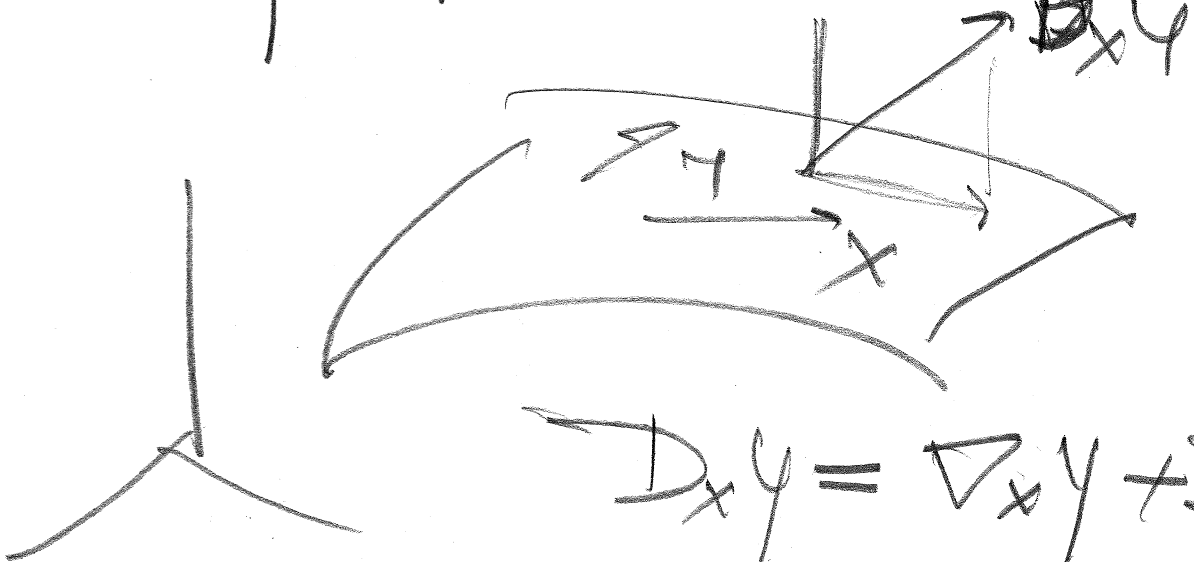
LECCIÓN 12

II forma fundamental y curvatura seccional

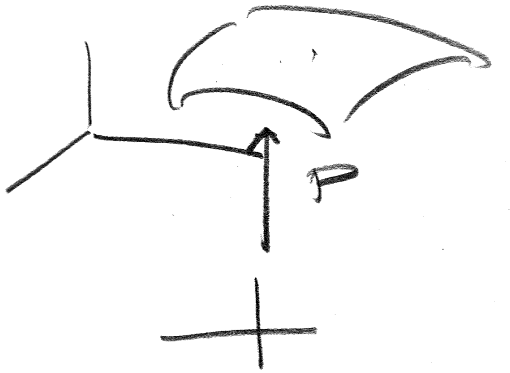


$$D_X Y = \nabla_X Y + \mathrm{II}(X,Y)$$

Sea $M^n \underset{v.e}{\subseteq} \mathbb{R}^n$ entonces para cada $p \in M$

$$T_pM \hookrightarrow T_p\mathbb{R}^n \simeq \{p\}\times\mathbb{R}^n$$

$$\left(\frac{\partial}{\partial u^i}\right)_p \longrightarrow \left(\frac{\partial P}{\partial u^i}\Big|_{\varphi(p)}\right)_p = \frac{\partial x^j}{\partial u^i}\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x^j}\right)_p$$



$$\Rightarrow T_pM \simeq \{p\}\times L\left(\frac{\partial P}{\partial u^1},\dots,\frac{\partial P}{\partial u^n}\right)\Big|_{\varphi(p)} < \{p\}\times\mathbb{R}^n.$$

punto a punto tenemos

$$\xi \in T_p\mathbb{R}^n = T_pM \oplus T_pM^\perp$$

$$\xi = \xi^T + \xi^\perp$$

se define la segunda forma fundamental

$$\mathrm{II} : \mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)^\perp$$

$$(X,Y) \longrightarrow (D_XY)^\perp = \mathrm{II}(X,Y)$$

siendo $\mathfrak{X}(M)^\perp = \{ Z : M \to T\mathbb{R}^n \mid Z(p) \in T_pM^\perp \ \forall p \}$ diferenciables

que es un $\mathfrak{F}(M)$-modulo.

Veamos que II es tensorial es decir es $\mathbb{R}$ bilineal 12-B

y $II(fX,Y) = II(X,fY) = f\,II(X,Y)$.

En efecto $II(X,fY) = (D_X(fY))^{\perp} = (X(f)Y + f D_X Y)^{\perp} =$

$= f(D_X Y)^{\perp} = f\,II(X,Y)$.

Esto significa que podemos escribir $II_p : T_pM \times T_pM \longrightarrow T_pM^{\perp}$

Como 2ª forma fundamental

Si ∇ es la conexión de Levi-Civita de M con

la métrica $g_M = g_{\mathbb{R}^n}|_M$ tenemos

$$D_X Y = \nabla_X Y + II(X,Y) \quad \text{es decir}$$

(Ecuación de Gauss)

$$\nabla_X Y = (D_X Y)^{T}$$

TEOREMA DE GAUSS $\quad \forall X, Y, Z \in \mathfrak{X}(M)$

$$\langle R(X,Y)Z, W\rangle = \mathrm{II}(X,Z)\cdot\mathrm{II}(Y,W) - \mathrm{II}(X,W)\cdot\mathrm{II}(Y,Z)$$

$$= -D_X D_Y Z + D_Y D_X Z + D_{[X,Y]} Z = 0$$

$$(-D_X D_Y Z)\cdot W = -X((D_Y Z)\cdot W) + (D_Y Z)\cdot(D_X W)$$

$$+(D_Y D_X Z)\cdot W = +Y((D_X Z)\cdot W) - (D_X Z)\cdot(D_Y W)$$

$$+ (D_{[X,Y]} Z)\cdot W =$$

$$= -X(\langle\nabla_Y Z, W\rangle + Y\langle\nabla_X Z, W\rangle + \langle\nabla_{[X,Y]} Z, W\rangle$$

$$+ (\nabla_Y Z + \mathrm{II}(Y,Z))\cdot(\nabla_X W + \mathrm{II}(X,W))$$

$$- (\nabla_X Z + \mathrm{II}(X,Z))\cdot(\nabla_Y W + \mathrm{II}(Y,W))$$

$$\langle\nabla_Y Z, \nabla_X W\rangle + \mathrm{II}(Y,Z)\cdot\mathrm{II}(X,W)$$

$$-\langle\nabla_X Z, \nabla_Y W\rangle - \mathrm{II}(X,Z)\cdot\mathrm{II}(Y,W)$$

$$-X\langle \nabla_Y Z, W\rangle + \langle \nabla_Y Z, \nabla_X W\rangle = -\langle \nabla_X \nabla_Y Z, W\rangle \qquad 12\text{-}D$$

$$Y\langle \nabla_X Z, W\rangle - \langle \nabla_X Z, \nabla_Y W\rangle = \langle \nabla_Y \nabla_X Z, W\rangle$$

así que:

$$\langle R(X,Y)Z, W\rangle = \mathrm{II}(Y,Z)\cdot \mathrm{II}(Y,W) - \mathrm{II}(Y,Z)\,\mathrm{II}(X,W)$$

En particular si $\sigma = L(u,v) \subset T_pM$ plano vect

$$K(p,\sigma) = \frac{\langle R(u,v)u,v\rangle}{|u\wedge v|^2} = \frac{\begin{vmatrix} \mathrm{II}(u,u) & \mathrm{II}(u,v) \\ \mathrm{II}(v,u) & \mathrm{II}(v,v) \end{vmatrix}}{\begin{vmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{vmatrix}}$$

Esto recuerda bastante a la curvatura de Gauss en superficies $\quad K(p) = \dfrac{\det \mathrm{II}_p}{\det \mathrm{I}_p}$

y efectivamente

si

Ecuación de Gauss $(\mathbb{R}^n, g_{\mathbb{R}^n})$

12-F

$M^m \underset{v.e}{\subset} \mathbb{R}^n$ tenemos $T_pM^m \subset T_p\mathbb{R}^n = \{p\}\times\mathbb{R}^n \quad \forall p \in M$

y así $T_p\mathbb{R}^n = T_pM \oplus T_pM^\perp$ $\qquad (M, g_M)$ ∇ conex. L.C.

$$\xi_p = \xi_p^T + \xi_p^\perp$$

$$II : \mathfrak{X}(M)\times\mathfrak{X}(M) \longrightarrow \mathfrak{X}^\perp(M) \qquad II_p : T_pM\times T_pM \longrightarrow T_pM^\perp$$
$$(X,Y) \longrightarrow (D_XY)^\perp$$

$$X,Y \in \mathfrak{X}(M) \Rightarrow D_XY = \nabla_XY + II(X,Y) \qquad \left(\nabla_XY = (D_XY)^T\right)$$

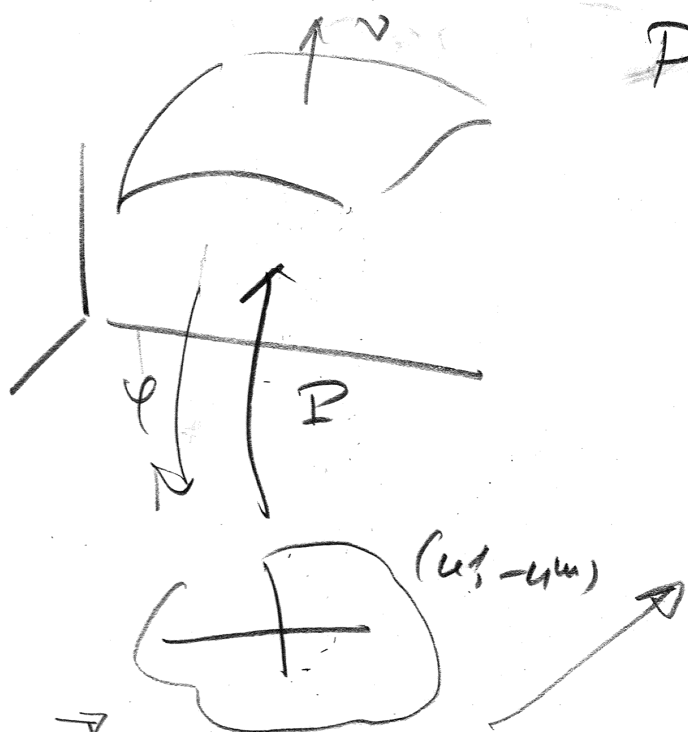
$$\langle R(X,Y)Z,W\rangle = II(X,Z)\cdot II(Y,W) - II(X,W)\cdot II(Y,Z)$$

así que $K(u,v,u,v) = II(u,u)\cdot II(v,v) - II(u,v)^2 = \begin{vmatrix} II(u,u) & II(u,v) \\ II(v,u) & II(v,v) \end{vmatrix}$

tenemos así $\sigma = L(u,v) \underset{\text{plano}}{\subset} T_pM$

$$K(p,\sigma) = \frac{\begin{vmatrix} II(u,u) & II(u,v) \\ II(v,u) & II(v,v) \end{vmatrix}}{\begin{vmatrix} \langle u,u\rangle & \langle u,v\rangle \\ \langle v,u\rangle & \langle v,v\rangle \end{vmatrix}} \overset{?}{=} \frac{\begin{vmatrix} e & f \\ f & g \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}$$

Pofams $M^n \subset \mathbb{R}^{n+1}$ (x



$P: \{x^i = x^i(u^1 \ldots u^n)\}$

$$\left(\frac{\partial}{\partial u^i}\right)_p = \frac{\partial x^j}{\partial u^i}\Big|_{\varphi(p)} \left(\frac{\partial}{\partial x^j}\right)_p = \left(\frac{\partial P}{\partial u^i}\Big|_{\varphi(p)}\right)_p .$$

$$g_{ij}^{\varphi} = \frac{\partial P}{\partial u^i} \cdot \frac{\partial P}{\partial u^j} \qquad \text{I } \cancel{FF}$$

$$h_{ij}^{\varphi} = \frac{\partial^2 P}{\partial u^i \partial u^j} \cdot \nu \qquad \text{II FF}$$

sieudo $\nu \cdot P$ un vector normal unitario

(*) $\vec{\mathrm{II}}(X,Y) = (D_X Y)^{\perp} = \mathrm{II}(X,Y)\,\nu$

$\mathrm{I}_p : T_pM \times T_pM \to \mathbb{R} \quad (\xi, \eta) \to \xi \cdot \eta = \langle \xi, \eta \rangle$

$\mathrm{II}_p : T_pM \times T_pM \to \mathbb{R} \quad (\xi, \eta) \to -(D_\xi \nu) \cdot \eta = \langle L\xi, \eta \rangle$

donde $L_p : T_pM \to T_pM$ es la aplicación de Weingarten
$\xi \longrightarrow -D_\xi \nu$

pero $\mathrm{II}(P_{u^i}, P_{u^j}) = \cdots = (D_{P_{u^i}}(P_{u^j})) \cdot \nu = \frac{\partial^2 P}{\partial u^i \partial u^j} \cdot \nu$

$P_{u^j} \cdot \nu = 0 \Rightarrow (D_{P_{u^i}} P_{u^j}) \cdot \nu + P_{u^j} \cdot L P_{u^i} = 0$

12-H

En el caso de las superficies tenemos $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = (g_{ij})$ $\quad \begin{pmatrix} e & f \\ f & g \end{pmatrix} = (h_{ij})$

y $K_P = \dfrac{eg - f^2}{EG - F^2}$ → la curvatura de Gauss.

pero la curvatura seccional es

$$\frac{K\left(\frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}; \frac{\partial}{\partial u_1}, \frac{\partial}{\partial u_2}\right)}{\left| \frac{\partial}{\partial u_1} \wedge \frac{\partial}{\partial u_2} \right|^2} = \frac{\begin{vmatrix} e & f \\ f & g \end{vmatrix}}{\begin{vmatrix} E & F \\ F & G \end{vmatrix}}.$$

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así la esfera S^2 tiene curvatura $K_{S^2} = 1$

En general $S^n_R = \{ (x_0, \ldots, x_n) \mid x_0^2 + \cdots + x_n^2 = 1 \}$

también tiene $K_{S^n} = 1/R^2$

y

Proposición.

Sea $S^n_R = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \,/\, x_0^2 + \dots + x_n^2 = R^2\}$ entonces $\nu = \frac{1}{R}(x_0, \dots, x_n)$ es un vector normal unitario saliente y si $\xi \in T_p S^2$ con $|\xi| = 1$ tenemos que $\vec{II}_p(\xi, \xi) = -\frac{1}{R}\nu(p)$

En particular $\vec{II}_p(\xi, \eta) = -\frac{1}{R}\langle \xi, \eta\rangle \nu(p)$

así que si $\sigma = L(\xi, \eta)$ tenemos

$$K(p, \sigma) = \frac{\begin{vmatrix} II(\xi,\xi) & II(\xi,\eta) \\ II(\xi,\eta) & II(\eta,\eta) \end{vmatrix}}{\begin{vmatrix} \langle \xi,\xi\rangle & \langle \xi,\eta\rangle \\ \langle \eta,\xi\rangle & \langle \eta,\eta\rangle \end{vmatrix}} = \frac{1}{R^2} \quad !!!$$

12-J

Demostración

$\xi \in T_p M$ tomamos $\alpha : I \to M$ con $\alpha(0) = p$ $\alpha'(0) = \xi$

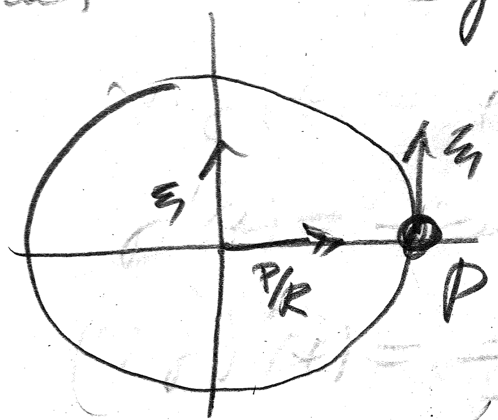
entonces $\left.\frac{d(\nu \circ \alpha)}{dt}\right|_{t=0} = -L(\xi) = -D_\xi \nu$

así $II_p(\xi, \xi) = \langle \alpha', (\nu \circ \alpha)' \rangle |_{t=0}$

En particular:

si $\xi \in T_p S_R^M$ con. $|\xi| = 1$ $p \in S_R^n \Rightarrow \left|\frac{p}{R}\right| = 1$

tomamos $\gamma(t) = R\cos(t/R)\frac{p}{R} + R\sin(t/R)\xi$



$\gamma'(t) = -\sin(t/R)\frac{p}{R} + \cos(t/R)\xi$

$\nu \circ \gamma(t) = \frac{1}{R}\gamma(t)$

$(\nu \circ \gamma)'(t) = \frac{1}{R}\gamma'(t)$

así

$\langle \gamma', \nu \circ \gamma' \rangle = \frac{-1}{R}\langle \gamma', \gamma' \rangle = -1/R$; II