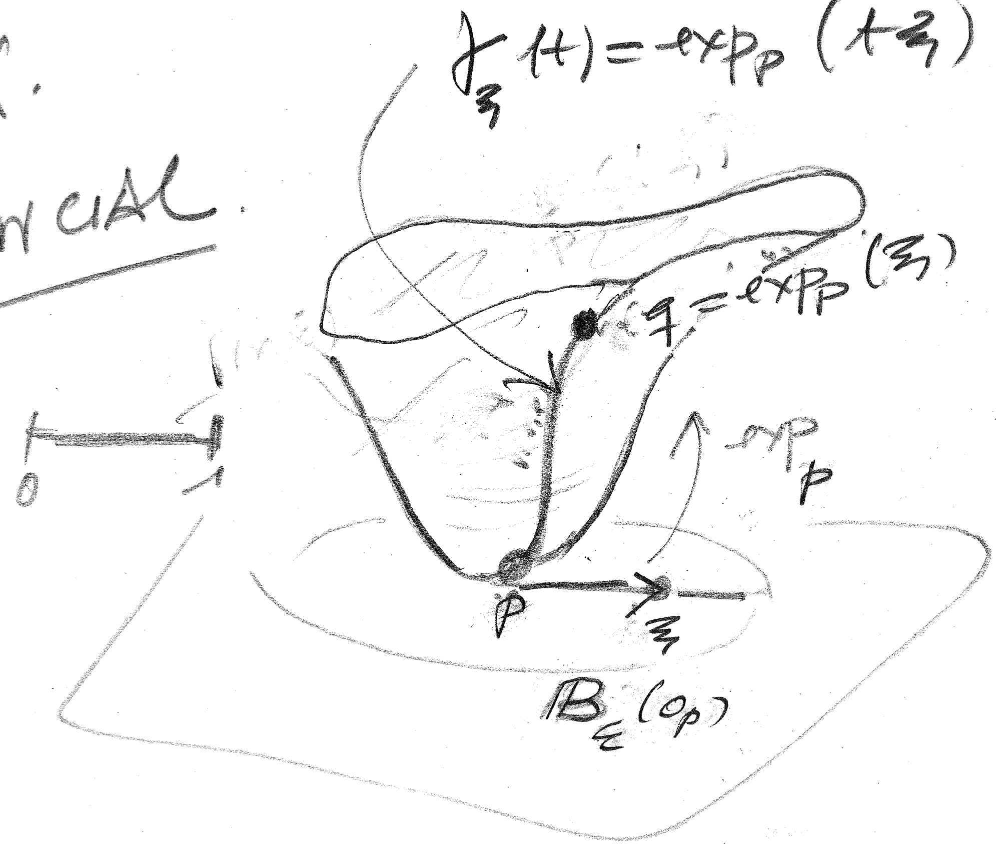
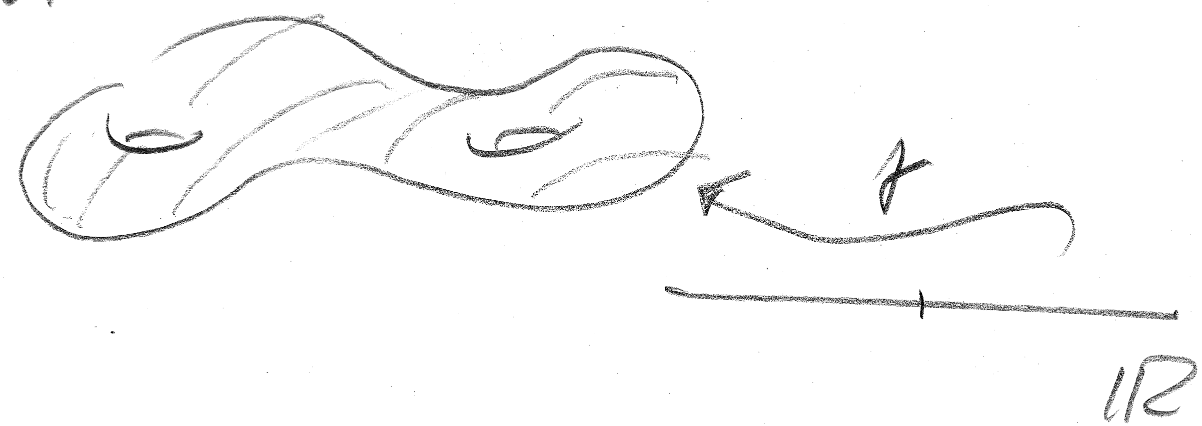
FUNCION EXPONENCIAL



COMPLETITUD GEODÉSICA



Sea (M, g) una VRC. Recordemos que las geodésicas verifican una ecuación diferencial del tipo: $\frac{d^2x^k}{dt^2} = -\Gamma^k_{ij}\frac{dx^i}{dt}\cdot\frac{dx^j}{dt}$ $\Big\}_{k=1\dots n}$ que es de 2º orden y puede escribirse en la forma

$$\left\{\begin{array}{l} \frac{dx^k}{dt} = \dot{x}^k \\ \frac{d\dot{x}^k}{dt} = -\Gamma^k_{ij}\,\dot{x}^i\dot{x}^j \end{array}\right.$$

de forma que para todas las condiciones iniciales $(x_0, \dot{x}_0, t_0)$, existe una (¡única!) solución $x = x(t)$ $\dot{x} = \dot{x}(t)$ definida para $|t - t_0| < \varepsilon$ de forma que $x(t_0) = x_0$ $\dot{x}(t_0) = \dot{x}_0$. Como $\dot{x}^k(t) = \frac{dx^k}{dt}$, podemos escribir que

existe una única geodésica $x = x(t)$ con $\dot{x}(t_0) = \dot{x}_0$

Si queremos hacer explícita la dependencia de la solución con las condiciones iniciales, tendríamos que escribir $x = x(t, t_0, x_0, \dot{x}_0)$ y podemos asegurar que esta dependencia es diferenciable.

Tenemos por tanto el siguiente resultado cuando se toma $t_0 = 0$, $\dot{x}_0 = 0$.

(6-B)

PROPOSICIÓN

Dado $x_0 \in M$, existe $\mathcal{V} \underset{\text{as}}{\subseteq} M$, $x_0 \in \mathcal{V}$, existen $\delta > 0$ $\varepsilon_1 > 0$ ($\mathcal{V}$ entorno de $(x_0, 0)$)

y una aplicación diferenciable $\gamma : (-\delta, \delta) \times \mathcal{V} \longrightarrow M$

donde $\mathcal{V} = \left\{ (x, \dot{x}) \,/\, \begin{matrix} x \in \mathcal{V} \\ \|\dot{x}\|_{g_x} < \varepsilon_1 \end{matrix} \right\} = \mathcal{V} \times \mathbb{B}^x_{\varepsilon_1}(0)$ tal que la curva

$(-\delta, \delta) \ni t \longrightarrow \gamma(t, x, \dot{x}) \Rightarrow \gamma_{(x,\dot{x})}(t)$ es la única geodésica

de M tal que $\gamma_{(x,\dot{x})}(0) = x$ y $\gamma'_{(x,\dot{x})}(0) = \dot{x}$

y se cumple la propiedad $\gamma_{(x,\lambda\dot{x})}(t) = \gamma_{(x,\dot{x})}(\lambda t)$

para $0 < \lambda < 1$.

Demostración. (de la última afirmación)

Nótese que haciendo $t = \lambda \bar{t}$, tenemos para $\gamma = \gamma_{(x,\dot{x})}$

$$\bar{\gamma}(\bar{t}) = \gamma(t(\bar{t})) = \gamma(\lambda \bar{t}) \quad : \quad \frac{d\bar{\gamma}}{d\bar{t}} = \frac{d\gamma}{dt} \cdot \frac{dt}{d\bar{t}} = \frac{d\gamma}{dt} \lambda$$

y en $t = 0$ $\left.\frac{d\bar{\gamma}}{d\bar{t}}\right|_{t=0} = \lambda \dot{x}$ así

$$\bar{\gamma}(\bar{t}) = \gamma_{(x,\dot{x})}(\lambda \bar{t}) = \gamma_{(x,\lambda\dot{x})}(\bar{t}).$$

(G-B')

Probemos ahora la primera parte:

Por dependencia diferenciable de las soluciones de la ecuación de las geodésicas, sabemos que existe una aplicación diferenciable $\gamma = \gamma(t, x, \dot{x})$ en torno a $t_0 = 0$ $x = x_0$ $\dot{x} = 0$

de forma que $\gamma(t, x, \dot{x}) = \gamma_{(x,\dot{x})}(t)$ es la única geodésica

tal que $\gamma_{(x,\dot{x})}(0) = x$ y $\gamma'_{(x,\dot{x})}(0) = \dot{x}$

$x_0 = \gamma(0, x_0, 0_0) \in M$, existe un entorno de $(0, x_0, 0)$ en $M \times \mathbb{R}^n$ que podemos describir como $(-\delta, \delta) \times V_\varepsilon$ tal que

$V_\varepsilon = \left\{ (x, \dot{x}) \;\middle|\; \begin{array}{l} x \in V \\ |\dot{x}| < \varepsilon \end{array} \right\}$

$\gamma : (-\delta, \delta) \times V_\varepsilon \longrightarrow M$

$\gamma : \mathbb{R} \times M \times \mathbb{R}^n \underset{ab}{\supset} A \longrightarrow M$

$(0, x_0, \dot{0}) \in A$ M

CORolARIO (6-C)

Dado $p_0 \in M$ existe $W \subseteq_{ab} M$ $x_0 \in W$, $\varepsilon > 0$ y

$\gamma : (-2,2) \times \mathcal{V} \longrightarrow M$ diferenciable $\mathcal{V} = \{(x,\dot{x}) \mid x \in W, |\dot{x}|_{g_x} < \varepsilon\} = W \times \mathbb{B}^x_\varepsilon(0)$

tal que $(-\delta,\delta) \ni t \longrightarrow \gamma(t,x,\dot{x}) = \gamma_{(x,\dot{x})}(t)$

geodesica tal que $\gamma_{(x,\dot{x})}(0) = x$ $\gamma'_{(x,\dot{x})}(0) = \dot{x}$.

Demostración.

Por el teorema anterior la curva.

$$t \longrightarrow \gamma_{(x,\frac{\delta}{2}\dot{x})}(t) = \gamma_{(x,\dot{x})}\left(\frac{\delta t}{2}\right) \quad \text{está definida}$$

cuando $\left|\frac{\delta t}{2}\right| < \delta \,(\Rightarrow |t| < 2)$, siempre que $\left|\frac{\delta}{2}\dot{x}\right|_{g_x} < \varepsilon_1$

y además $|\dot{x}|_{g_x} < \varepsilon_1$ es decir

$$|\dot{x}|_{g_x} < \min\left\{\varepsilon_1, \frac{2\varepsilon_1}{\delta}\right\}$$

6-D

entonces definimos $\gamma_{(x,\dot{x})}(1) = \exp_x(\dot{x})$ para $(x,\dot{x}) \in \mathcal{V}$

de forma que $\exp_x : T_x M \supset B^x_\varepsilon(0) \to M$, y está bien

definida cada vez que fijo $x \in V$. (Entorno del x_0)

se tiene entonces que $\gamma_{(x,\dot{x})} : [0,1] \to M$ es una geodésica

y $\gamma_{(x,\dot{x})}(t) = \gamma_{(x,t\dot{x})}(1) = \exp_x(t\dot{x})$.

Además $\gamma'_{(x,\dot{x})}(0) = \dot{x} = \left.\frac{d}{dt}\right|_{t=0} \exp_x(t\dot{x}) = D\exp_x\big|_0 \dot{x}$

Por tanto $D\exp_x\big|_{\dot{x}=0} = Id = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}$ y por el

teorema de la función inversa se tiene:

$\exists\, \varepsilon > 0$ tal que $\exp_x : B^x_\varepsilon(0) \to U^x \underset{as}{\subseteq} M$ es

un difeomorfismo. Hemos demostrado entonces

el siguiente:

Sea $(\mathbb{M}, g)$ VRC y sea $x \in \mathbb{M}$. (6-E

denotamos $\mathbb{B}^x_\varepsilon(0) = \{ \dot{x} \,/\, |\dot{x}|_{g_x} < \varepsilon \}$

TEOREMA

Fijado $x \in \mathbb{M}$, existe $\varepsilon > 0$ y un abierto $\mathcal{U}^x \subseteq_{ab} \mathbb{M}$, $x \in \mathcal{U}^x$ (denominado entorno normal de x) y un difeomorfismo $\exp_x : \mathbb{B}^x_\varepsilon(0) \longrightarrow \mathcal{U}^x$ denominado función exponencial, de forma que para cada $\dot{x} \in \mathbb{B}^x_\varepsilon(0)$ la curva

$[0,1] \ni t \longrightarrow \exp_x(t\dot{x}) = \gamma_{(x,\dot{x})}(t) \in \mathcal{U}^x$ es la única geodésica que une $\gamma_{(x,\dot{x})}(0) = x$ con $y = \exp_x(\dot{x}) \in \mathcal{U}$.

de longitud ($|\dot{x}|_{g_x} < \varepsilon$.) menor que ε.

Sea (M^n, g) variedad riemanniana.

TEOREMA.

Fijado $\xi \in T_pM$ $\exists\ \gamma_\xi : I \to M$ geodésica con $\gamma_\xi(0)=p$ y $\gamma_\xi'(0)=\xi$. Además si $\bar{\gamma}_\xi : J \to M$ es otra geodésica tal que $\bar{\gamma}_\xi(0)=p$ $\bar{\gamma}_\xi'(0)=\xi$ entonces $\gamma_\xi=\bar{\gamma}_\xi$ en $I\cap J$.

Demostración.

Tomemos $(U, \varphi=(x^1,\dots,x^n))$ carta de M con $p\in U$.

entonces $\xi = \sum_i \dot{x}_0^i \left(\frac{\partial}{\partial x^i}\right)_p$ $\qquad \varphi(p)=x_0=(x_0^1,\dots,x_0^n)$

sabemos entonces que $\exists\ x^i=x^i(t)$ $t\in I$ solución de las ecuaciones $\frac{d^2x^k}{dt^2}+\Gamma^k_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}=0$ con $x^i(0)=x_0^i$ y

$\left.\frac{dx^i}{dt}\right|_{t=0}=\dot{x}_0^i$ entonces tomamos

$\gamma(t)=\varphi^{-1}(x(t))$ o $\varphi\circ\gamma(t)=x(t)$, así $\gamma(0)=\varphi^{-1}(x_0)=p$

luego $\left.\frac{d\gamma}{dt}\right|_t = \left.\frac{dx^i}{dt}\right|_t \left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)}$ en particular $\left.\frac{d\gamma}{dt}\right|_{t=0} = \left.\frac{dx^i}{dt}\right|_{t=0}\left(\frac{\partial}{\partial x^i}\right)_p =$

$= \dot{x}_0^i\left(\frac{\partial}{\partial x^i}\right)_p = \xi.$

6-G

por la unicidad de soluciones tenemos que si

$\overline{\gamma}_\xi : J \to U$ es la otra entonces $\gamma_\xi = \overline{\gamma}_\xi$ en $I \cap J$

pero ¿qué pasa si ¿im γ_ξ no está contenida en una carta?

$\gamma_\xi : I \to M$ geodésica con $\gamma_\xi(0) = p$ $\gamma_\xi'(0) = \xi$

$\overline{\gamma}_\xi : J \to M$ geodésica con. $\overline{\gamma}_\xi(0) = p$ $\overline{\gamma}_\xi'(0) = \xi$.

consideremos $K = \{ t \in I \cap J \,/\, \gamma_\xi(t) = \overline{\gamma}_\xi(t), \gamma_\xi'(t) = \overline{\gamma}_\xi'(t) \}$

obviamente $0 \in K$ pues $\gamma_\xi(0) = \overline{\gamma}_\xi(0)$, y $\gamma_\xi'(0) = \overline{\gamma}_\xi'(0)$

además K es cerrado porque es donde coinciden dos aplicaciones continuas. Finalmente, se prueba que es abierto pues si $t_0 \in K$, entonces

$\gamma_\xi(t_0) = \overline{\gamma}_\xi(t_0)$ $\gamma_\xi'(t_0) = \overline{\gamma}_\xi'(t_0)$, y ...

Tomando $(\mathcal{U}, \varphi)$ carta por $p_0 = \gamma_\xi(t_0) = \overline{\gamma}_\xi(t_0)$ tendremos que $\exists\, \varepsilon > 0$ con $\gamma_\xi : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{U}$.

$\overline{\gamma}_\xi : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{U}$. si $\xi_0 = \gamma_\xi'(t_0) = \overline{\gamma}_\xi'(t_0)$

Tomando ahora $\xi_0 = \sum \dot{x}_0^i \left(\frac{\partial}{\partial x^i}\right)_{p_0}$ tendremos

que $\varphi \circ \overline{\gamma}_\xi = \varphi \circ \gamma_\xi : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \varphi(\mathcal{U})$ por

la unicidad de soluciones así

$\gamma_\xi = \overline{\gamma}_\xi : (t_0 - \varepsilon, t_0 + \varepsilon) \longrightarrow \mathcal{U} \subset M$ y por tanto

$(t_0 - \varepsilon, t_0 + \varepsilon) \subset K$.

Como K es abierto y cerrado se concluye

que $K = I \cap J$.

(6-I

<u>Definición</u>: Una geodésica $\gamma : I \to M$ se dice NO MAXIMAL si $\exists\, \bar{\gamma} : \bar{I} \to M$ geodésica con $\bar{I} \supsetneq I$ y $\bar{\gamma}|_I = \gamma$.
En otro caso se llama maximal.

<u>TEOREMA</u>.
$\forall p \in M \quad \xi \in T_pM \quad \exists!\ \gamma_\xi : I_\xi \to M$
geodésica maximal con $\gamma_\xi(0) = p \quad \gamma_\xi'(0) = \xi$.

<u>Dem</u>.
sea $\mathcal{I}_\xi = \left\{ I \underset{\text{int}}{\subset} \mathbb{R} \;\middle|\; \exists\, \gamma_I : I \to M \text{ geod con } \begin{array}{l} \gamma_I(0) = p \\ \gamma_I'(0) = \xi \end{array} \right\}$

entonces $I = \bigcup I_\xi \xrightarrow{\gamma_\xi} M$

$t \longrightarrow \gamma_I(t)$ si $t \in I \in \mathcal{I}_\xi$

(6-J)

(M, g) se dice geod. completa si las geodésicas maximales están definidas en todo $\mathbb{R}$, es decir $\gamma_\xi : \mathbb{R} \to M$ $\forall p \in M$ $\forall \xi \in T_pM$.

TEOREMA:

Si M es compacta, entonces M es geodésicamente completa.

TEOREMA: (Función exponencial)

$\forall p \in M$ $\exists \varepsilon > 0$ tq $\exp_p : T_pM \supset B_\varepsilon(0_p) \longrightarrow U^p \subseteq M$ es difeomorfismo tal que:

$\forall \xi \in B_\varepsilon(0_p)$ $\gamma_\xi(t) = \exp_p(t\xi)$ $0 \le t \le 1$ es geodésica. Además es la única geodésica de longitud $< \varepsilon$ que une $p \in U^p$ con $q = \exp_p(\xi) \in U^p$

$$B_\varepsilon(0_p) = \{\xi \in T_pM \mid |\xi| < \varepsilon\}$$

(Véase figura en la carátula)

Demostración: SIN COMENTARIOS !!

probemos la última afirmación: (6.K)

supongamos que $\bar{\gamma} : [0,1] \to M$ es una geodésica $L(\bar{\gamma}) < \varepsilon$ que une p con $q = \gamma_\xi(1) = \exp_p(\xi)$ con $|\xi| < \varepsilon$. (Nótese que $L(\gamma_\xi) = \int_0^1 |\gamma_\xi'(t)| = \int_0^1 |\xi| \, dt = |\xi| < \varepsilon$.) por tanto

entonces si $\bar{\gamma} : [0,1] \to M$ es $L(\bar{\gamma}) = \int_0^1 |\bar{\gamma}'| dt = \int_0^1 |\bar{\gamma}'(0)| \, dt =$

$= |\bar{\gamma}'(0)| < \varepsilon$; tomando $\bar{\xi} = \bar{\gamma}'(0)$ tenemos $\gamma_{\bar{\xi}} = \bar{\gamma}$ y por

tanto $\gamma_{\bar{\xi}}(1) = \exp_p(\bar{\xi}) = \exp_p(\xi) \Rightarrow \bar{\xi} = \xi \Rightarrow \bar{\gamma} = \gamma_\xi$.

Observación definimos $d(p,q) = \inf \{ L(\alpha) / \alpha \in \Omega(p,q) \}$

si Existiera una curva de longitud mínima que une p a q

$\gamma : [0,1] \to M$, con $|\gamma'| = cte$ entonces γ tendría que ser

una geodésica y $L(\gamma) < \varepsilon$ por tanto $\gamma = \gamma_\xi$:

En adelante, vamos a admitir que entre p, y $q \in U^p$

$\exists$ curva $\gamma \in \Omega(p,q)$ con $L(\gamma) = d(p,q)$.

así podemos suponer que las γ_ξ son minimizantes

denotemos $\exp_p^{-1} : U^p \ni q \longrightarrow \xi_q \in B_\varepsilon(0_p)$

(6-L

Dada (M,g) V.R. conexa. se define para $p,q \in M$

$d(p,q) = \inf \{ L(\alpha) \mid \alpha \in \Omega(p,q) \}$. [illegible]

En primer lugar obsérvese que si U^p es un entorno normal entonces

$\forall \delta$ con $0 < \delta < \varepsilon$ es $B(p,\delta) = \{ q \mid d(p,q) < \delta \} = \exp_p(\mathbb{B}_\delta(0_p))$ ya que

(1) si $q \in \exp_p(\mathbb{B}_\delta(0_p)) \Rightarrow q = \exp(\xi_q)$ $|\xi_q| < \delta$ y

$\gamma_{\xi_q} : [0,1] \to U^p \subset M$ es geodésica minimizante que une p a q

con longitud $L(\gamma_{\xi_q}) = |\xi_q| < \delta$ luego $q \in B(p,\delta)$

(2) si $q \in B(p,\delta)$, entonces existe $\alpha \in \Omega(p,q)$ tal que $L(\alpha) < \delta$

supuesto que $q \notin \exp_p(\mathbb{B}_\delta(0_p))$ encontraríamos un $t_1 \in [0,1]$

$\alpha(t_1) = q_1 \in \exp_p(\mathbb{S}_\delta(0_p)) \subset U^p$ y entonces $q_1 = \exp_p(\xi_{q_1})$

$L(\alpha|_{[0,t_1]}) \geq L(\gamma_{\xi_{q_1}}) = |\xi_{q_1}| = \delta$ en contradicción con

$L(\alpha) < \delta$.

Por tanto: $\mathcal{T}_d = \mathcal{T}_M$.

(6-M)

Es trivial ahora ver que $d(p,q)=0 \Rightarrow p=q$

y que $d(p,q)=d(q,p)$

Veamos la propiedad triangular

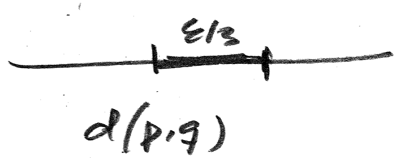
$p,q,r \in M$ entonces $d(p,q)+d(q,r) \geq d(p,r)$

Por reducción al absurdo:

si $\exists\, p,q,r$ con $d(p,q)+d(q,r) < d(p,r)$

sea $\varepsilon = d(p,r) - \{(d(p,q)+d(q,r)\}$ entonces

$\exists\, \alpha_{pq} \in \Omega(p,q)$ con $L(\alpha_{pq}) < d(p,q) + \frac{\varepsilon}{2}$ $\left(\varepsilon/3\right)$



$\exists\, \alpha_{q,r} \in \Omega(q,r)$ con $L(\alpha_{qr}) < d(q,r) + \frac{\varepsilon}{2}$ $\left(\varepsilon/3\right)$

así $L(\underbrace{\alpha_{p,q} \vee \alpha_{qr}}_{\text{diferenciable?}}) < d(p,q)+d(q,r)+\frac{2\varepsilon}{3} = d(p,r)\left(-\varepsilon/3\right)$

contradicción! ¡suavizar!

At this stage it is convenient to introduce a distance function on a Riemannian manifold (not necessarily complete) M as follows. Given two points $p, q \in M$, consider all the piecewise differentiable curves joining p to q. Since M is connected, such curves exist (cover a continuous curve joining p to q by a finite number of coordinate neighborhoods and replace each "piece" contained in a coordinate neighborhood by a differentiable curve).

2.4 DEFINITION. The *distance* $d(p,q)$ is defined by $d(p,q) =$ infimum of the lengths of all curves $f_{p,q}$, where $f_{p,q}$ is a piecewise differentiable curve joining p to q.

2.5 PROPOSITION. *With the distance d, M is a metric space, that is:*

1) $d(p,r) \leq d(p,q) + d(q,r)$,
2) $d(p,q) = d(q,p)$,
3) $d(p,q) \geq 0$, and $d(p,q) = 0 \Leftrightarrow p = q$.

Proof. (1), (2) and two of the assertions of (3) are immediate consequences of the definition of the infimum. It remains to show that if $d(p,q) = 0$ then $p = q$. Suppose to the contrary, and take a normal ball $B_r(p)$ that does not contain q. Since $d(p,q) = 0$, there exists a curve c joining p to q of length less than r. But the segment of c contained in $B_r(p)$ certainly has length greater than or equal to r, by Proposition 3.6 of Chapter 3, and that is a contradiction. □

Observe that if there exists a minimizing geodesic γ joining p to q (which is not always true) then $d(p,q) =$ length of γ.

2.6 PROPOSITION. *The topology induced by d on M coincides with the original topology on M.*

Proof. From the remark above, it follows that if r is sufficiently small, the normal ball $B_r(p)$ coincides with the metric ball of radius r, centered at p. Hence, metric balls contain normal balls, and conversely. □

2.7 COROLLARY. *If $p_o \in M$, the function $f: M \to \mathbf{R}$ given by $f(p) = d(p, p_o)$ is continuous.*

The fact which makes the concept of completeness relevant is the following theorem.

2.8 THEOREM. *(Hopf and Rinow [HR]). Let M be a Riemannian manifold and let $p \in M$. The following assertions are equivalent:*

a) $\exp_p$ *is defined on all of* $T_p(M)$.
b) *The closed and bounded sets of* M *are compact.*
c) M *is complete as a metric space.*
d) M *is geodesically complete.*
e) *There exists a sequence of compact subsets* $K_n \subset M$, $K_n \subset K_{n+1}$ *and* $\bigcup_n K_n = M$, *such that if* $q_n \notin K_n$ *then* $d(p, q_n) \to \infty$.

In addition, any of the statements above implies that

f) *For any* $q \in M$ *there exists a geodesic* γ *joining* p *to* q *with* $\ell(\gamma) = d(p, q)$.

Proof. a) $\Rightarrow$ f). Let $d(p,q) = r$, and let $B_\delta(p)$ be a normal ball at p, with $S_\delta(p) = S$ the boundary of $B_\delta(p)$. Let x_o be a point where the continuous function $d(q,x)$, $x \in S$, attains a minimum. Then $x_o = \exp_p \delta v$, where $v \in T_pM$ and $|v| = 1$. Let γ be a geodesic given by $\gamma(s) = \exp_p sv$ (See Fig. 1). We are going to show that $\gamma(r) = q$.

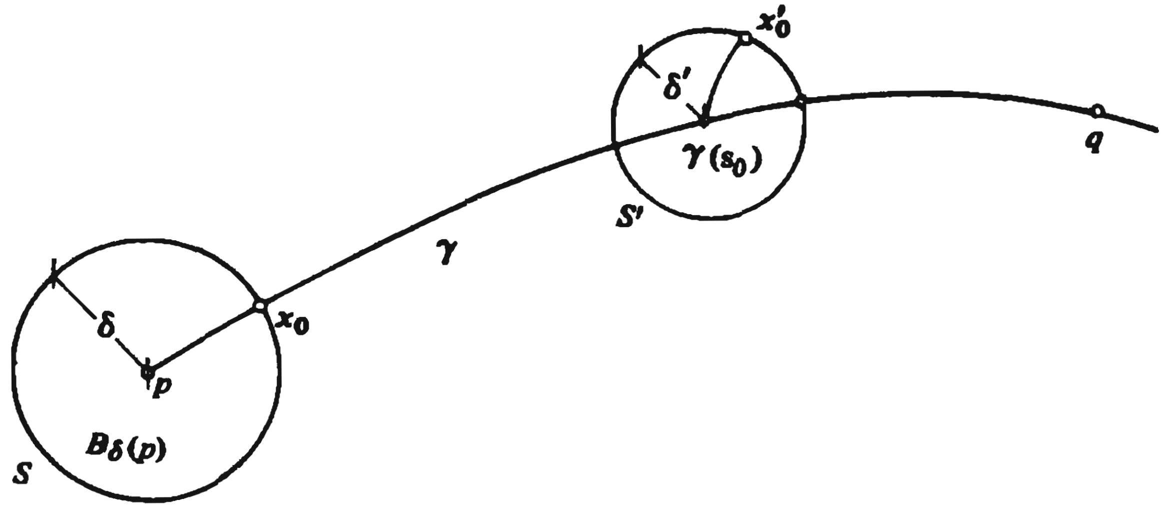


Figure 1

To prove this fact, consider the equation

$$d(\gamma(s), q) = r - s \tag{1}$$

and let $A = \{s \in [0,r]; (1) \text{ is valid}\}$. A is not empty, since (1) is true for $s = 0$. In addition, A is closed in $[0,r]$. Let $s_o \in A$. We are going to show that if $s_o < r$, then (1) is valid for $s_o + \delta'$, where $\delta' > 0$ is sufficiently small. This implies that $\sup A = r$; since A is closed then $r \in A$, which shows that $\gamma(r) = q$.

In order to prove that (1) is true for $s_o + \delta'$, let $B_{\delta'}(\gamma(s_o))$ be a normal ball at $\gamma(s_o)$, with $S' = \partial B_{\delta'}(\gamma(s_o))$ its boundary and let x'_o be a point where $d(x,q)$, $x \in S'$ has a minimum. It suffices to show that $x'_o = \gamma(s_o + \delta')$. Indeed, if $x'_o = \gamma(s_o + \delta')$, since

$$d(\gamma(s_o), q) = \delta' + \min d(x,q) = \delta' + d(x'_o, q)$$

and

$$d(\gamma(s_o), q) = r - s_o,$$

we have

$$r - s_o = \delta' + d(x'_o, q) = \delta' + d(\gamma(s_o + \delta'), q), \tag{2}$$

or that

$$d(\gamma(s_o + \delta'), q) = r - (s_o + \delta'),$$

which is (1) for $s_o + \delta'$.

To prove finally that $\gamma(s_o + \delta') = x'_o$, observe that, by the triangle inequality and by the first equality of (2),

$$d(p, x'_o) \geq d(p,q) - d(q, x'_o) = r - (r - s_o - \delta') = s_o + \delta'.$$

On the other hand, the broken curve joining p to x'_o, that goes from p to $\gamma(s_o)$ by the geodesic γ, and from $\gamma(s_o)$ to x'_o by the geodesic ray, has length equal to $s_o + \delta'$. Hence $d(p, x'_o) = s_o + \delta'$, and such a curve, by Corollary 3.9 of Chapter 3, is a geodesic. In particular, the curve is not broken, hence $\gamma(s_o + \delta') = x'_o$. This concludes the proof that a) $\Rightarrow$ f).

a) $\Rightarrow$ b). Let $A \subset M$ be closed and bounded. Since A is bounded, $A \subset B$, where B is a ball with center p in the metric d. By (f), there exists a ball $B_r(0) \subset T_pM$, such that $B \subset \exp_p \overline{B_r(0)}$. Being the continuous image of a compact set, $\exp_p \overline{B_r(0)}$ is compact. Hence, A is a closed set contained in a compact set, and is therefore compact.

b) $\Rightarrow$ c). It suffices to observe that a subset $\{p_n\}$ formed by a Cauchy sequence is bounded, therefore, has compact closure by (b). Thus $\{p_n\}$ contains a convergent subsequence and, being Cauchy, converges.

c) $\Rightarrow$ d). Suppose that M is not geodesically complete. Then some normalized geodesic γ of M is defined for $s < s_o$ and is not defined for s_o. Let $\{s_n\}$ be a convergent sequence, converging to s_o with $s_n < s_o$. Given $\varepsilon > 0$, there exists an index n_o such that if $n, m > n_o$ then $|s_n - s_m| < \varepsilon$. It follows that

$$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| < \varepsilon,$$

and hence the sequence $\{\gamma(s_n)\}$ is a Cauchy sequence in M. Since M is complete in the metric d, $\{\gamma(s_n)\} \to p_o \in M$.

Let (W, δ) be a totally normal neighborhood of p_o. Choose n_1 such that if $n, m > n_1$, then $|s_m - s_n| < \delta$ and $\gamma(s_n)$, $\gamma(s_m)$ belong to W. Then, there exists a unique geodesic g whose length is less than δ joining $\gamma(s_n)$ to $\gamma(s_m)$. It is clear that g coincides with γ, wherever γ is defined. Since $\exp_{\gamma(s_n)}$ is a diffeomorphism on $B_\delta(0)$ and $\exp_{\gamma(s_n)}(B_\delta(0)) \supset W$, g extends γ beyond s_0.

d) $\Rightarrow$ a). Obvious.

b) $\Leftrightarrow$ e). General topology. □

2.9 COROLLARY. *If M is compact then M is complete.*

2.10 COROLLARY. *A closed submanifold of a complete Riemannian manifold is complete in the induced metric; in particular, the closed submanifolds of Euclidean space are complete.*

3. The Theorem of Hadamard

As an application of the theorem of Hopf-Rinow, we are going to prove the following global fact.

3.1 THEOREM. *(Hadamard). Let M be a complete Riemannian manifold, simply connected, with sectional curvature $K(p,\sigma) \leq 0$, for all $p \in M$ and for all $\sigma \subset T_p(M)$. Then M is diffeomorphic to $\mathbf{R}^n$, $n = \dim M$; more precisely $\exp_p\colon T_pM \to M$ is a diffeomorphism.*

Before starting the proof, we need a few lemmas. The following lemma shows that the exponential map of a manifold with non-positive curvature is a local diffeomorphism.

3.2 LEMMA. *Let M be a complete Riemannian manifold with $K(p,\sigma) \leq 0$, for all $p \in M$ and for all $\sigma \subset T_pM$. Then for all $p \in M$, the conjugate locus $C(p) = \phi$; in particular the exponential map $\exp_p\colon T_pM \to M$ is a local diffeomorphism.*