TRANSVERSE RIEMANN-LORENTZ TYPE-CHANGING METRICS WITH POLAR END.

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Abstract

Consider a smooth manifold M with a smooth cometric g^* which changes the bilineal type by transverse way, on a hypersurface D^∞. Suppose that the radical annihilator hyperplane is tangent to D^∞. We examine the geometry of the (g^*-dual) covariant metric g on $M - D^\infty$, prove the existence of a canonical (polar-normal) vectorfield whose integral curves are $C^\infty-$pregeodesics crossing D^∞ transversely for each point, and analyze the curvature behavior using a natural coordinates. Finally we give an approach to the conformal geometry of such spaces and suggest some application as cosmological big-bang model.

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1 Introduction

There are several geometrical and physical reasons to study the metrics with signature type changing Lorentz to Riemann (see for example the introductions to [7], [5] and [8]). For physical reasons, two proposals for such spacetimes have been advanced:

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a) The metric everywhere is smooth but it is degenerate at the hypersurface of signature type change.

b) The metric is everywhere non degenerate but fails to be defined or to be continuous at the hypersurface that divides the Riemannian from Lorentzian region.

There are many articles devoted to the proposal a) (see .[6] ,[7], [1], [2]) and some others to proposal b) (see for example [3], [8])

In this article we analyze a particular version[1] of proposal b). Here the dual metric g^* (but not the metric g) is smooth and well defined on the whole space M and it is of transverse type changing from Lorentz to Riemann trough a hypersurface D^∞ (called polar). We refer these metric g (with certain annihilator condition added) as a *Lorentz-Riemann metric with polar end*. Its formal definition is displayed at the beginning of the Section 2. The main result of this Section is that the *geometry* of these space allow to define canonically a polar-normal transversal direction along D^∞.

In Section 3 we prove the existence of an unique pregeodesic crossing transversely D^∞ for any $p \in D^\infty$. Moreover these pregeodesics cross at the polar-normal direction. Then using as parameter of the pregeodesics, the square of the arc length to D^∞, we may establish by a standard process, a natural coordinate $(z_1, \ldots, z_m)$ system around any point to D^∞. The partial ∂_{z_m} is then a canonical polar-normal pregeodesic vectorfield. Its flow are called *the polar normal flow*.

Using the natural coordinates in Section 4, we analyze the behavior near of D^∞ of the semiriemannian curvatures. The conclusions are resumed in Theorem 14.

The *Lorentz-Riemann metric with polar end* character is preserved by a conformal change. Section 5 is devoted to analyze some aspects of this conformal geometry $(M, \mathcal{C})$. The main result is that fixing any (local) flow which moves D^∞, there exist a metric $g \in \mathcal{C}$ such that this flow is *the polar normal flow* with respect to g. Moreover g is univocally determined around D^∞ . Finally we consider the Lorentzian piece of $(M, \mathcal{C})$ as the causal structure support of a admissible cosmological model where D^∞means the big-bang singularity, then we speculate with the existence of a metric $g \in \mathcal{C}$, such that his *polar normal flow* moves D^∞ by (simultaneity) hypersurfaces with constant sectional curvature.

[1]This version has been inspired by a personal communication of Prof. M.Kossowski

2 Preliminaries.

2.1 Type-changing metrics with polar end.

A *type-changing metric space with polar end* is a $m-$dimensional manifold M ($m \geq 2$), endowed with a smooth, symmetric $(0,2)-$tensorfield g over an open set $M - D^{\infty}$of M. ($D^{\infty} \neq \varnothing$). For any $p \in M - D^{\infty}$, we construct the dual metric at p, $g^* : T_p^*M \times T_p^*M \to \mathbb{R}$, by

$$g^*(\alpha, \beta) = g(X_{\alpha}, X_{\beta}) \tag{1}$$

and g^* is a $(2,0)$ tensor over $M - D^{\infty}$. We demand:

D1) *The dual metric* g^* of g on $M - D^{\infty}$*, has smoothly extension to* M*, and it is transverse type-changing over* D^{∞}.
The *transverse type-changing property for the* extension means that if $(\theta^1, \ldots, \theta^m)$ is a coframe on a neighborhood U of $p \in D^{\infty}$, then $U \cap D^{\infty} = \left\{x \in U : \det\left(g\left(\theta^a, \theta^b\right)\right)\big|_x = 0\right\}$, and $\det\left(g^*\left(\theta^a, \theta^b\right)\right) = 0$ is a local equation for D^{∞} that is

$$d_x \det\left(g\left(\theta^a, \theta^b\right)\right) \neq 0, \forall x \in U \cap D^{\infty}$$

Of course this condition is frame-independent. In particular D^{∞} is an hypersurface (called *polar end*) and at each point $p \in D^{\infty}$ (called *polar point*) the radical $Rad_p(g^*) \subset T_p^*M$ is one dimensional. Therefore the annihilator

$$An\left(Rad_p(g^*)\right) = \{X \in T_pM : \mu(X) = 0, \forall \mu \in Rad_p(g^*)\}$$

is $(m-1)$-dimensional. Also the signature of g^* (or g) changes by $+1$ or -1 across D^{∞}.(see [6] for details)

The last condition required is:

D2) *The annihilator is tangent to* D^{∞}.
That is, $An\left(Rad_p(g^*)\right) = T_pD^{\infty}$ for any $p \in D^{\infty}$. In fact if $\mu \in Rad_p(g^*)$, and $\mu \neq 0$, the condition at p say : $\ker \mu = T_pD^{\infty}$.

We refer to (M, g) as a type-changing metric space with *polar end* (D^{∞})

Of course we replace the term *type-changing metric* by *Riemann-Lorentz*, if every component of $M - D^{\infty}$ is either Riemann or Lorentz.

Henceforth we restrict our attention to the Riemann-Lorentz type.

If N is manifold (possibly with boundary) denote $\mathfrak{X}(N)$ the $C^{\infty}(N)$-module of all vectorfields on N. If D is submanifold of N, then $\mathfrak{X}_N(D) = \{X \in \mathfrak{X}(N) : X|_D \in \mathfrak{X}(D)\}$ is the submodule the vectorfields tangent to D.

Also $\mathfrak{X}_D(N)$ or ($\mathfrak{X}_D$ if we understood N) is the $C^{\infty}(D)$ module of smooth $A : D \to TN$ with $A(x) \in T_x N$ for all $x \in D$. Finally $\Omega^1(N)$ is the module of 1-forms (dual module of $\mathfrak{X}(N)$)

We will use the following index conventions:$a, b, c \in \{1, \ldots, m\}$ varies 1 to m, and $i, j, k \in \{1, \ldots, m-1\}$. We also use Einstein's summation convention , unless the repeated index is m We will work on a fixed neighborhood of a polar point $p \in D^{\infty}$ of the Riemann-Lorentz space (M, g) with polar end. Without loss generality we will suppose that this neighborhood is the whole space M. Also we may suppose that $M - D^{\infty}$ have two connected component D^{+}(Riemannian) and D^{-} (Lorentzian).

Let us consider some function $\tau \in C^{\infty}(M)$ such that $\tau \mid_{D^{\infty}} = 0$ and $d\tau \mid_{D^{\infty}} \neq 0$ everywhere. We say that $\tau = 0$ *is an equation for* D^{∞}. Given another function $f \in C^{\infty}(M)$, it holds: $f \mid_{D^{\infty}} = 0 \;\Leftrightarrow\; f = h\tau$, for some $h \in C^{\infty}(M)$. When $f \mid_{D^{\infty}} = 0$, we write $\tau^{-1} f \cong 0$ and we say that $\tau^{-1} f$ *is extendible* as an element of $C^{\infty}(M)$.

2.2 Polar-adapted frames.

We say that a frame $(E_a) = (E_i, E_m)$ on M is polar-adapted if $(E_i \mid_{D^{\infty}})$ is a frame of D^{∞}, and

$$(g(E_a, E_b)) = \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & 1/\tau \end{pmatrix} \tag{2}$$

where $\tau = 0$ is a equation for D^{∞}.

We say that the coframe $(\theta^1, \ldots, \theta^m)$ on M is Rad^*-adapted if

$$\left(g^*\left(\theta^a, \theta^b\right)\right) = \begin{pmatrix} 1 & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 1 & 0 \\ 0 & \ldots & 0 & \tau \end{pmatrix} \tag{3}$$

where $\tau = 0$ is a equation for D^∞.

If (θ^a) is coframe Rad^*-adapted and (E_a) is the dual frame then (E_a) is polar-adapted frame. In fact (using notation of (**??**) and (**??**)) we have:

$$\delta_i^a = \theta^a(E_i) = g^*\left(\theta^i, \theta^a\right) = \theta^a\left(X_{\theta^i}\right) \tag{4}$$

since $\beta(X_\alpha) = g^*(\alpha, \beta)$. Analogously

$$\begin{cases} \theta^i(\tau E_m) = 0 = g^*(\theta^i, \theta^m) = \theta^i(X_{\theta^m}) \\ \theta^m(\tau E_m) = \tau = g^*(\theta^m, \theta^m) = \theta^m(X_{\theta^m}) \end{cases}$$

and we conclude that:

$$\begin{array}{ll} X_{\theta^i} = E_i & X_{\theta^m} = \tau E_m \\ \alpha_{E_i} = \theta^i & \alpha_{E_m} = (1/\tau)\,\theta^m \end{array} \tag{5}$$

Taking account that $g(X, Y) = g^*(\alpha_X, \alpha_Y)$, in $M - D^\infty$ we get that

$$g(E_i, E_j) = g^*(\theta^i, \theta^j) = \delta^i_j,\ g(E_i, E_m) = (1/\tau)\, g^*(\theta^i, \theta^m) = 0$$
$$g(E_m, E_m) = (1/\tau)^2\, g^*(\theta^m, \theta^m) = 1/\tau$$

and .$(g(E_a, E_b))$ is as (2)

Also $\theta^m(E_i) = 0$, $\theta^m(p) \in Rad_p(g^*)$. Using the tangent annihilator property we conclude that $\left(E_i|_{D^\infty}\right)$ is a frame for D^∞. Thus (E_a) is a polar-adapted frame.

Remark 1 *Since the existence of Rad^*-adapted coframes it is straightforward, we conclude the existence of polar-adapted frames. On the other and using a polar-adapted frame (E_a) is easy to see that g induces by restriction on D^∞ a canonical Riemannian structure such that $\left(E_i|_{D^\infty}\right)$ is orthonormal frame.*

The same arguments show that if (E_a) is a polar-adapted frame then the dual (θ^a) is a Rad^*-adapted coframe. We will prove now that it is possible to make a Rad^*-adapted coframe (θ^i, θ^m) for any fixed $\mu = \theta^m$:

Proposition 2 *Let μ be a $1-$form on M which is $Rad^{*}-$adapted (that is $\mu(x) \in Rad_x(g^{*}) - \{0\}$ for all $x \in D^{\infty \cdot}$). Then there exist (locally) a Rad^{*} $-$adapted coframe $(\mu^i, \mu^m = \mu)$.*

Proof. We start with an auxiliary Rad^{*}- adapted coframe (θ^a) as in (3) Without lost generality we may write

$$\mu = \sum (\tau h_i)\, \theta^i + \theta^m$$

since if $(\mu^i, \mu^m = \mu)$ is Rad^{*}- adapted then the same holds for $(\mu^i, h\mu)$ for any smooth everywhere non-null h. Thus $\alpha = \sum X_a \theta^a$ is orthogonal to μ iff $\sum h_j X_j + X_m = 0$, and the co-distribution $\mu^{\perp}$ is generated by the $(m-1)$-coframe $(\vartheta^i = \theta^i - h_i \theta^m)$ which is well-defined also over D^{∞}. By application of the the classical orthonormalization Graham Smith process to (ϑ^i) we obtain the desired coframe. ■

The dual result is the following:

Theorem 3 *Let N be any vectorfield on M transversal to D^{∞}. Then there exist (locally) a polar-adapted frame $(N_i, N_m = N)$.*

Proof. We start with an auxiliary Rad^{*}- adapted coframe (θ^a) with (polar-adapted) dual frame, (E_a). Thus (g_{ab}) is as (2). Writing $N = \sum h^a E_a$, transversality implies that $h^m(x) \neq 0$ for all $x \in D^{\infty}$.Now taking $\mu = \sum (\tau h^i)\, \theta^i + h^m \theta^m$ we have:

$$\begin{aligned} X_{\mu} &= \sum \tau h^i X_{\theta^i} + h^m X_{\theta^m} \\ &= \sum \tau h^i E_i + \tau h^m E_m \\ &= \tau N \end{aligned} \tag{6}$$

On the other hand we may construct $(\mu^i, \mu^m = \mu)$ coframe Rad^{*}-adapted as in proposition 2.Let (N_i, N_m) be his polar-adapted dual coframe. Then by (5) is $X_{\mu} = \tau_{\mu} N_m$ where $\tau_{\mu} = g^{*}(\mu, \mu)$. Since $(\tau_{\mu} = 0)$ is equation of D^{∞} then there exist a smooth everywhere non-null h such that $\tau_{\mu} = h\tau$. Therefore $X_{\mu} = \tau h N_m$. Comparing with (6) we get that $N = hN_m$. But if (N_i, N_m) is polar-adapted, the same is true for $(N_i, h^{-1} N_m = N)$.

Corollary 4 *Given any vectorfield N on M transversal to D^{∞} and $X \in \mathfrak{X}_M(D^{\infty})$, we have $g(X, N) \cong 0$*

■

Proof. By the theorem there exist a polar-adapted frame $(E_i, E_m = N)$ as in (2).Therefore we may write $X = \sum X^i E_i + \tau h E_m$ where h is some smooth function on M. Then $g(X,N) = \sum X^i g_{im} + h$ which it is a differentiable function. ■

Remark 5 *Using polar-adapted frames (E_a) (as in (2)) is easily to prove that $g(X,Y) \cong 0$ if X or Y belongs to $\mathfrak{X}_M(D^\infty)$. In fact if $X = \sum X^a E_a$ ...etc. then $X^m Y^m|_{D^\infty} = 0$ thus $g(X,Y) = \sum X^i Y^i + \tau^{-1}(X^m Y^m) \in C^\infty(M)$. On the other hand note that τg is defined on the whole space M, if $(\tau = 0)$ is an equation for D^∞.*

2.3 The dual connection near to $\mathbf{D}^\infty$.

First we remark the local nature of the work. In fact we should be replaced in any case M by a suitable neighborhood of a polar point $p \in D^\infty$.

We recall (see [4] for more details) that on the Riemann-Lorentz space $(M - D^\infty, g)$ there exists a unique torsion-free metric *dual connection*, which it is characterized as the unique map $\square : \mathfrak{X}(M - D^\infty) \times \mathfrak{X}(M - D^\infty) \to \mathfrak{X}^*(M - D^\infty)$ satisfying, for all $A, B, C \in \mathfrak{X}(M - D^\infty)$, the Koszul-like formula:

$$\begin{aligned} 2\square_A B(C) := {} & A\langle B,C\rangle + B\langle C,A\rangle - C\langle A,B\rangle \\ & - \langle A,[B,C]\rangle + \langle B,[C,A]\rangle + \langle C,[A,B]\rangle \quad . \end{aligned} \tag{7}$$

It follows that $\square$ is compatible with the Levi-Civita connection ∇ on $M - (D^0 \cup D^\infty)$, in the sense that it holds: $\square_A B(C) = \langle \nabla_A B, C\rangle$.

With respect to the frame (E_i, E_m) the dual connection is determined by the Christopher symbols $\Gamma_{cab} = \square_{E_a} E_b(E_c)$:

$$(\square_X Y)(Z) = X(Y^b) g_{bc} Z^c + \Gamma_{cab} Z^c X^a Y^b \ . \tag{8}$$

for $g_{ab} = g(E_a, E_{b)})$, $X = X^a E_a$,...etc. Explicitly:

$$\Gamma_{cab} = \frac{1}{2} \left\{ \begin{array}{c} E_a(g_{bc}) \quad + \quad E_b(g_{ca}) \quad - \quad E_c(g_{ab}) \\ -g(E_a,[E_b,E_c]) + g(E_b,[E_c,E_a]) + g(E_c,[E_a,E_b]) \end{array} \right\} \tag{9}$$

if the frame is polar-adapted then (g_{ab}) is as (2). In particular note that

$$\Gamma_{kij} \cong 0 \tag{10}$$

since $[E_j, E_k] \in \mathfrak{X}_M(D^\infty)$ and (by Remark 5) $g(E_i, [E_j, E_k]) \cong 0$. Moreover

$$\tau\Gamma_{mij} = \frac{1}{2}\{-\tau g(E_i, [E_j, E_m]) + \tau g(E_j, [E_m, E_i]) + \tau g(E_m, [E_i, E_j])\}$$

writing $[E_a, E_b] = \sum C_{ab}^c E_c$, and taking account the look of (τg_{ab}) we obtain

$$\begin{array}{c}\tau g(E_i, [E_j, E_m]) = \tau g\left(E_i, \sum C_{jm}^a E_a\right) = \tau C_{jm}^i \\ \tau g(E_m, [E_i, E_j]) = \tau g\left(E_m, \sum C_{ij}^a E_a\right) = C_{ij}^m\end{array}$$

but $\left.C_{ij}^m\right|_{D^\infty} = 0$ (since $[E_i, E_j] \in \mathfrak{X}_M(D^\infty)$. Therefore $\left.\tau\Gamma_{mij}\right|_{D^\infty} = 0$ and

$$\Gamma_{mij} \cong 0 \text{ analogous } \Gamma_{kmj} \cong 0,\ \Gamma_{kim} \cong 0 \tag{11}$$

because (again by Remark 5) τg is defined on the whole M. Taking account

$$\tau E_k\left(\frac{1}{\tau}\right) = -\frac{E_k(\tau)}{\tau} \cong 0 \text{ (since } \left.E_k(\tau)\right|_{D^\infty} = 0)$$

we conclude that

$$\begin{array}{l}\frac{1}{2}\left\{-\tau E_k\left(\frac{1}{\tau}\right) - 2\tau g(E_m, [E_m, E_k])\right\} \\ = \tau\Gamma_{kmm} \cong 0 \text{ analogous } \tau\Gamma_{mim} \cong 0,\ \tau\Gamma_{mmj} \cong 0\end{array} \tag{12}$$

However $\left.E_m(\tau)\right|_{D^\infty}$ is non null everywhere and

$$\frac{1}{2}\tau^2 E_m\left(\frac{1}{\tau}\right) = -\frac{E_m(\tau)}{2} = \tau^2\Gamma_{mmm} \tag{13}$$

A first consequence of these computations are:

Proposition 6 *Let $X, Y, Z \in \mathfrak{X}(M)$ then:*

1. *If two of these tree vectorfields are tangent to D^∞ then $(\Box_Z X)(Y) \cong 0$* .
2. *If Z is transversal to D^∞, $X \in \mathfrak{X}_M(D^\infty)$ and $g(X, Z) = 0$ then (locally) $g(X, X)^{-1}(\Box_Z Z)(X) \cong 0$*

3. *If Z is transversal to D^∞ then for any $V \in \mathfrak{X}(D^\infty)$ the function*

$$\beta_Z(V) = \left.\frac{(\Box_Z Z)(X)}{g(Z,Z)}\right|_{D^\infty} \quad (X|_{D^\infty} = V) \tag{14}$$

is independent of $X \in \mathfrak{X}_M(D^\infty)$ such that $X|_{D^\infty} = V$ and $g(X,Z) = 0$. Moreover $\beta_Z \in \Omega^1(D^\infty)$

Proof. First we take an auxiliary polar-adapted frame (E_a) as in (2).

The assert 1. is an easy consequence of the formula (8), using the general expression $X = \sum X^i E_i + \tau h E_m$, of any $X \in \mathfrak{X}_M(D^\infty)$, and taking account that τg, $\tau\Gamma_{kmm}$, $\tau\Gamma_{mij}$,...and $\tau^2\Gamma_{mmm}$ are defined on the whole space M.

To prove 2.and 3 note that by the Theorem 3 we may suppose as well (without lost generality) that $Z = E_m$. Then $Z^m = 1$, $0 = Z^i = X^m$ and $g(Z,Z)^{-1} = \tau$. Applying (8) gives

$$\frac{(\Box_Z Z)(X)}{g(Z,Z)} = \sum \tau\Gamma_{kmm} X^i \cong 0 \tag{15}$$

by (12). On the other hand if $\tau\Gamma_{kmm}|_{D^\infty} = \gamma_k$, we write the 1-form claimed in (14):

$$\beta_{E_m} = \sum \gamma_k \theta_\infty^k \text{ where } \gamma_k = \tau\Gamma_{kmm}|_{D^\infty} \tag{16}$$

where $\left(\theta_\infty^k\right)$ is the dual coframe of $\left(E_k|_{D^\infty}\right)$.This proves 3. ∎

We will require the following

Lemma 7 *For any generic transversal vectorfield $Z \in \mathfrak{X}(M)$ and any smooth ever non null function h we have $\beta_Z = \beta_{hZ}$.*

Proof. Let $X \in \mathfrak{X}(D^\infty)$ be a vectorfield and let $\overline{X} \in \mathfrak{X}(M)$ be such that $\overline{X}\big|_{D^\infty} = X$ and $g\left(\overline{X}, Z\right) = 0$ (thus $g\left(\overline{X}, hZ\right) = 0$). We have

$$\begin{aligned}
\beta_{hZ}(X) &= \left.\frac{(\Box_{hZ} hZ)\left(\overline{X}\right)}{g(hZ,hZ)}\right|_{D^\infty} \\
&= \left.\frac{hZ(h)\, g\left(\overline{X}, Z\right) + h^2 (\Box_Z Z)\left(\overline{X}\right)}{h^2 g(Z,Z)}\right|_{D^\infty} \\
&= \left.\frac{(\Box_Z Z)\left(\overline{X}\right)}{g(Z,Z)}\right|_{D^\infty} = \beta_Z(X)
\end{aligned}$$

∎

2.4 Polar-normal vectorfield.

We say that the vectorfield Z on M is a polar-normal vectorfield if it is transversal to D^{∞} and the associated $1-$form β_Z on D^{∞} is identically null. In order to prove the existence of polar-normal vectorfield, we start with an auxiliary polar-adapted frame (E_i, E_m) as (2) and let $(E_i^{\infty}) = (E_i|_{D^{\infty}})$ be the restriction frame on D^{∞}. As in (16) we have

$$\beta_{E_m}(E_k^{\infty}) = \gamma_k = \tau \Gamma_{kmm}|_{D^{\infty}} \tag{17}$$

We find a transversal to D^{∞} vectorfield

$$\overline{E}_m = \sum \lambda^i E_i + E_m = \widetilde{E}_m + E_m \in \mathfrak{X}(M) \text{ (where } g\left(\widetilde{E}_m, E_m\right) = 0) \tag{18}$$

such that $\beta_{\overline{E}_m}(E_k^{\infty}) = 0$. By the Theorem 3, There exist $\left(\overline{E}_i\right)$ such that $\left(\overline{E}_i, \overline{E}_m\right)$ is polar-adapted frame, and we may suppose without lost generality that $\overline{E}_i\big|_{D^{\infty}} = E_i^{\infty}$. (If not we may find an orthogonal functional matrix $\left(a_j^i\right)$ with $a_j^i \in C^{\infty}(D^{\infty})$ such that $E_j^{\infty} = \sum a_j^i \, \overline{E}_i\big|_{D^{\infty}}$ and we make $\left(\widehat{E}_i, \overline{E}_m\right)$ a polar-adapted frame, with $\widehat{E}_j = \sum A_j^i E_i$ where $A_j^i \in C^{\infty}(M)$ are the unique smooth function such that $A_j^i\big|_{D^{\infty}} = a_j^i$ and $\overline{E}_m\left(A_j^i\right) = 0$). Therefore we may write for some smooth μ_j and $\widetilde{E}_i$

$$\overline{E}_i = \widetilde{E}_i + \tau \mu_i E_m \text{ (where } g\left(\widetilde{E}_i, E_m\right) = 0, \text{ and } \widetilde{E}_i\Big|_{D^{\infty}} = E_i^{\infty}) \tag{19}$$

Taking account (18), (19) and (2) we get

$$0 = g\left(\overline{E}_i, \overline{E}_m\right) = g\left(\widetilde{E}_i, \widetilde{E}_m\right) + \mu_i$$

Since $\widetilde{E}_m\Big|_{D^{\infty}} = \sum g\left(\widetilde{E}_m, E_i^{\infty}\right) E_i^{\infty} = \sum g\left(\widetilde{E}_m, \widetilde{E}_i\right) E_i^{\infty} = \sum -\mu_i E_i|_{D^{\infty}}$, therefore

$$\lambda^i\big|_{D^{\infty}} = -\mu_i|_{D^{\infty}} \tag{20}$$

Now we compute $\beta_{\overline{E}_m}(E_k^{\infty})$ in order to find λ^i 'on D^{∞} which makes these values identically null.

Taking account (18) we have: $\Box_{\overline{E}_m}\overline{E}_m = \Box_{\widetilde{E}_m}\widetilde{E}_m + \Box_{\widetilde{E}_m}E_m + \Box_{E_m}\widetilde{E}_m + \Box_{E_m}E_m$, where

- $\Box_{\widetilde{E}_m}\widetilde{E}_m \cong 0$ by (10)

- $\square_{\widetilde{E}_m} E_m = \sum \lambda^i \square_{E_i} E_m = \sum \lambda^i \Gamma_{kim} \theta^k + \sum \lambda^i \Gamma_{mim} \theta^m$ (where (θ^a) is the dual coframe of (E_a))

- $\square_{E_m} \widetilde{E}_m = \square_{E_m} \left(\sum \lambda^i E_i \right) = \sum E_m \left(\lambda^i \right) \theta^i + \sum \lambda^i \Gamma_{kmi} \theta^k + \sum \lambda^i \Gamma_{mmi} \theta^m$

and we conclude using (11) that for some $\theta \in \Omega^1 (M)$:

$$\square_{\overline{E}_m} \overline{E}_m = \theta + \sum \lambda^i \left(\Gamma_{mim} + \Gamma_{mmi} \right) \theta^m + \square_{E_m} E_m$$

and taking account (14) and that $\overline{E}_k$ is extension of E_k^∞ which it is g-orthogonal to $\overline{E}_m$ we have:

$$\begin{aligned}
\beta_{\overline{E}_m} \left(E_k^\infty \right) &= \left. \frac{\square_{\overline{E}_m} \overline{E}_m}{g \left(\overline{E}_m, \overline{E}_m \right)} \left(\overline{E}_k \right) \right|_{D^\infty} \\
&= \left. \frac{\tau \square_{\overline{E}_m} \overline{E}_m}{1 + \tau \sum \left(\lambda^i \right)^2} \left(\widetilde{E}_k + \tau \mu_k E_m \right) \right|_{D^\infty} \\
&= \left. \frac{\tau \left(\theta + \sum \lambda^i \left(\Gamma_{mim} + \Gamma_{mmi} \right) \theta^m \right) + \tau \square_{E_m} E_m}{1 + \tau \sum \left(\lambda^i \right)^2} \left(\widetilde{E}_k + \tau \mu_k E_m \right) \right|_{D^\infty}
\end{aligned}$$

but

- $\theta^m \left(\widetilde{E}_k \right) = 0$ (by duality),

- $\tau^2 \left. \left(\Gamma_{mim} + \Gamma_{mmi} \right) \right|_{D^\infty} = 0$ and $\tau^2 \left. \square_{E_m} E_m \left(\widetilde{E}_k \right) \right|_{D^\infty} = 0$ (by (12)),

- $\tau \left. \square_{E_m} E_m \left(\widetilde{E}_k \right) \right|_{D^\infty} = \gamma_k$ (see (17)) and

- $\tau^2 \square_{E_m} E_m \left(E_m \right) = - E_m (\tau) / 2$ (by (13)), thus:

$$\beta_{\overline{E}_m} \left(E_k^\infty \right) = \gamma_k - \frac{1}{2} \left(\left. \mu_i \right|_{D^\infty} \right) \left. E_m (\tau) \right|_{D^\infty} \tag{21}$$

and if we want $\beta_{\overline{E}_m} \left(E_k^\infty \right) = 0$ we must to select λ^i such that (see(20))

$$\left. \lambda^i \right|_{D^\infty} = - \frac{2 \gamma_k}{\left. E_m (\tau) \right|_{D^\infty}} \quad \left(\text{where } \gamma_k = \left. \tau \Gamma_{kmm} \right|_{D^\infty} \right) \tag{22}$$

We are now ready to prove the following main theorem:

Theorem 8 *There exist $N \in \mathfrak{X}(M)$ polar-normal vectorfield (that is N is transversal to D^{∞} and $\beta_N = 0$). Moreover if $\overline{N}$ is other polar-normal vectorfield, then N and $\overline{N}$ are proportional along D^{∞}. (Thus a* polar-normal direction *along $D^{\infty}$is canonically determined).*

Proof. We have already proved the existence of polar-normal vectorfield. Only the second part need be to prove:

Continuing with the previous argument let $N = E_m$ be the first polar normal (that is $\beta_{E_m} = 0$, thus $\gamma_k = 0$). Using Lemma 7 we may suppose without lost generality that the other polar-normal $\overline{N} = \overline{E}_m = \widetilde{E}_m + E_m$ is as in (18). Then by (21) and (20) we have:

$$0 = \beta_{\overline{E}_m}\left(E_k^{\infty}\right) = -\frac{1}{2}\left(\lambda^i\big|_{D^\infty}\right) E_m(\tau)|_{D^\infty}$$

and this implies that $\lambda^i|_{D^\infty} = 0$, $\widetilde{E}_m = 0$ and $\overline{N}\big|_{D^\infty} = \overline{N}\big|_{D^\infty}$. ■

3 Polar-adapted coordinates.

The objective of this section is to prove that there exist a (essentially unique) C^{∞}-pregeodesic line traversing D^{∞} for everyone of their points. Moreover each one traverse in the polar normal direction. The proof is analogous to the proof in [6] of the similar result, in the singular context.

Next using these pregeodesics we construct an special C^{∞}-polar-adapted coordinates neighboring each point of D^{∞}.

3.1 Polar-normal pregeodesic.

We start with a polar-normal adapted frame. This means a polar adapted frame (E_a) as in (2) such that E_m is polar-normal. In particular if $\Gamma_{cab} = \square_{E_a} E_b(E_c)$ are the Christopher symbols, we make the *other* Christopher symbols Γ_{ab}^c defined by $\nabla_{E_a} E_b = \Gamma_{ab}^c E_c$ as:

$$\begin{pmatrix} \Gamma_{ab}^1 \\ \\ \Gamma_{ab}^{m-1} \\ \Gamma_{ab}^m \end{pmatrix} = \begin{pmatrix} 1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 1 & 0 \\ 0 & \dots & 0 & \tau \end{pmatrix} \begin{pmatrix} \Gamma_{1ab} \\ \\ \Gamma_{m-1,ab} \\ \Gamma_{mab} \end{pmatrix}$$

This symbols controls the Levi-Civita Connection by the formula

$$\nabla_X Y = \left\{ X^a E_a(Y^c) + \Gamma^c_{ab} X^a Y^b \right\} E_c$$

Since E_m is polar-normal we have $\gamma_k = \tau\Gamma_{kmm}|_{D^\infty} = 0$ and

$$\tau\Gamma_{kmm}\big|_{D^\infty} = \tau\Gamma^k_{mm}\big|_{D^\infty} = 0$$

Recollecting the information of (10), (12), (11) and (13) we obtain:

$$\begin{array}{l} \Gamma^c_{ij} \cong 0,\ \Gamma^m_{ij}\big|_{D^\infty} = 0 \\ \Gamma^c_{mj} \cong 0,\ \Gamma^c_{im} \cong 0 \\ \tau\Gamma^m_{mm} = -\frac{1}{2} E_m\left(\tau\right) \end{array} \tag{23}$$

Now we follow analogous argument that in Theorem 2 of [6]:

We fix a coordinate system (x^i, x^m) which we suppose global (without loss generality), and take on TM mixed coordinates (x^a, u^a), such that the following hold for any $\xi \in T_pM$ (and any $p \in M$)

$$x^a\left(\xi\right) = x^a\left(p\right), \xi = \sum u^a\left(\xi\right) E_a\left(p\right)$$

Also we have the induced $\left(x^a, \dot{x}^a\right)$ pure coordinates on TM with

$$\xi = \sum \dot{x}^a\left(\xi\right) \partial_{x^a}$$

Let $\pi : TM \rightarrow M$ be the canonical projection given locally by $(x^a, u^a) \rightarrow (x^a)$.

The geodesic spray is the vectorfield Γ on TM whose integral curves project down to the geodesics of M. Using mixed coordinates we may write:

$$\Gamma = \sum \dot{x}^a \partial_{x^a} - \sum \Gamma^c_{ab} u^a u^b \partial_{u^c}$$

The projection on M of the integral curves of $S = \tau\Gamma$ are pregeodesics and we get:

$$\begin{aligned} S = & \left(\tau\dot{x}^a\right)\partial_{x^a} - \left(\tau\Gamma^c_{ij}\right) u^i u^j \partial_{u^c} - \left(\tau\Gamma^a_{mj} + \tau\Gamma^a_{jm}\right) u^m u^j \partial_{u^a} \\ & - \tau\Gamma^k_{mm}\left(u^m\right)^2 \partial_{u^k} - \tau\Gamma^m_{mm}\left(u^m\right)^2 \partial_{u^m} \end{aligned}$$

and using (23) we obtain

$$S\left(E_m\right)|_{D^\infty} = \frac{1}{2}\, E_m\left(\tau\right)|_{D^\infty}$$

Let h be:

$$h = E_m\left(\tau\right)/2 \text{ (suppose for example } h < 0) \tag{24}$$

we consider now the vectorfields

$$\begin{array}{l} H = hu^a \partial_{u^a} \\ A = \dot{x}^a \partial_{x^a} - \sum_{(a,b,c) \neq (m,m,m)} \Gamma^c_{ab} u^a u^b \partial_{u^c} \\ B = h\left(u^m\right)^2 \partial_{u^m} \end{array}$$

and construct

$$\widetilde{S} = S - H = \tau A + B - H$$

and still the integral curves of $\widetilde{S}$ project down on pregeodesics in M.

Now fix $p \in D^\infty$ Then $\xi = E_m\left(p\right)$ is stationary point of $\widetilde{S}$ since

$$\tau\left(p\right) = 0, u^i\left(\xi\right) = 0, u^m\left(\xi\right) = 1 \tag{25}$$

and

$$\widetilde{S}\left(\xi\right) = \tau\left(p\right) A\left(\xi\right) + B\left(\xi\right) - H\left(\xi\right) = 0 + h\left(p\right)\partial_{u^m} - h\left(p\right)\partial_{u^m} = 0$$

We linearize $\widetilde{S}$ at ξ to obtain $D\widetilde{S}\Big|_\xi : T_\xi TM \to T_\xi TM$. Since $D\widetilde{S} = d\tau \otimes A + \tau DA + DB - DH$ and taking account (25) and

$$\begin{array}{c} DB = dh \otimes \left(u^m\right)^2 \partial_{u^m} + 2hu^m du^m \otimes \partial_{u^m} \\ DH = dh \otimes u^a \partial_{u^a} + h \otimes du^a \otimes \partial_{u^a} \end{array}$$

we have

$$\begin{aligned} D\widetilde{S}\Big|_\xi = & \left(d\tau|_p \circ \pi_*\right) \otimes \left(\xi - \sum \Gamma^k_{mm} \partial_{u^k}\right) \\ & + h\left(p\right) du^m \otimes \partial_{u^m} - h\left(p\right) du^i \otimes \partial_{u^i} \end{aligned}$$

We remark that we have identify $\xi \in TM$ with $\xi \in T_\xi TM$ through the (x^a)-canonical immersion $T_pM \hookrightarrow T_\xi TM$ such that $\partial_{x^a}|_p \to \partial_{x^a}|_\xi$. We compute now the eigenspaces to apply later an stable manifold theorem.

- If $\eta \in T_pD^\infty \hookrightarrow T_\xi TM$ then $0 = d\tau(\eta) = u^a(\eta)$ and $\left. D\widetilde{S} \right|_\xi (\eta) = 0$.

- If $\eta = \partial_{u^i}|_\xi$ then $0 = dx^i(\eta) = d\tau(\eta) = du^a(\eta)$ $(a \neq i)$ and $du^i(\eta) = 1$ thus $\left. D\widetilde{S} \right|_\xi (\eta) = -h(p)\eta$

- If $\eta = \partial_{u^m}|_\xi$ then $\left. D\widetilde{S} \right|_\xi (\eta) = h(p)\eta$

- $\left. D\widetilde{S} \right|_\xi (\xi) = 2h(p)\left(\xi - \sum \Gamma^k_{mm}(p)\partial_{u^k}\right)$. It is easy to prove that there exist an eigenvector associated to eigenvalue $2h(p)$ with the look $\eta = \xi - \sum c_i \partial_{u^i}$ for some constant c_i.

Collected the information we have eigenvalues $0, -h(p), h(p), 2h(p)$ with multiplicity $m-1$, $m-1$, 1 , 1respectively. (their are $2m$ counting their multiplicities). Because the eigenvalue $2h(p)$ is smaller than (and not equal to) any other negative eigenvalue we conclude (by certain refinement of the stable manifold theorem) that there exist a $\widetilde{S}$-stable line $\widetilde{L} \subset TM$ with $\xi \in \widetilde{L}$ and $T_\xi \widetilde{L} = Span\left(\xi - \sum c_i \partial_{u^i}\right)$. The projection $L = \pi\left(\widetilde{L}\right)$ of this stable line sweeps out the smoothly immersed pregeodesic.

3.2 The natural equation for D^∞.

We find a (locally) coordinate system (z^a) such that $(g_{ab}) = (g(\partial_{z^a}, \partial_{z^b}))$ is as

$$(g_{ab}) = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & 1/z^m \end{pmatrix} \tag{26}$$

Note that in these coordinates ∂_{z^m} be a polar normal vectorfield and their integral curves $\overline{\gamma} = \overline{\gamma}(s) : \{z^i = cte,\ z^m = s\}$ are the pregeodesics of the previous section. Thus the z^m $(= s)$-coordinate parametrize the pregeodesic γ in such way that

$$\left. g(\partial_{z^m}, \partial_{z^m}) \right|_\gamma = g\left(\frac{d\overline{\gamma}}{ds}, \frac{d\overline{\gamma}}{ds}\right) = \frac{1}{s} \tag{27}$$

Therefore as previous question, we analyze the existence of such (canonical) parametrization. Next we will construct from the parameter s (of obvious way) a natural equation $(z^m = 0)$ for D^∞.

We start with a fixed parametrization on the polar $D^\infty-$transversal pregeodesic $\gamma = \gamma(t)$ defined for $|t| < \varepsilon$, with $\gamma(0) = p \in D^\infty$. Using by example a polar normal vectorfield which has γ as integral curve, is easy to see that the function

$$\Phi(t) = \frac{1}{g(\gamma'(t), \gamma'(t))}, \text{ for } 0 < |t| < \varepsilon,\ \Phi(0) = 0$$

is a C^∞-function on the whole interval $(-\varepsilon, \varepsilon)$ .with $\Phi'(0) \neq 0$ and we may write

$$\Phi(t) = t\Psi(t)$$

for some C^∞-function Ψ such that $\Psi(t) \neq 0$ for $|t| < \varepsilon$ (suppose for example $\Psi(t) > 0$) . Let

$$\psi(t) = \frac{1}{2\sqrt{\Psi(t)}} \tag{28}$$

We find a new parameter $s = \mathbf{s}(t)$, with inverse $t = \mathbf{t}(s)$, such that the reparametrized curve $\overline{\gamma}(s) = \gamma(\mathbf{t}(s))$ verify (27). Thus

$$\frac{1}{s} = g\left(\frac{d\overline{\gamma}}{ds}, \frac{d\overline{\gamma}}{ds}\right) = \left(\frac{d\mathbf{t}}{ds}\right)^2 g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right)$$

and therefore the function $s = \mathbf{s}(t)$ satisfy the differential equation of separate variables

$$\frac{ds}{\sqrt{s}} = \frac{\psi dt}{2\sqrt{t}}$$

integrating both members we have an explicit solution:

$$\mathbf{s}(t) = sgn(t)\left(\int_0^t \frac{\psi dx}{\sqrt{x}}\right)^2 \tag{29}$$

Lemma 9 *Let $\psi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a C^∞-function. Then $\mathbf{s}(t)$ defined in $(-\varepsilon, \varepsilon)$ as (29) as well is a C^∞-function*

Proof. See appendix ■

Now there exist an unique smooth function z^m defined over a neighborhood of D^∞ such that $z^m(\overline{\gamma}(s)) = s$, where $\overline{\gamma} = \overline{\gamma}(s)$ is any normal pregeodesic and s is their natural parameter.

Remark 10 *Using the definition of ψ in (28) we have*

$$\frac{\psi(t)}{\sqrt{t}} = \pm\frac{1}{2}\sqrt{|g(\gamma'(t), \gamma'(t))|}$$

and for $x = \gamma(t_0)$ belonging to such neighborhood $4z^m(x) = \mathbf{s}(t_0)$ is the signed square of the the arc length (if $t_0 > 0$), or of the proper time (if $t_0 \leq 0$) to the normal pregeodesic segment between x and D^∞

3.3 The natural polar coordinates.

Rewriting with more formalism the end of the previous section we may say that:

For some $\varepsilon > 0$ there exist a smooth function $\zeta : D^\infty \times (-\varepsilon, \varepsilon) \to M$, such that $s \to \zeta(x, s)$ ($|s| < \varepsilon$) is the normal pregeodesic at $x = \zeta(x, 0) \in D^\infty$ with the natural parametrization. Since ζ is non singular at the points $(x, 0)$, we may suppose (replacing M by some neighborhood of D^∞) that ζ is diffeomorphism. We write the inverse as $\zeta^{-1} = (\sigma, z^m) : M \to D^\infty \times (-\varepsilon, \varepsilon)$, and the vectorfield ∂_{z^m} defined by

$$\partial_{z^m}|_p = \left.\frac{\partial\zeta}{\partial s}\right|_{(\sigma(p), z^m(p))}$$

is called polar-normal pregeodesic vectorfield. Note that using an auxiliary polar normal frame (E_i, ∂_{z^m}) and the last equation of (23) (here is $\tau = z^m$) we see that

$$\nabla_{\partial_{z^m}}\partial_{z^m} = -\frac{1}{2z^m}\partial_{z^m} \tag{30}$$

Moreover by (27) we have

$$g(\partial_{z^m}, \partial_{z^m}) = \frac{1}{z^m} \tag{31}$$

We start now with a coordinate system (x^i) on D^∞ We will prove that the coordinates (z^i, z^m) on M where $z^i = x^i \circ \sigma$, are natural coordinates, that is (g_{ab}) is as (26). Taking account (31) we have

$$0 = \frac{\partial}{\partial z^i}\left(\frac{1}{z^m}\right) = \frac{\partial}{\partial z^i} g(\partial_{z^m}, \partial_{z^m}) = 2\square_{\partial_{z^i}}\partial_{z^m}(\partial_{z^m})$$

and therefore, using also (30)

$$\frac{\partial g_{im}}{\partial z^m} = \frac{\partial}{\partial z^m} g\left(\partial_{z^i}, \partial_{z^m}\right) = g\left(\nabla_{\partial_{z^m}} \partial_{z^m}, \partial_{z^i}\right)$$
$$= -\frac{1}{2z^m} g_{im}$$

By the Corollary 4, g_{im} is a differentiable function on M. In order to prove that $g_{im} = 0$, we fix the variables z^i. The smooth function $\phi(t) = g_{im}\left(z^i, t\right)$ defined on some open interval around zero satisfy.

$$-2t\frac{d\phi}{dt} = \phi \text{ for all } t \tag{32}$$

the following Lemma proves that but always that ϕ is identically null

Lemma 11 *Let $\phi : I \to \mathbb{R}$ a smooth function defined over some open interval I around zero which satisfy (32). Then $\phi(t) = 0\ \forall t$.*

Proof. First note that by (32) is $\phi(0) = 0$. Suppose that there exist $t_0 \in I$ with $\phi(t_0) \neq 0$ (for example $\phi(t_0) > 0$.) Let $J = (a, b) \subset I \cap \mathbb{R}^+$ an open maximal interval containing t_0 such that $\phi(t) > 0$for all $t \in J$. Then using the differential equation (32) it is easy to see that there exist a constant $C > 0$ such that $\phi(t) = C/\sqrt{t}$. Then $a > 0$ (if not $\lim_{t\to 0^+} \phi(t) = +\infty \neq 0 = \phi(0)$), but then $\phi(a) = C/\sqrt{a} > 0$ and thus J is not maximal. ■

Therefore we have established the existence of polar-normal coordinates as explain in the following

Theorem 12 *Around each point of D^∞ there exist a coordinate system (z^i, z^m) such that $(g_{ij}) = ((\partial_{z^i}, \partial_{z^j}))$ is as (26)*

4 The Curvature near to D^∞.

In order to analyze the limiting behavior of the curvatures on $M - D^\infty$ as we approach the polar hypersurface D^∞, we fix a polar-normal coordinate system $(z^i, z^m = \tau)$ whose domain is the whole space M (if not we restrict to the domain). Let (g_{ab}) be as (26) with $z^m = \tau$. The inverse is

$$(g_{ab})^{-1} = \left(g^{ab}\right) = \begin{pmatrix} (g^{ij}) & 0 \\ 0 & \tau \end{pmatrix}$$

Recall that here the Christopher symbols are

$$\Gamma_{cab} = \left(\Box_{\partial_{z^a}} \partial_{z^b}\right)\left(\partial_{z^c}\right) = \frac{1}{2}\left\{\frac{\partial g_{ac}}{\partial z^b} + \frac{\partial g_{bc}}{\partial z^a} - \frac{\partial g_{ab}}{\partial z^c}\right\}$$
$$\Gamma^c_{ab} = \sum \Gamma_{dab} g^{dc}$$

The contravariant curvature is

$$R\left(A,B\right)C = \nabla_A \nabla_B C - \nabla_B \nabla_A C - \nabla_{[A,B]} C$$

their components R^d_{cab} are defined by $R\left(\partial_{z^a}, \partial_{z^b}\right)\partial_{z^c} = \sum R^d_{cab}\partial_{z^d}$ and are given by

$$R^d_{cab} = \frac{\partial \Gamma^d_{bc}}{\partial z^a} - \frac{\partial \Gamma^d_{ac}}{\partial z^a} + \sum \Gamma^e_{bc}\Gamma^d_{ae} - \sum \Gamma^e_{ac}\Gamma^d_{be}$$

The covariant Ricci tensor is $Ric\left(A,B\right) = tr\left\{V \to R\left(A,V\right)CB\right\}$ which has components

$$R_{ca} = \sum R^d_{cad},\ R^d_c = \sum R_{ca} g^{ad}$$

where R^d_c are the components of the contravariant Ricci tensor defined by the identity $Ric\left(A,B\right) = g\left(RIC\left(A\right), B\right)$.

Finally the components of the Weil curvature tensor W are:

$$W^d_{cab} = R^d_{cab} + \frac{1}{m-2}\left(\delta^d_a R_{cb} - \delta^d_b b R_{ca} + g_{cb} R^d_a - g_{ca} R^d_b\right) + \frac{S}{\left(m-1\right)\left(m-2\right)}\left(\delta^d_b g_{ca} - \delta^d_a g_{cb}\right)$$

where S is the scalar curvature.

$$S = tr\left(RIC\right) = \sum R^c_c$$

By inspection of previous formulas we have the following:

Lemma 13 *In the local natural coordinates we have:*

1. $\Gamma_{cab} \cong 0$ *except* $\Gamma_{mmm} = -1/\tau^2$
2. $\Gamma^c_{ab} \cong 0$ *except* $\Gamma^m_{mm} = -1/\tau$
3. $R^d_{abc} \cong 0$, *except perhaps* $-R^i_{mmj} = R^i_{mjm} \cong -\Gamma^i_{jm}/\tau$

4. $R_{ab} \cong 0$,*except perhaps* $R_{mm} = \sum R^j_{mmj}$, *but* $\tau R_{mm} \cong 0$

5. $R^b_a \cong 0$, $S \cong 0$

6. $W^d_{abc} \cong 0$ *except perhaps* W^d_{mjm} *but* $\tau W^d_{mjm} \cong 0$

As consequence we have the following

Theorem 14 *Let* R, Ric, RIC, W *be the previous curvature tensors of* $(M - D^\infty, g)$, *and* $X, Y, Z \in \mathfrak{X}(M)$, *and let* τ *be the natural equation of* D^∞. *Denote by* R^∞,*...etc. the corresponding tensors to the Riemannian space* D^∞

1. τR, τRic, RIC, τW *are* $C^\infty-$*tensorfields defined around* D^∞

2. $R(X,Y)Z \cong 0$ *if* $Z \in \mathfrak{X}_M(D^\infty)$, *or* $X, Y \in \mathfrak{X}_M(D^\infty)$. *Moreover if* $X, Y, Z \in \mathfrak{X}_{D^\infty}(M)$ *then* $R(X,Y)Z|_{D^\infty} = R^\infty(X,Y)Z$

3. $Ric(X,Y) \cong 0$ *if* X *or* Y *are tangent to* D^∞.

4. $W(X,Y)Z \cong 0$ *if* $Z \in \mathfrak{X}_M(D^\infty)$, *or* $X, Y \in \mathfrak{X}_{D^\infty}(M)$

Proof. It is straightforward. For example we will prove 2:
Note that $R(\partial_{z^a}, \partial_{z^b})\partial_{z^c} = \sum R^d_{cab}\partial_{z^d} \cong 0$ if $c \neq m$ or $c = m$, $a \neq m$,and $n \neq m$. Writing $X = \sum X^a \partial_{z^a}$ ...etc. then

$$\begin{aligned} R(X,Y)Z &= X^a Y^b Z^i R(\partial_{z^a}, \partial_{z^b})\partial_{z^i} + X^i Y^j Z^m R(\partial_{z^i}, \partial_{z^j})\partial_{z^m} \\ &\cong X^m Y^j Z^m R(\partial_{z^m}, \partial_{z^j})\partial_{z^m} + X^i Y^m Z^m R(\partial_{z^i}, \partial_{z^m})\partial_{z^m} \end{aligned}$$

But if Z is tangent to D^∞ is $Z^m = \tau C^m$ for some smooth C^m...etc. The proof finish taking account that $\tau R \cong 0$ ■

Remark 15 *It is easy to see that on the natural coordinates* (z^i, τ), *the condition to* $R \cong 0$ *is equivalent to*

$$\left.\frac{\partial g_{ij}}{\partial z^m}\right|_{z^m=0} = 0$$

and this condition may be surely expressed without coordinates.

5 Conformal Geometry.

We consider the conformal class $(M, \mathcal{C})$ of a Riemann-Lorentz space (M, g) with polar end D^∞. We recall that

$$\mathcal{C} = \left\{ e^{2\sigma} g : \sigma \in C^\infty (M) \right\}$$

We remark that $(M, \overline{g})$ is also a Riemann-Lorentz space with polar end D^∞ for any $\overline{g} \in \mathcal{C}$.

Of course the polar-normal pregeodesic introduced in the section 3.1 are not determined by the conformal class $\mathcal{C}$. Also the polar-normal direction on D^∞ is not determined by $\mathcal{C}$. In fact, if (z_i, z_m) are the canonical coordinates associated to g, then $\overline{g} = fg$ with $f = f(z_i, z_m) > 0$, then $(\partial_{z_i}, \partial_{z_m})$ is still a polar (not orthonormal) frame and (see subsection 2.4)

$$\overline{\Gamma}_{kmm} = \frac{1}{2} \frac{\partial (f/z_m)}{\partial z_k} = \frac{1}{2 z_m} \frac{\partial f}{\partial z_k}$$

therefore $z_m \overline{\Gamma}_{kmm}\big|_{z_m=0}$ may to take any arbitrary value moving f. Of course the polar normal direction remains invariant if $\left.\dfrac{\partial f}{\partial z_k}\right|_{z_m=0} = 0$.

However we will prove that the family of the polar-normal pregeodesics are the same for all the metrics of the conformal subclass

$$\mathcal{C}'_g = \left\{ e^{2\sigma} g : \sigma = f \circ \tau_g \text{ with } f \in C^\infty (\mathbb{R}) \right\} \tag{33}$$

where $\tau_g = 0$ is the canonical equation for D^∞ (with respect to (M, g)) established in section 3.2.Moreover $\mathcal{C}' = \mathcal{C}'_g$ depends only to the family of the polar-normal pregeodesic and not of the initial metric g. This means that: if $g, \overline{g} \in \mathcal{C}$ then $\overline{g}$ has the same polar normal pregeodesics that g, if and only if $\overline{g} \in \mathcal{C}'_g$.

On the other hand the family of hypersurfaces D^∞_t with equation $(\tau_g = t)$ depends only to $\mathcal{C}'_g$. This is the family of the *simultaneity hypersurfaces* which determines the *simultaneity distribution* $\mathcal{D}_g$. We define a (abstract) simultaneity distribution as a completely integrable $(m-1)$-distribution $\mathcal{D}$ such that D^∞ is an integral manifold.

Let N be an everywhere non isotropic vectorfield on M, transverse to D^∞. We define the distribution $N^\perp$ as $N^\perp(p) = N(p)^\perp$ if $p \in M - D^\infty$, and $N^\perp(p) = T_p D^\infty$ if $p \in D^\infty$.

Lemma 16 *If N is an everywhere non isotropic vectorfield on M transverse to D^{∞}, then $N^{\perp}$ is a smooth distribution, and for any $g \in \mathcal{C}$, the function $g(N,N)^{-1}$ extend to D^{∞}, and $g(N,N)^{-1} = 0$ is an equation for D^{∞}. Reciprocally if $\mathcal{D}$ is a simultaneity distribution on the conformal space $(M, \mathcal{C})$, then there exist N non isotropic vectorfield on M transverse to D^{∞}such that $\mathcal{D} = N^{\perp}$.*

Proof. Fix any $g \in \mathcal{C}$ and let (z_i, z_m) be polar normal coordinates around D^{∞} this means that

$$(g_{ab}) = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & 1/z_m \end{pmatrix}$$

suppose

$$N = \sum \lambda_i \partial_{z_i} + \lambda_m \partial_{z_m} \tag{34}$$

The non isotropic condition for N assure that $N^{\perp}$ is a $(m-1)$ −distribution away D^{∞}. We may describe $N^{\perp}$ on $M - D^{\infty}$ as the family of vectorfields $X = \sum X_i \partial_{z_i} + X_m \partial_{z_m}$, such that

$$\sum_j \Lambda_j X_j + \lambda_m X_m = 0 \text{ with } \Lambda_j = \sum_i z_m g_{ij} \lambda_i \tag{35}$$

Note that the coefficients Λ_j and λ_m are smooth on M. But for $z_m = 0$ are $\Lambda_j(z_i, 0) = 0$, and $\lambda_m(z_i, 0) \neq 0$ (by transversality). Previous equation gives $X_m = 0$. This means that the vectorfield X is tangent to D^{∞} and that the condition (35) define the whole distribution $N^{\perp}$. This proves that $N^{\perp}$ is smooth. Also

$$\frac{1}{g(N,N)} = \frac{z_m}{z_m \Lambda + \lambda_m^2} \text{ with } \Lambda = \sum_i g_{ij} \lambda_i \lambda_j$$

since $\lambda_m(z_i, 0) \neq 0$, this proves that $g(N,N)^{-1}$ extend to D^{∞}, and $g(N,N)^{-1} = 0$ is an equation for D^{∞}.

We may describe a simultaneity distribution $\mathcal{D}$ in the polar normal coordinates (z_i, z_m) as the vector fields $X = \sum X_i \partial_{z_i} + X_m \partial_{z_m}$ which satisfy a condition as

$$\sum_j \Lambda_j X_j + \lambda_m X_m = 0 \tag{36}$$

Since D^{∞} is integral manifold we conclude that $\Lambda_j(z_i, 0) = 0$ and there exist smooth functions μ_j such that

$$\Lambda_j = z_m \mu_j$$

since (g_{ij}) is nonsingular there exist λ_i such that $\mu_j = \sum g_{ij}\lambda_i$, and (36) becomes in (35), select N as in (34) we conclude $\mathcal{D} = N^{\perp}$. ■

Theorem 17 *Given an abstract simultaneity distribution $\mathcal{D}$ then there exist $\overline{g} \in \mathcal{C}$ such that $\mathcal{D} = \mathcal{D}_{\overline{g}}$.*

Proof. We take an auxiliary metric $g \in \mathcal{C}$. By previous Lemma we may select an everywhere non null vectorfield N such that $g(N,N)^{-1} = 0$ is an equation for D^{∞}, and $N^{\perp} = \mathcal{D}$. We fix a point $x_0 \in D^{\infty}$, and let $\gamma_{x_0} : (-c,c) \to M$ *any* regular parametrization of the integral curve α_{x_0} of N which $\alpha_{x_0}(0) = x_0 = \gamma_{x_0}(0)$ (we may take for example $\gamma_{x_0} = \alpha_{x_0}$). Let Σ_t the integral manifold of $\mathcal{D}$ by $\gamma_{x_0}(t)$. For any $x \in \Sigma$ we parametrize the integral curve α_x of N by $\gamma_x : (-c,c) \to M$ such that $\gamma_{x_0}(t) \in \Sigma_t$. Let $\Phi : \Sigma \times (-c,c) \to M$ be such that $\Phi(x,t) = \gamma_x(t)$. It is straightforward to see that Φ is not singular over points of $\Sigma \times \{0\}$, and we may suppose without lost generality that Φ is diffeomorphism. Using the inverse $\Phi^{-1} : M \to D^{\infty} \times (-c,c)$ we may make the unique coordinate system (x_i, x_m) on M, such that

$$\Phi : \begin{cases} x_i = u_i \\ x_m = t \end{cases}$$

are the equations of Φ (here (u_i) are a fixed coordinate system on D^{∞} and t is the coordinate on $(-c,c)$). Since the curves $\{x_i = cte,\, x_m = t\}$ are preintegral curves of N we conclude that $\partial_{x_m} = e^{-\varphi}N$ for some smooth φ. Also since (for a fixed t) $x_m = t$ is the equation of Σ_t then $\partial_{x_i} \in \mathcal{D} = \partial_{x_m}^{\perp}$ and the metric g in coordinates (x_i, x_m) has the matrix

$$(g_{ab}) = \begin{pmatrix} (g_{ij}) & 0 \\ 0 & e^{-2\varphi} g(N,N) \end{pmatrix}$$

Since $g(N,N)^{-1} = 0$ is an equation for D^{∞}, as $x_m = 0$ we conclude that $g(N,N) = h/x_m$ where $h > 0$ everywhere. then the matrix of $\overline{g} = e^{2\varphi}h^{-1}g$ with respect to such coordinates are

$$(\overline{g}_{ab}) = \begin{pmatrix} (e^{2\varphi} g_{ij}) & 0 \\ 0 & 1/x_m \end{pmatrix}$$

and we conclude that (x_i, x_m) are polar normal coordinates for $\overline{g}$, and $\mathcal{D} = \mathcal{D}_{\overline{g}}$. ■

Remark 18 *With the hypothesis of previous theorem, the same argument proves that the class* $\mathcal{C}_{\mathcal{D}}$ *of all* $\overline{g} \in \mathcal{C}$ *such that* $\mathcal{D} = \mathcal{D}_{\overline{g}}$ *it is equal to* $\mathcal{C}'_g$*. The key is that we are free to parametrize* γ_{x_0}*. This means that the previous coordinate* x_m *(and therefore* $\overline{g}$*) is determined up composition by arbitrary diffeomorphism.* $f \in C^{\infty}(\mathbb{R})$

5.1 Cosmological remarks.

We consider the conformal class $(M, \mathcal{C})$ of a Riemann-Lorentz space (M, g) with polar end D^{∞}. This is the support to a causality structure of the Lorentz component D^{-}. The aim of this section, is to know if it is possible to find a big-bang.*cosmologically* privileged metric (around D^{∞}). Of Course we must to impose to $(M, \mathcal{C})$ some initial restriction as for example that should be D^{∞} conformal flat.

We recall that a Robertson-Walker space is a warped product $I \times_f S = (I \times S, g_{RW})$ where $I = (0, t^*)$ is an open interval and $f : I \to \mathbb{R}$ is a smooth function and (S, g_S) is a Riemannian manifold with constant sectional curvature.C_0 Finally

$$g_{RW} = -dt^2 + f(t)^2 g_S = f(t)^2 g$$

where $g = -f(t)^{-2} dt^2 + g_S$. Recall that $\{t\} \times S$ are simultaneity hypersurfaces of constant curvature $C(t) = C_0 f(t)^{-2}$.

We remark that the flow $\zeta_t : S \to \{t\} \times S$, $x \to (t, x)$ are homoteties of ratio $f(t)^2$

This suggest that in (M, g), near to D^{∞} the metric $g_c = -(\tau_g) g$ is the cosmologically relevant one. In fact we have:

Proposition 19 *Let* (M, g) *be a four dimensional Riemann-Lorentz space with polar end* D^{∞} *the flow* $\zeta^g : D^{\infty} \times (-\varepsilon, \varepsilon) \to M$ *in section 3.3 moves* D^{∞} *by*

$$\zeta^g_t : D^{\infty} \to D^{\infty}_t = \zeta^g(D^{\infty} \times \{t\})$$

Suppose that D^{∞} *has constant (Riemannian) curvature and* $\zeta^g_t : D^{\infty} \to D^{\infty}_t$ *are homoteties. Then the simultaneity distribution* $\mathcal{D}_g$ *has integral manifolds which are of constant curvature*[2] *, and* $g_c = -(\tau_g) g$ *becomes (locally)* D^{-} *into a Robertson Walker space.*

[2]Note that such property depends only of the conformal subclass $\mathcal{C}'_g$.

Note that g_c induces the same causality structure as $[g]$ on D^-. However $g_c \notin \mathcal{C}_g$ since τ_g is null over D^∞.

Of course the physical relevant metric $g_c = -(\tau_g)\, g$ is not determined by the conformal structure $\mathcal{C} = \mathcal{C}_g$. but neither by the restricted conformal class $\mathcal{C}'_g$. However the physical relevant metric are determined up the selection of an *universal time* τ in $(M, \mathcal{C})$, as we was proved in Remark 18

Remark 20 *To say that the integral hypersurfaces of a simultaneity distribution $\mathcal{D}$ are of constant curvature has meaning into the conformal Riemann Lorentz space space $(M, \mathcal{C})$. This means that their integral hypersurfaces has constant curvature with respect to any (or some) g belonging to the restricted conformal class $\mathcal{C}'_{\mathcal{D}}$ induced by $\mathcal{D}$ according Theorem 17*

Finally we set out the following conjecture that explain the philosophical motivation mentioned at the beginning of this subsection.

Conjecture 21 *Let $(M, \mathcal{C})$ be a Riemann-Lorentz conformal space with polar end D^∞. Suppose that D^∞ is conformal- flat. Then there exist (locally) a simultaneity distribution $\mathcal{D}$ of constant curvature. Moreover $\mathcal{D}$ is univocally determined.*

Note that by Remark 18 we find (locally) a equation $(\tau = 0)$ of D^∞ whose level hypersurfaces $(\tau = cte)$ are of constant curvature (as we explain in the Remark 20). Then the conjecture says that the equation $(\tau = 0)$ of D^∞ is univocally determined by the constant curvature condition up diffeomorphism. $\phi : \mathbb{R} \to \mathbb{R}$.

6 Appendix

This appendix is devoted to prove the following result:

Lemma 22 *Let $\psi : I_\varepsilon = (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a C^∞-function. Then $F(t)$ defined in I_ε as*

$$F(t) = \epsilon(t) \left(\int_0^t \frac{\psi dx}{\sqrt{x}} \right)^2$$

is also a C^∞-function,.where ϵ is the sign function ($\epsilon(x) = 1$ if $x > 0$, $\epsilon(x) = -1$ if $x < 0$, and $\epsilon(0) = 0$)

Proof. To simplify we will denote the functions without reference to the variable, $I_\varepsilon^* = I_\varepsilon - \{0\}$, J denote the absolute value function (that is $J(x) = |x| = \epsilon(x)\,x$ and $\int g$ denote

$$\left(\int g\right)(t) = \int_0^t g \text{ for } |t| < \varepsilon, \text{ and } g \text{ integrable in } (-\varepsilon, \varepsilon)$$

$g^{(n)}$ is the n-th derivate, and g^r the r-th exponential of g. For example we get for $k = 0, 1, 2, \ldots$

$$\left(J^{-\frac{2k+1}{2}}\right)^{(1)} = -\frac{2k+1}{2}\epsilon J^{-\frac{2k+3}{2}} \tag{37}$$

and integrating by parts we have

$$\int \left(J^{\frac{2k-1}{2}}\Psi\right) = \frac{2\epsilon}{2k+1}\left(J^{\frac{2k+1}{2}}\Psi - \int \left(J^{\frac{2k-1}{2}}\Psi^{(1)}\right)\right) \tag{38}$$

It is suffice to prove that for any integer $k \geq 0$ there exist $G_k \in C^\infty(I_\varepsilon)$ and constant coefficients a_i such that

$$\begin{array}{c} B_k = \sum_{i=0}^k \epsilon^{i+1} a_i \psi^{(i)} J^{-\frac{2(k-i)+1}{2}} \text{ and} \\ F^{(k+1)} = G_k + F_k, \text{ where } F_k = B_k \int \left(J^{\frac{2k-1}{2}}\psi^{(k)}\right) \end{array} \tag{39}$$

since $\lim_{t\to 0} F_k(t)$ there exist and it is finite. In fact by L'Hôpital rule we see that

$$m \geq n-1 \geq 0 \Rightarrow \exists \lim_{t\to 0} \frac{\int (J^m\Psi)}{J^n} = \lim_{t\to 0}\frac{J^m\Psi}{nJ^{n-1}} \in \mathbb{R}$$

and (assuming (39)), this means that $\lim_{t\to 0} F_k(t) \in \mathbb{R}$ when $k > 0$.

Moreover for $k = 0$

$$F^{(1)} = B_0 \int \left(J^{-\frac{1}{2}}\psi\right) = F_0,\ (G_0 = 0,\ B_0 = 2\epsilon J^{-\frac{1}{2}}\psi) \tag{40}$$

Applying (38) $\int \left(J^{-\frac{1}{2}}\psi\right) = 2\epsilon\left(J^{\frac{1}{2}}\psi - \int J^{\frac{1}{2}}\psi^{(1)}\right)$ we see that

$$F_0 = 4\psi^2 - 4J^{-\frac{1}{2}}\psi \int J^{\frac{1}{2}}\psi^{(1)}$$

by L'Hôpital rule:

$$\lim_{t\to 0}\frac{\int\left(J^{\frac{1}{2}}\psi^{(1)}\right)}{J^{\frac{1}{2}}}=\frac{1}{2}\lim_{t\to 0}\frac{J^{\frac{1}{2}}\psi^{(1)}}{J^{-\frac{1}{2}}}=\frac{1}{2}\lim_{t\to 0}J\psi^{(1)}=0$$

and there exist $\lim_{t\to 0}F_0\in\mathbb{R}$. Also this proves the existence of F_k and G_k as in (39) for $k=0$.

Assuming the existence of B_k and G_k as in (39), we proceed by induction. In order to construct B_{k+1} we derive F_k and we get:

$$F_k^{(1)}=B_kJ^{\frac{2k-1}{2}}\psi^{(k)}+B_k^{(1)}\int\left(J^{\frac{2k-1}{2}}\psi^{(k)}\right)$$

a computation using (37), (38) gives for some constant coefficients b_i and c_i

$$B_k^{(1)}=-a_0\left(\frac{2k+1}{2}\right)J^{-\frac{2k+3}{2}}\psi+\sum_{i=1}^{k+1}\epsilon^ib_iJ^{\frac{2(k-i)+3}{2}}\psi^{(i)}$$

$$\int\left(J^{\frac{2k-1}{2}}\psi^{(k)}\right)=\frac{2\epsilon}{2k+1}\left(J^{\frac{2k+1}{2}}\psi^{(k)}-\int\left(J^{\frac{2k+1}{2}}\psi^{(k+1)}\right)\right)$$

$$\begin{aligned}B_k^{(1)}\int\left(J^{\frac{2k-1}{2}}\psi^{(k)}\right)=&-a_0\epsilon J^{-1}\psi\psi^{(k)}+\sum_{i=1}^{k+1}c_i\left(\epsilon J\right)^{i-1}\psi^{(i)}\psi^{(k)}\\&-\frac{2\epsilon}{2k+1}B_k^{(1)}\int\left(J^{\frac{2k+1}{2}}\psi^{(k+1)}\right)\end{aligned}\tag{41}$$

$$B_kJ^{\frac{2k-1}{2}}\psi^{(k)}=a_0\epsilon J^{-1}\psi\psi^{(k)}+\sum_{i=1}^{k}a_i\left(\epsilon J\right)^{i-1}\psi^{(i)}\psi^{(k)}\tag{42}$$

adding (41) and (42) we observe that cancel terms in J^{-1} and we get for

$$F_k^{(1)}=A_{k+1}+B_{k+1}\int J^{\frac{2k+1}{2}}\psi^{(k+1)}$$

where $A_{k+1}\in C^{\infty}\left(I_\varepsilon\right)$ and for some constant coefficients d_i:

$$B_{k+1}=-\frac{2\epsilon}{2k+1}B_k^{(1)}=\sum_{i=0}^{k+1}\epsilon^{i+1}d_iJ^{-\frac{2(k+1-i)+1}{2}}$$

an this end of the induction argument, and the proof. ■

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