manifold:

$$\widehat{E}_\lambda^+ := \inf\{E_\lambda(u) : u \in \mathcal{N}_\lambda^+\}, \tag{9}$$

$$\widehat{E}_\lambda^- := \inf\{E_\lambda(u) : u \in \mathcal{N}_\lambda^-\}. \tag{10}$$

Here

$$\mathcal{N}_\lambda^\pm := \{u \in H_0^1(\Omega) \setminus \{0\} : E_\lambda'(u) = 0,\ \pm E_\lambda''(u) > 0\},$$

while

$$\mathcal{N}_\lambda := \{u \in H_0^1(\Omega) \setminus \{0\} : E_\lambda'(u) = 0\}.$$

For $F \in C^1(H_0^1(\Omega))$, we use the notation

$$F'(u) := DF(u)[u] = \frac{d}{dt}F(tu)\bigg|_{t=1}.$$

Thus

$$E_\lambda'(u) = H_\lambda(u) - M(u), \qquad E_\lambda''(u) = H_\lambda(u) - \alpha M(u),$$

and, in particular,

$$u \in \mathcal{N}_\lambda \iff H_\lambda(u) = M(u).$$

Lemma 1. *Let $u \in H_0^1(\Omega)$ be a non-negative weak solution of (1). Then $u \in W^{2,p}(\Omega)$ for every $p < \infty$, and consequently $u \in C^{1,\gamma}(\overline{\Omega})$ for every $\gamma \in (0,1)$. If, in addition, m is locally Hölder continuous, then $u \in C^2(\Omega)$.*

Proof. A bootstrap argument, as in [22], gives $u \in L^\infty(\Omega)$. Hence $\lambda u + m(x)u^\alpha \in L^p(\Omega)$ for every $p < \infty$, and standard elliptic regularity yields $u \in W^{2,p}(\Omega)$. Sobolev embedding then gives $u \in C^{1,\gamma}(\overline{\Omega})$ for every $\gamma \in (0,1)$. If m is locally Hölder continuous, interior Schauder regularity implies $u \in C^2(\Omega)$. See also [1, 26] and [37, Appendix B]. $\square$

3 Solutions at the Nehari critical value

We now study the critical value

$$\lambda^* := \inf \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx} : u \in H_0^1(\Omega) \setminus \{0\}, \quad M(u) \geq 0 \right\}. \tag{11}$$

By definition, $0 < \lambda_{1,\Omega} \leq \lambda^*$, and $\lambda^* < +\infty$ whenever $|\Omega^0 \cup \Omega^+| > 0$. If $|\Omega^0 \cup \Omega^+| = 0$, we set $\lambda^* := +\infty$.

The position of λ^* relative to the first eigenvalue is characterized by

$$\lambda^* > \lambda_{1,\Omega} \quad \Leftrightarrow \quad \int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0,$$

whereas

$$\lambda^* = \lambda_{1,\Omega} \quad \leftrightarrow \quad \int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx \geq 0.$$

The definition of λ^* also gives, for every $\lambda < \lambda^*$,

$$M(u) \geq 0 \quad \Rightarrow \quad H_\lambda(u) > 0, \tag{12}$$

or, equivalently,

$$H_\lambda(u) \leq 0 \quad \Rightarrow \quad M(u) < 0, \tag{13}$$

for every $u \in H_0^1(\Omega) \setminus \{0\}$.

We shall use the following assumption.

(M) If $|\Omega^0| > 0$, then $\Omega^0 \cup \Omega^+$ is a domain and

$$\lambda_{1,\Omega^0\cup\Omega^+} < \lambda_{1,\Omega^0}. \tag{M}$$

Whenever $\lambda_{1,\Omega^0\cup\Omega^+}$ is used, we assume that $\Omega^0 \cup \Omega^+$ is a nonempty bounded Lipschitz domain.

Remark 1 (On condition (M) and its geometric meaning). *Condition* (M) *is essentially geometric. If Ω^0 is a proper subdomain of the connected domain $\Omega^0 \cup \Omega^+$, then strict domain monotonicity of the first Dirichlet eigenvalue gives $\lambda_{1,\Omega^0\cup\Omega^+} < \lambda_{1,\Omega^0}$. Thus, in this standard situation, the spectral inequality in* (M) *is automatic: the positive phase genuinely enlarges the zero phase. The connectedness assumption is important. If the positive phase is separated from Ω^0 by a region where $m < 0$, then $\Omega^0 \cup \Omega^+$ may be disconnected. For instance, in one dimension, if $\Omega^0 = (-1, 1)$, $I_\varepsilon = (2, 2 + \varepsilon)$, with a negative region between them, then*

$$\lambda_{1,\Omega^0} = \frac{\pi^2}{4}, \qquad \lambda_{1,I_\varepsilon} = \frac{\pi^2}{\varepsilon^2}.$$

For $0 < \varepsilon < 2$, the added positive component has a larger first eigenvalue and hence the lowest first eigenvalue of the disconnected union. Condition (M) *excludes precisely this geometric–spectral decoupling. See [27, 10].*

Lemma 2. *Let*

$$\mathcal{R}(u) := \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega |u|^2\,dx}, \qquad \mathcal{A} := \{u \in H_0^1(\Omega) \setminus \{0\} : M(u) \geq 0\}.$$

Suppose that $\bar{u} \in \mathcal{A}$ is a local minimizer of $\mathcal{R}$ on $\mathcal{A}$. Then there exist $\mu_0, \mu_1 \geq 0$, with $\mu_0 + \mu_1 > 0$, such that

$$\mu_0 D\mathcal{R}(\bar{u}) = \mu_1 DM(\bar{u}), \tag{14}$$

and

$$\mu_1 M(\bar{u}) = 0. \tag{15}$$

If $DM(\bar{u}) \neq 0$, then $\mu_0 > 0$, and consequently there exists $\mu \geq 0$ such that

$$D\mathcal{R}(\bar{u}) = \mu\, DM(\bar{u}), \qquad \mu M(\bar{u}) = 0. \tag{16}$$

Thus, under the constraint qualification $DM(\bar{u}) \neq 0$, the Fritz John conditions [34] reduce to the usual Kuhn–Tucker conditions.

Proof. Since $\mathcal{R}$ and M are continuously Fréchet differentiable on $H_0^1(\Omega) \setminus \{0\}$, the Fritz John theorem applied to the constraint $M(u) \geq 0$ gives (14) together with (15); see, e.g., [6].

If $DM(\bar{u}) \neq 0$ and $\mu_0 = 0$, then (14) and $\mu_0 + \mu_1 > 0$ imply $\mu_1 > 0$ and hence $DM(\bar{u}) = 0$, a contradiction. Thus $\mu_0 > 0$. Dividing (14) by μ_0 and setting $\mu = \mu_1/\mu_0$ gives (16). □

Theorem 4. *Assume that $|\Omega^0 \cup \Omega^+| > 0$, that the standing domain regularity assumptions hold, and*

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\, dx < 0.$$

Then the following assertions hold.

i) Problem (11) has a nonzero non-negative minimizer $\overline{u} \in H_0^1(\Omega)$. Moreover, $H_{\lambda^*}(\overline{u}) = 0, M(\overline{u}) = 0$.

ii) If $|\Omega^+| = 0$, then $\lambda^* = \lambda_{1,\Omega^0}$, and $\overline{u}$ is a positive multiple of φ_{1,Ω^0}, extended by zero outside Ω^0.

*iii) If, in addition, condition **(M)** holds, then there exists $\tau > 0$ such that $\tau\overline{u}$ is a weak solution of (1) for $\lambda = \lambda^*$, and $E''_{\lambda^*}(\tau\overline{u}) = 0$.*

Proof. We divide the proof into three steps.

i) The admissible set in (11) is nonempty, since the first eigenfunction of $-\Delta$ in $\Omega^0 \cup \Omega^+$, extended by zero to Ω, satisfies $M(\varphi_{1,\Omega^0\cup\Omega^+}) \geq 0$. Hence

$$\lambda^* \leq \lambda_{1,\Omega^0\cup\Omega^+}. \tag{17}$$

Let (u_n) be a minimizing sequence normalized by $\|u_n\|_{L^2(\Omega)} = 1$. Then $\int_\Omega |\nabla u_n|^2\, dx \to \lambda^*$, so (u_n) is bounded in $H_0^1(\Omega)$. Up to a subsequence,

$$u_n \rightharpoonup \overline{u} \quad \text{in } H_0^1(\Omega), \qquad u_n \to \overline{u} \quad \text{in } L^r(\Omega), \quad 1 \leq r < 2^*.$$

In particular, $\|\overline{u}\|_{L^2(\Omega)} = 1$, hence $\overline{u} \neq 0$. Since $m \in L^\infty(\Omega)$, $M(u_n) \to M(\overline{u})$, so $M(\overline{u}) \geq 0$. Weak lower semicontinuity gives

$$\mathcal{R}(\overline{u}) \leq \liminf_{n\to\infty} \mathcal{R}(u_n) = \lambda^*.$$

By the definition of λ^*, equality holds. Thus $\overline{u}$ is a minimizer, $H_{\lambda^*}(\overline{u}) = 0$, and, replacing it by $|\overline{u}|$, we may assume $\overline{u} \geq 0$.

It remains to show that $M(\overline{u}) = 0$. By Lemma 2, there exist $\mu_0, \mu_1 \geq 0$, $\mu_0 + \mu_1 > 0$, such that

$$\mu_0 D\mathcal{R}(\overline{u}) = \mu_1 DM(\overline{u}). \tag{18}$$

Since $\mathcal{R}(\overline{u}) = \lambda^*$,

$$D\mathcal{R}(\overline{u})[v] = \frac{1}{\|\overline{u}\|_{L^2(\Omega)}^2} DH_{\lambda^*}(\overline{u})[v].$$

Absorbing the positive factor into μ_0, we obtain

$$\mu_0 DH_{\lambda^*}(\overline{u}) = \mu_1 DM(\overline{u}). \tag{19}$$

Equivalently,

$$2\mu_0(-\Delta\overline{u}-\lambda^*\overline{u})=(\alpha+1)\mu_1 m(x)|\overline{u}|^{\alpha-1}\overline{u}$$

in the weak sense. By complementary slackness (see, e.g., [6]), $\mu_1 M(\overline{u})=0$. If $\mu_1=0$, then $\mu_0>0$ and $-\Delta\overline{u}=\lambda^*\overline{u}$ in Ω. Since $\overline{u}\geq 0$ and $\overline{u}\not\equiv 0$, the strong maximum principle yields $\lambda^*=\lambda_{1,\Omega}$ and $\overline{u}=c\varphi_{1,\Omega}$, contradicting

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx<0.$$

Hence $\mu_1>0$, and therefore $M(\overline{u})=0$.
ii) If $|\Omega^+|=0$, then $m\leq 0$ a.e. in Ω. Since $M(\overline{u})=0$, we have $\overline{u}=0$ a.e. on $\{m<0\}$. By the standing zero-set and regularity assumptions, $\overline{u}\in H_0^1(\Omega^0)$. Hence

$$\lambda_{1,\Omega^0}\leq\frac{\displaystyle\int_{\Omega^0}|\nabla\overline{u}|^2\,dx}{\displaystyle\int_{\Omega^0}|\overline{u}|^2\,dx}=\lambda^*.$$

Conversely, φ_{1,Ω^0} is admissible in (11), so $\lambda^*\leq\lambda_{1,\Omega^0}$. Thus $\lambda^*=\lambda_{1,\Omega^0}$, and simplicity of the first eigenvalue gives $\overline{u}=c\varphi_{1,\Omega^0}$ for some $c>0$.

iii) We first show that $\mu_0>0$. Otherwise $\mu_1>0$, and (19) gives $DM(\overline{u})=0$, hence $\overline{u}=0$ a.e. on $\{m\neq 0\}$. If $|\Omega^0|=0$, this contradicts $\overline{u}\neq 0$. If $|\Omega^0|>0$, the standing zero-set and regularity assumptions give $\overline{u}\in H_0^1(\Omega^0)$, and $H_{\lambda^*}(\overline{u})=0$ implies $\lambda_{1,\Omega^0}\leq\lambda^*\leq\lambda_{1,\Omega^0\cup\Omega^+}$, contrary to condition **(M)**. Thus $\mu_0>0$. Since also $\mu_1>0$, set

$$\tau:=\left(\frac{2\mu_0}{(\alpha+1)\mu_1}\right)^{\frac{1}{1-\alpha}}.$$

Then (19) implies that $\tau\overline{u}$ is a weak solution of (1) for $\lambda=\lambda^*$. Finally, $H_{\lambda^*}(\overline{u})=M(\overline{u})=0$, and homogeneity yields $H_{\lambda^*}(\tau\overline{u})=M(\tau\overline{u})=0$. Therefore $E''_{\lambda^*}(\tau\overline{u})=0$. This completes the proof. □

4 The case $\lambda<\lambda^*$

Let

$$S:=\{v\in H_0^1(\Omega):\|v\|=1\}.$$

We introduce

$$\Theta^+:=\{v\in H_0^1(\Omega)\setminus\{0\}:M(v)>0\},$$

and, for $\lambda>\lambda_{1,\Omega}$,

$$\Theta^-_\lambda:=\{v\in H_0^1(\Omega)\setminus\{0\}:H_\lambda(v)<0\}.$$

Clearly, $\Theta^+\neq\emptyset$ if $|\Omega^+|>0$, while $\Theta^-_\lambda\neq\emptyset$ for every $\lambda>\lambda_{1,\Omega}$, since

$$H_\lambda(\varphi_{1,\Omega})=(\lambda_{1,\Omega}-\lambda)\int_\Omega\varphi_{1,\Omega}^2\,dx<0.$$

If $v\in\Theta^+$ and $\lambda<\lambda^*$, then (12) gives $H_\lambda(v)>0$. Similarly, if $v\in\Theta^-_\lambda$ and $\lambda_{1,\Omega}<\lambda<\lambda^*$, then (13) gives $M(v)<0$.

In either case, the fibering equation

$$\phi_v'(t)=tH_\lambda(v)-t^\alpha M(v)=0 \tag{20}$$

has the unique positive solution

$$t_\lambda(v)=\left(\frac{M(v)}{H_\lambda(v)}\right)^{\frac{1}{1-\alpha}}, \tag{21}$$

where the quotient is positive. Hence

$$\mathcal{N}_\lambda^{\pm} = \{t_\lambda(v)v : v \in \Theta_\lambda^{\pm} \cap S\}, \tag{22}$$

with $\Theta_\lambda^+ := \Theta^+$. Set

$$c_\alpha := \frac{1-\alpha}{2(1+\alpha)}.$$

For $v \in \Theta^+$, define

$$J_\lambda^+(v) := E_\lambda(t_\lambda(v)v) = -c_\alpha \frac{M(v)^{\frac{2}{1-\alpha}}}{H_\lambda(v)^{\frac{1+\alpha}{1-\alpha}}}, \qquad \lambda < \lambda^*, \tag{23}$$

whereas, for $v \in \Theta_\lambda^-$,

$$J_\lambda^-(v) := E_\lambda(t_\lambda(v)v) = c_\alpha \frac{|M(v)|^{\frac{2}{1-\alpha}}}{|H_\lambda(v)|^{\frac{1+\alpha}{1-\alpha}}}, \qquad \lambda_{1,\Omega} < \lambda < \lambda^*. \tag{24}$$

Both functionals are homogeneous of degree zero. We therefore consider

$$\widehat{J}_\lambda^+ := \inf_{v \in \Theta^+ \cap S} J_\lambda^+(v), \qquad \lambda < \lambda^*, \tag{25}$$

$$\widehat{J}_\lambda^- := \inf_{v \in \Theta_\lambda^- \cap S} J_\lambda^-(v), \qquad \lambda_{1,\Omega} < \lambda < \lambda^*. \tag{26}$$

If $|\Omega^+| > 0$, then $\widehat{J}_\lambda^+ < 0$. Moreover, if $\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0$, then $\widehat{J}_\lambda^- < +\infty$ for every $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$.

Proposition 1.

i) If $|\Omega^+| > 0$, $\lambda < \lambda^*$, and $v_\lambda^+ \in \Theta^+ \cap S$ minimizes (25)*, then $u_\lambda^+ := t_\lambda(v_\lambda^+)v_\lambda^+$ is a non-negative weak solution of* (1)*. Moreover,*

$$E_\lambda(u_\lambda^+) = \widehat{E}_\lambda^+ = \widehat{J}_\lambda^+ < 0, \qquad E_\lambda''(u_\lambda^+) > 0.$$

ii) Assume that $|\Omega^0 \cup \Omega^+| > 0$ and

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

If $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$ and $v_\lambda^- \in \Theta_\lambda^- \cap S$ minimizes (26)*, then $u_\lambda^- := t_\lambda(v_\lambda^-)v_\lambda^-$ is a non-negative weak solution of* (1)*. Moreover,*

$$E_\lambda(u_\lambda^-) = \widehat{E}_\lambda^- = \widehat{J}_\lambda^- > 0, \qquad E_\lambda''(u_\lambda^-) < 0.$$

Proof. The proof follows from the fibering representation (22) and the standard natural-constraint argument for the Nehari manifold; see, e.g., [20]. Since the minimizers may be replaced by their absolute values, the resulting weak solutions are non-negative. □

Lemma 3. *The following assertions hold.*

i) If $|\Omega^+| > 0$, then for every $\lambda < \lambda^*$ there exists a minimizer $v_\lambda^+ \in \Theta^+ \cap S$ of (25)*.*

ii) Assume that $|\Omega^0 \cup \Omega^+| > 0$ and

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

Then, for every $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$, there exists a minimizer $v_\lambda^- \in \Theta_\lambda^- \cap S$ of (26)*.*

Proof. *i*) Fix $\lambda < \lambda^*$ and let $(v_n) \subset \Theta^+ \cap S$ be a minimizing sequence for (25). Thus $\|v_n\| = 1$, $M(v_n) > 0$, and

$$J_\lambda^+(v_n) \longrightarrow \widehat{J}_\lambda^+ < 0.$$

Up to a subsequence, there exists $v_\lambda^+ \in H_0^1(\Omega)$ such that

$$v_n \rightharpoonup v_\lambda^+ \quad \text{in } H_0^1(\Omega), \qquad v_n \to v_\lambda^+ \quad \text{in } L^q(\Omega), \quad 1 \le q < 2^*.$$

In particular, $M(v_n) \to M(v_\lambda^+)$. We first obtain a uniform lower bound for $H_\lambda(v_n)$. Since $M(v_n) > 0$, each v_n is admissible in the definition of λ^*, and hence

$$\lambda^* \le \frac{\displaystyle\int_\Omega |\nabla v_n|^2\,dx}{\displaystyle\int_\Omega |v_n|^2\,dx}.$$

Because $\|v_n\| = 1$, we have $\|v_n\|_2^2 \le 1/\lambda^*$. Consequently,

$$H_\lambda(v_n) \ge \begin{cases} 1 - \lambda/\lambda^*, & 0 \le \lambda < \lambda^*, \\ 1, & \lambda < 0. \end{cases}$$

Thus there exists $c_\lambda > 0$, independent of n, such that

$$H_\lambda(v_n) \ge c_\lambda > 0. \tag{27}$$

Suppose that $v_\lambda^+ = 0$. Then $M(v_n) \to 0$, and (23) together with (27) yields $J_\lambda^+(v_n) \to 0$, contradicting $\widehat{J}_\lambda^+ < 0$. Hence $v_\lambda^+ \neq 0$. Since $M(v_n) > 0$, we have $M(v_\lambda^+) \ge 0$. If $M(v_\lambda^+) = 0$, the same argument again gives $J_\lambda^+(v_n) \to 0$, a contradiction. Therefore $M(v_\lambda^+) > 0$, so $v_\lambda^+ \in \Theta^+$. By weak lower semicontinuity and strong L^2 convergence,

$$H_\lambda(v_\lambda^+) \le \liminf_{n\to\infty} H_\lambda(v_n).$$

Since $M(v_n) \to M(v_\lambda^+) > 0$, formula (23) gives

$$J_\lambda^+(v_\lambda^+) \le \liminf_{n\to\infty} J_\lambda^+(v_n) = \widehat{J}_\lambda^+.$$

If $\|v_\lambda^+\| < 1$, normalize by setting

$$\widetilde{v}_\lambda^+ := \frac{v_\lambda^+}{\|v_\lambda^+\|}.$$

Since J_λ^+ is homogeneous of degree zero, $\widetilde{v}_\lambda^+ \in \Theta^+ \cap S$ and $J_\lambda^+(\widetilde{v}_\lambda^+) = J_\lambda^+(v_\lambda^+)$. Hence

$$\widehat{J}_\lambda^+ \le J_\lambda^+(\widetilde{v}_\lambda^+) \le \widehat{J}_\lambda^+.$$

Thus $\widetilde{v}_\lambda^+$ is a minimizer of (25). Renaming it v_λ^+ proves *i*).

ii) The proof is analogous. Let $(v_n) \subset \Theta_\lambda^- \cap S$ be a minimizing sequence. Since $H_\lambda(v_n) < 0$, $\|v_n\|_2^2 > 1/\lambda$, and therefore its weak limit $\bar{v}$ is nonzero. Strong $L^{\alpha+1}$ convergence gives $M(v_n) \to M(\bar{v})$.

If $M(\bar{v}) = 0$, weak lower semicontinuity gives $H_\lambda(\bar{v}) \le 0$, contradicting (12), since $\lambda < \lambda^*$ and $\bar{v} \neq 0$. Thus $M(\bar{v}) < 0$. Since the minimizing sequence has bounded energy and $M(v_n)$ stays away from zero, (24) shows that $H_\lambda(v_n)$ stays uniformly away from zero, and consequently $H_\lambda(\bar{v}) < 0$. Formula (24), weak lower semicontinuity, and the homogeneity of J_λ^- then yield, after normalization, a minimizer $v_\lambda^- \in \Theta_\lambda^- \cap S$ of (26). □

Combining Proposition 1 and Lemma 3, we obtain the following existence result.

Corollary 1. *Assume that* $|\Omega^0 \cup \Omega^+| > 0$ *and*

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

i) If $|\Omega^+| > 0$, then for every $\lambda < \lambda^*$ there exists a non-negative weak solution u_λ^+ of (1) such that

$$E_\lambda(u_\lambda^+) < 0, \qquad E''_\lambda(u_\lambda^+) > 0.$$

ii) For every $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$, there exists a non-negative weak solution u_λ^- of (1) such that

$$E_\lambda(u_\lambda^-) > 0, \qquad E''_\lambda(u_\lambda^-) < 0.$$

Remark 2. *By Lemma 1, $u_\lambda^- \in W^{2,p}(\Omega)$ for every $p < \infty$.*

4.1 Dependence on the parameter λ

The next result is based on the comparison argument introduced in [29].

Proposition 2. *Let λ_0 belong to a compact subinterval of $(\lambda_{1,\Omega}, \lambda^*)$ for the branch u_λ^-, or of $(-\infty, \lambda^*)$ for the branch u_λ^+. Then there exist $\delta > 0$ and $r_i^\pm(\lambda, \lambda_0) = o(|\lambda - \lambda_0|)$, $i = 1, 2$, such that, whenever $0 < \lambda - \lambda_0 < \delta$,*

$$\begin{aligned} -\frac{\lambda-\lambda_0}{2}\|u_\lambda^-\|_2^2 + r_1^-(\lambda,\lambda_0) &\le E_\lambda(u_\lambda^-) - E_{\lambda_0}(u_{\lambda_0}^-) \\ &\le -\frac{\lambda-\lambda_0}{2}\|u_{\lambda_0}^-\|_2^2 + r_2^-(\lambda,\lambda_0). \end{aligned} \tag{28}$$

and

$$\begin{aligned} -\frac{\lambda-\lambda_0}{2}\|u_\lambda^+\|_2^2 + r_1^+(\lambda,\lambda_0) &\le E_\lambda(u_\lambda^+) - E_{\lambda_0}(u_{\lambda_0}^+) \\ &\le -\frac{\lambda-\lambda_0}{2}\|u_{\lambda_0}^+\|_2^2 + r_2^+(\lambda,\lambda_0). \end{aligned} \tag{29}$$

Moreover, for every fixed $\lambda_0 \in (\lambda_{1,\Omega}, \lambda^*)$, there exist $\delta, c_0, C_0 > 0$ such that, whenever $|\lambda - \lambda_0| < \delta$,

$$|H_\lambda(u_\lambda^-)| \ge c_0, \qquad \|u_\lambda^-\| \le C_0.$$

Proof. We give the proof for u_λ^-; the argument for u_λ^+ is analogous. Fix $\lambda_0 \in (\lambda_{1,\Omega}, \lambda^*)$ and choose

$$K := [\lambda_0 - \eta, \lambda_0 + \eta] \Subset (\lambda_{1,\Omega}, \lambda^*)$$

for some $\eta > 0$. For each $\lambda \in K$, let $v_\lambda^- \in \Theta_\lambda^- \cap S$ be a minimizer of (26) and write $u_\lambda^- = t_\lambda(v_\lambda^-)v_\lambda^-$.

Step 1: uniform estimates. There exists $c_M > 0$ such that

$$M(v) \le -c_M < 0 \tag{30}$$

for every $\lambda \in K$ and every $v \in S$ with $H_\lambda(v) \le 0$. Indeed, otherwise there would exist $\lambda_n \in K$ and $v_n \in S$ such that $H_{\lambda_n}(v_n) \le 0$ and $M(v_n) \to 0$. Up to a subsequence, $\lambda_n \to \bar\lambda \in K$, $v_n \rightharpoonup \bar v$ in $H_0^1(\Omega)$, and $v_n \to \bar v$ in $L^q(\Omega)$ for $1 \le q < 2^*$. Since

$$1 - \lambda_n\|v_n\|_2^2 = H_{\lambda_n}(v_n) \le 0,$$

we have $\|v_n\|_2^2 \ge 1/\lambda_n \ge 1/\max K$, so $\bar v \ne 0$. Moreover, $M(\bar v) = 0$ and

$$H_{\bar\lambda}(\bar v) \le \liminf_{n\to\infty} H_{\lambda_n}(v_n) \le 0,$$

contradicting (12). Thus (30) holds. The minimum levels are uniformly bounded above on K. Indeed, $M(\varphi_{1,\Omega}) < 0$, while

$$H_\lambda(\varphi_{1,\Omega}) = (\lambda_{1,\Omega} - \lambda)\|\varphi_{1,\Omega}\|_2^2 < 0$$

for all $\lambda \in K$. Hence $w := \varphi_{1,\Omega}/\|\varphi_{1,\Omega}\|$ belongs to $\Theta_\lambda^- \cap S$, and $\widehat{J}_\lambda^- \le J_\lambda^-(w) \le C_K$. Using (30) and (24), we obtain

$$|H_\lambda(v_\lambda^-)| \ge c_H > 0, \qquad \lambda \in K. \tag{31}$$

Since $v_\lambda^- \in S$, the values $M(v_\lambda^-)$ are uniformly bounded. Thus (21) and (31) imply

$$\|u_\lambda^-\| = t_\lambda(v_\lambda^-) \le C, \qquad \lambda \in K. \tag{32}$$

Moreover, the lower bound (30) and the boundedness of $|H_\lambda(v_\lambda^-)|$ show that $t_\lambda(v_\lambda^-)$ is bounded away from zero. Therefore

$$|H_\lambda(u_\lambda^-)| \ge c_H > 0, \qquad \lambda \in K. \tag{33}$$

Step 2: differentiability of the reduced energy. For $v \neq 0$ and parameters μ such that $M(v)H_\mu(v) > 0$, define

$$\mathcal{J}(\mu, v) := E_\mu(t_\mu(v)v).$$

Since $t_\mu(v)v \in \mathcal{N}_\mu$, differentiation gives

$$\frac{\partial \mathcal{J}}{\partial \mu}(\mu, v) = -\frac{1}{2} t_\mu(v)^2 \|v\|_2^2. \tag{34}$$

Furthermore,

$$\frac{\partial t_\mu(v)}{\partial \mu} = \frac{t_\mu(v)\|v\|_2^2}{(1-\alpha)H_\mu(v)},$$

and hence

$$\frac{\partial^2 \mathcal{J}}{\partial \mu^2}(\mu, v) = -\frac{t_\mu(v)^2\|v\|_2^4}{(1-\alpha)H_\mu(v)}. \tag{35}$$

By Step 1, for $\lambda, \lambda_0 \in K$ sufficiently close, the quantities $t_\mu(v_\lambda^-)$ and $\|v_\lambda^-\|_2$ remain uniformly bounded for μ between λ_0 and λ, while $|H_\mu(v_\lambda^-)| \ge c_H/2$. Hence the second derivative above is uniformly bounded, and Taylor's formula gives

$$\mathcal{J}(\lambda_0, v_\lambda^-) = \mathcal{J}(\lambda, v_\lambda^-) + \frac{\lambda - \lambda_0}{2}\|u_\lambda^-\|_2^2 + R_1^-(\lambda, \lambda_0), \tag{36}$$

and

$$\mathcal{J}(\lambda, v_{\lambda_0}^-) = \mathcal{J}(\lambda_0, v_{\lambda_0}^-) - \frac{\lambda - \lambda_0}{2}\|u_{\lambda_0}^-\|_2^2 + R_2^-(\lambda, \lambda_0), \tag{37}$$

where $|R_i^-(\lambda, \lambda_0)| \le C|\lambda - \lambda_0|^2$, $i = 1, 2$.

Step 3: comparison of the minimum levels. Let $0 < \lambda - \lambda_0 < \delta$, with δ sufficiently small. By (31), $v_\lambda^- \in \Theta_{\lambda_0}^-$ and $v_{\lambda_0}^- \in \Theta_\lambda^-$. Therefore, the minimizing properties at λ_0 and λ, combined with (36) and (37), yield

$$E_\lambda(u_\lambda^-) - E_{\lambda_0}(u_{\lambda_0}^-) \ge -\frac{\lambda - \lambda_0}{2}\|u_\lambda^-\|_2^2 - R_1^-(\lambda, \lambda_0)$$

and

$$E_\lambda(u_\lambda^-) - E_{\lambda_0}(u_{\lambda_0}^-) \le -\frac{\lambda - \lambda_0}{2}\|u_{\lambda_0}^-\|_2^2 + R_2^-(\lambda, \lambda_0).$$

Setting

$$r_1^- := -R_1^-, \qquad r_2^- := R_2^-,$$

we obtain (28), with $r_i^-(\lambda, \lambda_0) = O(|\lambda - \lambda_0|^2) = o(|\lambda - \lambda_0|)$. For u_λ^+, choose $K \Subset (-\infty, \lambda^*)$. Since $M(v) > 0$ for $v \in \Theta^+ \cap S$, the definition of λ^* yields $H_\lambda(v) \ge c_K > 0$ uniformly for $\lambda \in K$. A fixed admissible direction in $\Theta^+ \cap S$, together with the minimizing property, prevents $M(v_\lambda^+)$ from tending to zero. Hence $t_\lambda(v_\lambda^+)$ is bounded above and below on K, and the same Taylor argument applies. Since Θ^+ is independent of λ, the comparison is even simpler and gives (29), with $r_i^+(\lambda, \lambda_0) = O(|\lambda - \lambda_0|^2) = o(|\lambda - \lambda_0|)$. □

Corollary 2. *The maps*

$$(-\infty, \lambda^*) \ni \lambda \mapsto E_\lambda(u_\lambda^+)$$

and

$$(\lambda_{1,\Omega}, \lambda^*) \ni \lambda \mapsto E_\lambda(u_\lambda^-)$$

are continuous and strictly decreasing.

Proof. Let $\lambda > \lambda_0$. Proposition 2 gives

$$E_\lambda(u_\lambda^\pm) - E_{\lambda_0}(u_{\lambda_0}^\pm) \le -\frac{\lambda - \lambda_0}{2}\|u_{\lambda_0}^\pm\|_2^2 + o(\lambda - \lambda_0).$$

Since $u_{\lambda_0}^\pm \neq 0$, the right-hand side is negative for $\lambda - \lambda_0 > 0$ sufficiently small. Thus the energy levels are locally strictly decreasing, hence strictly decreasing on their whole domains by subdivision of compact parameter intervals. Continuity follows from the two-sided estimates in Proposition 2. $\square$

Remark 3. *The maps $\lambda \mapsto E_\lambda(u_\lambda^\pm)$ are single-valued because they represent minimum energy levels. The minimizers themselves, and hence the quantities $\|u_\lambda^\pm\|_2$, need not be unique.*

4.2 Asymptotic behavior as $\lambda \uparrow \lambda^*$

Lemma 4. *Assume that $\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0$.*

a) If $|\Omega^+| > 0$, then there exists a non-negative weak solution $u_{\lambda^*}^+$ of (1) such that

$$E_{\lambda^*}(u_{\lambda^*}^+) < 0, \qquad E''_{\lambda^*}(u_{\lambda^*}^+) > 0.$$

Moreover,

$$E_\lambda(u_\lambda^+) \to E_{\lambda^*}(u_{\lambda^*}^+) \qquad \text{as } \lambda \uparrow \lambda^*.$$

*b) If $|\Omega^0 \cup \Omega^+| > 0$ and condition **(M)** holds, then*

$$E_\lambda(u_\lambda^-) \to 0 \qquad \text{as } \lambda \uparrow \lambda^*.$$

Proof. *a*) By Corollary 2, $\lambda \mapsto E_\lambda(u_\lambda^+)$ is strictly decreasing on $(-\infty, \lambda^*)$. Hence

$$\overline{E}_{\lambda^*}^+ := \lim_{\lambda\uparrow\lambda^*} E_\lambda(u_\lambda^+)$$

exists in $[-\infty, 0)$. We first prove that $\overline{E}_{\lambda^*}^+ > -\infty$. Suppose otherwise that, for some $\lambda_n \uparrow \lambda^*$, $E_{\lambda_n}(u_{\lambda_n}^+) \to -\infty$. Since $u_{\lambda_n}^+ \in \mathcal{N}_{\lambda_n}^+$,

$$H_{\lambda_n}(u_{\lambda_n}^+) = M(u_{\lambda_n}^+) > 0,$$

and

$$E_{\lambda_n}(u_{\lambda_n}^+) = -c_\alpha H_{\lambda_n}(u_{\lambda_n}^+) = -c_\alpha M(u_{\lambda_n}^+), \qquad c_\alpha = \frac{1-\alpha}{2(1+\alpha)}. \tag{38}$$

Thus $M(u_{\lambda_n}^+) \to +\infty$, and therefore $\|u_{\lambda_n}^+\| \to +\infty$. Set

$$\rho_n := \|u_{\lambda_n}^+\|, \qquad v_n := \frac{u_{\lambda_n}^+}{\rho_n}.$$

Then $\|v_n\| = 1$ and $v_n \ge 0$. Up to a subsequence,

$$v_n \rightharpoonup \bar{v} \quad \text{in } H_0^1(\Omega), \qquad v_n \to \bar{v} \quad \text{in } L^q(\Omega), \quad 1 \le q < 2^*.$$

Dividing the Nehari identity by ρ_n^2 gives

$$H_{\lambda_n}(v_n) = \rho_n^{\alpha-1}M(v_n) \longrightarrow 0.$$

Hence $1-\lambda_n\|v_n\|_2^2 \to 0$, and consequently $\|\bar{v}\|_2^2 = 1/\lambda^* > 0$. Dividing the equation for $u_{\lambda_n}^+$ by ρ_n and passing to the limit yields

$$-\Delta \bar{v} = \lambda^* \bar{v} \qquad \text{in } \Omega.$$

Since $\bar{v} \geq 0$ and $\bar{v} \not\equiv 0$, the strong maximum principle implies $\lambda^* = \lambda_{1,\Omega}$, contrary to

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

Thus $\overline{E}_{\lambda^*}^+ > -\infty$. We next prove compactness as $\lambda \uparrow \lambda^*$. For any sequence $\lambda_n \uparrow \lambda^*$, the family $(u_{\lambda_n}^+)$ is bounded in $H_0^1(\Omega)$; otherwise the same normalization argument would again give $\lambda^* = \lambda_{1,\Omega}$. Hence, up to a subsequence,

$$u_{\lambda_n}^+ \rightharpoonup u_{\lambda^*}^+ \quad \text{in } H_0^1(\Omega), \qquad u_{\lambda_n}^+ \to u_{\lambda^*}^+ \quad \text{in } L^q(\Omega), \quad 1 \leq q < 2^*.$$

In particular, $M(u_{\lambda_n}^+) \to M(u_{\lambda^*}^+)$. The limit is nonzero. Indeed, for any fixed $\lambda_0 < \lambda^*$, monotonicity gives

$$E_{\lambda_n}(u_{\lambda_n}^+) \leq E_{\lambda_0}(u_{\lambda_0}^+) < 0$$

for large n. If $u_{\lambda^*}^+ = 0$, then $M(u_{\lambda_n}^+) \to 0$, and (38) would imply $E_{\lambda_n}(u_{\lambda_n}^+) \to 0$, a contradiction. Passing to the limit in the weak formulation, using the strong $L^{\alpha+1}$ convergence, shows that $u_{\lambda^*}^+$ is a non-negative weak solution of (1) for $\lambda = \lambda^*$. Testing the limit equation by $u_{\lambda^*}^+$ gives

$$H_{\lambda^*}(u_{\lambda^*}^+) = M(u_{\lambda^*}^+).$$

Moreover, from the Nehari identities,

$$\|u_{\lambda_n}^+\|^2 = \lambda_n\|u_{\lambda_n}^+\|_2^2 + M(u_{\lambda_n}^+) \longrightarrow \|u_{\lambda^*}^+\|^2.$$

Together with weak convergence, this yields

$$u_{\lambda_n}^+ \longrightarrow u_{\lambda^*}^+ \qquad \text{strongly in } H_0^1(\Omega).$$

Consequently,

$$E_{\lambda_n}(u_{\lambda_n}^+) \longrightarrow E_{\lambda^*}(u_{\lambda^*}^+) = \overline{E}_{\lambda^*}^+ < 0.$$

Hence $H_{\lambda^*}(u_{\lambda^*}^+) = M(u_{\lambda^*}^+) > 0$, and therefore

$$E''_{\lambda^*}(u_{\lambda^*}^+) = (1-\alpha)M(u_{\lambda^*}^+) > 0.$$

It remains to verify the minimizing property at λ^*. Let $w \in \mathcal{N}_{\lambda^*}^+$. Then $H_{\lambda^*}(w) = M(w) > 0$. For large n, set

$$s_n := \left(\frac{M(w)}{H_{\lambda_n}(w)}\right)^{\frac{1}{1-\alpha}}.$$

Then $s_n w \in \mathcal{N}_{\lambda_n}^+$ and $s_n \to 1$. By minimality,

$$E_{\lambda_n}(u_{\lambda_n}^+) \leq E_{\lambda_n}(s_n w).$$

Passing to the limit gives $E_{\lambda^*}(u_{\lambda^*}^+) \leq E_{\lambda^*}(w)$. Thus $u_{\lambda^*}^+$ minimizes E_{λ^*} on $\mathcal{N}_{\lambda^*}^+$, and the limit is independent of the chosen sequence. This proves *a*).

b) Let u_* be the nonzero non-negative minimizer associated with λ^* in Theorem 4. Under condition **(M)**, Theorem 4(3) allows us, after a positive rescaling, to assume that u_* is a weak solution of (1) for $\lambda = \lambda^*$. Moreover,

$$H_{\lambda^*}(u_*) = M(u_*) = 0.$$

Let $\varphi_1 = \varphi_{1,\Omega} > 0$. Testing the equation for u_* by φ_1 gives

$$\int_\Omega m(x)u_*^\alpha\varphi_1\,dx = (\lambda_{1,\Omega} - \lambda^*)\int_\Omega u_*\varphi_1\,dx < 0.$$

Hence

$$DH_{\lambda^*}(u_*)[\varphi_1] < 0, \qquad DM(u_*)[\varphi_1] < 0.$$

For $v_\varepsilon := u_* + \varepsilon\varphi_1$, we therefore have

$$H_{\lambda^*}(v_\varepsilon) = -c_1\varepsilon + o(\varepsilon), \qquad M(v_\varepsilon) = -c_2\varepsilon + o(\varepsilon)$$

for some $c_1, c_2 > 0$. Choose $\varepsilon(\lambda) = \sqrt{\lambda^* - \lambda}$. Since

$$H_\lambda(v_\varepsilon) = H_{\lambda^*}(v_\varepsilon) + (\lambda^* - \lambda)\|v_\varepsilon\|_2^2,$$

for λ sufficiently close to λ^*,

$$H_\lambda(v_{\varepsilon(\lambda)}) < 0, \qquad M(v_{\varepsilon(\lambda)}) < 0.$$

Thus $v_{\varepsilon(\lambda)}$ is admissible for the negative Nehari level. By minimality,

$$0 < E_\lambda(u_\lambda^-) \leq J_\lambda^-(v_{\varepsilon(\lambda)}).$$

Using (24) and the preceding expansions,

$$J_\lambda^-(v_{\varepsilon(\lambda)}) \leq C\varepsilon(\lambda) \longrightarrow 0.$$

Therefore $E_\lambda(u_\lambda^-) \longrightarrow 0$ as $\lambda \uparrow \lambda^*$. □

For every weak solution u_λ of (1),

$$E_\lambda(u_\lambda) = -\frac{1-\alpha}{2(1+\alpha)} H_\lambda(u_\lambda) = -\frac{1-\alpha}{2(1+\alpha)} M(u_\lambda). \tag{39}$$

Corollary 3. *Assume that* $\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0$.

a) If $|\Omega^+| > 0$, *then, as* $\lambda \uparrow \lambda^*$,

$$H_\lambda(u_\lambda^+) \to H_{\lambda^*}(u_{\lambda^*}^+), \qquad M(u_\lambda^+) \to M(u_{\lambda^*}^+).$$

b) If $|\Omega^0 \cup \Omega^+| > 0$ *and condition* ***(M)*** *holds, then*

$$H_\lambda(u_\lambda^-) \longrightarrow 0, \qquad M(u_\lambda^-) \longrightarrow 0 \qquad \text{as } \lambda \uparrow \lambda^*.$$

Lemma 5. *Assume that* $0 < \alpha < 1$ *and* $m \in L^\infty(\Omega)$.

a) If $\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0$, *then, as* $\lambda \downarrow \lambda_{1,\Omega}$,

$$E_\lambda(u_\lambda^-) \to +\infty, \qquad \|u_\lambda^-\| \to +\infty,$$

and, after normalizing $\|\varphi_{1,\Omega}\| = 1$,

$$\frac{u_\lambda^-}{\|u_\lambda^-\|} \longrightarrow \varphi_{1,\Omega} \qquad \text{strongly in } H_0^1(\Omega).$$

b) If $|\Omega^+| > 0$, *then, as* $\lambda \to -\infty$,

$$E_\lambda(u_\lambda^+) \to 0, \qquad \|u_\lambda^+\| \to 0.$$

Proof. *a*) Let $\lambda_n \downarrow \lambda_{1,\Omega}$ and set

$$v_n := \frac{u_{\lambda_n}^-}{\|u_{\lambda_n}^-\|}.$$

Then $\|v_n\| = 1$ and $H_{\lambda_n}(v_n) < 0$. Up to a subsequence, $v_n \rightharpoonup \bar{v}$ in $H_0^1(\Omega)$ and $v_n \to \bar{v}$ in $L^q(\Omega)$ for $1 \leq q < 2^*$. Since $1 < \lambda_n \|v_n\|_2^2$, passing to the limit gives $H_{\lambda_{1,\Omega}}(\bar{v}) \leq 0$. The variational characterization of $\lambda_{1,\Omega}$ gives the reverse inequality; hence $H_{\lambda_{1,\Omega}}(\bar{v}) = 0$. Thus $\bar{v}$ is proportional to $\varphi_{1,\Omega}$, and under the chosen normalization

$$v_n \longrightarrow \varphi_{1,\Omega} \qquad \text{strongly in } H_0^1(\Omega).$$

Since $M(\varphi_{1,\Omega}) < 0$, formula (21) yields $\|u_{\lambda_n}^-\| = t_{\lambda_n}(v_n) \to +\infty$, and (24) gives $E_{\lambda_n}(u_{\lambda_n}^-) \to +\infty$. Since (λ_n) was arbitrary, the assertions hold for the whole family.
b) By Corollary 2, the limit

$$\overline{E}_{-\infty}^{+} := \lim_{\lambda \to -\infty} E_\lambda(u_\lambda^+) \leq 0$$

exists. From (39), $H_\lambda(u_\lambda^+)$ remains bounded as $\lambda \to -\infty$. Since

$$H_\lambda(u_\lambda^+) = \|u_\lambda^+\|^2 + |\lambda| \|u_\lambda^+\|_2^2,$$

both terms on the right are bounded, and hence $\|u_\lambda^+\|_2 \to 0$. Since $\alpha + 1 < 2$ and Ω is bounded, $\|u_\lambda^+\|_{\alpha+1} \to 0$, so

$$|M(u_\lambda^+)| \leq \|m\|_\infty \|u_\lambda^+\|_{\alpha+1}^{\alpha+1} \longrightarrow 0.$$

By (39), $E_\lambda(u_\lambda^+) \to 0$ and $H_\lambda(u_\lambda^+) \to 0$. Finally,

$$0 \leq \|u_\lambda^+\|^2 \leq H_\lambda(u_\lambda^+),$$

and therefore $\|u_\lambda^+\| \to 0$. □

5 Degenerate case: $m(x) \leq 0$ and $|\Omega^0| > 0$

Theorem 5. *Assume that $0 < \alpha < 1$, $m \in L^\infty(\Omega)$, $m \leq 0$ a.e. in Ω, and that*

$$\Omega^0 := \text{int}\{x \in \Omega : m(x) = 0\}$$

is a nonempty connected $C^{1,1}$ domain compactly contained inΩ. Assume also that $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$. Then the following assertions hold.

i) For every $\lambda \in (\lambda_{1,\Omega}, \lambda_{1,\Omega^0})$, problem (1) possesses a non-negative weak solution u_λ^- such that

$$E_\lambda(u_\lambda^-) > 0, \qquad E_\lambda''(u_\lambda^-) < 0,$$

and

$$E_\lambda(u_\lambda^-) \to 0 \qquad \text{as } \lambda \uparrow \lambda_{1,\Omega^0}.$$

ii) If $\lambda \geq \lambda_{1,\Omega^0}$, then problem (1) has no positive weak solution.

iii) As $\lambda \uparrow \lambda_{1,\Omega^0}$,

$$\|u_\lambda^-\| \to 0.$$

Moreover, after normalizing $\|\varphi_{1,\Omega^0}\| = 1$,

$$\frac{u_\lambda^-}{\|u_\lambda^-\|} \to \widetilde{\varphi}_{1,\Omega^0} \qquad \text{strongly in } H_0^1(\Omega),$$

where $\widetilde{\varphi}_{1,\Omega^0}$ denotes the zero extension of φ_{1,Ω^0} to Ω.

Proof. *i*) Since $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$ and $\varphi_{1,\Omega} > 0$ in Ω, we have

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

By Theorem 4, $\lambda^* = \lambda_{1,\Omega^0}$, and an extremal function corresponding to λ^* is, up to a positive constant, $\widetilde{\varphi}_{1,\Omega^0}$. Hence Corollary 1 yields, for every $\lambda \in (\lambda_{1,\Omega}, \lambda_{1,\Omega^0})$, a non-negative weak solution u_λ^- satisfying

$$E_\lambda(u_\lambda^-) > 0, \qquad E_\lambda''(u_\lambda^-) < 0.$$

It remains to prove that

$$E_\lambda(u_\lambda^-) \to 0 \qquad \text{as } \lambda \uparrow \lambda_{1,\Omega^0}.$$

Set

$$\lambda^* := \lambda_{1,\Omega^0}, \qquad \phi := \widetilde{\varphi}_{1,\Omega^0}.$$

Then

$$H_{\lambda^*}(\phi) = 0, \qquad M(\phi) = 0.$$

Consider the bilinear form

$$B_*(v,w) := \int_\Omega \nabla v \cdot \nabla w\,dx - \lambda^* \int_\Omega vw\,dx.$$

We claim that there exists $\zeta \in H_0^1(\Omega)$ such that

$$B_*(\phi,\zeta) < 0, \qquad M(\zeta) < 0. \tag{40}$$

Indeed, the functional $\zeta \mapsto B_*(\phi,\zeta)$ cannot vanish identically on $H_0^1(\Omega)$. Otherwise ϕ would satisfy

$$-\Delta\phi = \lambda^*\phi \qquad \text{weakly in } \Omega,$$

and the strong maximum principle would imply $\phi > 0$ in Ω, contrary to $\phi = 0$ in $\Omega \setminus \overline{\Omega^0}$. Thus there exists $\zeta_0 \in H_0^1(\Omega)$ with $B_*(\phi,\zeta_0) < 0$. Choose a nonzero $\eta \in C_c^\infty(\Omega \setminus \overline{\Omega^0})$. Since $m < 0$ a.e. there, $M(\eta) < 0$, while, because $\phi = 0$ on the support of η, $B_*(\phi,\eta) = 0$. Hence, for $\zeta = \zeta_0 + s\eta$ with $s > 0$ sufficiently large, $M(\zeta) < 0$, whereas $B_*(\phi,\zeta) = B_*(\phi,\zeta_0) < 0$. This proves (40). Let

$$\delta := \lambda^* - \lambda > 0, \qquad \varepsilon_\lambda := K\delta, \qquad w_\lambda := \phi + \varepsilon_\lambda\zeta,$$

where $K > 0$ will be chosen below. Since $m = 0$ on Ω^0 and $\phi = 0$ outside Ω^0,

$$M(w_\lambda) = K^{\alpha+1}\delta^{\alpha+1}M(\zeta) < 0. \tag{41}$$

Moreover,

$$H_\lambda(w_\lambda) = 2\varepsilon_\lambda B_*(\phi,\zeta) + \varepsilon_\lambda^2 H_{\lambda^*}(\zeta) + \delta\|\phi + \varepsilon_\lambda\zeta\|_2^2,$$

and therefore

$$H_\lambda(w_\lambda) = \delta\big[\|\phi\|_2^2 + 2KB_*(\phi,\zeta)\big] + O(\delta^2). \tag{42}$$

Since $B_*(\phi,\zeta) < 0$, we may choose K sufficiently large so that

$$\|\phi\|_2^2 + 2KB_*(\phi,\zeta) < 0.$$

Hence there exist $c > 0$ and $\delta_0 > 0$ such that

$$H_\lambda(w_\lambda) \le -c\delta < 0 \qquad \text{for } 0 < \delta < \delta_0. \tag{43}$$

Thus $w_\lambda \in \Theta_\lambda^-$ for λ sufficiently close to λ^*. By the minimizing property of u_λ^- and the homogeneity of J_λ^-,

$$0 < E_\lambda(u_\lambda^-) \le J_\lambda^-(w_\lambda).$$

Using (24), (41), and (43), we obtain

$$J_\lambda^-(w_\lambda) \le C\frac{\delta^{2(\alpha+1)/(1-\alpha)}}{\delta^{(1+\alpha)/(1-\alpha)}} = C\delta^{(1+\alpha)/(1-\alpha)}.$$

Therefore

$$E_\lambda(u_\lambda^-) \to 0 \qquad \text{as } \lambda \uparrow \lambda_{1,\Omega^0}.$$

ii) Suppose that $u_\lambda > 0$ is a weak solution of (1) for some $\lambda \ge \lambda_{1,\Omega^0}$. Since $m = 0$ a.e. in Ω^0,

$$-\Delta u_\lambda = \lambda u_\lambda \qquad \text{in } \Omega^0.$$

For $\phi \in C_0^\infty(\Omega^0)$, Picone's inequality gives

$$\int_{\Omega^0} |\nabla \phi|^2\,dx \ge \lambda \int_{\Omega^0} \phi^2\,dx.$$

Taking the infimum over $\phi \in H_0^1(\Omega^0)\setminus\{0\}$ yields $\lambda \le \lambda_{1,\Omega^0}$, and hence $\lambda = \lambda_{1,\Omega^0}$. To exclude equality, multiply $-\Delta u_\lambda = \lambda_{1,\Omega^0} u_\lambda$ by φ_{1,Ω^0} and use Green's identity:

$$\int_{\partial\Omega^0} u_\lambda \frac{\partial \varphi_{1,\Omega^0}}{\partial n}\,dS = 0.$$

By the Hopf lemma, $\partial_n \varphi_{1,\Omega^0} < 0$ on $\partial\Omega^0$, whereas $u_\lambda > 0$ there since $\Omega^0 \Subset \Omega$. The integral is therefore strictly negative, a contradiction. (iii) Let

$$\lambda_n \uparrow \lambda_{1,\Omega^0}, \qquad u_n := u_{\lambda_n}^-, \qquad t_n := \|u_n\|, \qquad v_n := \frac{u_n}{t_n}.$$

Then $\|v_n\| = 1$ and $H_{\lambda_n}(u_n) = M(u_n) < 0$. By part (i) and the Nehari energy identity,

$$E_{\lambda_n}(u_n) = -\frac{1-\alpha}{2(1+\alpha)} H_{\lambda_n}(u_n) = -\frac{1-\alpha}{2(1+\alpha)} M(u_n),$$

we have

$$H_{\lambda_n}(u_n) \longrightarrow 0, \qquad M(u_n) \longrightarrow 0.$$

We first prove that $t_n \to 0$. Suppose otherwise. Up to a subsequence, $t_n \to a \in (0, +\infty]$, so t_n is bounded away from zero. Since (v_n) is bounded in $H_0^1(\Omega)$, up to a subsequence,

$$v_n \rightharpoonup \bar v \quad \text{in } H_0^1(\Omega), \qquad v_n \to \bar v \quad \text{in } L^q(\Omega), \quad 1 \le q < 2^*.$$

Since

$$M(v_n) = \frac{M(u_n)}{t_n^{\alpha+1}} \to 0,$$

we obtain $M(\bar v) = 0$. Since $m \le 0$ and $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$, $\bar v = 0$ a.e. outside $\overline{\Omega^0}$, and therefore

$$\bar v \in H_0^1(\Omega^0).$$

Furthermore,

$$H_{\lambda_n}(v_n) = \frac{H_{\lambda_n}(u_n)}{t_n^2} \to 0.$$

Since $\|v_n\| = 1$,

$$1 - \lambda_n \|v_n\|_2^2 \to 0,$$

and hence

$$\|\bar v\|_2^2 = \frac{1}{\lambda_{1,\Omega^0}} > 0.$$

The variational characterization of λ_{1,Ω^0} and weak lower semicontinuity give

$$1 \le \|\bar{v}\|^2 \le \liminf_{n\to\infty} \|v_n\|^2 = 1.$$

Thus $\bar{v}$ is the normalized positive first eigenfunction of Ω^0. Consequently,

$$\bar{v} = \widetilde{\varphi}_{1,\Omega^0}, \qquad v_n \to \bar{v} \quad \text{strongly in } H_0^1(\Omega).$$

Dividing (1) by t_n gives

$$-\Delta v_n - \lambda_n v_n = t_n^{\alpha-1} m v_n^\alpha \qquad \text{in } \Omega. \tag{44}$$

If $a \in (0, +\infty)$, then $u_n \to a\widetilde{\varphi}_{1,\Omega^0}$ strongly in $H_0^1(\Omega)$, and

$$m u_n^\alpha \to 0 \qquad \text{in } L^{(\alpha+1)/\alpha}(\Omega).$$

Passing to the limit in the equation for u_n would therefore give

$$-\Delta\widetilde{\varphi}_{1,\Omega^0} = \lambda_{1,\Omega^0}\widetilde{\varphi}_{1,\Omega^0} \qquad \text{weakly in } \Omega.$$

If $a = +\infty$, then $t_n^{\alpha-1} \to 0$, and passing to the limit in (44) gives the same conclusion. Thus, in either case, $\widetilde{\varphi}_{1,\Omega^0}$ would be a non-negative nontrivial weak solution of the eigenvalue equation in the whole connected domain Ω. The strong maximum principle would then imply $\widetilde{\varphi}_{1,\Omega^0} > 0$ in Ω, contradicting its vanishing in $\Omega \setminus \overline{\Omega^0}$. Therefore

$$t_n = \|u_n\| \to 0.$$

Finally, the preceding compactness argument applies to every sequence $\lambda_n \uparrow \lambda_{1,\Omega^0}$. Since the normalized positive first eigenfunction is unique, every such sequence satisfies

$$\frac{u_{\lambda_n}^-}{\|u_{\lambda_n}^-\|} \to \widetilde{\varphi}_{1,\Omega^0} \qquad \text{strongly in } H_0^1(\Omega).$$

Hence, as $\lambda \uparrow \lambda_{1,\Omega^0}$,

$$\frac{u_{\lambda}^-}{\|u_{\lambda}^-\|} \to \widetilde{\varphi}_{1,\Omega^0} \qquad \text{strongly in } H_0^1(\Omega).$$

□

The following estimate yields, in particular, $\|u_\lambda^-\|_{L^\infty(\Omega)} \to 0$ as $\lambda \uparrow \lambda_{1,\Omega^0}$ in any dimension.

Proposition 3. *Assume that $0 < \alpha < 1$, $m \in L^\infty(\Omega)$, and $m \le 0$ a.e. in Ω. Let $u_\lambda \in H_0^1(\Omega)$ be a non-negative weak solution of (1). If $\lambda \le \Lambda$, then there exists $C = C(\Omega, N, \Lambda) > 0$, independent of λ, such that*

$$\|u_\lambda\|_{L^\infty(\Omega)} \le C\|u_\lambda\|. \tag{45}$$

Proof. Since $u_\lambda \ge 0$, $m \le 0$, and $\lambda \le \Lambda$,

$$-\Delta u_\lambda = \lambda u_\lambda + m(x)u_\lambda^\alpha \le \Lambda_+ u_\lambda, \qquad \Lambda_+ := \max\{\Lambda, 0\}.$$

Thus u_λ is a non-negative weak subsolution of $-\Delta u = \Lambda_+ u$. The standard L^∞ estimate gives

$$\|u_\lambda\|_{L^\infty(\Omega)} \le C_0\|u_\lambda\|_{L^2(\Omega)} \le C_0 C_P \|u_\lambda\|,$$

where the last inequality follows from Poincaré's inequality. This proves (45). □

Taking $\Lambda = \lambda_{1,\Omega^0}$, the constant in (45) is uniform for $\lambda < \lambda_{1,\Omega^0}$. Hence Theorem 5(iii) immediately gives:

Corollary 4. *Under the assumptions of Theorem 5, for $\lambda \in (\lambda_{1,\Omega}, \lambda_{1,\Omega^0})$,*

$$\|u_\lambda^-\|_{L^\infty(\Omega)} \to 0 \qquad \textit{as } \lambda \uparrow \lambda_{1,\Omega^0}.$$

Proof. By Proposition 3,

$$\|u_\lambda^-\|_{L^\infty(\Omega)} \le C\|u_\lambda^-\|,$$

with C independent of $\lambda < \lambda_{1,\Omega^0}$. The conclusion follows from Theorem 5 *iii*). □

Remark 4. *The preceding L^∞ convergence will be crucial for proving compactness of* $\operatorname{supp} u_\lambda^-$ *when* $\lambda \in [\lambda_{1,\Omega^0} - \varepsilon, \lambda_{1,\Omega^0}]$*; see Corollary 5.*

Proof of Theorem 1.

Part i). If $|\Omega^+| > 0$, then $\Theta^+ \neq \varnothing$. For every $\lambda < \lambda^*$, Lemma 3(i) yields a minimizer $v_\lambda^+ \in \Theta^+ \cap S$ of

$$\widehat{J}_\lambda^+ = \inf_{v \in \Theta^+ \cap S} J_\lambda^+(v).$$

Set

$$u_\lambda^+ := t_\lambda(v_\lambda^+) v_\lambda^+, \qquad t_\lambda(v) = \left(\frac{M(v)}{H_\lambda(v)}\right)^{1/(1-\alpha)}.$$

By Proposition 1 *i*), u_λ^+ is a non-negative weak solution of (1), with

$$E_\lambda(u_\lambda^+) = \widehat{E}_\lambda^+ = \widehat{J}_\lambda^+ < 0, \qquad E_\lambda''(u_\lambda^+) > 0.$$

Lemma 1 gives the required regularity. Continuity and strict monotonicity of $\lambda \mapsto E_\lambda(u_\lambda^+)$ follow from Corollary 2, while Lemma 5(b) yields

$$E_\lambda(u_\lambda^+) \longrightarrow 0 \qquad \text{as } \lambda \to -\infty.$$

This proves *i*).

Part ii). The bifurcation at the infinity is consequence from [11] and [28]. Assume

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0, \qquad |\Omega^+ \cup \Omega^0| > 0.$$

Then

$$0 < \lambda_{1,\Omega} < \lambda^* < +\infty.$$

For every $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$, Lemma 3(ii) gives a minimizer $v_\lambda^- \in \Theta_\lambda^- \cap S$. Setting

$$u_\lambda^- := t_\lambda(v_\lambda^-) v_\lambda^-,$$

Proposition 1(ii) gives a non-negative weak solution with

$$E_\lambda(u_\lambda^-) = \widehat{E}_\lambda^- = \widehat{J}_\lambda^- > 0, \qquad E_\lambda''(u_\lambda^-) < 0.$$

Regularity follows from Lemma 1. Corollary 2 yields continuity and strict monotonicity of

$$(\lambda_{1,\Omega}, \lambda^*) \ni \lambda \longmapsto E_\lambda(u_\lambda^-).$$

Moreover, Lemma 5(a) gives

$$E_\lambda(u_\lambda^-) \longrightarrow +\infty \qquad \text{as } \lambda \downarrow \lambda_{1,\Omega},$$

while, under condition **(M)**, Lemma 4(b) gives

$$E_\lambda(u_\lambda^-) \longrightarrow 0 \qquad \text{as } \lambda \uparrow \lambda^*.$$

It remains to justify the endpoint limit in the degenerate case without condition **(M)**. Assume $m \le 0$ a.e. in Ω and that $\Omega^0 = \operatorname{int}\{m = 0\}$ is a nonempty connected Lipschitz domain compactly contained in Ω.Assume also that

$$m < 0 \qquad \text{a.e. in } \Omega \setminus \overline{\Omega^0}.$$

By Theorem 4(2),

$$\lambda^* = \lambda_{1,\Omega^0}. \tag{46}$$

Let $\phi := \widetilde{\varphi}_{1,\Omega^0}$ be the zero extension of a first eigenfunction of Ω^0. Then

$$H_{\lambda^*}(\phi) = 0, \qquad M(\phi) = 0. \tag{47}$$

We claim that there exists $\zeta \in H_0^1(\Omega)$ such that

$$B_*(\phi, \zeta) < 0, \qquad M(\zeta) < 0, \tag{48}$$

where

$$B_*(v, w) := \int_\Omega \nabla v \cdot \nabla w \, dx - \lambda^* \int_\Omega vw \, dx.$$

Indeed, the functional $\zeta \mapsto B_*(\phi, \zeta)$ cannot vanish identically; otherwise ϕ would satisfy

$$-\Delta\phi = \lambda^* \phi$$

weakly in all of Ω, contradicting the strong maximum principle, since ϕ vanishes identically in $\Omega \setminus \overline{\Omega^0}$. Thus there exists $\zeta_0 \in H_0^1(\Omega)$ such that $B_*(\phi, \zeta_0) < 0$. Choose a nonzero function $\eta \in C_c^\infty(\Omega \setminus \overline{\Omega^0})$. Since $m < 0$ a.e. in $\Omega \setminus \overline{\Omega^0}$, $M(\eta) < 0$, whereas, because $\phi = 0$ a.e. on the support of η, $B_*(\phi, \eta) = 0$. Set $\zeta := \zeta_0 + s\eta$. For $s > 0$ sufficiently large,

$$M(\zeta) < 0, \qquad B_*(\phi, \zeta) = B_*(\phi, \zeta_0) < 0.$$

This proves (48). Set

$$\delta := \lambda^* - \lambda > 0, \qquad \varepsilon_\lambda := K\delta, \qquad w_\lambda := \phi + \varepsilon_\lambda \zeta.$$

Since $m = 0$ on Ω^0 and $\phi = 0$ outside Ω^0,

$$M(w_\lambda) = K^{\alpha+1} \delta^{\alpha+1} M(\zeta) < 0. \tag{49}$$

Moreover,

$$H_\lambda(w_\lambda) = 2\varepsilon_\lambda B_*(\phi, \zeta) + \varepsilon_\lambda^2 H_{\lambda^*}(\zeta) + \delta \|\phi + \varepsilon_\lambda \zeta\|_2^2,$$

and hence

$$H_\lambda(w_\lambda) = \delta \big[\|\phi\|_2^2 + 2K B_*(\phi, \zeta) \big] + O(\delta^2). \tag{50}$$

Choosing K sufficiently large gives

$$H_\lambda(w_\lambda) \le -c\delta < 0 \qquad (0 < \delta < \delta_0) \tag{51}$$

for suitable $c, \delta_0 > 0$. Thus $w_\lambda \in \Theta_\lambda^-$ for λ close to λ^*. By minimality and the homogeneity of J_λ^-,

$$0 < E_\lambda(u_\lambda^-) \le J_\lambda^-(w_\lambda).$$

Using (24), (49), and (51),

$$J_\lambda^-(w_\lambda) \le C\delta^{(1+\alpha)/(1-\alpha)}.$$

Therefore

$$E_\lambda(u_\lambda^-) \longrightarrow 0 \qquad \text{as } \lambda \uparrow \lambda^* = \lambda_{1,\Omega^0}. \tag{52}$$

Thus the degenerate endpoint relation holds without condition **(M)**. This completes the proof of Theorem 1. □

6 Compact support solutions

As pointed out in the Introduction, our main goal is to identify, among the non-negative solutions obtained by the Nehari manifold method, flat solutions satisfying $u > 0$ in Ω and $\partial u/\partial n = 0$ on $\partial\Omega$, as well as solutions $u \geq 0$ with $\operatorname{supp} u \Subset \Omega$.

In the one-dimensional case $N = 1$, classical energy methods provide a complete description of the solution set. For (1) with $\Omega = (-1, 1)$ and $m \equiv -1$, the branch of non-negative solutions bifurcating from λ_1 passes from positive solutions with $u'(\pm 1) \neq 0$ to compactly supported solutions. All these solutions are unstable. At a unique intermediate value λ^*, the branch contains a flat positive solution satisfying $u'(\pm 1) = 0$; see [15, 16]. Interestingly, for $m \equiv -1$ and $N \geq 3$, stable compactly supported solutions may also occur; see [18].

For $N > 1$, only partial results seem to be available. In [17], for $m \leq 0$ and star-shaped domains, we derived through a Pohozaev-type identity a necessary condition for solutions satisfying $\partial u/\partial n = 0$ on $\partial\Omega$; see also [18]. We now obtain sufficient conditions for compact support, using the local supersolution method of [12]. The following result extends [17, Theorem 5.1].

For $q_0 \in (0, \|m^-\|_{L^\infty(\Omega)}]$, where $m^-(x) := \max\{-m(x), 0\}$, set

$$\Omega^-_{q_0} := \{x \in \Omega : m^-(x) \geq q_0\}.$$

For a non-negative solution u_λ of (1) and $M > 0$, we also write

$$[0 \leq u_\lambda \leq M] := \{x \in \Omega : 0 \leq u_\lambda(x) \leq M\}.$$

Since $0 < \alpha < 1$, the function $f_{\lambda,q_0}(s) := q_0 s^\alpha - \lambda s$ is non-decreasing on $[0, \delta_{\lambda,q_0}]$, where

$$\delta_{\lambda,q_0} := \left(\frac{\alpha q_0}{\lambda}\right)^{\frac{1}{1-\alpha}} \quad \text{if } \lambda > 0, \qquad \delta_{\lambda,q_0} := +\infty \quad \text{if } \lambda \leq 0. \tag{53}$$

Indeed, $f'_{\lambda,q_0}(s) = \alpha q_0 s^{\alpha-1} - \lambda \geq 0$ for $0 < s \leq \delta_{\lambda,q_0}$.

For $\mu > 0$, define

$$\psi_{\mu,q_0}(\tau) := \frac{1}{\sqrt{2\mu}} \int_0^\tau \frac{ds}{\sqrt{\dfrac{q_0}{\alpha+1} s^{\alpha+1} - \dfrac{\lambda}{2} s^2}}, \qquad 0 \leq \tau < \delta_{\lambda,q_0}.$$

The integrand is positive on $(0, \delta_{\lambda,q_0})$, and its singularity at 0 is integrable. Hence ψ_{μ,q_0} is well defined, continuous, and strictly increasing.

We shall use the following local dead-core criterion.

Theorem 6. *Let $u_\lambda \in H^1_0(\Omega) \cap C(\overline{\Omega})$ be a non-negative weak solution of (1), and let $G \subset \Omega$ be open. Assume*

$$-m(x) \geq q_0 \qquad \text{for a.e. } x \in G \tag{54}$$

and

$$0 \leq u_\lambda(x) \leq \delta_{\lambda,q_0} \qquad \text{for every } x \in G. \tag{55}$$

If $\lambda > 0$, assume additionally that $\|u_\lambda\|_{L^\infty(\Omega)} < \delta_{\lambda,q_0}$.

For $x_0 \in G$, set

$$d_G(x_0) := \operatorname{dist}(x_0, \partial G \setminus \partial\Omega),$$

with $d_G(x_0) = +\infty$ if $\partial G \setminus \partial\Omega = \emptyset$. If

$$d_G(x_0) \geq \psi_{1/N,q_0}\big(\|u_\lambda\|_{L^\infty(\Omega)}\big), \tag{56}$$

then $u_\lambda(x_0) = 0$. Consequently,

$$\{u_\lambda > 0\} \cap G \subset \left\{x \in G : d_G(x) < \psi_{1/N,q_0}\big(\|u_\lambda\|_{L^\infty(\Omega)}\big)\right\},$$

and therefore

$$\operatorname{supp} u_\lambda \cap G \subset \left\{x \in G : d_G(x) \leq \psi_{1/N,q_0}\big(\|u_\lambda\|_{L^\infty(\Omega)}\big)\right\}.$$

Proof. Set
$$R := \psi_{1/N,q_0}\big(\|u_\lambda\|_{L^\infty(\Omega)}\big).$$
If $R = 0$, the conclusion is immediate. Assume $R > 0$. By (56), $D_R(x_0) := B_R(x_0) \cap \Omega \subset G$. Let $\eta_{1/N,q_0} := \psi^{-1}_{1/N,q_0}$ and define

$$U(x) := \eta_{1/N,q_0}(|x - x_0|), \qquad x \in D_R(x_0).$$

Then $U(x_0) = 0$, $0 \le U \le \|u_\lambda\|_{L^\infty(\Omega)}$ in $D_R(x_0)$, and $U = \|u_\lambda\|_{L^\infty(\Omega)}$ on $\partial B_R(x_0) \cap \Omega$. By the local barrier construction in [12, Theorem 1.5],

$$-\Delta U + q_0 U^\alpha - \lambda U \ge 0 \qquad \text{in } D_R(x_0). \tag{57}$$

On the other hand, (1) and (54) give

$$-\Delta u_\lambda + q_0 u_\lambda^\alpha - \lambda u_\lambda = (m(x) + q_0) u_\lambda^\alpha \le 0 \qquad \text{in } D_R(x_0).$$

Moreover, $u_\lambda \le U$ on $\partial D_R(x_0)$: this follows from the definition of U on the spherical part and from the homogeneous Dirichlet condition on $\partial\Omega$.

Since $s \mapsto q_0 s^\alpha - \lambda s$ is non-decreasing on $[0, \delta_{\lambda,q_0}]$, the comparison principle gives $0 \le u_\lambda \le U$ in $D_R(x_0)$. Hence $0 \le u_\lambda(x_0) \le U(x_0) = 0$, proving the assertion. □

We can now prove Theorem 2.

Proof of Theorem 2 Choose $q_0 \in (0, m^-_{\rho_0})$. By (6), $-m(x) \ge q_0$ a.e. in $\mathcal{C}_{\rho_0}$. Set

$$\rho := \frac{\rho_0}{4}, \qquad G := \mathcal{C}_{\rho_0/2} = \left\{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \frac{\rho_0}{2}\right\}.$$

Then $G \subset \mathcal{C}_{\rho_0} \subset \Omega^-_{q_0}$ and, for every $x_0 \in \mathcal{C}_\rho$,

$$\operatorname{dist}(x_0, \partial G \setminus \partial\Omega) \ge \frac{\rho_0}{4}. \tag{58}$$

By Theorem 5(iii) and Corollary 4,

$$\|u_\lambda^-\|_{L^\infty(\Omega)} \longrightarrow 0 \qquad \text{as } \lambda \uparrow \lambda_{1,\Omega^0}.$$

Hence, for λ sufficiently close to λ_{1,Ω^0}, $\|u_\lambda^-\|_\infty < \delta_{\lambda,q_0}$ and

$$\psi_{1/N,q_0}\big(\|u_\lambda^-\|_{L^\infty(\Omega)}\big) < \frac{\rho_0}{4}. \tag{59}$$

Thus there exists
$$0 < \varepsilon < \lambda_{1,\Omega^0} - \lambda_{1,\Omega}$$
such that these inequalities hold whenever $\lambda \in (\lambda_{1,\Omega^0} - \varepsilon, \lambda_{1,\Omega^0})$. For such λ and $x_0 \in \mathcal{C}_\rho$, (58) and (59) allow us to apply Theorem 6, yielding $u_\lambda^-(x_0) = 0$. Therefore

$$u_\lambda^- = 0 \quad \text{in } \mathcal{C}_\rho, \qquad \operatorname{supp} u_\lambda^- \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \rho\} \Subset \Omega.$$

□

As announced in Remark 13 of [20], we next give a useful L^∞ estimate.

Lemma 6. *Assume that $0 < \alpha < 1$, $m \in L^\infty(\Omega)$, and $\lambda < 0$. If u_λ is a non-negative weak solution of (1), then*

$$\|u_\lambda\|_{L^\infty(\Omega)} \le \left(\frac{\|m^+\|_{L^\infty(\Omega)}}{|\lambda|}\right)^{\frac{1}{1-\alpha}}. \tag{60}$$

In particular, $\|u_\lambda\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to -\infty$.

Proof. Set

$$K_\lambda := \left(\frac{\|m^+\|_{L^\infty(\Omega)}}{|\lambda|} \right)^{\frac{1}{1-\alpha}}$$

and let $w := (u_\lambda - K_\lambda)^+ \in H_0^1(\Omega)$. Testing (1) by w gives

$$\int_{\{u_\lambda > K_\lambda\}} |\nabla w|^2 \, dx \le \int_{\{u_\lambda > K_\lambda\}} \big(-|\lambda| u_\lambda + \|m^+\|_\infty u_\lambda^\alpha\big) w \, dx.$$

On $\{u_\lambda > K_\lambda\}$,

$$u_\lambda^{1-\alpha} > K_\lambda^{1-\alpha} = \frac{\|m^+\|_\infty}{|\lambda|},$$

so the integrand on the right is negative. Hence $\int |\nabla w|^2 \le 0$, and therefore $w = 0$. Thus $u_\lambda \le K_\lambda$ a.e. in Ω, proving (60). □

We can now complete the proof of Theorem 3.

Proof of Theorem 3 Choose $q_0 \in (0, m_\partial^-)$. By (7) and (8), $-m(x) \ge q_0$ a.e. in $\mathcal{C}_{\rho_0}$. Set

$$\rho := \frac{\rho_0}{4}, \qquad G := \mathcal{C}_{\rho_0/2}.$$

Then $G \subset \mathcal{C}_{\rho_0} \subset \Omega_{q_0}^-$ and, for every $x_0 \in \mathcal{C}_\rho$,

$$\operatorname{dist}(x_0, \partial G \setminus \partial\Omega) \ge \frac{\rho_0}{4}. \tag{61}$$

By Lemma 6, $\|u_\lambda^+\|_{L^\infty(\Omega)} \to 0$ as $\lambda \to -\infty$. Therefore

$$\psi_{1/N, q_0}\big(\|u_\lambda^+\|_{L^\infty(\Omega)}\big) \longrightarrow 0.$$

Hence there exists $\sigma < 0$ such that, for every $\lambda < \sigma$,

$$\psi_{1/N, q_0}\big(\|u_\lambda^+\|_{L^\infty(\Omega)}\big) < \frac{\rho_0}{4}. \tag{62}$$

Combining (61) and (62), Theorem 6 gives $u_\lambda^+ = 0$ in $\mathcal{C}_\rho$. Consequently,

$$\operatorname{supp} u_\lambda^+ \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \rho\} \Subset \Omega,$$

so u_λ^+ has compact support in Ω. □

Corollary 5. *Assume that $0 < \alpha < 1$, $m \le 0$ a.e. in Ω, $|\Omega^0| > 0$, and $m \in L^\infty(\Omega)$. Suppose that there exist $\rho_0 > 0$ and $m_{\rho_0}^-$ such that*

$$\mathcal{C}_{\rho_0} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho_0\} \subset \Omega^-, \qquad -m(x) \ge m_{\rho_0}^- \quad \text{a.e. in } \mathcal{C}_{\rho_0},$$

and assume that $\lambda_{1,\Omega} < \lambda_{1,\Omega^0}$. Then there exist $\varepsilon > 0$ and $\rho \in (0, \rho_0)$ such that, for every $\lambda \in (\lambda_{1,\Omega^0} - \varepsilon, \lambda_{1,\Omega^0})$,

$$u_\lambda^- = 0 \quad \text{in } \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho\}.$$

In particular,

$$\operatorname{supp} u_\lambda^- \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \ge \rho\} \Subset \Omega.$$

Proof. This is an immediate consequence of Theorem 2. □

The following result concerns sign-changing weights. In this case, compact support requires, in addition, a negative boundary collar and a suitable smallness condition on the solution.

Corollary 6. *Assume that* $0 < \alpha < 1$, $m \in L^\infty(\Omega)$, $|\Omega^0| > 0$, *and*

$$\int_\Omega m(x)\varphi_{1,\Omega}^{\alpha+1}\,dx < 0.$$

Suppose that there exist $\rho_0 > 0$ *and* $m_{\rho_0}^- > 0$ *such that*

$$\mathcal{C}_{\rho_0} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < \rho_0\} \subset \{m < 0\}, \qquad -m(x) \geq m_{\rho_0}^- \quad \text{a.e. in } \mathcal{C}_{\rho_0}.$$

Assume, in addition, that for some $\lambda_c \in (\lambda_{1,\Omega}, \lambda_{1,\Omega^0})$, *equation* (1) *has a non-negative weak solution* $u_{\lambda_c}^-$ *satisfying* $E''_{\lambda_c}(u_{\lambda_c}^-) < 0$ *and, for some* $q_0 \in (0, m_\rho^-)$,

$$\psi_{1/N,q_0}\big(\|u_{\lambda_c}^-\|_{L^\infty(\Omega)}\big) < \frac{\rho_0}{4}, \qquad \|u_{\lambda_c}^-\|_{L^\infty(\Omega)} < \delta_{\lambda_c,q_0}.$$

Then $u_{\lambda_c}^-$ *has compact support in* Ω.

Proof. Set

$$G := \mathcal{C}_{\rho_0/2}, \qquad \rho := \frac{\rho_0}{4}.$$

Since $q_0 < m_{\rho_0}^-$, we have $-m(x) \geq q_0$ a.e. in G, and for every $x_0 \in \mathcal{C}_\rho$,

$$\operatorname{dist}(x_0, \partial G \setminus \partial\Omega) \geq \frac{\rho_0}{4}.$$

The remaining assumptions of Theorem 6 follow from the hypotheses. Hence $u_{\lambda_c}^- = 0$ in $\mathcal{C}_\rho$, and therefore

$$\operatorname{supp} u_{\lambda_c}^- \subset \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq \rho\} \Subset \Omega.$$

□

6.1 A one-dimensional threshold for the formation of compact support

The preceding results suggest a qualitative connection between the bifurcation diagram for the strictly negative weight $m \equiv -1$ and the diagrams corresponding to non-positive weights with a nontrivial zero region. This connection is especially transparent in the one-dimensional setting. Let $\Omega = (-1, 1)$ and for $n \geq 2$, consider

$$m_n(x) = \begin{cases} 0, & |x| < 1/n, \\ -1, & 1/n \leq |x| < 1. \end{cases}$$

We denote

$$\Omega_n^0 := \left(-\frac{1}{n}, \frac{1}{n}\right).$$

For later use, we also set

$$M_n(u) := \int_{-1}^{1} m_n(x)|u|^{\alpha+1}\,dx.$$

Then

$$\lambda_n^* = \lambda_{1,\Omega_n^0} = \left(\frac{n\pi}{2}\right)^2.$$

Theorem 7 (Explicit threshold, compact-support branch and onset amplitude)**.** *Let* $0 < \alpha < 1$ *and* $n \geq 2$. *Consider the problem*

$$\begin{cases} -u'' = \lambda u + m_n(x)u^\alpha, & -1 < x < 1, \\ u(-1) = u(1) = 0. \end{cases}$$

Define

$$\lambda_{c,n} := \frac{\pi^2}{\left((1-\alpha) + \dfrac{1+\alpha}{n}\right)^2}.$$

Then

$$\frac{\pi^2}{4} < \lambda_{c,n} < \lambda_n^*.$$

At $\lambda = \lambda_{c,n}$ *there exists an even positive flat solution* $u_{c,n}$ *satisfying* $u_{c,n}(\pm 1) = u'_{c,n}(\pm 1) = 0$. *For every* $\lambda \in (\lambda_{c,n}, \lambda_n^*)$ *there exists a unique even non-negative solution in the above class whose connected support is*

$$\operatorname{supp} u_{\lambda,n} = [-b_{\lambda,n}, b_{\lambda,n}] \Subset (-1,1),$$

where

$$b_{\lambda,n} = \frac{1}{1-\alpha}\left(\frac{\pi}{\sqrt{\lambda}} - \frac{1+\alpha}{n}\right).$$

For $x \geq 0$ *the solution is*

$$u_{\lambda,n}(x) = \begin{cases} A_{\lambda,n}\cos(\sqrt{\lambda}\,x), & 0 \leq x \leq \dfrac{1}{n}, \\ \left(\dfrac{2}{(1+\alpha)\lambda}\right)^{\frac{1}{1-\alpha}} \sin^{\frac{2}{1-\alpha}}\left[\dfrac{1-\alpha}{2}\sqrt{\lambda}\,(b_{\lambda,n} - x)\right], & \dfrac{1}{n} \leq x \leq b_{\lambda,n}, \\ 0, & b_{\lambda,n} \leq x \leq 1, \end{cases}$$

and it is extended evenly to $(-1,0)$. *Moreover,*

$$A_{\lambda,n} = \|u_{\lambda,n}\|_{L^\infty(-1,1)} = \left(\frac{2}{(1+\alpha)\lambda}\right)^{\frac{1}{1-\alpha}} \cos^{\frac{1+\alpha}{1-\alpha}}\left(\frac{\sqrt{\lambda}}{n}\right).$$

For each fixed n, *the mapping* $\lambda \longmapsto A_{\lambda,n}$ *is strictly decreasing on* $(\lambda_{c,n}, \lambda_n^*)$, *and* $A_{\lambda,n} \longrightarrow 0$ *as* $\lambda \uparrow \lambda_n^*$. *If*

$$A_n := \|u_{c,n}\|_{L^\infty(-1,1)},$$

then

$$A_n = \left[\frac{2}{(1+\alpha)\pi^2}\left((1-\alpha) + \frac{1+\alpha}{n}\right)^2\right]^{\frac{1}{1-\alpha}} \cos^{\frac{1+\alpha}{1-\alpha}}\left(\frac{\pi}{n(1-\alpha) + (1+\alpha)}\right),$$

and

$$\lambda_{c,n} \uparrow \lambda_{c,\infty} := \frac{\pi^2}{(1-\alpha)^2},$$

while

$$A_n \longrightarrow A_\infty := \left[\frac{2(1-\alpha)^2}{(1+\alpha)\pi^2}\right]^{\frac{1}{1-\alpha}} > 0.$$

Furthermore,

$$\frac{\lambda_{c,n}}{\lambda_n^*} \longrightarrow 0.$$

Moreover, $H_\lambda(u_{\lambda,n}) = M_n(u_{\lambda,n}) < 0$, *and therefore* $E_\lambda(u_{\lambda,n}) > 0$, $E''_\lambda(u_{\lambda,n}) < 0$. *Thus, these explicit solutions have the variational character of the branch* $u^-_{\lambda,n}$.

Proof. Set $a_n := \frac{1}{n}$. We work on $[0,1]$ and use even reflection. Assume that the positivity set is $[0,b)$, with $a_n < b \leq 1$, and that the solution is flat at the free boundary: $u(b) = u'(b) = 0$. On (a_n, b) one has $m_n = -1$, and therefore

$$-u'' + u^\alpha = \lambda u.$$

Multiplying by u' and integrating from x to b gives

$$\frac{1}{2}(u')^2 = \frac{1}{1+\alpha}u^{1+\alpha} - \frac{\lambda}{2}u^2.$$

Hence

$$(u')^2 = \frac{2}{1+\alpha}u^{1+\alpha} - \lambda u^2.$$

Since $u' < 0$ in (a_n, b),

$$-u' = u^{(1+\alpha)/2}\left(\frac{2}{1+\alpha} - \lambda u^{1-\alpha}\right)^{1/2}.$$

Therefore,

$$b - x = \int_0^{u(x)} \frac{ds}{s^{(1+\alpha)/2}\sqrt{\frac{2}{1+\alpha} - \lambda s^{1-\alpha}}}.$$

With

$$y = s^{(1-\alpha)/2}, \qquad s^{-(1+\alpha)/2}\,ds = \frac{2}{1-\alpha}\,dy,$$

we obtain

$$b - x = \frac{2}{(1-\alpha)\sqrt{\lambda}}\arcsin\left[\sqrt{\frac{(1+\alpha)\lambda}{2}}\,u(x)^{(1-\alpha)/2}\right].$$

Equivalently,

$$u(x) = \left(\frac{2}{(1+\alpha)\lambda}\right)^{\frac{1}{1-\alpha}}\sin^{\frac{2}{1-\alpha}}\left[\frac{1-\alpha}{2}\sqrt{\lambda}\,(b-x)\right].$$

Thus

$$u(x) = C_\lambda \sin^p[\kappa_\lambda(b-x)],$$

where

$$C_\lambda = \left(\frac{2}{(1+\alpha)\lambda}\right)^{1/(1-\alpha)}, \qquad p = \frac{2}{1-\alpha}, \qquad \kappa_\lambda = \frac{1-\alpha}{2}\sqrt{\lambda}.$$

On $(0, a_n)$, where $m_n = 0$, $-u'' = \lambda u$. The symmetry condition $u'(0) = 0$ yields

$$u(x) = A\cos(\sqrt{\lambda}\,x).$$

We now explain carefully the matching at $x = a_n$. Since the coefficient m_n is discontinuous there, the two formulas above are obtained independently on the two subintervals. In order that their union define a global weak solution, no Dirac mass may appear in $-u''$ at $x = a_n$. Thus both u and u' must be continuous at the interface

$$u_{\text{in}}(a_n) = u_{\text{out}}(a_n), \qquad u'_{\text{in}}(a_n) = u'_{\text{out}}(a_n).$$

Instead of solving these two equations simultaneously, it is more convenient to divide the second by the first. This eliminates the unknown amplitudes and gives equality of the logarithmic derivatives

$$\frac{u'_{\text{in}}(a_n)}{u_{\text{in}}(a_n)} = \frac{u'_{\text{out}}(a_n)}{u_{\text{out}}(a_n)}.$$

For the inner profile,

$$\frac{u'_{\text{in}}(a_n)}{u_{\text{in}}(a_n)} = -\sqrt{\lambda}\tan(\sqrt{\lambda}\,a_n).$$

For the outer profile,

$$\frac{u'_{\text{out}}(a_n)}{u_{\text{out}}(a_n)} = -p\kappa_\lambda\cot\left[\kappa_\lambda(b-a_n)\right].$$

Since $p\kappa_\lambda = \sqrt{\lambda}$, the matching condition becomes

$$\cot\left[\kappa_\lambda(b - a_n)\right] = \tan(\sqrt{\lambda}\, a_n).$$

Along the first positive branch the two arguments lie in the relevant principal interval, and hence

$$\kappa_\lambda(b - a_n) + \sqrt{\lambda}\, a_n = \frac{\pi}{2}.$$

Therefore

$$b = \frac{1}{1-\alpha}\left(\frac{\pi}{\sqrt{\lambda}} - \frac{1+\alpha}{n}\right).$$

The boundary dead core is born when $b = 1$, and this yields

$$\lambda_{c,n} = \frac{\pi^2}{\left((1-\alpha) + \dfrac{1+\alpha}{n}\right)^2}.$$

Moreover,

$$b < 1 \quad\Longleftrightarrow\quad \lambda > \lambda_{c,n},$$

while

$$b > \frac{1}{n} \quad\Longleftrightarrow\quad \lambda < \lambda_n^*.$$

Thus, compact support occurs precisely for $\lambda \in (\lambda_{c,n}, \lambda_n^*)$. To compute the amplitude, use again the phase relation

$$\kappa_\lambda(b - a_n) = \frac{\pi}{2} - \frac{\sqrt{\lambda}}{n}.$$

Hence

$$\sin\left[\kappa_\lambda(b - a_n)\right] = \cos\left(\frac{\sqrt{\lambda}}{n}\right).$$

The continuity of u at $x = a_n$ gives

$$A_{\lambda,n}\cos\left(\frac{\sqrt{\lambda}}{n}\right) = C_\lambda \cos^{2/(1-\alpha)}\left(\frac{\sqrt{\lambda}}{n}\right).$$

Since $\sqrt{\lambda}/n < \pi/2$,

$$A_{\lambda,n} = C_\lambda \cos^{(1+\alpha)/(1-\alpha)}\left(\frac{\sqrt{\lambda}}{n}\right).$$

For fixed n,

$$\log A_{\lambda,n} = -\frac{1}{1-\alpha}\log\lambda + \frac{1+\alpha}{1-\alpha}\log\cos\left(\frac{\sqrt{\lambda}}{n}\right) + \text{constant},$$

so

$$\frac{d}{d\lambda}\log A_{\lambda,n} = -\frac{1}{(1-\alpha)\lambda} - \frac{1+\alpha}{1-\alpha}\,\frac{\tan(\sqrt{\lambda}/n)}{2n\sqrt{\lambda}} < 0.$$

Thus $\lambda \mapsto A_{\lambda,n}$ is strictly decreasing. Since

$$\frac{\sqrt{\lambda}}{n} \uparrow \frac{\pi}{2} \quad \text{as } \lambda \uparrow \lambda_n^*,$$

one has $A_{\lambda,n} \to 0$. Taking $\lambda = \lambda_{c,n}$ gives the formula for A_n. Since

$$\lambda_{c,n} \uparrow \frac{\pi^2}{(1-\alpha)^2}$$

and

$$\frac{\sqrt{\lambda_{c,n}}}{n} = \frac{\pi}{n(1-\alpha)+(1+\alpha)} \longrightarrow 0,$$

we obtain

$$A_n \to A_\infty = \left[\frac{2(1-\alpha)^2}{(1+\alpha)\pi^2}\right]^{1/(1-\alpha)}.$$

Furthermore,

$$\frac{\lambda_{c,n}}{\lambda_n^*} = \frac{4}{\left(n(1-\alpha)+(1+\alpha)\right)^2} \longrightarrow 0 \qquad \text{as } n \to \infty.$$

Finally, testing the equation by $u_{\lambda,n}$ gives

$$H_\lambda(u_{\lambda,n}) = M_n(u_{\lambda,n}).$$

Since $m_n = -1$ on a set where $u_{\lambda,n} > 0$, $M_n(u_{\lambda,n}) < 0$. The usual Nehari identities therefore yield

$$E_\lambda(u_{\lambda,n}) > 0, \qquad E''_\lambda(u_{\lambda,n}) < 0.$$

□

Remark 5. *The limiting values*

$$\lambda_{c,\infty} = \frac{\pi^2}{(1-\alpha)^2}, \qquad A_\infty = \left[\frac{2(1-\alpha)^2}{(1+\alpha)\pi^2}\right]^{1/(1-\alpha)}$$

coincide with the one-dimensional flat-solution constants obtained in [13], after the identifications $R = 1, \quad V_0 = 1, \quad m = \alpha$. *Thus, at the level of the critical parameter and the onset amplitude, the limit* $n \to \infty$ *recovers exactly the flat-solution constants of the strictly negative problem* $m \equiv -1$.

It is important to get some additional information about the amplitudes A_n. We have

Corollary 7 (Comparison of the onset amplitudes for $n \geq 3$)**.** *Let* $0 < \alpha < 1$, *and let* A_n *denote the* L^∞*-amplitude of the flat positive solution at the threshold* $\lambda = \lambda_{c,n}$, *as in the preceding theorem. Then,* $A_n < A_\infty$ *if and only if*

$$\left(1 + \frac{1+\alpha}{n(1-\alpha)}\right)^2 \cos^{1+\alpha}\left(\frac{\pi}{n(1-\alpha)+(1+\alpha)}\right) < 1. \tag{63}$$

In particular, $A_n > A_\infty$ *for every integer* $n \geq 3$.

Proof. To compare A_n with A_∞, divide the two explicit formulas and raise to the power $1-\alpha$. This gives

$$\left(\frac{A_n}{A_\infty}\right)^{1-\alpha} = \left(1 + \frac{1+\alpha}{n(1-\alpha)}\right)^2 \cos^{1+\alpha}\left(\frac{\pi}{n(1-\alpha)+(1+\alpha)}\right).$$

Therefore $A_n < A_\infty$ if and only if (63) holds. This also shows that the convergence of A_n towards A_∞ is not forced to be monotone: the algebraic prefactor and the cosine factor have opposite effects as n varies. From the above explicit formula

$$R_n := \left(\frac{A_n}{A_\infty}\right)^{1-\alpha} = \left(1 + \frac{1+\alpha}{n(1-\alpha)}\right)^2 \cos^{1+\alpha}\left(\frac{\pi}{n(1-\alpha)+(1+\alpha)}\right).$$

Clearly,

$$R_n \longrightarrow 1 \qquad \text{as } n \to \infty.$$

We shall prove that $R_n > 1$ for every $n \geq 3$. Put

$$\varepsilon := 1 - \alpha \in (0,1), \qquad 1 + \alpha = 2 - \varepsilon,$$

and regard n temporarily as a real variable $s \geq 3$. Define

$$D(s) := s\varepsilon + 2 - \varepsilon, \qquad \theta(s) := \frac{\pi}{D(s)}.$$

Then

$$R(s) = \left(\frac{D(s)}{s\varepsilon}\right)^2 \cos^{2-\varepsilon}\theta(s).$$

A direct logarithmic differentiation gives

$$\frac{R'(s)}{R(s)} = \frac{2-\varepsilon}{D(s)}\left[-\frac{2}{s} + \varepsilon\theta(s)\tan\theta(s)\right].$$

Since

$$s\varepsilon = \frac{\pi}{\theta} - (2-\varepsilon),$$

the sign of $R'(s)$ is the sign of

$$G(\theta(s)) - 2, \qquad G(\theta) := [\pi - (2-\varepsilon)\theta]\tan\theta.$$

We claim that G is strictly increasing on $(0, \pi/2)$. Indeed,

$$G'(\theta) = -(2-\varepsilon)\tan\theta + [\pi - (2-\varepsilon)\theta]\sec^2\theta,$$

and therefore

$$\cos^2\theta\, G'(\theta) = \pi - (2-\varepsilon)\big(\theta + \sin\theta\cos\theta\big).$$

The function

$$h(\theta) := \theta + \sin\theta\cos\theta$$

is strictly increasing on $(0, \pi/2)$, because

$$h'(\theta) = 2\cos^2\theta > 0,$$

and

$$h(\theta) < h(\pi/2) = \frac{\pi}{2}.$$

Consequently,

$$\cos^2\theta\, G'(\theta) > \pi - (2-\varepsilon)\frac{\pi}{2} = \frac{\pi\varepsilon}{2} > 0,$$

which proves the claim. Now $\theta(s)$ is strictly decreasing in s. Since G is strictly increasing, the function $G(\theta(s)) - 2$ is strictly decreasing. Hence $R'(s)$ can change sign at most once, and, if it does, it changes from positive to negative. Thus R is either decreasing on $[3, \infty)$, or first increasing and then decreasing. In particular, R has no local minimum on $[3, \infty)$. Assume that for the case $n = 3$ we have

$$R(3) = \left(\frac{A_3}{A_\infty}\right)^{1-\alpha} > 1.$$

Together with $\lim_{s\to\infty} R(s) = 1$, the preceding implies $R(s) > 1$ for every finite $s \geq 3$. Indeed, if R is decreasing, it decreases from $R(3) > 1$ to its limit 1; if it first increases and then decreases, its decreasing part again approaches 1 from above. Therefore, for every integer $n \geq 3$,

$$\left(\frac{A_n}{A_\infty}\right)^{1-\alpha} > 1.$$

Thus, we only need to prove that $R(3) > 1$. We recall that

$$\left(\frac{A_3}{A_\infty}\right)^{1-\alpha} = \left[\frac{2(2-\alpha)}{3(1-\alpha)}\right]^2 \cos^{1+\alpha}\left(\frac{\pi}{2(2-\alpha)}\right). \tag{64}$$

Put $\varepsilon := 1-\alpha \in (0,1)$. Then (64) can be rewritten as

$$\left(\frac{A_3}{A_\infty}\right)^{1-\alpha} = \left[\frac{2(1+\varepsilon)}{3\varepsilon}\right]^2 \sin^{2-\varepsilon}\left(\frac{\pi\varepsilon}{2(1+\varepsilon)}\right). \tag{65}$$

Indeed,

$$\cos\left(\frac{\pi}{2(1+\varepsilon)}\right) = \sin\left(\frac{\pi\varepsilon}{2(1+\varepsilon)}\right).$$

Introduce

$$y := \frac{\pi\varepsilon}{2(1+\varepsilon)}.$$

Since $0<\varepsilon<1$,

$$0 < y < \frac{\pi}{4},$$

and

$$\frac{2(1+\varepsilon)}{\varepsilon} = \frac{\pi}{y}.$$

Hence (65) takes the particularly simple form

$$\left(\frac{A_3}{A_\infty}\right)^{1-\alpha} = \frac{\pi^2}{9y^2}\sin^{2-\varepsilon} y. \tag{66}$$

We claim that

$$\sin^{2-\varepsilon} y > y^2. \tag{67}$$

Indeed,

$$\sin y = y\,\frac{\sin y}{y},$$

so (67) is equivalent to

$$\left(\frac{\sin y}{y}\right)^{2-\varepsilon} y^{-\varepsilon} > 1.$$

Taking logarithms, it is enough to prove

$$(2-\varepsilon)\log\frac{\sin y}{y} + \varepsilon\log\frac{1}{y} > 0. \tag{68}$$

From

$$y = \frac{\pi\varepsilon}{2(1+\varepsilon)}$$

we obtain

$$\frac{\varepsilon}{2-\varepsilon} = \frac{y}{\pi - 3y}. \tag{69}$$

On the other hand, for $0 < y \le \pi/4$,

$$\frac{\sin y}{y} \ge 1 - \frac{y^2}{6}.$$

Since $y^2/6 < 1/2$, we have

$$-\log\frac{\sin y}{y} \le -\log\left(1-\frac{y^2}{6}\right) \le \frac{y^2}{3}. \tag{70}$$

We next observe that

$$\frac{y^2}{3} < \frac{y}{\pi-3y}\log\frac{1}{y}, \qquad 0 < y \le \frac{\pi}{4}. \tag{71}$$

Indeed, (71) is equivalent to

$$\log\frac{1}{y} > \frac{y(\pi-3y)}{3}.$$

Define

$$q(y) := \log \frac{1}{y} - \frac{\pi y}{3} + y^2.$$

Then

$$q'(y) = -\frac{1}{y} - \frac{\pi}{3} + 2y < 0 \qquad \text{for } 0 < y \leq \frac{\pi}{4}.$$

Therefore q is decreasing on this interval, and

$$q(y) \geq q\left(\frac{\pi}{4}\right) = \log \frac{4}{\pi} - \frac{\pi^2}{48} > 0.$$

This proves (71). Combining (70), (71), and (69), we obtain

$$-\log \frac{\sin y}{y} < \frac{\varepsilon}{2-\varepsilon} \log \frac{1}{y}.$$

Multiplication by $2-\varepsilon$ gives precisely

$$(2-\varepsilon) \log \frac{\sin y}{y} + \varepsilon \log \frac{1}{y} > 0.$$

Thus (67) holds. Returning to (66), we conclude that

$$\left(\frac{A_3}{A_\infty}\right)^{1-\alpha} > \frac{\pi^2}{9}.$$

Since $\frac{\pi^2}{9} > 1$ and $1-\alpha > 0$, it follows that

$$A_3 > A_\infty \qquad \text{for every } 0 < \alpha < 1.$$

□

Remark 6. *In fact, the preceding argument gives the stronger estimate*

$$\frac{A_3}{A_\infty} > \left(\frac{\pi^2}{9}\right)^{\frac{1}{1-\alpha}}.$$

Notice that the limiting case $\alpha \uparrow 1$ is delicate, since the two factors in (64) compensate each other. The restriction $n \geq 3$ is essential for this simple uniform statement. The case $n = 2$ is exceptional: depending on α, the onset amplitude A_2 may lie either above or below A_∞.

Remark 7 (Why the degenerate profile may lie below the profile for $m \equiv -1$)**.** *At first sight, the inequality between the amplitudes may seem counter-intuitive. Indeed, since $m_n(x) \geq -1$, one might expect the solution corresponding to the degenerate weight m_n to be larger than the solution for the strictly negative weight $m \equiv -1$: in the central region $\Omega_n^0 = (-1/n, 1/n)$ the absorption term has disappeared completely. This intuition would in fact be natural for a problem with a prescribed positive source. For instance, consider*

$$-u'' + a(x)u^\alpha = f(x), \qquad f(x) > 0,$$

with the same boundary conditions, and compare two non-negative coefficients $a_1 \leq a_2$. Under the usual assumptions ensuring the comparison principle, the equation with the smaller coefficient has less absorption, and one expects the corresponding solution to be larger. In particular, replacing the coefficient $a \equiv 1$ by a coefficient which vanishes in a subinterval has the natural interpretation

$$\textit{less absorption} \quad \Longrightarrow \quad \textit{larger solution.}$$

The present problem is essentially different, because the right-hand side is not prescribed. In the region where $m = -1$ the equation reads

$$-u'' + u^\alpha = \lambda u.$$

Thus the term which plays the role of a source is λu itself. Changing the solution changes simultaneously the right-hand side, so the preceding comparison argument is no longer available. Equivalently, the zero-order term $s \longmapsto s^\alpha - \lambda s$ is not globally monotone on $\mathbb{R}_+$, since

$$\frac{d}{ds}\big(s^\alpha - \lambda s\big) = \alpha s^{\alpha-1} - \lambda$$

changes sign. Hence the pointwise ordering $m_n \geq -1$ does not imply the analogous ordering of the particular solutions selected on the free-boundary branch. There is, moreover, a second and more precise explanation in the present one-dimensional problem. For $m \equiv -1$, the zero-energy flat orbit satisfies

$$\frac{1}{2}(u')^2 = \frac{1}{1+\alpha}u^{1+\alpha} - \frac{\lambda}{2}u^2,$$

and its turning point has amplitude

$$A_\infty(\lambda) = \left(\frac{2}{(1+\alpha)\lambda}\right)^{1/(1-\alpha)}.$$

For the degenerate weight m_n, this nonlinear orbit is interrupted at the interface $x = 1/n$. Inside Ω_n^0 the nonlinear absorption term disappears and the even solution is necessarily

$$u(x) = A_{\lambda,n}\cos(\sqrt{\lambda}\,x).$$

The global solution is therefore not obtained merely by "removing some absorption" from the solution for $m \equiv -1$; it must satisfy the additional matching conditions at $x = 1/n$. Matching the linear core with the outer flat orbit gives

$$A_{\lambda,n} = \left(\frac{2}{(1+\alpha)\lambda}\right)^{1/(1-\alpha)} \cos^{(1+\alpha)/(1-\alpha)}\left(\frac{\sqrt{\lambda}}{n}\right).$$

Consequently, whenever $0 < \frac{\sqrt{\lambda}}{n} < \frac{\pi}{2}$, we have

$$A_{\lambda,n} = A_\infty(\lambda)\cos^{(1+\alpha)/(1-\alpha)}\left(\frac{\sqrt{\lambda}}{n}\right) < A_\infty(\lambda).$$

Thus the apparently paradoxical inequality is not a direct monotonicity effect with respect to the coefficient m. It is a global free-boundary and matching effect. The zero region forces the insertion of a linear oscillatory core, and the phase accumulated in that core produces the factor

$$\cos^{(1+\alpha)/(1-\alpha)}\left(\frac{\sqrt{\lambda}}{n}\right) < 1.$$

In phase-plane language, the nonlinear zero-energy orbit for $m = -1$ is stopped before reaching its turning point and is joined to the linear orbit generated inside Ω_n^0. The sublinear character $0 < \alpha < 1$ plays a double role. First, it is responsible for the non-Lipschitz behavior at the origin which permits flat solutions and free boundaries. Second, it amplifies the loss created by the matching, because

$$\frac{1+\alpha}{1-\alpha} > 1, \qquad \frac{1+\alpha}{1-\alpha} \longrightarrow +\infty \quad \text{as } \alpha \uparrow 1.$$

Thus even a modest phase loss in the cosine may produce a substantial decrease of the amplitude. In the present equation, the term λu reverses the naive ordering suggested by the absorption coefficient.

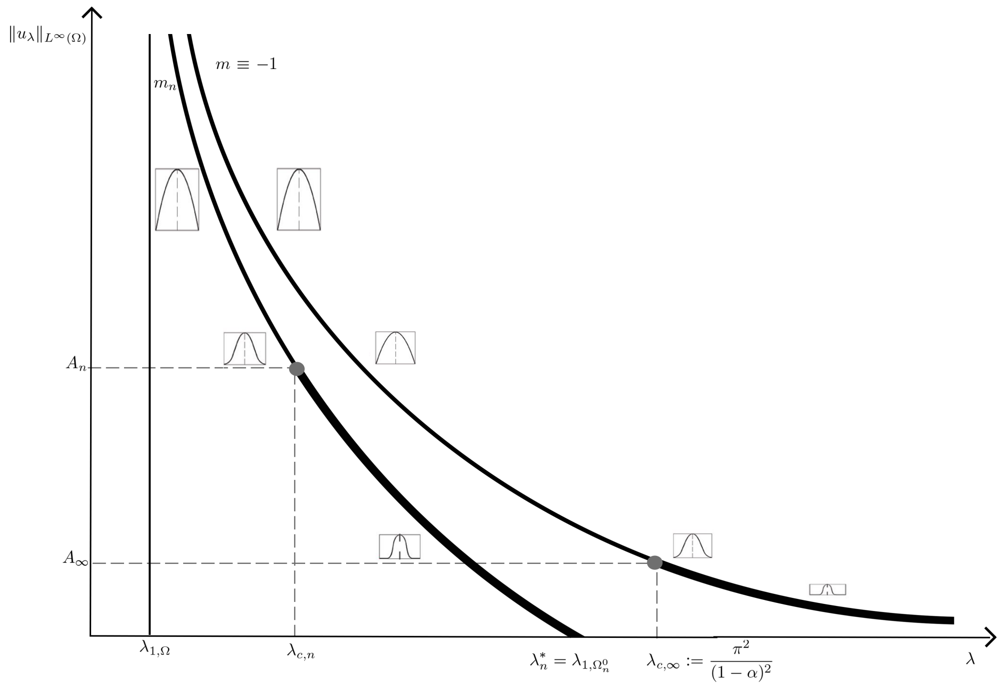

Figure 3: Qualitative behavior of the amplitude bifurcating curves for $m = -1$ and m_n with $n > 2$.

6.2 Convergence of the supports of u_λ^- and of the free boundaries as $\lambda \to \lambda^* = \lambda_{1,\Omega_0}$

We keep the notation and assumptions of Theorem 5. In particular, $0 < \alpha < 1$, $m \leq 0$ a.e. in Ω, and Ω^0 is assumed to be a connected domain compactly contained in Ω such that $|\Omega^0| > 0$. Moreover, $\lambda^* = \lambda_{1,\Omega_0}$, and u_λ^- denotes the positive-energy branch constructed for $\lambda \in (\lambda_{1,\Omega}, \lambda^*)$.

Theorem 8. *Assume the hypotheses of Theorem 5. Assume also that, for every $\varepsilon > 0$ such that*

$$G_\varepsilon := \{x \in \Omega : \operatorname{dist}(x, \Omega^0) > \varepsilon\} \neq \varnothing,$$

there exists a constant $q_\varepsilon > 0$ satisfying

$$-m(x) \geq q_\varepsilon \qquad \text{for a.e. } x \in G_\varepsilon. \tag{72}$$

Set

$$K_\lambda := \operatorname{supp} u_\lambda^-, \qquad \Gamma_\lambda := \partial K_\lambda \cap \Omega.$$

Then, as $\lambda \uparrow \lambda^* = \lambda_{1,\Omega^0}$,

$$d_H\big(K_\lambda, \overline{\Omega^0}\big) \longrightarrow 0, \tag{73}$$

and

$$d_H\big(\Gamma_\lambda, \partial\Omega^0\big) \longrightarrow 0. \tag{74}$$

Here d_H denotes the Hausdorff distance between compact subsets of $\overline{\Omega}$.

Proof. We divide the proof into three steps.

Step 1. The zero region Ω^0 is contained in the positivity set of u_λ^-. By Theorem 5,

$$\frac{u_\lambda^-}{\|u_\lambda^-\|_{H_0^1(\Omega)}} \to \widetilde{\varphi}_{1,\Omega^0} \qquad \text{strongly in } H_0^1(\Omega),$$

where $\widetilde{\varphi}_{1,\Omega^0}$ denotes the zero extension to Ω of the first Dirichlet eigenfunction of $-\Delta$ in Ω^0. In particular, for λ sufficiently close to λ^*, the restriction of u_λ^- to Ω^0 is not identically zero. Since $m = 0$ a.e. in Ω^0,

$$-\Delta u_\lambda^- = \lambda u_\lambda^- \qquad \text{in } \Omega^0.$$

Therefore, by the strong maximum principle,

$$u_\lambda^- > 0 \qquad \text{in } \Omega^0.$$

Consequently, for every λ sufficiently close to λ^*,

$$\overline{\Omega^0} \subset K_\lambda. \tag{75}$$

Step 2. The support cannot remain at a positive distance from Ω^0. Fix $\varepsilon > 0$ such that $G_\varepsilon \neq \emptyset$. By (72), there exists $q_\varepsilon > 0$ such that

$$-m(x) \geq q_\varepsilon \qquad \text{a.e. in } G_\varepsilon.$$

Moreover, by Corollary 4,

$$\|u_\lambda^-\|_{L^\infty(\Omega)} \to 0 \qquad \text{as } \lambda \uparrow \lambda^*.$$

Hence, for λ sufficiently close to λ^*, all the smallness assumptions required in the local dead-core criterion of Theorem 6 are satisfied in G_ε. Let

$$R_{\lambda,\varepsilon} := \psi_{1/N,q_\varepsilon}\left(\|u_\lambda^-\|_{L^\infty(\Omega)}\right),$$

where $\psi_{\mu,q}$ is the function introduced before Theorem 6. Since $\psi_{1/N,q_\varepsilon}(0) = 0$, we have

$$R_{\lambda,\varepsilon} \to 0 \qquad \text{as } \lambda \uparrow \lambda^*. \tag{76}$$

Theorem 6 gives

$$K_\lambda \cap G_\varepsilon \subset \left\{x \in G_\varepsilon : \operatorname{dist}\big(x, \partial G_\varepsilon \setminus \partial\Omega\big) \leq R_{\lambda,\varepsilon}\right\}.$$

Since the interior boundary of G_ε is contained in the level set

$$\{x \in \Omega : \operatorname{dist}(x, \Omega^0) = \varepsilon\},$$

it follows that

$$K_\lambda \subset \left\{x \in \Omega : \operatorname{dist}(x, \Omega^0) \leq \varepsilon + R_{\lambda,\varepsilon}\right\}. \tag{77}$$

Combining (75) and (77), we obtain

$$d_H\big(K_\lambda, \overline{\Omega^0}\big) \leq \varepsilon + R_{\lambda,\varepsilon}.$$

Taking the upper limit as $\lambda \uparrow \lambda^*$ and using (76), we get

$$\limsup_{\lambda\uparrow\lambda^*} d_H\big(K_\lambda, \overline{\Omega^0}\big) \leq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this proves (73). Moreover, since $\Omega^0 \Subset \Omega$, we may choose

$$0 < \varepsilon < \frac{1}{2}\operatorname{dist}(\overline{\Omega^0}, \partial\Omega).$$

Then, by (76) and (77), $K_\lambda \Subset \Omega$ for all λ sufficiently close to λ^*. In particular, Γ_λ is compact for such λ.

Step 3. Convergence of the free boundaries. By Step 1,

$$u_\lambda^- > 0 \qquad \text{in } \Omega^0,$$

and hence $\Gamma_\lambda \cap \Omega^0 = \emptyset$. Together with (73), this yields

$$\sup_{x\in\Gamma_\lambda} \operatorname{dist}(x, \partial\Omega^0) \to 0. \tag{78}$$

It remains to prove the reverse semidistance. Suppose, by contradiction, that there exist $\eta > 0$, a sequence $\lambda_n \uparrow \lambda^*$, and points $y_n \in \partial\Omega^0$ such that

$$\operatorname{dist}(y_n, \Gamma_{\lambda_n}) \geq \eta \qquad \text{for every } n. \tag{79}$$

Passing to a subsequence, we may assume that

$$y_n \to y \in \partial\Omega^0.$$

Choose

$$0 < r < \min\left\{\frac{\eta}{4}, \frac{1}{4}\operatorname{dist}(\overline{\Omega^0}, \partial\Omega)\right\}.$$

Since $\partial\Omega^0$ is of class $C^{1,1}$, it satisfies a uniform exterior ball condition. Hence, for all sufficiently large n, there exist points

$$z_n \in \Omega \setminus \overline{\Omega^0}, \qquad |z_n - y_n| < r,$$

such that

$$\operatorname{dist}(z_n, \Omega^0) \geq \frac{r}{2}.$$

By the already proved Hausdorff convergence (73),

$$z_n \notin K_{\lambda_n}$$

for all sufficiently large n. On the other hand, choose

$$x_n \in \Omega^0, \qquad |x_n - y_n| < r.$$

By Step 1, $x_n \in \operatorname{int} K_{\lambda_n}$. Since $z_n \notin K_{\lambda_n}$, the line segment joining x_n to z_n must meet ∂K_{λ_n}. By the choice of r, this segment is contained in Ω, and therefore it meets

$$\partial K_{\lambda_n} \cap \Omega = \Gamma_{\lambda_n}$$

at some point ξ_n. Since both x_n and z_n belong to $B_r(y_n)$ and this ball is convex,

$$|\xi_n - y_n| < r < \eta.$$

Thus

$$\operatorname{dist}(y_n, \Gamma_{\lambda_n}) < \eta,$$

contradicting (79). Therefore

$$\sup_{y\in\partial\Omega^0} \operatorname{dist}(y, \Gamma_\lambda) \longrightarrow 0.$$

Together with (78), this proves (74). □

Remark 8 (A quantitative version). *Assume, more strongly, that there exists $q_* > 0$ such that*

$$-m(x) \geq q_* \qquad \text{for a.e. } x \in \Omega \setminus \overline{\Omega^0}.$$

Then Theorem 6 may be applied directly to $G = \Omega \setminus \overline{\Omega^0}$. Hence

$$K_\lambda \setminus \overline{\Omega^0} \subset \left\{x \in \Omega \setminus \overline{\Omega^0} : dist(x, \partial\Omega^0) < R_\lambda\right\},$$

where $R_\lambda = \psi_{1/N,q_*}\left(\|u_\lambda^-\|_{L^\infty(\Omega)}\right)$. Since, as $s \downarrow 0$,

$$\psi_{1/N,q_*}(s) \sim C_{N,\alpha,q_*}\, s^{(1-\alpha)/2},$$

one obtains the quantitative localization estimate

$$\sup_{x\in K_\lambda} dist(x, \overline{\Omega^0}) \leq C_{N,\alpha,q_*} \|u_\lambda^-\|_{L^\infty(\Omega)}^{(1-\alpha)/2}$$

for λ sufficiently close to λ^*. In particular, the same rate controls the outer semidistance of the free boundary from $\partial\Omega^0$.

6.3 Localization of the support of u_λ^+ as $\lambda \to -\infty$

Theorem 9. *Let $0 < \alpha < 1$, let $\Omega \subset \mathbb{R}^N$ be bounded and connected, and let $m \in L^\infty(\Omega)$. Set $\Omega^+ = \{m > 0\}$ and assume that $\Omega^+ \Subset \Omega$ is nonempty and connected, $\partial\Omega^+ \in C^{1,1}$, and, for some $m_0, q_0 > 0$,*

$$m \geq m_0 > 0 \quad \text{a.e. in } \Omega^+, \qquad m \leq -q_0 < 0 \quad \text{a.e. in } \Omega \setminus \overline{\Omega^+}.$$

Let u_λ^+ be the non-negative solution of (1) given by Theorem 1 for $\lambda < 0$, and put

$$K_\lambda = \operatorname{supp} u_\lambda^+, \qquad \Gamma_\lambda = \partial K_\lambda \cap \Omega.$$

Then there exists $C > 0$ such that, for all sufficiently negative λ,

$$\overline{\Omega^+} \subset K_\lambda \subset \left\{ x \in \Omega : \operatorname{dist}(x, \overline{\Omega^+}) < C|\lambda|^{-1/2} \right\}. \tag{80}$$

Consequently,

$$d_H(K_\lambda, \overline{\Omega^+}) \leq C|\lambda|^{-1/2} \to 0, \tag{81}$$

and, possibly enlarging C,

$$d_H(\Gamma_\lambda, \partial\Omega^+) \leq C|\lambda|^{-1/2} \to 0. \tag{82}$$

Before presenting the proof, it is useful to obtain a comparison result for solutions to this non-monotone elliptic equation when the boundary traces are ordered.

Lemma 7. *Let $D \subset \mathbb{R}^N$ be a bounded Lipschitz domain and let $F : D \times (0, +\infty) \to \mathbb{R}$ be a Carathéodory function such that, for a.e. $x \in D$, the map*

$$s \mapsto \frac{F(x,s)}{s} \tag{83}$$

is strictly decreasing on $(0, +\infty)$. Let $U, V \in H^1(D) \cap L^\infty(D)$, with $U > 0$, $V > 0$ a.e. in D, and assume the one-sided trace condition

$$(U - V)^+ \in H_0^1(D). \tag{84}$$

Suppose that U is a weak subsolution and V a weak supersolution of

$$-\Delta w = F(x, w) \qquad \text{in } D, \tag{85}$$

that is,

$$\int_D \nabla U \cdot \nabla \varphi \, dx \leq \int_D F(x, U)\varphi \, dx, \tag{86}$$

$$\int_D \nabla V \cdot \nabla \varphi \, dx \geq \int_D F(x, V)\varphi \, dx \tag{87}$$

for every non-negative $\varphi \in H_0^1(D) \cap L^\infty(D)$ for which the right-hand sides are finite. Assume, moreover, that

$$F(\cdot, U), F(\cdot, V) \in L^1_{\mathrm{loc}}(D) \tag{88}$$

and

$$\left(\frac{|F(x,U)|}{U} + \frac{|F(x,V)|}{V} \right) (U^2 - V^2)^+ \in L^1(D). \tag{89}$$

Then

$$U \leq V \qquad \text{a.e. in } D.$$

Proof. Set $E := \{x \in D : U(x) > V(x)\}$. We will prove that $|E| = 0$. We use the pointwise algebraic identity associated with the works of Brezis and Oswald [7], Díaz and Saá [21], and, in a pioneering form and a different framework, Picone [36]

$$\begin{aligned}\nabla U \cdot \nabla \left(\frac{U^2 - V^2}{U}\right) - \nabla V \cdot \nabla \left(\frac{U^2 - V^2}{V}\right) \\ = \left|\nabla U - \frac{U}{V}\nabla V\right|^2 + \left|\nabla V - \frac{V}{U}\nabla U\right|^2 \geq 0.\end{aligned} \tag{90}$$

This formal computation is classical, so we do not reproduce its derivation. The delicate point here is that the two functions may have different boundary traces and the quotients $1/U$ and $1/V$ may become singular. We therefore give the regularization and passage to the limit. For $\varepsilon > 0$, set

$$U_\varepsilon := U + \varepsilon, \qquad V_\varepsilon := V + \varepsilon,$$

and define

$$W_\varepsilon := (U_\varepsilon^2 - V_\varepsilon^2)^+. \tag{91}$$

Since $U_\varepsilon - V_\varepsilon = U - V$,

$$\{U_\varepsilon > V_\varepsilon\} = E.$$

Moreover, on E,

$$W_\varepsilon = (U - V)(U + V + 2\varepsilon) = (U^2 - V^2) + 2\varepsilon(U - V). \tag{92}$$

Because $U, V \in H^1(D) \cap L^\infty(D)$, $U_\varepsilon^2, V_\varepsilon^2 \in H^1(D)$. Condition (84) means that the trace of U does not exceed that of V. Hence the trace of W_ε is zero, and $W_\varepsilon \in H_0^1(D)$. We may therefore introduce

$$\varphi_\varepsilon := \frac{W_\varepsilon}{U_\varepsilon}, \qquad \psi_\varepsilon := \frac{W_\varepsilon}{V_\varepsilon}. \tag{93}$$

Since $U_\varepsilon, V_\varepsilon \geq \varepsilon$, both are non-negative elements of $H_0^1(D) \cap L^\infty(D)$ and are admissible in (86)–(87). Using φ_ε in the subsolution inequality and ψ_ε in the supersolution inequality, and subtracting, we obtain

$$\begin{aligned}I_\varepsilon &:= \int_D \{\nabla U \cdot \nabla \varphi_\varepsilon - \nabla V \cdot \nabla \psi_\varepsilon\}\, dx \\ &\leq \int_D \left[\frac{F(x, U)}{U_\varepsilon} - \frac{F(x, V)}{V_\varepsilon}\right] W_\varepsilon\, dx.\end{aligned} \tag{94}$$

Since $\nabla U_\varepsilon = \nabla U$ and $\nabla V_\varepsilon = \nabla V$, identity (90), applied to $(U_\varepsilon, V_\varepsilon)$, gives on E

$$\begin{aligned}\nabla U \cdot \nabla \varphi_\varepsilon - \nabla V \cdot \nabla \psi_\varepsilon \\ = \left|\nabla U - \frac{U_\varepsilon}{V_\varepsilon}\nabla V\right|^2 + \left|\nabla V - \frac{V_\varepsilon}{U_\varepsilon}\nabla U\right|^2 \geq 0,\end{aligned} \tag{95}$$

whereas both sides vanish outside E. Hence

$$I_\varepsilon \geq 0. \tag{96}$$

We now pass to the limit $\varepsilon \downarrow 0$. For a.e. $x \in E$, since $U(x), V(x) > 0$,

$$\frac{U_\varepsilon}{V_\varepsilon} \to \frac{U}{V}, \qquad \frac{V_\varepsilon}{U_\varepsilon} \to \frac{V}{U}.$$

The integrand in (95) is non-negative. Therefore Fatou's lemma gives

$$\begin{aligned}\int_E \left[\left|\nabla U - \frac{U}{V}\nabla V\right|^2 + \left|\nabla V - \frac{V}{U}\nabla U\right|^2\right] dx \\ \leq \liminf_{\varepsilon \downarrow 0} I_\varepsilon.\end{aligned} \tag{97}$$

This is why no a priori bound on U/V or V/U is needed (in contrast to [7]). It remains to pass to the limit in the right-hand side of (94). On E, by (92),

$$W_\varepsilon = W + 2\varepsilon(U-V), \qquad W := (U^2-V^2)^+.$$

For the first term,

$$\begin{aligned}\left|\frac{F(x,U)}{U_\varepsilon}W_\varepsilon\right| &\le \frac{|F(x,U)|}{U}\,W + 2|F(x,U)|(U-V)\\ &\le 3\,\frac{|F(x,U)|}{U}\,W,\end{aligned}\tag{98}$$

because

$$U(U-V) \le (U-V)(U+V) = W \qquad \text{on } E.$$

Similarly,

$$\begin{aligned}\left|\frac{F(x,V)}{V_\varepsilon}W_\varepsilon\right| &\le \frac{|F(x,V)|}{V}\,W + 2|F(x,V)|(U-V)\\ &\le 3\,\frac{|F(x,V)|}{V}\,W,\end{aligned}\tag{99}$$

because $V(U-V) \le W$ on E. The right-hand sides of (98) and (99) are integrable by (89). Thus dominated convergence yields

$$\begin{aligned}&\lim_{\varepsilon\downarrow 0}\int_D \left[\frac{F(x,U)}{U_\varepsilon} - \frac{F(x,V)}{V_\varepsilon}\right] W_\varepsilon\,dx\\ &\qquad = \int_E \left[\frac{F(x,U)}{U} - \frac{F(x,V)}{V}\right](U^2-V^2)\,dx.\end{aligned}\tag{100}$$

Combining (94), (97), and (100), we obtain the one-sided Díaz–Saá inequality [21] (see also [14]):

$$\begin{aligned}0 &\le \int_E \left[\left|\nabla U - \frac{U}{V}\nabla V\right|^2 + \left|\nabla V - \frac{V}{U}\nabla U\right|^2\right] dx\\ &\le \int_E \left[\frac{F(x,U)}{U} - \frac{F(x,V)}{V}\right](U^2-V^2)\,dx.\end{aligned}\tag{101}$$

Finally, on E, $U > V > 0$. By the strict monotonicity assumption (83),

$$\frac{F(x,U)}{U} < \frac{F(x,V)}{V} \qquad \text{for a.e. } x \in E,$$

while $U^2 - V^2 > 0$. Hence the last integral in (101) is strictly negative whenever $|E| > 0$, contradicting its non-negativity. Therefore $|E| = 0$, and $U \le V$ a.e. in D. □

Remark 9. *The structural condition* (83) *is the classical Brezis–Oswald subhomogeneity condition. The point of Lemma 7 is that the two functions need not have the same Dirichlet trace; the only boundary assumption is the ordered-trace condition* $(U-V)^+ \in H_0^1(D)$*. For this reason, the usual symmetric Brezis–Oswald identity (see expression (10) in [7]) cannot be invoked directly. If, for instance,* $U = 0$ *and* $V \ge 0$ *on* ∂D*, with* V *positive on a portion of the boundary, the quotient* V^2/U *may be singular there and need not be an admissible test function. The regularization*

$$U_\varepsilon = U + \varepsilon, \qquad V_\varepsilon = V + \varepsilon, \qquad W_\varepsilon = (U_\varepsilon^2 - V_\varepsilon^2)^+$$

avoids this difficulty and tests only the region where the desired ordering could fail. Thus (101) *is the appropriate one-sided version of the Brezis–Oswald/Díaz–Saá/Picone mechanism for ordered, possibly different, boundary traces.*

Proof of Theorem 9. For $\lambda < 0$, let U_λ be the unique positive solution of

$$-\Delta U + |\lambda| U = m(x) U^\alpha \quad \text{in } \Omega^+, \qquad U = 0 \quad \text{on } \partial\Omega^+$$

(see, e.g., [14]). The restriction of u_λ^+ to Ω^+ satisfies the same equation and has non-negative boundary trace. We first note that $u_\lambda^+ > 0$ in Ω^+. Indeed, $M(u_\lambda^+) > 0$, and hence u_λ^+ is not identically zero in Ω^+. Moreover,

$$-\Delta u_\lambda^+ + |\lambda| u_\lambda^+ = m(x)(u_\lambda^+)^\alpha \geq 0 \qquad \text{in } \Omega^+.$$

Since Ω^+ is connected, the strong maximum principle yields

$$u_\lambda^+ > 0 \qquad \text{in } \Omega^+.$$

To apply Lemma 7, set

$$F(x,s) := -|\lambda| s + m(x) s^\alpha.$$

Then

$$\frac{F(x,s)}{s} = -|\lambda| + m(x) s^{\alpha-1}$$

is strictly decreasing in $s > 0$ for a.e. $x \in \Omega^+$, since $m(x) \geq m_0 > 0$ and $0 < \alpha < 1$. Moreover, $U_\lambda = 0$ on $\partial\Omega^+$ and the trace of u_λ^+ is non-negative, so

$$(U_\lambda - u_\lambda^+)^+ \in H_0^1(\Omega^+).$$

It remains to verify (89). Put

$$E_\lambda := \{x \in \Omega^+ : U_\lambda(x) > u_\lambda^+(x)\}.$$

On E_λ,

$$\frac{|F(x,U_\lambda)|}{U_\lambda} \leq |\lambda| + \|m\|_\infty U_\lambda^{\alpha-1}.$$

Since

$$\left(U_\lambda^2 - (u_\lambda^+)^2\right)^+ \leq U_\lambda^2,$$

we obtain

$$\frac{|F(x,U_\lambda)|}{U_\lambda}\left(U_\lambda^2 - (u_\lambda^+)^2\right)^+ \leq |\lambda| U_\lambda^2 + \|m\|_\infty U_\lambda^{\alpha+1},$$

which belongs to $L^1(\Omega^+)$. For the term involving u_λ^+, we claim that U_λ/u_λ^+ is bounded on E_λ. Away from $\partial\Omega^+$ this follows from the positivity and continuity of u_λ^+. Near a boundary point at which the trace of u_λ^+ is positive, the set E_λ is empty in a sufficiently small neighborhood, since $U_\lambda = 0$ on $\partial\Omega^+$. At a boundary point at which the trace of u_λ^+ vanishes, the Hopf boundary lemma, applied to

$$-\Delta u_\lambda^+ + |\lambda| u_\lambda^+ = m(x)(u_\lambda^+)^\alpha > 0 \qquad \text{in } \Omega^+,$$

gives a linear lower bound for u_λ^+ in terms of $\operatorname{dist}(x, \partial\Omega^+)$, whereas $U_\lambda \in C^1(\overline{\Omega^+})$ and $U_\lambda = 0$ on $\partial\Omega^+$ give a corresponding linear upper bound. By compactness of the zero-trace part of $\partial\Omega^+$, a finite covering yields

$$U_\lambda \leq C_\lambda u_\lambda^+ \qquad \text{on } E_\lambda$$

for some $C_\lambda > 0$. Consequently,

$$\begin{aligned} &\frac{|F(x,u_\lambda^+)|}{u_\lambda^+}\left(U_\lambda^2 - (u_\lambda^+)^2\right)^+ \\ &\qquad \leq |\lambda| U_\lambda^2 + \|m\|_\infty (u_\lambda^+)^{\alpha-1} U_\lambda^2 \\ &\qquad \leq |\lambda| U_\lambda^2 + C_\lambda^2 \|m\|_\infty (u_\lambda^+)^{\alpha+1}, \end{aligned}$$

which is integrable in Ω^+. Thus (89) holds. Therefore Lemma 7, with

$$D = \Omega^+, \qquad U = U_\lambda, \qquad V = u_\lambda^+|_{\Omega^+},$$

gives

$$u_\lambda^+ \geq U_\lambda > 0 \qquad \text{in } \Omega^+.$$

Hence

$$\overline{\Omega^+} \subset K_\lambda. \tag{102}$$

Set $G = \Omega \setminus \overline{\Omega^+}$. By the assumptions of the theorem,

$$-m(x) \geq q_0 \qquad \text{for a.e. } x \in G.$$

Lemma 6 gives

$$\|u_\lambda^+\|_\infty \leq \left(\frac{\|m^+\|_\infty}{|\lambda|}\right)^{1/(1-\alpha)}. \tag{103}$$

Applying Theorem 6 in G yields

$$K_\lambda \cap G \subset \left\{x \in G : \operatorname{dist}(x, \partial\Omega^+) \leq R_\lambda\right\},$$

where

$$R_\lambda = \psi_{1/N,q_0}(\|u_\lambda^+\|_\infty).$$

Since

$$\psi_{1/N,q_0}(\tau) \leq \frac{\sqrt{2N(\alpha+1)/q_0}}{1-\alpha}\,\tau^{(1-\alpha)/2},$$

(103) gives

$$R_\lambda \leq C_0|\lambda|^{-1/2}. \tag{104}$$

Choose a fixed $C > C_0$. Then, for all sufficiently negative λ,

$$K_\lambda \subset \left\{x \in \Omega : \operatorname{dist}(x, \overline{\Omega^+}) < C|\lambda|^{-1/2}\right\}.$$

Together with (102), this proves (80) and (81).

Put

$$\varepsilon_\lambda = C|\lambda|^{-1/2}.$$

Then

$$\overline{\Omega^+} \subset K_\lambda \subset (\overline{\Omega^+})_{\varepsilon_\lambda},$$

where

$$(\overline{\Omega^+})_{\varepsilon_\lambda} := \left\{x \in \Omega : \operatorname{dist}(x, \overline{\Omega^+}) < \varepsilon_\lambda\right\}.$$

Since $u_\lambda^+ > 0$ in Ω^+, no point of Γ_λ lies in Ω^+, and for $x \in \Gamma_\lambda$,

$$\operatorname{dist}(x, \overline{\Omega^+}) = \operatorname{dist}(x, \partial\Omega^+).$$

Hence

$$\sup_{x \in \Gamma_\lambda} \operatorname{dist}(x, \partial\Omega^+) \leq \varepsilon_\lambda. \tag{105}$$

Conversely, let $y \in \partial\Omega^+$ and let $\nu(y)$ denote the exterior unit normal. Since $\partial\Omega^+ \in C^{1,1}$, there exists a uniform tubular neighborhood of $\partial\Omega^+$. Moreover, $\Omega^+ \Subset \Omega$. Hence, for all sufficiently negative λ,

$$z_\lambda := y + 2\varepsilon_\lambda \nu(y) \in \Omega$$

and

$$\operatorname{dist}(z_\lambda, \overline{\Omega^+}) = 2\varepsilon_\lambda.$$

Thus $z_\lambda \notin K_\lambda$, whereas $y \in K_\lambda$ by (102). If $y \in \Gamma_\lambda$, there is nothing to prove. Otherwise, the segment joining y to z_λ must leave the closed set K_λ at some point $\xi_\lambda \in \Gamma_\lambda$, and

$$|\xi_\lambda - y| \le 2\varepsilon_\lambda.$$

Therefore

$$\sup_{y\in\partial\Omega^+} \operatorname{dist}(y, \Gamma_\lambda) \le 2\varepsilon_\lambda. \tag{106}$$

Equations (105)–(106) prove (82). □

Remark 10. *Setting*

$$\mu = |\lambda| = -\lambda, \qquad v_\mu = \mu^{1/(1-\alpha)} u^+_{-\mu},$$

the equation in Ω^+ becomes exactly

$$-\frac{1}{\mu}\Delta v_\mu + v_\mu = m(x) v_\mu^\alpha. \tag{107}$$

Thus, away from $\partial\Omega^+$, the formal dominant balance is $v = m(x)v^\alpha$, so that on the positive branch

$$v = m(x)^{1/(1-\alpha)}. \tag{108}$$

Equivalently, the expected leading-order interior behavior is

$$u^+_\lambda(x) \sim |\lambda|^{-1/(1-\alpha)} m(x)^{1/(1-\alpha)}. \tag{109}$$

The same rescaling shows that the natural transition-layer thickness is $\mu^{-1/2} = |\lambda|^{-1/2}$, exactly the scale appearing in (80)–(82)*. Indeed, write $\lambda = -\mu$, $\mu \to +\infty$, and set*

$$u^+_{-\mu} = \mu^{-1/(1-\alpha)} v_\mu.$$

Substitution into

$$-\Delta u^+_{-\mu} + \mu u^+_{-\mu} = m(x)(u^+_{-\mu})^\alpha$$

gives the exact equation (107)*. The zero-order and nonlinear terms therefore have the same order, whereas diffusion carries the small coefficient $1/\mu$. Formally letting $\mu \to +\infty$ on compact subsets of Ω^+ gives* (108)*, and hence* (109)*. Finally, writing $\varepsilon^2 = 1/\mu$ gives*

$$-\varepsilon^2 \Delta v_\mu + v_\mu = m(x) v_\mu^\alpha,$$

so the natural length scale on which diffusion competes with the remaining terms is

$$\varepsilon = \mu^{-1/2} = |\lambda|^{-1/2}.$$

This coincides with the rigorous localization scale. Note that the support and free-boundary convergence in (81)–(82)*, including the rate $O(|\lambda|^{-1/2})$, are rigorous consequences of Lemma 6, Theorem 6, and the interior comparison. Formulae* (107)–(109) *explain the dominant balance and the natural boundary-layer scale.*

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