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Transverse Riemann–Lorentz type-changing metrics with tangent radical

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Abstract

Consider a smooth manifold with a smooth metric which changes bilinear type on a hypersurface Σ and whose radical line field is everywhere tangent to Σ. We describe two natural tensors on Σ and use them to describe "integrability conditions" which are similar to the Gauss–Codazzi conditions. We show that these forms control the smooth extendibility to Σ of ambient curvatures.

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1. Preliminaries

Let M be a m-dimensional connected manifold ($m > 2$) endowed with a smooth, symmetric $(0, 2)$-tensorfield g which fails to have maximal rank on a (nonempty) subset $\Sigma \subset M$. Thus, at each point $p \in \Sigma$, there exists a nontrivial subspace (the *radical*) $Rad_p \subset T_pM$, which is orthogonal to the whole T_pM. We say that (M, g) is a *singular* (*semiriemannian*) *manifold*. Geodesics in these spaces were first analyzed in [6]. We say moreover that (M, g) is a *transverse type-changing* (*singular*) *manifold* if, for any local coordinate system $(x_1, \dots, x_m)$, the function $\det(g_{ab})_{a,b=1,\dots,m}$ has non-zero differential at the points of Σ (here g_{ab} are the components of g in the coordinate frame). This implies: (i) the subset Σ is a smooth hypersurface in M, called the *type-changing hypersurface*, (ii) at each point $p \in \Sigma$ the radical Rad_p is one-dimensional, and (iii) the signature of g changes by $+1$ or -1 across Σ (see [4] for details); when this change is from riemannian to Lorentzian, we say that (M, g) is a *Riemann–Lorentz* (*transverse type-changing*) *manifold*. We say moreover that (M, g) is *radical transverse* (respectively *radical tangent*) *on* Σ if $Rad_p \cap T_p\Sigma = \{0\}$ (respectively $Rad_p \subset T_p\Sigma$) for all $p \in \Sigma$. We will not consider the intermediate cases where the radical is tangent to Σ on a submanifold of Σ. There are several geometric and physical reasons to study transverse type-changing manifolds (see the Introduction to [4]) and there are many articles devoted to the case with

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transverse radical (see [3–5] and references therein). *In this article we analyze Riemann–Lorentz manifolds with tangent radical*.

In Section 2 we study the induced metric on the hypersurface Σ (which is degenerate since the radical is assumed to be tangent). In the familiar case of a null hypersurface in a semiriemannian manifold, the Levi-Civita connection remains well-defined at the points of the hypersurface (however it does not induce a connection on the hypersurface), here the hypersurface has a one-dimensional normal vector bundle, everywhere tangent to the hypersurface, and the differential geometry (both intrinsic and extrinsic) can be studied using the Levi-Civita connection. This has been carried out by comparing (screen) distributions on the hypersurface complementary to the normal bundle (the selected screen is not unique) and focusing attention on those properties of the resulting connection which are screen-independent (see e.g. [1]).

In contrast, for our setting the metric g fails to have maximal rank at the points of the hypersurface Σ and the Levi-Civita connection fails to exist at such points. Here, the suitable tool to analyze the geometry of Σ will be the canonically defined, torsion-free, metric, "dual connection" on the whole (M, g) (first defined in [2]) which, in the case of one-dimensional radical, induces a (conformally defined) symmetric $(0, 2)$-tensorfield II on Σ. All this occurs without any assumption on the radical. If we moreover assume that the radical is tangent to Σ, the g-normal vector bundle of Σ is two-dimensional and there exists a (locally determined up to a sign) *canonical* smooth vectorfield N transverse to Σ which is normal, unit length, and II-isotropic. This vectorfield N allows us to construct a second fundamental form $\mathcal{H}$ on Σ, which in turn gives rise to a *canonical* screen distribution S and also to a *canonical* vectorfield R in the radical distribution. Vectorfields tangent to Σ are uniquely decomposable in S- and R-components. We then describe a natural family of *admissible* torsion-free connections on Σ. In case of II-flatness (i.e. the tensorfield II vanishes on the whole Σ), all such connections are metric and have the same covariant curvatures.

In Section 3 we analyze the limiting behaviour of well-defined semiriemannian objects on $M - \Sigma$ as we approach the hypersurface Σ. By a theorem in [4] (see also [6]) the transverse, II-isotropic vectorfield N along Σ has a *canonical* (local) extension to M which is Levi-Civita geodesic on $M - \Sigma$. We use the flow of this extension to (locally) extend every vectorfield defined on Σ to a neighborhood of Σ. We then apply this extension construction to analyze limiting behaviours, specifically the dependence of limiting values on the vector fields used in their construction. Our main results indicate that the symmetric $(0, 2)$-tensorfields $\mathcal{H}$ and II control these limit properties. The tangent radical case gives rise to some unavoidable divergences, which are not present in the transverse radical case. When Σ is II-flat, we establish a "Gauss–Codazzi equation" relating the curvature of the admissible connections on Σ with the limit of the Levi-Civita curvature on $M - \Sigma$.

It would be interesting to find: (i) natural occurrences of Riemann–Lorentz manifolds (M, g) with tangent radical, and (ii) local isometric embeddings of a given (M, g) into a lorentzian manifold. Concerning (i), orbit submanifolds of indefinite isometry groups provide examples. More specifically, given a regular curve $\alpha : \mathbb{R} \ni t \mapsto (w = 0,\ x(t),\ y = 0,\ z(t)) \in \mathbb{R}^4_1$ in Minkowski 4-space, we get (for fixed $a \neq 0$) a 3-dimensional parametrized "helicoid" $\varphi(t, s, r) = (-as \sin r, x \cosh s + z \sinh s, as \cos r, x \sinh s + z \cosh s)$. If $\alpha(0) = a\alpha'(0)$ and $\langle \alpha'(0), \alpha'(0)\rangle = 1$, this helicoid turns out to be (for $s \neq 0$) a Riemann–Lorentz manifold with radical tangent to the type-changing surface $\varphi(0, s, r)$. Concerning (ii), by Remark 3 we can (locally) isometrically embed (M, g) into a lorentzian ambient $(\bar{M}, \bar{g})$ (i.e. via local coordinates $(x^0, \ldots, x^m)$, in such a way that $x^0|_M = 0$, $dx^0|_M \neq 0$ and it holds:

$$(\bar{g}_{\bar{a}\bar{b}}) = \begin{pmatrix} x^1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (\bar{g}_{\lambda\mu}) & x^1\bar{g}_\lambda \\ 1 & 0 & x^1\bar{g}_\lambda & x^1\bar{g}_m \end{pmatrix},$$

with $\bar{g}_a, \bar{g}_{\lambda\mu}$ smooth extensions of $g_a, g_{\lambda\mu}$ in formula (10)). However $(\bar{M}, \bar{g})$ is not flat in general; the existence of an isometric embedding into a flat lorentzian manifold is a very subtle singular initial value problem.

Let (M, g) be transverse type-changing on a hypersurface Σ. Vectorfields on M are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots \in \mathfrak{X}(M)$; we use $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \ldots$ to denote vectorfields on M tangent to Σ. Vectorfields along Σ are denoted by capital letters $A, B, C, \ldots \in \mathfrak{X}_\Sigma$; if they are tangent to Σ we write $X, Y, Z, \ldots \in \mathfrak{X}(\Sigma)$. Given $\mathcal{A} \in \mathfrak{X}(M)$, we denote $A = \mathcal{A}|_\Sigma \in \mathfrak{X}_\Sigma$. In that case, we say that $\mathcal{A}$ is an *extension* of A.

Let us consider some function $\tau \in C^\infty(M)$ such that $\tau|_\Sigma = 0$ and $d\tau|_\Sigma \neq 0$ everywhere. We say that (locally, around Σ) $\tau = 0$ *is an equation for* Σ. Given another function $f \in C^\infty(M)$, it holds: $f|_\Sigma = 0 \Leftrightarrow f = k_f\tau$, for some $k_f \in C^\infty(M)$. When $f|_\Sigma = 0$, we write $\tau^{-1} f \cong 0$ and we say that $\tau^{-1} f$ *is extendible* as an element of $C^\infty(M)$.

Let $(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ be a (local, C^∞) $\mathfrak{X}(M)$-basis around some point of Σ. We say that $(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ is a *radical adapted frame* if $\mathcal{E}_m|_\Sigma$ spans the radical distribution and it holds (orthonormality): $g_{ab} := \langle \mathcal{E}_a, \mathcal{E}_b \rangle = \delta_{ab}(\pm(1 - \delta_{am}) + \delta_{am}\tau)$, for some $\tau := \langle \mathcal{E}_m, \mathcal{E}_m \rangle \in C^\infty(M)$ $(a, b = 1, \ldots, m)$. Thus $\tau = 0$ is an equation for Σ. The existence of radical adapted frames around any point $p \in \Sigma$ can be easily proved, e.g., starting with an orthonormal basis $(e_1, \ldots, e_m)$ of T_pM with $e_m \in Rad_p$ (i.e., $g(e_a, e_b) = \pm\delta_{ab}(1 - \delta_{am})$), using a local chart of M around p adapted to Σ, and applying a slight modification of the Gram–Schmidt orthonormalization procedure. If (M, g) is radical transverse (respectively, radical tangent), all $\mathcal{E}_i$'s $(i = 1, \ldots, m - 1)$ can be chosen to be tangent (respectively, one of the $\mathcal{E}_i$'s must be transverse) to Σ.

We use $\langle \mathcal{A}, \mathcal{B} \rangle$ and $g(\mathcal{A}, \mathcal{B})$ interchangeably.

2. Local geometry of the type-changing hypersurface

On a singular manifold (M, g) there exists [2] a unique torsion-free metric *dual connection*, which can be characterized as the unique map $\Box : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^*(M)$ satisfying, for all $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathfrak{X}(M)$, the Koszul-like formula:

$$2\Box_{\mathcal{A}}\mathcal{B}(\mathcal{C}) := \mathcal{A}\langle \mathcal{B}, \mathcal{C} \rangle + \mathcal{B}\langle \mathcal{C}, \mathcal{A} \rangle - \mathcal{C}\langle \mathcal{A}, \mathcal{B} \rangle + \big\langle [\mathcal{A}, \mathcal{B}], \mathcal{C} \big\rangle - \big\langle [\mathcal{B}, \mathcal{C}], \mathcal{A} \big\rangle + \big\langle [\mathcal{C}, \mathcal{A}], \mathcal{B} \big\rangle. \tag{1}$$

It follows that $\Box$ is compatible with the Levi-Civita connection ∇ on $M - \Sigma$, in the sense that it holds: $\Box_{\mathcal{A}}\mathcal{B}(\mathcal{C}) = \langle \nabla_{\mathcal{A}}\mathcal{B}, \mathcal{C} \rangle$.

Let (M, g) be transverse type-changing on a hypersurface Σ. Then $\Box_A\mathcal{B} := \Box_{\mathcal{A}}\mathcal{B}|_\Sigma \in \mathfrak{X}^*_\Sigma$ is well-defined (we denote $A = \mathcal{A}|_\Sigma$). This implies: (i) the dual connection has a good restriction $\Box : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}^*(\Sigma)$, which can also be characterized as the unique torsion-free metric dual connection on the singular manifold $(\Sigma, g|_\Sigma)$; and (ii) given *any* vectorfield $R \in \mathfrak{X}_\Sigma$ spanning the radical distribution, $\Box_A\mathcal{B}(R)$ depends only on A and $B = \mathcal{B}|_\Sigma$, thus $\Box_A B(R) := \Box_A\mathcal{B}(R)|_\Sigma \in C^\infty(\Sigma)$ becomes well-defined and we obtain [2] a $C^\infty(\Sigma)$-bilinear map $II_R : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma)$, $(A, B) \mapsto \Box_A B(R)$, which is moreover symmetric (see [5] for details). In a similar way, given *any* vectorfield $N \in \mathfrak{X}_\Sigma$ orthogonal to Σ, we obtain a $C^\infty(\Sigma)$-bilinear, symmetric map $\mathcal{H}_N : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$, $(X, Y) \mapsto \Box_X Y(N) := \Box_X \mathcal{Y}(N)|_\Sigma$. We use these general constructions right now.

Let (M, g) be radical tangent on Σ. At each point $p \in \Sigma$, the g-orthogonal subspace $T_p^\perp\Sigma \subset T_pM$ is a 2-plane and it holds: $T_p^\perp\Sigma \cap T_p\Sigma = Rad_p$. Let $(\mathcal{E}_1, \ldots, \mathcal{E}_m)$ be a radical adapted frame around p. Thus E_m spans the radical distribution (we denote $E_a = \mathcal{E}_a|_\Sigma \in \mathfrak{X}_\Sigma$, $a = 1, \ldots, m$), $\langle \mathcal{E}_m, \mathcal{E}_m \rangle = 0$ is an equation for Σ and (without loss of generality) E_1 is transverse to Σ. Formula (1) leads to: $2II_{E_m}(E_m(p), E_m(p)) = E_m(p)\langle \mathcal{E}_m, \mathcal{E}_m \rangle = 0$ and $2II_{E_m}(E_1(p), E_m(p)) = E_1(p)\langle \mathcal{E}_m, \mathcal{E}_m \rangle \neq 0$. It follows that II_{E_m} turns $T_p^\perp\Sigma$ into a Lorentz plane. One of the two II_{E_m}-isotropic directions at p is determined by $E_m(p)$. Since the other cannot be g-isotropic, it determines a unique (up to a sign) unit vector in T_pM normal to Σ. Moving from p to the neighboring points in Σ we locally obtain a canonical (up to a sign) smooth vectorfield $N \in \mathfrak{X}_\Sigma$ satisfying: $\langle N, T\Sigma \rangle = 0$, $II_{Rad}(N, N) = 0$ and $\langle N, N \rangle = \pm 1$. If g changes from riemannian to Lorentzian, it must hold: $\langle N, N \rangle = 1$. We call N the *normal vectorfield* on Σ.

On the type-changing hypersurface Σ we have a *first* (*degenerate*) *fundamental form*, namely the restriction $g|_\Sigma$. As indicated above, the normal N on Σ allows us to define the *second fundamental form* $\mathcal{H} \equiv \mathcal{H}_N$ by: $\mathcal{H}(X, Y) := \Box_X Y(N)$, for $X, Y \in \mathfrak{X}(\Sigma)$. This is a symmetric $(0, 2)$-tensor field over Σ, locally determined up to a sign.

At each $p \in \Sigma$, since $2\mathcal{H}(E_m(p), E_m(p)) = -2II_{E_m}(E_m(p), N(p)) = -N(p)\langle \mathcal{E}_m, \mathcal{E}_m \rangle \neq 0$, the II_{E_m}-isotropic direction determined by $E_m(p)$ cannot be $\mathcal{H}$-isotropic. Thus we can select a vectorfield $R \in \mathfrak{X}(\Sigma)$ which spans the radical distribution and such that $\mathcal{H}(R, R) = \pm 1$. *Choosing the sign of N such that* $\mathcal{H}(R, R) = -1$, we locally obtain a canonical (up to a sign) smooth vectorfield $R \in \mathfrak{X}(\Sigma)$ satisfying: $R(p) \in Rad_p$, for all $p \in \Sigma$, and $\mathcal{H}(R, R) = -1$. We call R the *radical vectorfield* on Σ.

The radical vectorfield R induces a canonical $C^\infty(\Sigma)$-bilinear symmetric map $II \equiv II_R : \mathfrak{X}_\Sigma \times \mathfrak{X}_\Sigma \to C^\infty(\Sigma)$, $(A, B) \mapsto \Box_A B(R)$, whose restriction to $\mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma)$ yields another *symmetric* $(0, 2)$-*tensorfield* II *on* Σ. Note that it holds:

$$\begin{cases} II(N, N) = 0, \\ II(N, X) = -\mathcal{H}(X, R), \quad \text{for all } X \in \mathfrak{X}(\Sigma), \\ II(A, R) = 0 \Leftrightarrow A \in \mathfrak{X}(\Sigma). \end{cases} \tag{2}$$

A *screen distribution* is a distribution on the type-changing hypersurface Σ which yields, at each $p \in \Sigma$, a hyperplane of $T_p\Sigma$ transversal to Rad_p. We now define the *canonical screen distribution* by choosing (at each $p \in \Sigma$)

$$S_p := \{v \in T_p\Sigma\colon\ \mathcal{H}(v, Rad_p) = 0\}. \tag{3}$$

We shall denote by S either the set $\{S_p\colon\ p \in \Sigma\}$ or the corresponding vector subbundle (the *screen bundle*) $S \subset T\Sigma$. We denote by $\Gamma(S)$ the $C^\infty(\Sigma)$-module of sections of S, which is a submodule of $\mathfrak{X}(\Sigma)$.

From now on, we only consider Riemann–Lorentz manifolds with tangent radical. This means that g is semi-definite on Σ and the screen bundle S becomes a riemannian vector bundle.

We say that Σ is *II-flat* (respectively, *$\mathcal{H}$-flat*) if it holds: $II(V, W) = 0$ (respectively , $\mathcal{H}(V, W) = 0$), for all $V, W \in \Gamma(S)$. Thus II-flatness is equivalent to $II(X, Y) = 0$, for all $X, Y \in \mathfrak{X}(\Sigma)$, and $\mathcal{H}$-flatness is equivalent to $\mathcal{H}(V, X) = 0$, for all $V \in \Gamma(S)$ and $X \in \mathfrak{X}(\Sigma)$. Both definitions become equivalent to the vanishing of the corresponding self-adjoint endomorphisms of S (Weingarten screen maps) induced by II and $\mathcal{H}$. Since (1) leads to: $R\langle X, Y\rangle = \langle [R, X], Y\rangle + \langle X, [R, Y]\rangle - 2II(X, Y)$, for all $X, Y \in \mathfrak{X}(\Sigma)$, we obtain the following conclusion: *Σ is II-flat if and only if R is a Killing vectorfield on Σ.*

A vectorfield $A \in \mathfrak{X}_\Sigma$ can now be decomposed in normal-, screen- and radical-components, as follows

$$A = \nu(A)N + A^S + \rho(A)R, \tag{4}$$

where $\nu(A) := \langle A, N\rangle$ and $\rho(A) := -\mathcal{H}(A - \nu(A)N, R)$. Thus $\rho \in \mathfrak{X}^*_\Sigma$ is completely determined by the 1-form $\rho = -\mathcal{H}(.,R) \in \mathfrak{X}^*(\Sigma)$. Since $d\rho(V, W) = -\rho([V, W])$, for all $V, W \in \Gamma(S)$, *the form ρ is closed only if S is integrable*. Of course, the converse is not true: given $V \in \Gamma(S)$, the Lie bracket $[V, R]$ needs not belong to $\Gamma(S)$.

We want to describe some natural connections on Σ. Let us first introduce the *screen connection-operator* as the map $D^S\colon \mathfrak{X}(\Sigma) \times \mathfrak{X}_\Sigma \to \Gamma(S), (X, A) \mapsto D^S_X A$, given by:

$$\langle D^S_X A, V\rangle := \Box_X A(V), \quad \text{for all } V \in \Gamma(S). \tag{5}$$

Thus the screen connection-operator D^S gives a metric connection $D^S\colon \mathfrak{X}(\Sigma) \times \Gamma(S) \to \Gamma(S)$ on the riemannian vector bundle $S \to \Sigma$, and satisfies:

(i) $\langle D^S_X A, V\rangle = \Box_X A(V)$ (compatibility with $\Box$),
(ii) $\langle D^S_V R, W\rangle = -II(V, W)$, and
(iii) $\langle D^S_V N, W\rangle = -\mathcal{H}(V, W)$, for all $V, W \in \Gamma(S)$.

However the restriction $D^S\colon \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \Gamma(S)$ *does not* give a connection on Σ, since it holds: $D^S_X(fR) = fD^S_X R$, for all $f \in C^\infty(\Sigma)$.

We say that a connection D on Σ is *admissible* if it holds: (i) $(D_X Y)^S = D^S_X Y$ (for all $X, Y \in \mathfrak{X}(\Sigma)$), or equivalently: $\langle D_X Y, V\rangle = \Box_X Y(V)$ (for all $V \in \Gamma(S)$), and (ii) D is torsion-free. The most obvious connection on Σ satisfying the first condition is the one defined by: $\tilde{D}_X Y := D^S_X Y + X(\rho(Y))R$ (for all $X, Y \in \mathfrak{X}(\Sigma)$), which has the following properties:

Proposition 1. *The connection $\tilde{D}$ on the hypersurface Σ*: (a) *has torsion $\widetilde{Tor} = R \otimes d\rho$, and* (b) *is metric if and only if Σ is II-flat.*

Proof. Let X, Y, Z be arbitrary in $\mathfrak{X}(\Sigma)$. (a) Since the screen operator D^S is compatible with the torsion free dual connection $\Box$, one immediately sees that: $(\widetilde{Tor}(X, Y))^S = 0$. Therefore, $\widetilde{Tor}(X, Y) = \rho(\widetilde{Tor}(X, Y))R = d\rho(X, Y)R$.

(b) Again because D^S is compatible with $\Box$, one gets: $\langle \tilde{D}_X Y, Z\rangle = \langle D^S_X Y, Z^S\rangle = \langle D^S_X Y^S, Z^S\rangle + \rho(Y)\langle D^S_X R, Z^S\rangle = \langle D^S_X Y^S, Z^S\rangle - \rho(Y)II(X, Z)$. And since $D^S\colon \mathfrak{X}(\Sigma) \times \Gamma(S) \to \Gamma(S)$ is metric, one gets: $\langle \tilde{D}_X Y, Z\rangle + \langle Y, \tilde{D}_X Z\rangle = X\langle Y, Z\rangle - II(X, \rho(Y)Z + \rho(Z)Y)$.

Now ($\Leftarrow$) is trivial. Let us prove ($\Rightarrow$): If $\tilde{D}$ is metric, last formula yields: $\rho(X)II(X, X) = 0$, for all $X \in \mathfrak{X}(\Sigma)$. Because $T_p\Sigma - S_p$ is dense in $T_p\Sigma$ (for all $p \in \Sigma$), it follows that $II(X, X) = 0$, for all $X \in \mathfrak{X}(\Sigma)$. Since II is symmetric, this implies that Σ is II-flat. $\Box$

Thus, unless $d\rho = 0$, the connection $\tilde{D}$ is not admissible. However, it is straightforward to check that the connection defined by $\dot{D}_X Y := \tilde{D}_X Y - \frac{1}{2} d\rho(X,Y)R$ (for all $X, Y \in \mathfrak{X}(\Sigma)$) *is always admissible*. Now if D is an admissible connection, it must satisfy (for all $X, Y \in \mathfrak{X}(\Sigma)$):

$$D_X Y = \dot{D}_X Y + \sigma(X,Y)R, \tag{6}$$

where σ is some symmetric $(0,2)$-tensorfield on Σ. Indeed, the difference of the torsion-free connections D and $\dot{D}$ must be some symmetric $(1,2)$-tensorfield on Σ, and the fact that $(D_X Y)^S = D_X^S Y = (\dot{D}_X Y)^S$ leads to the result. Moreover, admissible connections have following properties:

Theorem 2. (a) *If there exists a torsion-free metric connection on Σ, then*: (i) *it is admissible, and* (ii) *Σ is II-flat.* (b) *If Σ is II-flat, all admissible connections are metric and have the same covariant curvature.*

Proof. Let X, Y, Z be arbitrary in $\mathfrak{X}(\Sigma)$. (a) (i) If D is a torsion-free metric connection on Σ, the induced dual connection $\square^D$ on the singular manifold $(\Sigma, g|_\Sigma)$, defined by: $\square_X^D Y(Z) := \langle D_X Y, Z\rangle$, becomes torsion-free and metric; by uniqueness, $\square^D = \square$ and D becomes admissible. (ii) Admissible connections, having necessarily the form (6), are metric if and only if $\dot{D}$ is metric, if and only if $\tilde{D}$ is metric, if and only if (Proposition 1(b)) Σ is II-flat.

(b) The first assert was proved in (a). Let D be an admissible connection on Σ with covariant curvature R^D defined by: $\langle R^D(X,Y)Z, T\rangle := \langle D_X(D_Y Z) - D_Y(D_X Z) - D_{[X,Y]}Z, T\rangle$. Since we have: $\langle D_X(D_Y Z), T\rangle = \langle D_X(\dot{D}_Y Z + \sigma(Y,Z)R), T\rangle = \langle \dot{D}_X(\dot{D}_Y Z) + \sigma(Y,Z)\dot{D}_X R, T\rangle = \langle \dot{D}_X(\dot{D}_Y Z), T\rangle + \sigma(Y,Z)\langle D_X^S R, T\rangle = \langle \dot{D}_X(\dot{D}_Y Z), T\rangle - \sigma(Y,Z)II(X,T)$, we finally obtain:

$$\langle R^D(X,Y)Z, T\rangle = \langle R^{\dot{D}}(X,Y)Z, T\rangle - \det\begin{pmatrix} \sigma(Y,Z) & II(Y,Z) \\ \sigma(X,T) & II(X,T)\end{pmatrix},$$

and the result follows. □

3. Near the type-changing hypersurface

We analyze in this section the limiting behaviour of some well-defined Levi-Civita objects on $M - \Sigma$ as we approach the type-changing hypersurface Σ, to which the radical is tangent. Thus we can replace M by a neighborhood of Σ in M. Typically, we start with a semiriemannian differentiable object, say $\bigcirc$ (for example, the Levi-Civita connection ∇, or the curvature R) and vectorfields $\mathcal{A}, \mathcal{B}, \ldots \in \mathfrak{X}(M)$, construct $\bigcirc(\mathcal{A}, \mathcal{B}, \ldots)$ on $M - \Sigma$, and ask under what circumstances: (i) $\bigcirc(\mathcal{A}, \mathcal{B}, \ldots) \cong 0$, that is, $\bigcirc(\mathcal{A}, \mathcal{B}, \ldots)$ has a differentiable extension (denoted also by $\bigcirc(\mathcal{A}, \mathcal{B}, \ldots)$) to the whole M, in that case we say that "$\bigcirc(\mathcal{A}, \mathcal{B}, \ldots)$ is extendible"; and (ii) the restriction $\bigcirc(\mathcal{A}, \mathcal{B}, \ldots)|_\Sigma$ only depends on $A = \mathcal{A}|_\Sigma$, $B = \mathcal{B}|_\Sigma, \ldots \in \mathfrak{X}_\Sigma$, in that case we say that "$\bigcirc(A, B, \ldots)$ is well-defined". When dealing with two such objects $\bigcirc_1$ and $\bigcirc_2$, we write $\bigcirc_1(\mathcal{A}, \mathcal{B}, \ldots) \cong \bigcirc_2(\mathcal{A}, \mathcal{B}, \ldots)$ to mean $\bigcirc_1(\mathcal{A}, \mathcal{B}, \ldots) - \bigcirc_2(\mathcal{A}, \mathcal{B}, \ldots) \cong 0$.

We first analyze extensions of vectorfields in $\mathfrak{X}_\Sigma$. Given any extension $\mathcal{R}$ of R, we obtain (using (1) and (2))

$$N\langle \mathcal{R}, \mathcal{R}\rangle = 2\square_R N(R) =: -2II(N,R) = -2\mathcal{H}(R,R) = 2; \tag{7}$$

it follows that $\langle \mathcal{R}, \mathcal{R}\rangle = 0$ is an equation for Σ and that $\langle \mathcal{R}, \mathcal{R}\rangle^{-1}(\mathcal{N}\langle \mathcal{R}, \mathcal{R}\rangle - 2) \cong 0$ (for any extension $\mathcal{N}$ of N). Moreover, given $A \in \mathfrak{X}_\Sigma$, it holds: $\langle \mathcal{R}, \mathcal{R}\rangle^{-1}\langle \mathcal{A}, \mathcal{R}\rangle \cong 0$ (for any extension $\mathcal{A}$) and, given $X \in \mathfrak{X}(\Sigma)$, it holds: $\langle \mathcal{R}, \mathcal{R}\rangle^{-1}\mathcal{X}\langle \mathcal{R}, \mathcal{R}\rangle \cong 0$ (for any extension $\mathcal{X}$).

Because N is nowhere tangent to Σ and $II(N,N) = 0$, it follows from Theorem 1 in [4] (see also [6]) that there exists a (local) *canonical extension* $\mathbf{N} \in \mathfrak{X}(M)$ of N which is Levi-Civita-geodesic on $M - \Sigma$, thus $\square_{\mathbf{N}}\mathbf{N} = 0 \in \mathfrak{X}^*(M)$ and $\langle \mathbf{N}, \mathbf{N}\rangle = \pm 1$. This induces, for each $A \in \mathfrak{X}_\Sigma$, a (local) *canonical extension* $\mathbf{A} \in \mathfrak{X}(M)$ (we always use boldface types to denote such extensions) such that: $[\mathbf{N}, \mathbf{A}] = 0$ ($\mathbf{A}$ is generated from A by the flow of $\mathbf{N}$), and it holds: $\mathbf{N}\langle \mathbf{N}, \mathbf{A}\rangle = \square_{\mathbf{N}}\mathbf{A}(\mathbf{N}) = \square_{\mathbf{A}}\mathbf{N}(\mathbf{N}) = \mathbf{A}\langle \mathbf{N}, \mathbf{N}\rangle = 0$. *We denote in what follows* $\tau \equiv \langle \mathbf{R}, \mathbf{R}\rangle$, thus $\tau = 0$ is an equation for Σ and $N(\tau) = 2$. Any extension $\mathcal{A}$ of A can be written in the form $\mathcal{A} = \mathbf{A} + \tau\bar{\mathcal{A}}$, for some $\bar{\mathcal{A}} \in \mathfrak{X}(M)$.

Let be $X \in \mathfrak{X}(\Sigma)$. Since $\langle \mathbf{N}, \mathbf{X}\rangle$ is constant along the integral curves of $\mathbf{N}$, it follows that: $\langle \mathbf{N}, \mathbf{X}\rangle = 0 \in C^\infty(M)$. Since $N(\mathbf{X}(\tau)) = X(\mathbf{N}(\tau)) = X(2) = 0$, it follows: $\tau^{-1}\mathbf{N}(\mathbf{X}(\tau)) \cong 0$, and we have: $\{\tau^{-1}\mathbf{N}(\mathbf{X}(\tau))\}\tau = \mathbf{N}(\mathbf{X}(\tau)) = \mathbf{N}(\{\tau^{-1}\mathbf{X}(\tau)\}\tau) = \mathbf{N}\{\tau^{-1}\mathbf{X}(\tau)\}\tau + \{\tau^{-1}\mathbf{X}(\tau)\}\mathbf{N}(\tau)$. Therefore $\{\tau^{-1}\mathbf{X}(\tau)\}|_\Sigma = 0$ and we finally obtain:

$$\tau^{-2}\mathbf{X}(\tau) \cong 0. \tag{8}$$

If $\mathcal{X} \in \mathfrak{X}(M)$ is *any* extension of X, a direct computation using (1) and (7) leads to: $\mathcal{H}(X, R) := \Box_{\mathcal{X}}\mathbf{R}(\mathbf{N})|_{\Sigma} = -\frac{1}{2}(\mathbf{N}\langle\mathcal{X}, \mathbf{R}\rangle)|_{\Sigma} = -\frac{1}{2}N(\{\tau^{-1}\langle\mathcal{X}, \mathbf{R}\rangle\}\tau) = -\frac{1}{2}\{\tau^{-1}\langle\mathcal{X}, \mathbf{R}\rangle\}|_{\Sigma}N(\tau) = -\{\tau^{-1}\langle\mathcal{X}, \mathbf{R}\rangle\}|_{\Sigma}$. Therefore, given $A \in \mathfrak{X}_{\Sigma}$, and for any extensions $\mathcal{A}$ of A and $f \in C^{\infty}(M)$ of $\nu(A)$, we have:

$$\rho(A) := -\mathcal{H}\big(A - \nu(A)N, R\big) = \big\{\tau^{-1}\langle\mathcal{A} - f\mathbf{N}, \mathbf{R}\rangle\big\}\big|_{\Sigma} = \big\{\tau^{-1}\langle\mathcal{A}, \mathbf{R}\rangle\big\}\big|_{\Sigma}. \tag{9}$$

Remark 3. The following construction gives all local examples of Riemann–Lorentz manifolds with tangent radical. Around each point of Σ, there exist *adapted coordinates* $(x^1, \ldots, x^m)$ in M such that: (i) $\partial_{x^1} = \mathbf{N}$ and $\partial_{x^m} = \mathbf{R}$, (ii) $x^1 = 0$ is an equation for Σ, and (iii) it holds:

$$(g_{ab}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (g_{\lambda\mu}) & x^1 g_{\lambda} \\ 0 & x^1 g_{\lambda} & x^1 g_m \end{pmatrix} \quad (a, b = 1, \ldots, m;\ \lambda, \mu = 2, \ldots, m-1), \tag{10}$$

for some $g_i \in C^{\infty}(M)$ $(i = 2, \ldots, m)$ with $g_m(0, x^2, \ldots, x^m) = 2$. To see this, at $p \in \Sigma$, we first choose coordinates $(x^2, \ldots, x^m)$ in Σ such that $\partial_{x^m} = R$. Using the flow of $\mathbf{N}$, we construct coordinates $(x^1, \ldots, x^m)$ in M such that $\partial_{x^1} = \mathbf{N}$ (thus $g_{11} = 1$) and ∂_{x^i} is the canonical extension of the (equally denoted) vectorfield $\partial_{x^i} \in \mathfrak{X}(\Sigma)$ $(i = 2, \ldots, m)$; in particular , $\partial_{x^m} = \mathbf{R}$. Obviously , $x^1 = 0$ is an equation for Σ. It follows that: $g_{1i} = 0$ $(i = 2, \ldots, m)$. On the other hand, $g_{im}|_{\Sigma} = 0$ implies: $g_{im} = x^1 g_i$, for some $g_i \in C^{\infty}(M)$ $(i = 2, \ldots, m)$. And finally, because $\tau = g_{mm}$, it follows from (7): $g_m|_{\Sigma} = 2$.

Note that, in these adapted coordinates, a vectorfield $\sum f_a \partial_{x^a} \in \mathfrak{X}(M)$ is the canonical extension $\mathbf{A}$ of $A = \sum f_a|_{\Sigma}\partial_{x^a} \in \mathfrak{X}_{\Sigma}$ if and only if all f_a's $(a = 1, \ldots, m)$ do not depend on x^1. Setting $\Gamma_{cab} := \Box_{\partial_{x^a}}\partial_{x^b}(\partial_{x^c})$, the first Christoffel symbols of $\Box$, it is straightforward to see from (1) that the components of $\mathcal{H}$ and II are given by: $\mathcal{H}_{ij} = \Gamma_{1ij}|_{\Sigma} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^1}|_{x^1=0}$ and $II_{ij} = \Gamma_{mij}|_{\Sigma} = -\frac{1}{2}\frac{\partial g_{ij}}{\partial x^m}|_{x^1=0}$ $(i, j = 2, \ldots, m)$.

We now analyze the limiting behaviour of some Levi-Civita objects. Around each point $p \in \Sigma$, there exist radical adapted frames $(\mathcal{E}_1, \ldots, \mathcal{E}_m = \mathbf{R})$ such that $E_1 = N$ and $E_2, \ldots, E_{m-1} \in \Gamma(S)$. *In what follows, we always use such frames*.

Let us first consider covariant derivatives. Let be $\mathcal{A}, \mathcal{B} \in \mathfrak{X}(M)$. Since we have (on $M - \Sigma$):

$$\nabla_{\mathcal{A}}\mathcal{B} = \sum_{i=1}^{m-1} \Box_{\mathcal{A}}\mathcal{B}(\mathcal{E}_i)\mathcal{E}_i + \tau^{-1}\Box_{\mathcal{A}}\mathcal{B}(\mathbf{R})\mathbf{R}, \tag{11}$$

it follows that: $\nabla_{\mathcal{A}}\mathcal{B} \cong 0$ if and only if $II(A, B) = 0$. Let now be $A, B \in \mathfrak{X}_{\Sigma}$ with $II(A, B) = 0$ and let $\mathcal{A}, \mathcal{B}$ be extensions. If the restriction $\nabla_{\mathcal{A}}\mathcal{B}|_{\Sigma}$ does not depend on the extensions, we say that $\nabla_A B$ *is well-defined*. Writing $\mathcal{A} = \mathbf{A} + \tau\bar{\mathcal{A}}, \mathcal{B} = \mathbf{B} + \tau\bar{\mathcal{B}}$, for some $\bar{\mathcal{A}}, \bar{\mathcal{B}} \in \mathfrak{X}(M)$, it is straightforward to see that it holds (we denote $\bar{A} = \bar{\mathcal{A}}|_{\Sigma}$, $\bar{B} = \bar{\mathcal{B}}|_{\Sigma}$): $\{\tau^{-1}\Box_{\mathcal{A}}\mathcal{B}(\mathbf{R})\}|_{\Sigma} = \{\tau^{-1}\Box_{\mathbf{A}}\mathbf{B}(\mathbf{R})\}|_{\Sigma} + A(\tau)\{\tau^{-1}\langle\bar{\mathcal{B}}, \mathbf{R}\rangle\}|_{\Sigma} + II(A, \bar{B}) + II(\bar{A}, B)$. Then (11) leads to:

$$\nabla_{\mathcal{A}}\mathcal{B}|_{\Sigma} = \nabla_{\mathbf{A}}\mathbf{B}_{\Sigma} + A(\tau)\sum_{i=1}^{m-1}\langle\bar{B}, E_i\rangle E_i + \big(A(\tau)\big\{\tau^{-1}\langle\bar{\mathcal{B}}, \mathbf{R}\rangle\big\}\big|_{\Sigma} + II(A, \bar{B}) + II(\bar{A}, B)\big)R \tag{12}$$

from which part (b) in the following proposition easily follows.

Proposition 4. (a) $\nabla_{\mathcal{A}}\mathcal{B} \cong 0$ *if and only if* $II(A, B) = 0$. *In particular,* $\nabla_{\mathcal{A}}\mathcal{B} \cong 0$, *for all* $\mathcal{A}, \mathcal{B}$ *tangent to* Σ, *if and only if* Σ *is II-flat.* (b) $\nabla_A B$ *is not well-defined, if one vectorfield is either* N *or* R. (c) $\nabla_A B$ *is well-defined, for all* $A, B \in \Gamma(S)$, *if and only if* Σ *is II-flat.*

The following two formulas are very useful in dealing with covariant derivatives. Let $\mathcal{X} \in \mathfrak{X}(M)$ be tangent to Σ and let $\mathcal{R}$ be any extension of R. Using (7), (8) and (9) we easily obtain the following: restricted to Σ

$$\begin{cases} \langle\nabla_{\mathcal{X}}\mathcal{R}, W\rangle = \Box_X R(W) = -II(X, W), \quad \text{for all } W \in \Gamma(S), \\ \rho(\nabla_{\mathcal{X}}\mathcal{R}) = \tau^{-1}\langle\nabla_{\mathcal{X}}\mathcal{R}, \mathbf{R}\rangle = \frac{1}{2}\tau^{-1}\mathcal{X}(\tau) = \frac{1}{2}(\bar{\mathcal{X}}(\tau)) = \nu(\bar{X}), \end{cases} \tag{13}$$

where $\bar{\mathcal{X}} \in \mathfrak{X}(M)$ is such that $\mathcal{X} = \mathbf{X} + \tau\bar{\mathcal{X}}$. Moreover, given $\mathcal{Y} \in \mathfrak{X}(M)$ tangent to Σ with $II(X, Y) = 0$, we obtain: restricted to Σ

$$\begin{cases} \nu(\nabla_{\mathcal{X}}\mathcal{Y}) := \langle \nabla_{\mathcal{X}}\mathcal{Y}, N\rangle = \square_X Y(N) =: \mathcal{H}(X, Y), \\ \rho([X, Y]) = \tau^{-1}\langle [\mathcal{X}, \mathcal{Y}], \mathbf{R}\rangle = 2\tau^{-1}\langle \nabla_{\mathcal{X}}\mathcal{Y}, \mathbf{R}\rangle_{Ant}, \end{cases} \tag{14}$$

where *Ant* means antisymmetrization under the permutation of $\mathcal{X}$ and $\mathcal{Y}$.

If Σ is II-flat, a natural map $III : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to C^\infty(\Sigma)$, $(X, Y, Z) \mapsto II(\nabla_{\mathcal{X}}\mathcal{Y}|_\Sigma, Z)$ arises, which turns out to be (straightforward computation) $C^\infty(\Sigma)$-*trilinear* and *symmetric in its first two entries*. It follows from (14) that $III(X, Y, R) = \nu(\nabla_{\mathcal{X}}\mathcal{Y}|_\Sigma) = \mathcal{H}(X, Y)$, for all $X, Y \in \mathfrak{X}(\Sigma)$. In particular, $III(V, R, R) = 0$ and $III(R, R, R) = -1$. We say that Σ is *III-flat* if it is *II*-flat and moreover it holds: $III(V, W, R) = 0$, for all $V, W \in \Gamma(S)$. Thus Σ is *III*-flat if and only if it is *II*-flat and $\mathcal{H}$-flat.

Let us now consider covariant curvatures. Let be $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D} \in \mathfrak{X}(M)$. First of all, we have (on $M - \Sigma$):

$$\begin{aligned} \big\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\big\rangle &:= \square_{\mathcal{A}}(\nabla_{\mathcal{B}}\mathcal{C})(\mathcal{D}) - \square_{\mathcal{B}}(\nabla_{\mathcal{A}}\mathcal{C})(\mathcal{D}) - \square_{[\mathcal{A},\mathcal{B}]}\mathcal{C}(\mathcal{D}) \\ &= \sum_{i=1}^{m-1} \big\{\mathcal{A}\big(\square_{\mathcal{B}}\mathcal{C}(\mathcal{E}_i)\big) - \mathcal{B}\big(\square_{\mathcal{A}}\mathcal{C}(\mathcal{E}_i)\big)\big\}\langle \mathcal{E}_i, \mathcal{D}\rangle \\ &\quad + \sum_{i=1}^{m-1} \big\{\square_{\mathcal{B}}\mathcal{C}(\mathcal{E}_i)\square_{\mathcal{A}}\mathcal{E}_i(\mathcal{D}) - \square_{\mathcal{A}}\mathcal{C}(\mathcal{E}_i)\square_{\mathcal{B}}\mathcal{E}_i(\mathcal{D})\big\} \\ &\quad + \mathcal{A}\big(\square_{\mathcal{B}}\mathcal{C}(\mathbf{R})\big\{\tau^{-1}\langle \mathbf{R}, \mathcal{D}\rangle\big\}\big) - \mathcal{B}\big(\square_{\mathcal{A}}\mathcal{C}(\mathbf{R})\big\{\tau^{-1}\langle \mathbf{R}, \mathcal{D}\rangle\big\}\big) - \square_{[\mathcal{A},\mathcal{B}]}\mathcal{C}(\mathcal{D}) \\ &\quad + \tau^{-1}\big\{\square_{\mathcal{A}}\mathcal{C}(\mathbf{R})\square_{\mathcal{B}}\mathcal{D}(\mathbf{R}) - \square_{\mathcal{B}}\mathcal{C}(\mathbf{R})\square_{\mathcal{A}}\mathcal{D}(\mathbf{R})\big\}. \end{aligned} \tag{15}$$

Observe that $\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\rangle \cong \tau^{-1}\{\square_{\mathcal{A}}\mathcal{C}(\mathbf{R})\square_{\mathcal{B}}\mathcal{D}(\mathbf{R}) - \square_{\mathcal{B}}\mathcal{C}(\mathbf{R})\square_{\mathcal{A}}\mathcal{D}(\mathbf{R})\}$, whereas: $\nabla_{\mathcal{B}}\mathcal{C} \cong \tau^{-1}\square_{\mathcal{B}}\mathcal{C}(\mathbf{R})\mathbf{R}$. Thus it may happen that $\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\rangle$ is extendible but $\nabla_{\mathcal{B}}\mathcal{D}$ is not. As an example, $\langle R(\mathcal{N}, \mathcal{V})\mathcal{N}, \mathcal{V}\rangle \cong 0$, for any extension $\mathcal{N}$ of N and for any $\mathcal{V} \in \mathcal{X}(M)$ tangent to the screen S (see next theorem); however, if Σ is not *II*-flat, it may be $\nabla_{\mathcal{V}}\mathcal{V} \ncong 0$.

It follows from (15) that $\Upsilon(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) := \tau\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\rangle$ defines a *tensorfield* $\Upsilon \in \mathfrak{T}_4^0(M)$, whose restriction to Σ is given by:

$$\begin{aligned} \Upsilon(A, B, C, D) &= \det\begin{pmatrix} II(A, C) & II(A, D) \\ II(B, C) & II(B, D) \end{pmatrix} \\ &= \Upsilon(A^S, B^S, C^S, D^S) - \det\begin{pmatrix} \nu(A) & \nu(B) \\ \rho(A) & \rho(B) \end{pmatrix} \cdot \det\begin{pmatrix} \nu(C) & \nu(D) \\ \rho(C) & \rho(D) \end{pmatrix} \\ &\quad + II\big(\nu(A)B^S - \nu(B)A^S, \rho(C)D^S - \rho(D)C^S\big) \\ &\quad + II\big(\rho(A)B^S - \rho(B)A^S, \nu(C)D^S - \nu(D)C\big)^S. \end{aligned} \tag{16}$$

Theorem 5.

(a) *If we consider the assertions*:

(i)

$$\big\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\big\rangle \cong 0 \Leftarrow \det\begin{pmatrix} \nu(A) & \nu(B) \\ \rho(A) & \rho(B) \end{pmatrix} \cdot \det\begin{pmatrix} \nu(C) & \nu(D) \\ \rho(C) & \rho(D) \end{pmatrix} = 0,$$

(ii) *Σ is II-flat,*

(iii)

$$\big\langle R(\mathcal{A}, \mathcal{B})\mathcal{C}, \mathcal{D}\big\rangle \cong 0 \Rightarrow \det\begin{pmatrix} \nu(A) & \nu(B) \\ \rho(A) & \rho(B) \end{pmatrix} \cdot \det\begin{pmatrix} \nu(C) & \nu(D) \\ \rho(C) & \rho(D) \end{pmatrix} = 0,$$

then it holds: (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii).

(b) *$\langle R(A,B)C,D\rangle$ is well-defined if and only if it holds*: $\Upsilon(\cdot,B,C,D)=\Upsilon(A,\cdot,C,D)=\Upsilon(A,B,\cdot,D)=\Upsilon(A,B,C,\cdot)=0$. *If Σ is II-flat, this occurs if and only if*: *either* $\det\begin{pmatrix}\nu(A) & \nu(B)\\ \rho(A) & \rho(B)\end{pmatrix}=0=\det\begin{pmatrix}\nu(C) & \nu(D)\\ \rho(C) & \rho(D)\end{pmatrix}$, *or one of these two matrices vanishes.*

Proof. (a) First of all, $\langle R(\mathcal{A},\mathcal{B})\mathcal{C},\mathcal{D}\rangle\cong 0$ if and only if $\Upsilon(A,B,C,D)=0$. Implications (i) $\Leftarrow$ (ii) and (ii) $\Rightarrow$ (iii) are now immediate from (16). To prove (i) $\Rightarrow$ (ii), note that $0=\Upsilon(N,V,R,W)=II(V,W)$, for all $V,W\in\Gamma(S)$.

(b) Starting with $\mathcal{A}=\mathbf{A}+\tau\bar{\mathcal{A}}$, $\mathcal{B}=\mathbf{B}+\tau\bar{\mathcal{B}}$, $\mathcal{C}=\mathbf{C}+\tau\bar{\mathcal{C}}$, $\mathcal{D}=\mathbf{D}+\tau\bar{\mathcal{D}}$, for some $\bar{\mathcal{A}},\bar{\mathcal{B}},\bar{\mathcal{C}},\bar{\mathcal{D}}\in\mathfrak{X}(M)$, one immediately sees that it holds (we denote $\bar{A}=\bar{\mathcal{A}}|_\Sigma,\ldots$): $\langle R(\mathcal{A},\mathcal{B})\mathcal{C},\mathcal{D}\rangle|_\Sigma-\langle R(\mathbf{A},\mathbf{B})\mathbf{C},\mathbf{D}\rangle|_\Sigma=\Upsilon(\bar{A},B,C,D)+\Upsilon(A,\bar{B},C,D)+\Upsilon(A,B,\bar{C},D)+\Upsilon(A,B,C,\bar{D})$, and the result follows. □

The property that four vectorfields on Σ make the covariant curvature well-defined has a *pointwise character*, in the following sense: Given a point $p\in\Sigma$, four vectors $a,b,c,d\in T_pM$ and two sets $A,B,C,D\in\mathfrak{X}_\Sigma$ and $A',B',C',D'\in\mathfrak{X}_\Sigma$ of extensions of a,b,c,d such that both $\langle R(A,B)C,D\rangle$ and $\langle R(A',B')C',D'\rangle$ are well-defined, then it holds: $\langle R(A',B')C',D'\rangle(p)=\langle R(A,B)C,D\rangle(p)$. The proof uses the fact that canonical extensions of vectorfields on Σ giving the same tangent vector at p must coincide along the integral curve trough p of the canonical extension of N.

To discuss the limiting behaviour of sectional curvatures, we should consider all pairs of $C^\infty(M)$-linearly independent vectorfields $\mathcal{A},\mathcal{B}\in\mathfrak{X}(M)$ such that the plane(field) $\mathcal{A}\wedge\mathcal{B}$ is non-degenerate on $M-\Sigma$.

Let us first assume that the plane $A\wedge B$ never degenerates. Then the behaviour of the sectional curvature $K_{\mathcal{A}\wedge\mathcal{B}}:=\langle R(\mathcal{A},\mathcal{B})\mathcal{A},\mathcal{B}\rangle/\det(g_{(\mathcal{A},\mathcal{B})})$ near Σ can be directly read out from the behaviour of the covariant curvature $\langle R(\mathcal{A},\mathcal{B})\mathcal{A},\mathcal{B}\rangle$, thus Theorem 5(a) leads to the following result: $K_{\mathcal{A}\wedge\mathcal{B}}\cong 0$, for all $\mathcal{A},\mathcal{B}$ with $A^{NR}\wedge B^{NR}=0$ (we denote $A^{NR}\equiv\nu(A)N+\rho(A)R$), if and only if Σ is II-flat. And Theorem 5(b) implies moreover that, if Σ is II-flat, $K_{A\wedge B}$ is well-defined, for all A,B with $A^{NR}\wedge B^{NR}=0$ (here "$\wedge$" denotes the exterior product).

Let us now assume that the plane $A\wedge B$ degenerates everywhere on Σ. Then R must belong to the plane $A\wedge B$ and, since $\langle A,B\rangle=\langle A^{NS},B^{NS}\rangle$ (we denote $A^{NS}\equiv\nu(A)N+A^S$), it must hold: $A^{NS}\wedge B^{NS}=0$. Thus (16) leads to: $\Upsilon(A,B,A,B)=-(\nu(A)\rho(B)-\nu(B)\rho(A))^2$, which vanishes if and only if $A^{NR}\wedge B^{NR}=0$, what implies (since $A\wedge B\neq 0$) $\nu(A)=0=\nu(B)$. We arrive to the conclusion (Theorem 5(a)) that: if $K_{\mathcal{A}\wedge\mathcal{B}}\cong 0$, then it must hold: $A^S\wedge B^S=0$ and $\nu(A)=0=\nu(B)$. For the converse we prove:

Proposition 6. *The following assertions are equivalent*: (i) $K_{\mathcal{X}\wedge\mathcal{Y}}\cong 0$, *if* $\mathcal{X},\mathcal{Y}\in\mathfrak{X}(\Sigma)$ *and* $X^S\wedge Y^S=0$, *and* (ii) Σ *is III-flat.*

Proof. Assertion (i) reads equivalently: $K_{\mathcal{V}\wedge\mathcal{R}}\cong 0$, for any extension $\mathcal{R}$ of R and for any vectorfield $\mathcal{V}\in\mathfrak{X}(M)$ tangent to S, or in other words (take into account that $\tau^{-1}\det(g_{(\mathcal{V},\mathcal{R})})$ is a nowhere vanishing regular function, for the radical distribution is one-dimensional), $\langle R(\mathcal{V},\mathcal{R})\mathcal{V},\mathcal{R}\rangle$ is proportional to τ, for all $\mathcal{R},\mathcal{V}$. This implies (Theorem 5(b)): $0=\Upsilon(V,N,V,R)=II(V,V)$, for all $V\in\Gamma(S)$. Thus, not only (ii) but also (i) implies that Σ is II-flat.

To prove the equivalence (i) $\Leftrightarrow$ (ii), let us start with the expression: $\langle R(\mathcal{V},\mathcal{R})\mathcal{V},\mathcal{R}\rangle=\Box_\mathcal{V}(\nabla_\mathcal{R}\mathcal{V})(\mathcal{R})-\Box_\mathcal{R}(\nabla_\mathcal{V}\mathcal{V})(\mathcal{R})-\Box_{[\mathcal{V},\mathcal{R}]}\mathcal{V}(\mathcal{R})$. Because Σ is in either case II-flat, we know that the first and third terms in the right-hand side are proportional to τ: the first one, because it follows from (14) that $\nu(\nabla_\mathcal{R}\mathcal{V}|_\Sigma)=\mathcal{H}(V,R)=0$, thus $\nabla_\mathcal{R}\mathcal{V}$ is tangent to Σ and $II(V,\nabla_\mathcal{R}\mathcal{V}|_\Sigma)=0$; and the third one, because $II([V,R],V)=0$. Now the second term will be proportional to τ if and only if it holds: $0=II(R,\nabla_\mathcal{V}\mathcal{V}|_\Sigma)=:III(V,V,R)$. Therefore $\langle R(\mathcal{V},\mathcal{R})\mathcal{V},\mathcal{R}\rangle$ will be proportional to τ, for all $\mathcal{R},\mathcal{V}$, if and only if $III(V,V,R)=0$, for all $V\in\Gamma(S)$; and the result follows. □

On the other hand, it is easy to prove that $K_{A\wedge B}$ cannot be well-defined if $A\wedge B$ degenerates everywhere on Σ.

We finally consider Ricci curvatures. Starting with two vectorfields $\mathcal{A},\mathcal{B}\in\mathfrak{X}(M)$, we have (on $M-\Sigma$):

$$Ric(\mathcal{A},\mathcal{B})=\sum_{i=1}^{m-1}\bigl\langle R(\mathcal{A},\mathcal{E}_i)\mathcal{B},\mathcal{E}_i\bigr\rangle+\tau^{-1}\bigl\langle R(\mathcal{A},\mathbf{R})\mathcal{B},\mathbf{R}\bigr\rangle$$

$$\Rightarrow\tau Ric(\mathcal{A},\mathcal{B})=\sum_{i=1}^{m-1}\Upsilon(\mathcal{A},\mathcal{E}_i,\mathcal{B},\mathcal{E}_i)+\bigl\langle R(\mathcal{A},\mathbf{R})\mathcal{B},\mathbf{R}\bigr\rangle. \tag{17}$$

Proposition 7. *Let be $V \in \Gamma(S)$. Given any extensions $\mathcal{N} = \mathbf{N} + \tau\bar{\mathcal{N}}$, $\mathcal{V} = \mathbf{V} + \tau\bar{\mathcal{V}}$, $\mathcal{R} = \mathbf{R} + \tau\bar{\mathcal{R}}$ of N, V, R (with $\mathbf{N}, \mathbf{V}, \mathbf{R}$ the canonical extensions of N, V, R and $\bar{\mathcal{N}}, \bar{\mathcal{V}}, \bar{\mathcal{R}} \in \mathfrak{X}(M)$), it holds:*

(i) $Ric(\mathcal{N}, \mathcal{N}) \ncong 0$.
(ii) $Ric(\mathcal{N}, \mathcal{V}) \cong 0 \Leftrightarrow \rho([V, R]) + \nu(\bar{V}) = 0$. *In particular*: $d\rho = 0 \Rightarrow Ric(\mathcal{N}, \mathbf{V}) \cong 0$, *for all* $V \in \Gamma(S)$.
(iii) $Ric(\mathcal{N}, \mathcal{R}) \cong 0 \Leftrightarrow trace(II^S) - \nu(\bar{R}) = 0$, *where II^S is the self-adjoint endomorphism of S induced by II. In particular*: Σ *is II-flat* $\Rightarrow Ric(\mathcal{N}, \mathbf{R}) \cong 0$.
(iv) $Ric(\mathcal{V}, \mathcal{R}) \cong 0$, *for all* $V \in \Gamma(S)$.
(v) $Ric(\mathcal{R}, \mathcal{R}) \ncong 0$.
(vi) Σ *is II-flat* $\Rightarrow (Ric(\mathcal{V}, \mathcal{W}) \cong 0 \Leftrightarrow \mathcal{H}(V, W) = 0)$. *In particular*: Σ *is III-flat* $\Rightarrow Ric(\mathcal{V}, \mathcal{W}) \cong 0$, *for all* $V, W \in \Gamma(S)$.

Proof. Since: $Ric(\mathcal{A}, \mathcal{B}) \cong 0 \Leftrightarrow \{\tau Ric(\mathcal{A}, \mathcal{B})\}|_\Sigma = 0$, it follows from (17): $Ric(\mathcal{A}, \mathcal{B}) \cong 0 \Leftrightarrow \sum_{i=1}^{m-1} \Upsilon(A, E_i, B, E_i) + \langle R(\mathcal{A}, \mathbf{R})\mathcal{B}, \mathbf{R}\rangle|_\Sigma = 0 \Rightarrow \Upsilon(A, R, B, R) = 0$.

(i) $\Upsilon(N, R, N, R) = -1$, thus $\langle R(\mathcal{N}, \mathbf{R})\mathcal{N}, \mathbf{R}\rangle$ diverges like τ^{-1} and $Ric(\mathcal{N}, \mathcal{N})$ diverges like τ^{-2}.

(ii) We use (15) and (2) to compute: $\langle R(\mathcal{N}, \mathbf{R})\mathcal{V}, \mathbf{R}\rangle = k\tau - \Box_{[\mathcal{V},\mathbf{R}]}\mathcal{N}(\mathbf{R}) - \Box_{\mathbf{R}}\mathcal{N}(\mathbf{R})\{\tau^{-1}\langle\nabla_{\mathcal{V}}\mathbf{R}, \mathbf{R}\rangle\}$, for some $k \in C^\infty(M)$. This leads, using again (2), (13) and (9), to: $\langle R(\mathcal{N}, \mathbf{R})\mathcal{V}, \mathbf{R}\rangle|_\Sigma = \mathcal{H}([V, R], R) - \nu(\bar{V}) = -\rho([V, R]) - \nu(\bar{V})$. Finally take into account that $\Upsilon(N, E_i, V, E_i) = 0$ $(i = 1, \ldots, m-1)$.

(iii) As in (ii), we get: $\langle R(\mathcal{N}, \mathbf{R})\mathcal{R}, \mathbf{R}\rangle = k\tau - \Box_{\mathbf{R}}\mathcal{N}(\mathbf{R})\{\tau^{-1}\langle\nabla_{\mathcal{R}}\mathbf{R}, \mathbf{R}\rangle\}$, which leads to: $\langle R(\mathcal{N}, \mathbf{R})\mathcal{R}, \mathbf{R}\rangle|_\Sigma = -\nu(\bar{R})$. But $\sum_{i=1}^{m-1} \Upsilon(N, E_i, R, E_i) = trace(II^S)$.

(iv) Since $\langle R(V, R)R, R\rangle$ is well-defined, it follows: $\langle R(\mathcal{V}, \mathbf{R})\mathcal{R}, \mathbf{R}\rangle|_\Sigma = \langle R(\mathcal{V}, \mathbf{R})\mathbf{R}, \mathbf{R}\rangle|_\Sigma = 0$. But $\Upsilon(V, E_i, R, E_i) = 0$ $(i = 1, \ldots, m-1)$.

(v) As in (iv), we get: $\langle R(\mathcal{R}, \mathbf{R})\mathcal{R}, \mathbf{R}\rangle|_\Sigma = 0$. But $\Upsilon(R, N, R, N) = -1$ and $\Upsilon(R, E_\lambda, R, E_\lambda) = 0$ $(\lambda = 2, \ldots, m-1)$, thus $Ric(\mathcal{R}, \mathcal{R})$ diverges like τ^{-1}.

(vi) As in (ii), we get: $\langle R(\mathcal{V}, \mathbf{R})\mathcal{W}, \mathbf{R}\rangle = k\tau - \Box_{\mathcal{V}}\mathcal{W}(\mathcal{E}_1)\Box_{\mathbf{R}}\mathcal{E}_1(\mathbf{R}) + \sum_{\lambda=2}^{m-1} \Box_{\mathbf{R}}\mathcal{W}(\mathcal{E}_\lambda)\Box_{\mathcal{V}}\mathcal{E}_\lambda(\mathbf{R}) - \mathbf{R}(\Box_{\mathcal{V}}\mathcal{W}(\mathbf{R})) - \Box_{[\mathcal{V},\mathbf{R}]}\mathcal{W}(\mathbf{R})$, which leads to: $\langle R(\mathcal{V}, \mathbf{R})\mathcal{W}, \mathbf{R}\rangle|_\Sigma = -\mathcal{H}(V, W) - \langle II^S(V), II^S(W)\rangle - (L_R II)(V, W)$ (we denote by $L_R II$ the Lie-derivative of the tensorfield $II \in \mathfrak{T}_2^0(\Sigma)$). If Σ is II-flat, we have: $\langle R(\mathcal{V}, \mathbf{R})\mathcal{W}, \mathbf{R}\rangle|_\Sigma = -\mathcal{H}(V, W)$ and $\sum_{i=1}^{m-1} \Upsilon(V, E_i, W, E_i) = 0$. □

Moreover, it is possible to prove that $Ric(A, B)$ is never well-defined, for all $A, B \in \mathfrak{X}_\Sigma$.

Concerning differences between the radical transverse and radical (everywhere) tangent cases, Theorem 5a and Propositions 6 and 7 are missing in the case of transverse radical, as indicated by Theorem 3 in [5], which uses the same notions of II- and III-flatness. The intermediate cases of radical tangent to Σ on a submanifold of Σ are more subtle.

Finally we have the following analogue of the "Gauss–Codazzi equation". *Let Σ be II-flat*. Then we get (Proposition 4(a)) a natural connection (the *tangential connection*) on Σ, given by

$$\nabla^\Sigma : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma),$$
$$(X, Y) \mapsto \nabla^\Sigma_X Y := \nabla_{\mathbf{X}}\mathbf{Y}|_\Sigma - \nu(\nabla_{\mathbf{X}}\mathbf{Y}|_\Sigma)N,$$

where $\mathbf{X}, \mathbf{Y} \in \mathfrak{X}(M)$ are the canonical extensions of X, Y. From (13) and (14) we obtain: $\nabla_{\mathbf{R}}\mathbf{R}|_\Sigma = -N$, thus *the radical vectorfield R is ∇^Σ-geodesic*. Moreover, if Σ is III-flat, it follows from (14) that: $\nabla^\Sigma_X Y = \nabla_{\mathbf{X}}\mathbf{Y}|_\Sigma$, for all $X, Y \in \mathfrak{X}(\Sigma)$.

Using (5) we get $(\nabla^\Sigma_X Y)^S = D^S_X Y$ and a straightforward argument leads to the conclusion that ∇^Σ is torsion-free, thus it is admissible. From Theorem 2(b) it follows that ∇^Σ *is metric and all admissible connections on Σ have the same covariant curvature R^Σ as ∇^Σ*. Let us compute this curvature.

Theorem 8. *Let Σ be II-flat and let be $X, Y, Z, T \in \mathfrak{X}(\Sigma)$. Then it holds (Gauss–Codazzi equation)*:

$$\langle R^\Sigma(X, Y)Z, T\rangle = \langle R(X, Y)Z, T\rangle - \det\begin{pmatrix} \mathcal{H}(X, Z) & \mathcal{H}(Y, Z) \\ \mathcal{H}(X, T) & \mathcal{H}(Y, T) \end{pmatrix}.$$

Proof. By Theorem 5(b), $\langle R(X,Y)Z,T\rangle$ is well-defined. To compute it, we use: (i) canonical extensions, (ii) the fact that (∇^{Σ} is admissible) $\Box_X Y(Z) = \langle \nabla^{\Sigma}_X Y, Z\rangle$, and (iii) the following consequence of (14): $\nabla_{\mathbf{Y}}\mathbf{Z}|_{\Sigma} = \nabla^{\Sigma}_Y Z + \mathcal{H}(Y,Z)N$, thus: $\Box_X(\nabla_{\mathbf{Y}}\mathbf{Z}|_{\Sigma})(T) = \Box_X(\nabla^{\Sigma}_Y Z)(T) - \mathcal{H}(Y,Z)\mathcal{H}(X,T) = \langle \nabla^{\Sigma}_X(\nabla^{\Sigma}_Y Z), T\rangle - \mathcal{H}(Y,Z)\mathcal{H}(X,T)$. Therefore we obtain:

$$\begin{aligned}
\bigl\langle R(X,Y)Z,T\bigr\rangle &:= \Box_X(\nabla_{\mathbf{Y}}\mathbf{Z}|_{\Sigma})(T) - \Box_Y(\nabla_{\mathbf{X}}\mathbf{Z}|_{\Sigma})(T) - \Box_{[X,Y]}Z(T)\\
&= \bigl\langle \nabla^{\Sigma}_X(\nabla^{\Sigma}_Y Z), T\bigr\rangle - \mathcal{H}(Y,Z)\mathcal{H}(X,T) - \bigl\langle \nabla^{\Sigma}_Y(\nabla^{\Sigma}_X Z), T\bigr\rangle + \mathcal{H}(X,Z)\mathcal{H}(Y,T) - \bigl\langle \nabla^{\Sigma}_{[X,Y]}Z, T\bigr\rangle\\
&= \bigl\langle R^{\Sigma}(X,Y)Z,T\bigr\rangle + \det\begin{pmatrix} \mathcal{H}(X,Z) & \mathcal{H}(Y,Z)\\ \mathcal{H}(X,T) & \mathcal{H}(Y,T)\end{pmatrix}. \qquad \Box
\end{aligned}$$

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