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Causality and skies: is non-refocussing necessary?

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Abstract

The causal structure of a strongly causal, null pseudo-convex, space-time M is completely characterized in terms of a partial order on its space of skies defined by means of a class of non-negative Legendrian isotopies called sky isotopies. It is also shown that such partial order is determined by the class of future causal celestial curves, that is, curves in the space of light rays which are tangent to skies and such that they determine non-negative sky isotopies. It will also be proved that the space of skies Σ equipped with Low's (or reconstructive) topology is homeomorphic and diffeomorphic to M under the only additional assumption that M separates skies, that is, that different events determine different skies. The sky-separating property of M is sharp and the previous result provides an answer to the question about the class of space-times whose causal structure, topological and differentiable structure can be reconstructed from their spaces of light rays and skies. These results can be understood as a Malament–Hawking-like theorem stated in terms of the partial order defined on the space of skies.

Keywords: causality, strongly causal space-time, Legendrian isotopies, light rays, sky-separating, refocussing

(Some figures may appear in colour only in the online journal)

1. Introduction

The celebrated Malament and Hawking–McCarthy–King (MH) theorem provides an important relationship between the causal structure $\prec$ of a space-time M and its topological, smooth and metric structures [Ha76, Ma77]: 'if there exists a causal bijection between two n-

dimensional ($n > 2$) space-times which are both future and past distinguishing, then these space-times are conformally isometric'.

The causal structure $(M, \prec)$ of a space-time $(M, \mathbf{g})$ is a derived construction which relies on the underlying differentiable structure of M and the local lightcone structure provided by the conformal class of its metric. The relation $x \prec y$, 'y is in the causal future set of x', that is, there exists a future-oriented causal curve joining x with y, is a partial order if the space-time M is causal, i.e., there are no closed causal curves on it. A space-time is said to be future and past distinguishing if the chronological (time-like) future and past sets are unique for every space-time event. Thus MH theorem implies that the causal structure of future and past distinguishing space-times characterize both its conformal geometry and topology.

MH theorem also provides a strong motivation for the 'causal sets programme' to quantum gravity [Br91, Ri00], in which (locally finite) partially ordered sets replace the space-time M. Kronheimer and Penrose started the study of abstract causal spaces, that is, sets equipped with a partial order relation $\prec$ (and two additional relations $\ll$ and $\to$, corresponding to the chronological ordering and horisms relation in standard Lorentzian space-time geometry) satisfying a family of axioms kept as small and 'physically reasonable' as possible [Kr67].

The natural topology defined on such causal spaces, generated by the intersections of the chronological past and future sets of events, $I^+(x) \cap I^-(y)$, $x, y \in M$, known as Alexandrov topology, is equivalent to the topology defined by its smooth structure when M is a strongly causal space-time [Pe72], i.e., a space-time $(M, \mathbf{g})$ such that given any neighborhood U of $p \in M$ there exists a neighborhood $V \subset U$, $p \in V$, such that any future-directed causal curve $\gamma \colon I \to M$ with endpoints in V is entirely contained in U. It is equivalent to state that a space-time is strongly causal if given any neighborhood U of p there exists a neighborhood $V \subset U$, $p \in V$ (which can be chosen globally hyperbolic), such that V is causally convex in M. Again if M is strongly causal, its Alexandrov topology is Hausdorff (see for instance, theorem 3.27 [Mi08]). Strongly causal space-times provide a stronger version of MH theorem as it can be readily proved that if there is a causal bijection between to strongly causal space-times, then they both are homeomorphic and diffeomorphic, as well as conformally isometric, i.e., being causally isomorphic implies that they must have the same dimension.

Once the causal partial order structure $\prec$ is promoted to the centre of the stage it makes sense to consider different realizations of the abstract causal space determined by a space-time. In physical space-times events can be determined by performing local measurements but, dually, they are characterized by the congruence of light rays passing through them or, in other words, by the light rays arriving (and leaving) from a given observer. Such congruence of light rays passing through a given event $x \in M$ is called the sky $S(x)$ of x and is diffeomorphic to a $(n - 2)$–dimensional sphere. It is clear that the collection of skies could offer an alternative realization of the abstract causal space determined by the causal structure of M, hence an alternative way of studying its properties, provided that it carries a topological and differentiable structure that makes it homeomorphic and diffeomorphic to the original space-time M.

R Low initiated a program to systematically explore the causality structure of a space-time in terms of the properties of its space of skies or, more precisely, its space of light rays, that provide the appropriate framework to study them. Such project is rooted in Penrose's twistor program and represents a real version of it that, contrary to the twistor transform, is defined on any space-time regardless of its dimension and geometry. Thus to any space-time $(M, \mathbf{g})$ we can associate its space of light rays $\mathcal{N}$, i.e., the set of unparametrized oriented null geodesics. Such space depends only on the conformal class of the metric $\mathbf{g}$ and inherits a natural topology from the topology of TM.

The family of skies $\Sigma = \{S(x) | x \in M\}$, being submanifolds of $\mathcal{N}$, inherit a topology from the ambient space $\mathcal{N}$. Thus, the sky map S: $M \to \Sigma$, that maps each event $x \in M$ to its sky $S(x) \in \Sigma$, becomes a map between topological spaces. One of the basic questions raised by Lows's programme is the reconstruction problem for the topology and differentiable structure of M: under what conditions the sky map S is a homeomorphism (diffeomorphism), that is, under what conditions the space of skies Σ is homeomorphic (diffeomorphic) to M. This is one of the problems that will be discussed along the paper.

It is clear that a positive answer to the reconstruction problem for the topology of a given space-time would imply that the sky map is a bijection. Space-times with this property will be called sky-separating spaces, that is, a sky-separating space-time is such that given two different events $x \neq y$ their skies are different $S(x) \neq S(y)$. Notice that simple examples, like the Einstein cylinder $\mathbb{R} \times \mathbb{S}^{m-1}$ equipped with the standard product metric $\mathbf{g} = -dt^2 \oplus \mathbf{h}$, where $\mathbf{h}$ is the induced Euclidean metric on $\mathbb{S}^{m-1}$, show that even globally hyperbolic spaces could have many-to-one sky maps, hence are not sky-separating.

It is not difficult to show that for a sky-separating strongly causal space-time the sky map S is continuous (see for instance proposition 3, [Ba14]). Hence it would remain to show what extra condition, if any, is needed to guarantee that its inverse map P (called the 'parachute map') is continuous.

Moreover the space of light rays $\mathcal{N}$ of a strongly causal space-time M is a smooth manifold of dimension $2m - 3$ if M has dimension m (see for instance [Lo89, proposition 2.1. and ff.]) and the skies $S(x) \subset \mathcal{N}$ are immersed $(m-2)$–submanifolds. Thus strongly causal space-times are natural candidates for a reconstruction theorem of the differentiable structure too. We may try to construct a smooth structure on the space of skies Σ induced from the smooth structure on $\mathcal{N}$ and check that the sky map S is a diffeomorphism. This will be the main contribution of the present paper and will be discussed at length on section 4.

It is relevant to point out that the topology of a space-time is assumed to be Hausdorff, reflecting the locality of physical theories defined on it. Hence a proper reconstruction theorem of the topological structure would require a Hausdorff space of light rays. However it is easy to exhibit examples, like Minkowski space with a point removed, whose space of light rays is not Hausdorff.

It was proved by Low [Lo90a] that for a strongly causal space-time, if the space of light rays $\mathcal{N}$ is not Hausdorff than M must have naked singularities, that is M must contain some point whose past contains a future endless causal curve (actually it was proved that if the space of causal curves is not Hausdorff the space-time must be naked singular). The lack of such pathology is equivalent to M being globally hyperbolic [Pe78] (see also [Ge72]). Thus we could be tempted to assume that globally hyperbolic space-times are the only good candidates for a reconstruction theorem. However even if the absence of naked singularities guarantees that $\mathcal{N}$ is Hausdorff, there are simple examples of spaces with naked singularities whose space of light rays is Hausdorff (a strip $(\mathbb{R} \times (-1, 1), -\mathrm{d}t^2 + \mathrm{d}x^2)$ would be an example, see also example 3.1 in [Lo90a]).

Thus we may conclude this discussion by observing that a natural condition to be added to the sky-separating property discussed before on any reasonable reconstruction theorem is that the space of light rays is Hausdorff. Such topological property of the space of light rays can be characterized in terms of causal properties of M. It was proved by Low [Lo90a, proposition 3.2 and ff.] that on a strongly causal space-time M, $\mathcal{N}$ is Hausdorff iff M is causally pseudo-convex, i.e., if given any compact set $K \subset M$, there is a compact set $K' \subset M$ such that any causal geodesic segment with endpoints in K lies in K' and the same property holds if we restrict ourselves to consider just null geodesics. We will refer then to this property as null pseudo-convexity. Thus being null pseudo-convex is the required property on M that guarantees that $\mathcal{N}$ is Hausdorff and conversely.

However the fundamental interest of MH theorem is that causal structures determine the topology and geometry of the corresponding spaces, thus the question that raises naturally is if there exists a natural partial order $\prec_\Sigma$ in the space of skies Σ such that an analogue to MH theorem will hold, that is, if the sky map is a causal bijection between M and Σ then it is a homeomorphism (and a diffeomorphism too).

An analogous reconstruction theorem for the causal relation $\prec$ would be provided by a partial order relation $\prec_\Sigma$ on the space of skies Σ that will induce it by means of the sky map S. One fundamental advantage of considering the space of skies is that they are embodied in the manifold $\mathcal{N}$ which provides an additional geometrical structure, a contact structure, not present when considering the standalone space of events M.

It is even of greater importance that the contact structure on the space of light rays will allow for the construction of the partial order $\prec_\Sigma$. A contact structure is the odd-dimensional counterpart of a symplectic structure and is defined as a maximally non-integrable distribution of hyperplanes $\mathcal{H}$. A contact structure $\mathcal{H}$ is said to be co-orientable (or exact) if there exists a 1-form α such that $\ker \alpha = \mathcal{H}$ (notice that the maximal non-integrability of $\mathcal{H}$ means that $d\alpha$ restricted to $\mathcal{H}$ is of maximal rank).

Tangent vectors to the space of light rays $\mathcal{N}$ at the light ray γ are defined by geodesic variations by null geodesics of γ up to reparametrizations, that is, equivalence classes of Jacobi fields J along the null geodesic γ (two Jacobi fields being equivalent module γ'). The contact structure $\mathcal{H}$, equipping $\mathcal{N}$ with the structure of an odd structure phase space, is defined as the hyperplanes of Jacobi fields along γ orthogonal to the direction defined by the geodesic itself, i.e., such that $\mathbf{g}(J, \gamma') = 0$.

Notice that skies are Legendrian submanifolds of $\mathcal{N}$, i.e., maximally isotropic submanifold of the contact structure, because the Jacobi field determined by any geodesic variation defining a tangent vector to a sky $S(x)$ will vanish at the event x defining it. Legendrian submanifolds constitute again the contact analogue of Lagrangian submanifolds.

A co-oriented contact manifold induces a natural relation in the space of Legendrian submanifolds. Two Legendrian submanifolds L_0, L_1 are related if there exists a definite Legendrian isotopy joining them, that is, a family of Legendrian submanifolds L_s such that $\alpha(F_* \partial/\partial s)$ is definite in sign, where $F\colon [0, 1] \times L_0 \to \mathcal{N}$ is a diffeomorphism such that $F(s, L_0) = L_s$. The family of Legendrian spheres in $\mathcal{N}$ is very large (see for instance [El98]) containing a small subspace, the space of skies Σ, of Legendrian spheres which are skies or, in other words determined by events (in this sense it is possible to think of arbitrary Legendrian spheres on $\mathcal{N}$ as skies associated to 'virtual events'). Hence there is a natural relation induced from the natural relation in the space of Legendrian spheres in the space of skies. However the corresponding induced relation in the space of skies is too coarse and we should restrict to the relation induced by Legendrian isotopies consisting on skies that will be denoted by $\prec_\Sigma$.

In section 3 it will be shown that for sky-separating strongly causal space-times the sky map is an order preserving bijection between the space-time $(M, \prec)$ and the space of skies with its natural partial order relation $(\Sigma, \prec_\Sigma)$, hence a natural extension of MH theorem would assert that S should be a homeomorphism (and a diffeomorphism too). However that it is not possible to use the original MH theorem in this context because it is not known *a priori* that the natural topology on Σ is the topology defined by a strongly causal structure. In spite of this the last assertion is true and constitutes the main result of this work.

Before discussing the structure of the proof we should mention that it was conjectured by R Low that two events in a (2 + 1)-dimensional space-time are causally related iff their corresponding skies are topologically linked [Lo88], however the conjecture fails to be true in higher-dimensions. Actually Low proved that if the linking number of two skies $S(x)$, $S(y)$ in a globally hyperbolic $(2 + 1)$-dimensional space-time is non-trivial, then the events x, y must be causally related. This motivated the problem (communicated by Penrose) on Arnold's 1998 problem list asking to apply knot theory to the study of causality.

Low's topological linking conjecture was refined by Natario and Tod [Na04] as: 'let $(M, \mathbf{g})$ be a globally hyperbolic $(3 + 1)$-space-time with Cauchy surface diffeomorphic to a subset of $\mathbb{R}^3$, and let $\mathcal{N}$ be its manifold of light rays. Then two spacetime points are causally related in M iff their skies either intersect or are Legendrian linked in $\mathcal{N}$', and it is referred now as Low's Legendrian conjecture. Recently it was shown by Chernov and Rudyak [Ch08] and Chernov and Nemirovski [Ch10] that Low's Legendrian conjecture is actually true in a globally hyperbolic space with a Cauchy surface whose universal covering is diffeomorphic to an open domain in $\mathbb{R}^n$. Even more, Chernov and Nemirovski [Ch14] had extended the previous ideas to show that the causal structure of a simply connected globally hyperbolic space-time M can be reconstructed from the partial ordering in the universal covering of Legendrian isotopy class of the fibres of the sphere bundle of a smooth Cauchy surface.

In the previous paper by the authors [Ba14] it was shown that for a class of strongly causal space-times their causal structures were determined by the family of causal celestial curves. Let us recall that a celestial curve Γ is a differentiable curve in the space of light rays which is tangent everywhere to a sky. A celestial curve is called past (future) causal if it defines a non-negative (non-positive) Legendrian isotopy of skies. The class of causal celestial curves emerges as the relevant geometrical structure on $\mathcal{N}$ characterizing the original conformal class of the Lorentzian metric on M. However this theorem was proved under the assumption that the space-times were non-refocussing (and non-null conjugate).

The non-refocussing property of space-times was introduced by Low in [Lo01, Lo93] and [Lo06] to guarantee that the sky map S is open: a space-time M is refocussing at $x \in M$ if there exists an open neighbourhood U of x such that for every open neighbourhood $V \subset U$ of x there exists $y \notin U$ such that every null geodesic through y enters V. This property has been studied in depth in [Ki11] and, as it was mentioned before, it plays an important role in the proofs given in [Ba14, proposition 3] and [Ki11, proposition 4.1] that the sky map $S: M \rightarrow \Sigma$ is a homeomorphism. It is important to point out here that the notion of non-refocussing was also used by Chernov and Rudyak [Ch08] to prove that in a non-refocussing globally hyperbolic space-time, two events are causally unrelated iff there skies can be deformed by a sky isotopy to two standard fibres of the sphere fibration of a Cauchy hypersurface.

In the present paper, we hope, the role of the different hypothesis used in the discussions before are clarified. Actually we will be able to reproduce some the results already presented in [Ba14] without recurring to the property of non-refocussing, only strongly causal sky-separating space-times will be required, of course, and the assumption that the manifold of light rays $\mathcal{N}$ is Hausdorff or, equivalently, that M is null pseudo-convex.

In section 3.4 it will be shown that the causal structure of M can be recovered from a partial ordering introduced in the space of skies by a restricted class of non-negative Legendrian isotopies called sky isotopies. Without entering in the analysis of Low's Legendrian conjecture here, it will be shown that the analysis of the causal structure of M in terms of Σ is deeply related to the study of celestial curves. It will be shown that celestial curves are in correspondence with a class of null curves that will be called twisted null curves. The causal structure of the original space-time will be characterized completely at the end of section 3 in terms of the partial order relation induced in the space of skies by future (past) directed twisted null curves.

Finally, the proof that the space of skies of a sky-separating space-time is homeomorphic (and diffeomorphic) to the original space proceeds by constructing a basis for the reconstructive (or Low's) topology by means of regular open subsets of Σ, where 'regular' here means that the corresponding tangent spaces to the skies elements of the open set 'pile up' nicely and define a regular submanifold in the tangent space to $\mathcal{N}$. The method of the proof is novel. The definition and discussion of the main properties of regular sets constitutes the core of section 4, where again the properties of twisted causal null curves will be used in a critical way.

Summarizing, we offer an answer to the question of characterizing a large class of space-times M such that the pair $(\mathcal{N}, \Sigma)$ is capable of reconstructing the causal, topological and differentiable structures of M. However the question of what is the largest class of space-times for which Low's Legendrian conjecture holds is still open.

2. The space of light rays and the space of skies

2.1. The space of light rays

Let M be a second countable paracompact m-dimensional smooth manifold ($m \geqslant 3$) and $\mathcal{C}$ a conformal class of Lorentzian metrics of signature $(- + \cdots +)$ such that M becomes a time-orientable strongly causal space-time. We will denote by $\mathbf{g}$ a representative metric on $\mathcal{C}$ and by T a fixed time-like vector field determining a time-orientation on M.

Let $\mathcal{N}$ denote the space of unparametrized inextensible future-directed null geodesics, called in what follows light rays, i.e., $\mathcal{N}$ is the space of equivalence classes of inextensible smooth null curves $\gamma: I \to M$, with I an interval in $\mathbb{R}$, such that $\nabla_{\gamma'}\gamma' = 0$, $g(\gamma', T) < 0$, and two such curves are equivalent if they are related by a reparametrization for some representative $\mathbf{g}$ of the conformal class $\mathcal{C}$.

We will consider in what follows the fibre bundle $\mathbb{N}$ over M consisting of non-zero null vectors, and the corresponding components of future (past) null vectors $\mathbb{N}^{\pm}$. If we denote by $\mathbb{N}_x^+ = \{v \in \mathbb{N}_x | v \neq 0,\, g_x(v, T(x)) < 0\}$ and $\mathbb{N}_x^- = \{v \in \mathbb{N}_x | v \neq 0,\, g_x(v, T(x)) > 0\}$ the components of the fiber over $x \in M$ of the bundle $\mathbb{N}$, then $\mathbb{N}^{\pm} = \bigcup_{x\in M} \mathbb{N}_x^{\pm}$ and $\mathbb{N} = \mathbb{N}^+ \cup \mathbb{N}^-$. We will denote by $\pi: \mathbb{N} \to M$ the restriction of the canonical tangent bundle projection $TM \to M$ to $\mathbb{N}$ (and $\mathbb{N}^{\pm}$).

Consider now the quotient space $\mathbb{PN}^+$ of $\mathbb{N}^+$ by the action of the multiplicative group of positive real numbers $\mathbb{R}^+$ by scalar multiplication. We will denote again by π the projection of $\mathbb{PN}^+$ onto M induced by the projection $\pi: \mathbb{N}^+ \to M$, that is $\pi: \mathbb{PN}^+ \to M$, and $\pi([u]) = \pi(u)$ where $[u] \in \mathbb{PN}^+$ denotes the equivalence class $\{\lambda u | \lambda > 0\}$ defined by the future null vector $u \in \mathbb{N}_+$.

Notice that there is a canonical surjection $\sigma: \mathbb{PN}^+ \to \mathcal{N}$, given by $\sigma([u]) = \gamma_{[u]}$, where $\gamma_{[u]}$ (or $[\gamma_u]$ as it will be used in what follows too) denotes the unparametrized geodesic containing the unique future-directed parametrized null geodesic $\gamma_u(t)$ such that $\gamma_u(0) = \pi(u)$, and $\gamma_u'(0) = u$. Moreover, because $\gamma_{\lambda u}(t) = \gamma_u(\lambda t)$, $u \in \mathbb{N}^+$ the previous notation is consistent.

2.2. The smooth structure of $\mathcal{N}$

The space of light rays $\mathcal{N}$ can be equipped with the structure of a second countable paracompact smooth manifold of dimension $2m - 3$ ($\dim M = m$) such that the map σ becomes a submersion in two different ways. We will succinctly describe them in the following paragraphs.

First, we can use the local structure of M. Because M is strongly causal, given any event $x \in M$, there exists a globally hyperbolic neighbourhood U_x of x and a local smooth Cauchy hypersurface $C_x \subset U_x$ [Mi08]. We can take U_x small enough such that it is contained in a local chart of M. Then we can define an atlas for $\mathcal{N}$ as follows: Select for any event $x \in M$ a globally hyperbolic open neighbourhood U_x as before with Cauchy hypersurface C_x. Consider the restriction of the projective bundle $\mathbb{PN}^+$ to C_x and we denote it by $\mathbb{PN}^+(C_x)$. There is a natural embedding $i_x\colon \mathbb{PN}^+(C_x) \to \mathbb{PN}^+$. The compositions $\sigma \circ i_x\colon \mathbb{PN}^+(C_x) \to \mathcal{N}$ will provide the charts of the atlas we are looking for and we denote the open sets $\sigma \circ i_x(\mathbb{PN}^+(C_x)) \subset \mathcal{N}$ by $\mathcal{U}_x$ (see [Ba14, section 2.3] for more details).

Alternatively, we can induce a smooth structure on $\mathcal{N}$ from the smooth structure of the bundle $\mathbb{N}^+$ by considering the foliation defined by the leaves of the integrable distribution generated by the vector fields $X_{\mathbf{g}}$ and Δ, where $X_{\mathbf{g}}$ denotes the geodesic spray of a fixed representative metric in the conformal class $\mathcal{C}$, and Δ is the dilation or Euler field. Because $[X_{\mathbf{g}}, \Delta] = X_{\mathbf{g}}$, the distribution $D = \operatorname{span}\{\Delta, X_{\mathbf{g}}\}$ is integrable and denoting by $\mathcal{D}$ the corresponding foliation, its space of leaves can be identified canonically with the space of light rays, $\mathbb{N}^+/\mathcal{D} \cong \mathcal{N}$. If M is strongly causal it can be shown that $\mathcal{D}$ is a regular foliation and the space of leaves inherits a smooth structure from $\mathbb{N}^+$. Again, it is not hard to show that both ways of defining smooth structures on $\mathcal{N}$ coincide.

2.3. The tangent bundle $T\mathcal{N}$ and the contact structure of $\mathcal{N}$

Let $\Gamma\colon (-\epsilon, \epsilon) \to \mathcal{N}$ be a differentiable curve such that $\Gamma(0) = \gamma$ and let $\chi(s, t)\colon (-\epsilon, \epsilon) \times I \to M$ be a geodesic variation by null geodesics of a parametrization $\gamma(t)$ of the null geodesic γ, that is, χ is a smooth function such that the curves $\gamma_s(\,\cdot\,) = \chi(s, \cdot\,)$ are null geodesics, $\gamma_0(t)$ is a parametrization of γ, and $[\gamma_s] = \Gamma(s)$ where $[\gamma_s]$ denotes the unparametrized geodesic containing γ_s. Then the vector field along $\gamma(t)$ defined by $J(t) = \partial\chi(s, t)/\partial s\,|_{s=0}$ is a Jacobi field. The set of Jacobi fields along $\gamma(t)$ will be denoted by $\mathcal{J}(\gamma)$ and they satisfy the second order differential equation:

$$J'' = R\big(\gamma', J\big)\gamma', \tag{2.1}$$

where J' denotes the covariant derivative of J along $\gamma'(t)$. Notice that since the geodesic variation χ is by null geodesics, we have $\langle J, \gamma' \rangle = \text{constant}$. Actually from equation (2.1) we get immediately $\langle J, \gamma' \rangle'' = \langle J'', \gamma' \rangle = 0$ and $\langle J, \gamma' \rangle$ is an affine function on the affine parameter t, however because $\chi(s, t)$ is a geodesic variation by null geodesics we have:

$$\begin{aligned} 0 = \frac{\partial}{\partial s}\left\langle \frac{\partial}{\partial t}\chi(s, t), \frac{\partial}{\partial t}\chi(s, t) \right\rangle &= 2\left\langle \frac{\nabla}{\partial s}\frac{\partial}{\partial t}\chi(s, t), \frac{\partial}{\partial t}\chi(s, t) \right\rangle \\ = 2\left\langle \frac{\nabla}{\partial t}\frac{\partial}{\partial s}\chi(s, t), \frac{\partial}{\partial t}\chi(s, t) \right\rangle &= 2\frac{\partial}{\partial t}\left\langle \frac{\partial}{\partial s}\chi(s, t), \frac{\partial}{\partial t}\chi(s, t) \right\rangle, \end{aligned}$$

but then, evaluating the previous expression at $s = 0$, we get:

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\langle J(t), \gamma'(t) \big\rangle = 0,$$

as claimed. We denote by $\mathcal{L}(\gamma)$ the linear space of Jacobi fields satisfying this property.

Equivalence classes of curves $\Gamma(s)$ possessing a first order contact define tangent vectors to $\mathcal{N}$ at γ, hence tangent vectors at γ correspond to equivalence classes of Jacobi fields with respect to the equivalence relation defined by reparametrization of the geodesic variation χ. Such reparametrizations will correspond to Jacobi fields of the form $(at + b)\gamma'(t)$, then there is a canonical projection $\mathcal{L}(\gamma) \to T_\gamma\mathcal{N}$, mapping each Jacobi field J into a tangent vector

$[J] = J \mod \gamma'$ whose kernel is given by Jacobi fields proportional to γ'. In what follows the tangent vectors $[J]$ will be denoted again as J unless there is risk of confusion.

There is a canonical contact structure on $\mathcal{N}$ defined by the maximally non-integrable hyperplane distribution $\mathcal{H}_\gamma \subset T_\gamma \mathcal{N}$ formed by the vectors orthogonal to their supporting light ray, i.e.,

$$\mathcal{H}_\gamma = \left\{ J \in T_\gamma \mathcal{N} \,|\, \langle J, \gamma' \rangle = 0 \right\}. \tag{2.2}$$

It is easy to show that $\mathcal{H}$ does not depend on the representative metric $\mathbf{g}$ used before to define it, the representative J chosen for the tangent vector $[J]$, or the parametrization $\gamma(t)$ we chose for the light ray γ.

An alternative way of realizing this is by observing that the contact distribution $\mathcal{H}$ can be obtained as the quotient of the canonical contact distribution $\mathcal{H}^+ = \ker \theta_{\mathbf{g}}$ on $\mathbb{N}^+$, where $\theta_{\mathbf{g}}$ is the restriction to $\mathbb{N}^+$ of the pull-back of the canonical 1-form θ on T^*M by means of the canonical bundle isomorphism $\hat{\mathbf{g}}\colon TM \to T^*M$ defined by the metric $\mathbf{g}$. Notice that $\mathcal{H}^+$ just depends on the conformal class defined by $\mathbf{g}$ because $\theta_{\lambda \mathbf{g}} = \lambda \theta_{\mathbf{g}}$. Then it is easy to check that the distribution D is contained in $\mathcal{H}^+$ and it quotients down under the canonical projection $\mathbb{N}^+ \to \mathcal{N}$ to the contact structure on $\mathcal{N}$, i.e., $\mathcal{H} \cong \mathcal{H}^+/D$.

The same argument shows that the contact structure $\mathcal{H}$ is co-orientable. Let us recall that if X is a contact manifold with contact distribution a maximally non-integrable codimension one distribution $\mathcal{H}$, the contact structure is said to be exact or co-orientable if there exists a globally defined 1-form α, such that $\mathcal{H} = \ker \alpha$ and such 1-form is called a contact 1-form for the contact structure $\mathcal{H}$. Thus the contact structure $\mathcal{H}^+$ on $\mathbb{N}^+$ is co-orientable and $\theta_{\mathbf{g}}$ defines a co-orientation, that, equivalenttly, determines a reduction of its principal fiber bundle of co-orientations by selecting its positive co-orientations (positive multiples of the 1-form $\theta_{\mathbf{g}}$). But then, this bundle pass to the quotient $\mathcal{N}$ and is trivial because its fibre is contractible.

Alternatively, we observe that the canonical contact structure $\mathcal{H}$ on $\mathcal{N}$, equation (2.2), can be locally defined by the family of 1-forms α^x defined on the open sets $\mathcal{U}_x$ described in section 2.2 above, that are given by explicitly as:

$$\alpha_\gamma^x\colon T_\gamma \mathcal{N} \to \mathbb{R}, \qquad \alpha_\gamma^x(J) = \langle J, \gamma' \rangle, \qquad \forall\, J \in T_\gamma \mathcal{N},$$

where the parametrization $\gamma(t)$ of the light ray γ is determined by the Cauchy surface $C_x \subset U_x$ and the orientation provided by the temporal vector field T, $\langle T(x), \gamma'(0) \rangle = -1$. The local 1-forms α^x do not define a global 1-form, (unless M is globally hyperbolic, in which case a global smooth Cauchy surface C can be chosen [Be05], then $\mathbb{PN}^+(C)$ and the 1-form α^x above is globally defined[3]). Because $\mathcal{N}$ is paracompact we can use a partition of the unity $\{\rho_x | 0 \leqslant \rho_x \leqslant 1\}$ subordinated to a locally finite refinement of the open covering $\{\mathcal{U}_x\}$ of $\mathcal{N}$ defined by family of globally hyperbolic open neighbourhoods $\{U_x | x \in M\}$, and paste the local 1-forms to define a globally defined 1-form $\alpha = \sum_x \rho_x \alpha^x$ whose kernel is $\mathcal{H}$.

It is remarkable that the contact structure $\mathcal{H}$ is co-orientable because the existence of the decomposition $\mathbb{N} = \mathbb{N}^+ \cup \mathbb{N}^-$, that is, because M is time-orientable. Notice however that in general the space of non-oriented unparametrized null geodesics still carries a canonical contact structure (defined by the same formula above, equation (2.2)) which is not co-orientable.

[3] In such case $\mathbb{PN}^+(C)$ is diffeomorphic to the spheres bundle $S(TC)$ over C and is a simple exercise to check that the 1-form α defined above agrees with the canonical 1-form defined on $S(TC)$ by restricting the pull-back of $\theta_{\mathbf{g}}$ to C.

3. Reconstruction of the causal structure

3.1. The space of skies and its topology

As it was explained in the introduction, the sky of an event is the congruence of light rays passing through it. Thus if $x \in M$ denotes an event, the corresponding sky will be denoted either by $S(x)$ or X and $S(x) = \{\gamma \in \mathcal{N} | x \in \gamma\}$. Notice that there is a canonical diffeomorphism $\sigma_x: \mathbb{PN}_x^+ \to S(x)$, $\sigma_x([u]) = \gamma_{[u]}$, $[u] \in \mathbb{PN}_x^+$. Clearly the sky $S(x)$ as a submanifold of $\mathcal{N}$ is diffeomorphic to the sphere of dimension $m - 2$ because the fibre $\mathbb{PN}_x^+$ over x of the fibre bundle $\mathbb{PN}^+ \to M$ is the sphere of dimension $m - 2$. The family of all skies will be denoted by Σ, that is,

$$\Sigma = \{X = S(x) | x \in M\},$$

and the canonical map $S: M \to \Sigma$, $x \mapsto S(x)$, is called the sky map. The sky map is clearly surjective, however it doesn't have to be injective as indicated in the introduction.

We will say that M separates skies if S is injective, that is, whenever $x \neq y$, then $S(x) \neq S(y)$. If M separates skies, the map $P: \Sigma \to M$, inverse to the sky map, is well defined and will be called the parachute map.

The space of skies Σ carries a canonical topology, called the reconstructive topology, defined as follows. Let $\mathcal{U} \subset \mathcal{N}$ be an open set, then consider the set of all skies X such that $X \subset \mathcal{U}$. We will denote this set by $\Sigma(\mathcal{U})$. It is clear that the family of sets $\Sigma(\mathcal{U})$ satisfies $\Sigma(\mathcal{U}) \cap \Sigma(\mathcal{V}) = \Sigma(\mathcal{U} \cap \mathcal{V})$, then they constitute a basis for a topology on Σ called the reconstructive topology.

It is easy to prove that the sky map S is continuous with respect to the reconstructive topology. However it is not obvious if it is open or not. As it was discussed in the introduction it is one of the objectives of this paper to determine under what conditions S is open, i.e., P continuous.

We will end these remarks by observing that if $X = S(x)$ is a sky, then given $\gamma \in X$, a tangent vector J to X at γ is determined by a geodesic variation such that all their geodesics pass through the point x (at time 0 in some parametrization), then $J(0) = 0$. This implies that $\langle J, \gamma' \rangle = 0$ for all $J \in T_\gamma X$ and $TX \subset \mathcal{H}$. Thus skies are Legendrian spheres because $2\dim T_\gamma X = 2(m-2) = 2m - 4 = \dim \mathcal{H}_\gamma$.

3.2. The partial order in the space of skies

The canonical contact structure on the space of light rays allows to define a natural partial ordering in the space of skies.

Let us recall first that if X is a contact manifold with contact distribution $\mathcal{H}$, a differentiable family Λ_s, $s \in [0, 1]$, of diffeomorphic embedded Legendrian submanifolds is called a Legendrian isotopy. If Λ_0 is closed there is always an ambient compactly supported contact isotopy $\Psi: X \times [0, 1] \to X$ with $\Psi_s(\Lambda_0) = \Psi(\Lambda_0, s) = \Lambda_s$ (see for instance [Ge06, theorem 2.41]). Thus we will describe a Legendrian isotopy Λ_s via a parametrization $F: \Lambda_0 \times [0, 1] \to X$ verifying $F_s(\Lambda_0) = F(\Lambda_0, s) = \Lambda_s \subset X$ (the map F being the restriction of the ambient isotopy Ψ to Λ_0) and the map $F_s: \Lambda_0 \to \Lambda_s$, given by $F_s(\lambda) = F(\lambda, s)$, $\lambda \in \Lambda_0$, is a diffeomorphism for all $s \in [0, 1]$. The Legendrian isotopy Λ_s is said to be non-negative (non-positive) if $(F^*\alpha)(\partial/\partial s) \geqslant 0$ (respect. $(F^*\alpha)(\partial/\partial s) \leqslant 0$) with F a parametrization of Λ_s. It is easy to check that the previous definition does not depend on the chosen parametrization.

If we consider now the class $\mathcal{S}$ of Legendrian spheres on the contact manifold X, we can define a relation on $\mathcal{S}$ by saying that $S_0 \prec S_1$, $S_0, S_1 \in \mathcal{S}$, if there exists a non-negative

Legendrian isotopy $F: S_0 \times [0, 1] \to X$, joining S_0 and S_1, i.e., such that $F_0(S_0) = S_0$, $F_1(S_0) = S_1$. Notice, however, that in general the relation $\prec$ fails to define a partial order because the group of contactomorphisms of X could have non-trivial closed loops based at the identity (see for instance [El00]).

We will apply now the previous ideas to the contact manifold $\mathcal{N}$ of light rays of a given space-time M. The class $\mathcal{S}$ of Legendrian spheres in M contains the space of skies Σ. Then the relation $\prec$ described before induces a relation in Σ. However we would like to restrict the previous relation because it could happen that two skies $X_0 = S(x_0)$ and $X_1 = S(x_1)$ would be related, $X_0 \prec X_1$, but the non-negative Legendrian isotopy X_s joining X_0 and X_1 will fall out of Σ, that is, not all Legendrian spheres X_s will be the sky of a point $x_s \in M$.

Let us remark that this is often what happens. Consider, for instance, the simple case of Minkowski space $\mathbb{M}^m$, whose space of light rays is contactomorphic to the bundle of spheres $S(TN)$ over a Cauchy surface that, in this case, can be chosen to be the standard $t = 0$ section, i.e., $N = \{(0, \mathbf{x}) | \mathbf{x} \in \mathbb{R}^{m-1}\}$. Now the sky $S(x) \subset S(TN)$ of an event $x = (t, \mathbf{x})$ will project onto the geometrical circle $C(x)$ traced on $N = \{t = 0\}$ by the congruence of geodesics passing through it (for instance if $x = (0, \mathbf{x})$, then $C(x) = \{\mathbf{x}\}$). Let x, y be two events lying in the same geodesic γ. The projections on $N = \{t = 0\}$ of the skies $S(\gamma(s))$ of events lying in the geodesic segment $[x = \gamma(0), y = \gamma(1)]$ will define an isotopy by geometrical circles $C_s = C(\gamma(s))$ on $\mathbb{R}^{m-1} \cong N$ that will join the projections $C(x)$ and $C(y)$ of the skies $S(x)$, $S(y)$ respectively. Notice that any knot L in N determines a Legendrian lifting to $S(TN)$, thus any deformation $\tilde{C}_s$ of the isotopy C_s will define a isotopy by Legendrian circles S_s joining $S(x)$ and $S(y)$. However if the deformed submanifolds $\tilde{C}_s$ are not geometrical circles, then the corresponding Legendrian liftings S_s will not be skies. See for instance [Na04] for an explicit description of projections of skies on a Cauchy surface for a 3-dimensional globally hyperbolic space.

Thus we will weaken the relation $\prec$ by restricting the class of Legendrian isotopies to those consisting of skies. Hence let $F: X_0 \times [0, 1] \to \mathcal{N}$ be a Legendrian isotopy such that $X_s = F_s(X_0)$ is the sky of $x_s \in M$, i.e., $X_s = S(x_s)$ and defines a differentiable curve $\mu: [0, 1] \to M$, given by $\mu(s) = x_s$. Conversely, let $x_0 \in M$ be an event and $X_0 = S(x_0)$ its sky. Then any differentiable curve $\mu: [0, 1] \to M$ with $\mu(0) = x_0$ defines a Legendrian isotopy parametrized by the function $F^\mu: X_0 \times [0, 1] \to \mathcal{N}$ given by $F^\mu(\gamma_{[u]}, s) = \gamma_{[u_s]}$, and $u_s \in \mathbb{N}^+_{\mu(s)}$ is the parallel transport of $u \in \mathbb{N}^+_{x_0}$ along the curve μ. Notice that F^μ is a Legendrian isotopy of skies and $F_s(X_0) = S(\mu(s))$, $s \in [0, 1]$.

We will call 'sky isotopies' the Legendrian isotopies consisting of skies and the corresponding relation in the space of skies will be denoted by $\prec_\Sigma$.

On the other hand there is a natural partial order relation in M defined by the conformal class of the Lorentzian metric. Given two events $x, y \in M$, we say that y is in the causal future of x, and it will be denoted by $x \prec y$, if $y \in J^+(x)$, i.e., y can be reached by a future-directed causal curve starting at x.

Now it is simple to show that the curve $\mu: [0, 1] \to M$ is causal past (future) iff F^μ is a non-negative (respect. non-positive) sky isotopy. Hence we have the following characterization of causality in terms of definite sky isotopies ([Ba14, proposition 4]).

Proposition 1. *Let M a strongly causal space-time, $x, y \in M$ and $X, Y \in \Sigma$ their corresponding skies, then $x \prec y$ iff $X \prec_\Sigma Y$.*

The previous observations and results lead naturally to the following:

Definition 2. A continuous curve χ: $[0, 1] \to \Sigma$ will be causal past (future) if it defines a non-negative (respect. non-positive) Legendrian isotopy in $\mathcal{N}$. Two skies $X, Y \in \Sigma$ are said to be past (future) causally related if there is a causal past (future) curve χ such that $\chi(0) = X$ and $\chi(1) = Y$.

Notice that the causal relation $\prec$ defines a partial order in M, then because of the previous proposition, proposition 1, the relation $\prec_\Sigma$ defines a partial order in the space of skies Σ, provided that M is sky-separating. Thus the sky map S becomes trivially a continuous order preserving bijection. In order to obtain a characterization similar to that provided by MH theorem, it would remain to show that S is a diffeomorphism. But if M is null pseudo-convex, then as a consequence of the 'Twisted Curve Theorem', theorem 9, the 'μ-Lemma', lemma 8, and corollary 17 below it will follow that the space-time M is diffeomorphic and order isomorphic to the space of skies Σ equipped with the partial order $\prec_\Sigma$ and the natural differentiable structure induced from the space of light rays $\mathcal{N}$.

Corollary 3. *Let M be a strongly causal, null pseudo-convex and sky-separating space-time, then M is diffeomorphic and order isomorphic to its space of skies Σ.*

3.3. Celestial curves and twisted null curves

As stated in the introduction the reconstruction theorem in [Ba14] asserts that the conformal structure of M is captured by the class of causal celestial curves, i.e., by curves in $\mathcal{N}$ that are everywhere tangent to skies. More formally:

Definition 4. A non-zero tangent vector[4] $J \in \widehat{T}_\gamma \mathcal{N}$ will be called a celestial vector if there exists a sky $S \in \Sigma$ such that $J \in \widehat{T}_\gamma S$. A differentiable curve Γ: $I \to \mathcal{N}$ is called a celestial curve if $\Gamma'(s)$ is a celestial vector for all $s \in I$.

We will analyze in this section the relation existing between celestial curves and the causality properties of M and of the space of skies Σ. To do that we will introduce first the notion of directed (future of past) twisted null curve that will prove to be useful in the arguments to follow.

Definition 5. A continuous curve μ: $[a, b] \to M$ will be called a *piecewise twisted null curve* if there exists a partition $a = s_0 < s_1 < \ldots < s_k = b$ such that for every $i = 1, \ldots, k$:

(i) $\mu|_{(s_{i-1}, s_i)}$ is differentiable.
(ii) $\mathbf{g}(\mu'(s), \mu'(s)) = 0$ for all $s \in (s_{i-1}, s_i)$.
(iii) $\mu'(s)$ and $\frac{D\mu'}{ds}(s)$ are linearly independent for all $s \in (s_{i-1}, s_i)$.

We say that μ is future-directed (past-directed) if $\mu|_{(s_{i-1}, s_i)}$ is future-directed (respect. past-directed) for all $i = 1, \ldots, k$. If $k = 1$ then μ will be simply called *twisted null curve*.

Now it is clear that if we are given a parametrized null geodesic γ: $[0, 1] \to M$, a curve λ: $(-\epsilon, \epsilon) \to M$ verifying that $\lambda(0) = \gamma(0)$, and $W(s)$ a null vector field along λ such that $W(0) = \gamma'(0)$, the family of curves:

[4] In what follows we will use the notation $\widehat{T}N$ to indicate the bundle TN –or its fibres– with the zero-section removed, that is, in the present situation $\widehat{T}_\gamma \mathcal{N} = T_\gamma \mathcal{N} - \{\mathbf{0}\}$.

$$\mathbf{f}(s, t) = \exp_{\lambda(s)}(tW(s)) \tag{3.1}$$

is a geodesic variation of $\gamma(t)$ formed by null geodesics with $\mathbf{f}(0, t) = \gamma(t)$ and $J(t) = \frac{\partial \mathbf{f}}{\partial s}(0, t)$.

If μ is a null curve then we may use $W(s) = \mu'(s)$ and obtain a geodesic variation of γ that, in addition, defines a celestial curve in $\mathcal{N}$. Actually more is true as it is shown by the following:

Proposition 6. *[Ba14] If the curve Γ: $[0, 1] \to \mathcal{N}$ with $\Gamma(s) = \gamma_s \in \mathcal{N}$ is celestial then there exists a differentiable null curve μ: $[0, 1] \to M$ such that $\gamma_s(\tau) = \exp_{\mu(s)}(\tau\sigma(s))$ where $\sigma(s) \in \mathbb{N}^+_{\mu(s)}$ is a differentiable curve proportional to $\mu'(s)$ wherever μ is regular.*

In fact, by construction, the curve μ in proposition 6 runs the points in M such that the celestial curve Γ is tangent to their skies, in other words, $\Gamma'(s) \in \widehat{T}S(\mu(s))$ for all $s \in [0, 1]$.

As a consequence of the previous result, we have the following corollary.

Corollary 7. *Given a celestial curve Γ: $[0, 1] \to \mathcal{N}$ such that $\Gamma'(s_0) \in \widehat{T}S(p_0)$, $0 \leqslant s_0 \leqslant 1$, then the curve μ: $[0, 1] \to M$ of the previous Proposition, proposition 6, is unique verifying $\mu(s_0) = p_0 \in M$.*

Proof. Consider that there exists μ_1, μ_2: $[0, 1] \to M$ associated to Γ in the sense of proposition 6 and verifying $\mu_1(s_0) = \mu_2(s_0) = p_0$ for $s_0 \in [0, 1]$. Let us define the set $A = \left\{ s \in [0, 1]: \mu_1(s) = \mu_2(s) \right\}$. Clearly, A is not empty and closed in $[0, 1]$. Consider a causally convex and normal neighbourhood $U \subset M$ of p_0. Since U is open, then there exist $\delta > 0$ such that $\mu_i\left(\left(s_0 - \delta, s_0 + \delta \right) \right) \subset U$ for $i = 1, 2$ (eventually if $s_0 = 0$ then we consider $\mu_i([0, \delta)) \subset U$ and analogously for $s_0 = 1$). Let us suppose that for $s \in \left(s_0 - \delta, s_0 + \delta \right)$ we have that $\mu_1(s) \neq \mu_2(s)$ and since U is causally convex, then the segment of the light ray $\Gamma(s) = \gamma_s \in \mathcal{N}$ connecting $\mu_1(s)$ and $\mu_2(s)$ is totally contained in U and, moreover since $\Gamma'(s) \in \widehat{T}S\left(\mu_1(s) \right) \cap \widehat{T}S\left(\mu_2(s) \right)$, then the points $\mu_1(s)$ and $\mu_2(s)$ are mutually conjugated along γ_s but, in virtue of [On83, proposition 10.10], this is not possible in a normal neighbourhood contradicting U is normal. Then we have that $\mu_1(s) = \mu_2(s)$ and hence the set A is also open in $[0, 1]$. Since A is open, closed and not empty in $[0, 1]$ then $A = [0, 1]$ and we conclude that $\mu_1 = \mu_2$. □

Given a celestial curve Γ the unique curve μ associated to it in the sense of proposition 6 passing by $p_0 \in S^{-1}(X_0)$ will be called the 'dust' of Γ by X_0 and denoted by $\mu^{\Gamma}_{X_0}$. The previous arguments can be made more precise by proving that the dust of a celestial curve is a twisted null curve. This is the content of the next lemma.

Lemma 8 ($\boldsymbol{\mu}$-Lemma). *Let Γ: $[0, 1] \to \mathcal{N}$ be a celestial curve such that $\Gamma'(0) \in \widehat{T}X_0$ with $X_0 \in \Sigma$. Then there exists a unique curve $\chi^{\Gamma}_{X_0}$: $[0, 1] \to \Sigma$ such that it is continuous in Low's topology and verifies $\chi^{\Gamma}_{X_0}(0) = X_0$ and $\Gamma'(s) \in \widehat{T}\chi^{\Gamma}_{X_0}(s)$. Moreover, the dust curve $\mu^{\Gamma}_{X_0}$ is a piecewise twisted null curve in M running along the image of $S^{-1} \circ \chi^{\Gamma}_{X_0}$.*

Conversely, given a regular twisted null curve μ: $[0, 1] \to M$ such that $\mu(0) = x_0 = S^{-1}\left(X_0 \right)$, $\mu'(0) \neq 0 \neq \mu'(1)$, then the curve Γ^{μ}: $[0, 1] \to \mathcal{N}$ defined by the variation of null geodesics $\mathbf{x}$: $[0, 1] \times I \to M$ such that

$$\mathbf{x}(s, t) = \exp_{\mu(s)}(t\mu'(s)) = \Gamma^{\mu}(s)\big|_t$$

is celestial with $\Gamma'(0) \in \widehat{T}X_0$ *and* $\chi^{\Gamma}_{X_0}(s) = S(\mu(s))$.

Proof. Let Γ: $[0, 1] \to \mathcal{N}$ be a celestial curve such that $\Gamma(s) = \gamma_s \in \mathcal{N}$ and $\Gamma'(0) \in \widehat{T}X_0$ with $X_0 = S(x_0) \in \Sigma$. By corollary 7, there exists a unique differentiable curve μ: $[0, 1] \to M$ and a partition

$$\left\{0 = a_1 \leqslant b_1 < a_2 \leqslant b_2 < \cdots < a_{n-1} \leqslant b_{n-1} < a_n \leqslant b_n = 1\right\} \subset [0, 1]$$

such that

$$\gamma_s(\tau) = \exp_{\mu(s)}(t\sigma(s)), \tag{3.2}$$

where σ: $[0, 1] \to \mathbb{N}$ is a differentiable curve verifying $\sigma(s) = \lambda_k(s)\mu'(s)$ for $s \in \left(b_k, a_{k+1}\right)$ and λ_k differentiable with $k = 1, \ldots, n-1$. This curve μ also verifies $\mu(s) = p_k \in M$ for all $s \in \left[a_k, b_k\right]$.

Now, we can define the curve $\chi^{\Gamma}_{X_0} = S \circ \mu$: $[0, 1] \to \Sigma$. Recall that for an open set $\mathcal{U} \subset \mathcal{N}$ containing a sky $X \in \Sigma$, the set of all skies contained in $\mathcal{U}$ is denoted as $\Sigma(\mathcal{U})$. By the definition of the Low's topology, the set $\Sigma(\mathcal{U})$ is open in Σ and these collection of open sets forms a basis at X.

In order to show that $\chi^{\Gamma}_{X_0}$ is continuous, we will show that, given any $\mathcal{U} \subset \mathcal{N}$ containing a sky $S(\mu(s)) \in \Sigma$ then $\left(\chi^{\Gamma}_{X_0}\right)^{-1}(\Sigma(\mathcal{U}))$ is open in $[0, 1]$ is verified. So, take any $s \in [0, 1]$ and consider an open set $\mathcal{U} \subset \mathcal{N}$ such that $\chi^{\Gamma}_{X_0}(s) \subset \mathcal{U}$ and then $\chi^{\Gamma}_{X_0}(s) \in \Sigma(\mathcal{U})$. Choose a collection of nested intervals $I^s_n \subset \mathbb{R}$ such that $\{s\} = \bigcap_n I^s_n$. Let us suppose that there exists $s_n \in I^s_n$ such that $\chi^{\Gamma}_{X_0}\left(s_n\right) \notin \Sigma(\mathcal{U})$. Then there is a light ray $\gamma_n \in \chi^{\Gamma}_{X_0}\left(s_n\right) \in \Sigma$ such that $\gamma_n \notin \mathcal{U}$. Recall that a light ray is fully determined by a point $p \in M$ and a direction $[v] \in \mathbb{PN}^+_p$, so γ_n can be defined by $\mu\left(s_n\right) \in \gamma_n \subset M$ and a null direction $[v_n] \in \mathbb{PN}^+_{\mu\left(s_n\right)}$. Since $\lim \mu\left(s_n\right) = \mu(s)$ and due to the compactness of the fibres $\mathbb{PN}^+_{\mu\left(s_n\right)}$, then with no lack of generality taking a subsequence of $[v_n]$ if necessary, there exists a direction $[v] \in \mathbb{PN}^+_{\mu(s)}$ defining, together with $\mu(s)$, the light ray γ such that $\lim \gamma_n = \gamma \in \chi^{\Gamma}_{X_0}(s) \subset \mathcal{U}$.

But since $\mathcal{U}$ is open, there exists an integer K such that for every $n > K$ we have that $\gamma_n \in \mathcal{U}$ contradicting that $\chi^{\Gamma}_{X_0}\left(s_n\right) \notin \Sigma(\mathcal{U})$. Therefore there exist I^s_n such that $\chi^{\Gamma}_{X_0}\left(s_n\right) \in \Sigma(\mathcal{U})$ and hence $\left(\chi^{\Gamma}_{X_0}\right)^{-1}(\Sigma(\mathcal{U}))$ is open in $[0, 1]$.

To obtain the dust $\mu^{\Gamma}_{X_0}$, we will cut off the segments $\mu|_{\left(a_k, b_k\right)}$ from μ and glue together the segments $\mu|_{\left[b_k, a_{k+1}\right]}$. We call $c_1 = 0$ and for every $k = 1, \ldots, n-1$, let us define $c_{k+1} = a_{k+1} - \sum_{i=1}^{k}\left(b_i - a_i\right) \in [0, 1]$ and consider the change of parameter h_k: $\left[c_k, c_{k+1}\right] \to \left[b_k, a_{k+1}\right]$ defined by $h_k(\tau) = \tau + a_{k+1} - c_{k+1}$. Since μ is differentiable and h_k is a diffeomorphism for every $k = 1, \ldots, n-1$ then $\overline{\mu}_k(\tau) = \mu \circ h_k(\tau)$ is differentiable for $\tau \in \left(c_k, c_{k+1}\right)$. Moreover, since $\overline{\mu}'_k(\tau) = \mu'\left(h_k(\tau)\right)$ then

$$\mathbf{g}\left(\overline{\mu}'_k(\tau), \overline{\mu}'_k(\tau)\right) = \mathbf{g}\left(\mu'_k\left(h_k(\tau)\right), \mu'_k\left(h_k(\tau)\right)\right) = 0$$

for $\tau \in \left(c_k, c_{k+1}\right)$. Also, the covariant derivatives verify

$$\frac{D\overline{\mu}_k'(\tau)}{\mathrm{d}\tau} = h_k''(\tau)\mu'\left(h_k(\tau)\right) + \left(h_k'(\tau)\right)^2 \frac{D\mu'\left(h_k(\tau)\right)}{\mathrm{d}s} = \frac{D\mu'\left(h_k(\tau)\right)}{\mathrm{d}s}$$

then denoting J_s as the Jacobi field along γ_s defined by the variation 3.2, we have $J_s(0) = \mu'(s)$ and

$$J_s'(0) = \frac{D\sigma(s)}{\mathrm{d}s} = \frac{D\left(\lambda_k(s)\mu'(s)\right)}{\mathrm{d}s} = \lambda_k'(s)\mu'(s) + \lambda_k(s)\frac{D\mu'(s)}{\mathrm{d}s}$$

for $s \in \left(b_k, a_{k+1}\right)$. Since Γ is celestial, then $J_s \neq 0\left(\operatorname{mod} \gamma_s'\right)$ and so, $\frac{D\mu'(s)}{\mathrm{d}s}$ is not proportional to $\mu'(s)$ for $s \in \left(b_k, a_{k+1}\right)$, therefore $\frac{D\overline{\mu}_k'(\tau)}{\mathrm{d}\tau}$ and $\overline{\mu}_k'(\tau)$ are linearly independent for $\tau \in \left(c_k, c_{k+1}\right)$. We have shown that for any $k = 1, \ldots, n-1$ the curves $\overline{\mu}_k$ are twisted null curves. Since $h_k^{-1}\left(a_{k+1}\right) = h_{k+1}^{-1}\left(b_{k+1}\right)$ then all the segments $\overline{\mu}_k$ glue together continuously. Therefore we can define, with no ambiguity, the curve $\mu_{X_0}^{\Gamma}\colon [0, a] \to M$ such that $\mu_{X_0}^{\Gamma}(\tau) = \overline{\mu}_k(\tau)$ if $\tau \in \left[c_k, c_{k+1}\right]$ for $k = 1, \ldots, n-1$ and $[0, a] = \cup_{k=1}^{n-1}\left[c_k, c_{k+1}\right]$. This curve $\mu_{X_0}^{\Gamma}$ is then a piecewise twisted null curve associated to the partition $\left\{0 = c_1 < c_2 < \cdots < c_n = a\right\} \subset [0, a]$ and it is unique except by reparametrization.

Conversely, let us consider a twisted null curve $\mu\colon [0, 1] \to M$ such that $\mu(0) = x_0 = S^{-1}\left(X_0\right)$. Then, we can define the variation of null geodesics $\mathbf{x}\colon [0, 1] \times I \to M$ such that

$$\mathbf{x}(s, t) = \exp_{\mu(s)}(t\mu'(s)) = \gamma_s(t)$$

which verifies $\gamma_s'(0) = \mu'(s)$. Now, define the curve $\Gamma^{\mu}(s) = \gamma_s \in \mathcal{N}$ for every $s \in [0, 1]$. The Jacobi field J_s of the variation $\mathbf{x}$ along γ_s verifies $J_s(0) = \mu'(s) = \gamma_s'(0)$ and $J_s'(0) = \frac{D\mu'}{\mathrm{d}s}(s)$ and, since μ is twisted null then $\frac{D\mu'}{\mathrm{d}s}$ is not proportional to γ_s'. Therefore $\left(\Gamma^{\mu}\right)'(s) = J_s\left(\operatorname{mod} \gamma_s'\right) \neq 0\left(\operatorname{mod} \gamma_s'\right)$ and hence

$$\left(\Gamma^{\mu}\right)'(s) \in \widehat{T}S\left(\gamma_s(0)\right) = \widehat{T}S(\mu(s))$$

then Γ^{μ} is celestial. □

3.4. Celestial curves and the partial order in the space of skies

We have already pointed it out that if $x \prec y$, then their corresponding skies are related $S(x) \prec_{\Sigma} S(y)$. The discussion to follow will show that such relation can actually be refined by proving that in case of $y \in I^{+}(x)$[5], there exists a directed piecewise twisted null curve joining x and y, hence relating the causal properties of Σ to the existence of appropriate celestial curves.

Theorem 9 **(Twisted null curve theorem).** *Let $p, q \in M$ such that $q \in I^{+}(p)$, then there exists a future-directed piecewise twisted null curve μ joining p to q.*

To prove the previous theorem we will need some lemmas.

[5] Recall that $y \in I^{+}(x)$ means that there exists a future-directed time-like curve from x to y.

Lemma 10. *Let M be a 3-dimensional space-time and $\gamma: I \to M$ be a future-directed time-like geodesic. Then there exists $\delta > 0$ such that for any $t \in (t_0, t_0 + \delta]$, there exists a future-directed twisted null curve μ joining $\gamma(t_0)$ to $\gamma(t)$.*

Proof. Given the future-directed time-like geodesic $\gamma: I \to M$ and $t_0 \in I$, it is known, e.g. by [La03, section 97] and [Pe72, definiton 7.13], that there exists a synchronous coordinate system $(U, \phi = (t, x, y))$ with $\gamma\left(t_0\right) \subset U$ in which the metric $\mathbf{g}$ of M can be written as

$$\left(g_{ij}\right) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & g_{11} & g_{12} \\ 0 & g_{12} & g_{22} \end{pmatrix},$$

where $g_{ij} \equiv g_{ij}(t, x, y)$ for $i, j = 1, 2$, U is causally convex and the expression of the geodesic γ in these coordinates is $\phi(\gamma(s)) = (s, 0, 0) \in \mathbb{R}^3$. For a point $\gamma\left(\bar{t}\right) \in U$, it is possible to find $R > 0$ such that the compact set

$$U_0 = \left\{ (t, x, y): x^2 + y^2 \leqslant R,\ t_0 \leqslant t \leqslant \bar{t} \right\}$$

is contained in U.

As candidates for the required twisted null curve, we will study curves μ_r such that

$$\phi\left(\mu_r(s)\right) = \left(f_r(s),\ r(1 - \cos s),\ r \sin s\right),$$

where $0 \leqslant r \leqslant R/2$ and $f_r = f_r(s)$ is a function. If μ_r is a null curve, then $\mathbf{g}\left(\mu_r', \mu_r'\right) = 0$ and therefore

$$-\left(f_r'(s)\right)^2 + r^2 g_{11} \sin^2 s + 2r^2 g_{12} \sin s \cos s + r^2 g_{22} \cos^2 s = 0,$$

where $g_{ij} = g_{ij}\left(\phi\left(\mu_r(s)\right)\right)$. Thus, we have a first order ordinary differential equation which describes a null curve passing through $\gamma\left(t_0\right)$

$$\begin{cases} f_r'(s) = r\sqrt{g_{11} \sin^2 s + 2g_{12} \sin s \cos s + g_{22} \cos^2 s} \\ f_r(0) = t_0. \end{cases} \tag{3.3}$$

Since the metric in the hypersurfaces $\{t = c\}$ with $t_0 \leqslant c \leqslant \bar{t}$ is positive definite, then the term under the square root in (3.3) is always positive. Moreover, since $f_r' > 0$ then μ_r is future.

Let us show that we can find $r > 0$ such that μ_r is twisted. A simple calculation gives

$$(\mathrm{d}\phi)_{\mu_r(s)}\left(\frac{D\mu_r'}{\mathrm{d}s}(s)\right) = \left(f_r'' + r^2\varphi_0(r, s),\ r \cos s + r^2\varphi_1(r, s),\ - r \sin s + r^2\varphi_2(r, s)\right),$$

where $\varphi_i = \varphi_i(r, s)$, $i = 0, 1, 2$, are continuous functions in U depending on the Christoffel symbols and the components of μ_r'. In order to show that $\frac{D\mu_r'}{\mathrm{d}s}$ and μ_r' are linearly independent, it is enough to see that the determinant of their components x, y does not cancel out, so

$$\begin{vmatrix} r \cos s + r^2\varphi_1(r, s) & r \sin s \\ -r \sin s + r^2\varphi_2(r, s) & r \cos s \end{vmatrix} = r^2\left(1 + r\left(\varphi_1(r, s) \cos s + \varphi_2(r, s) \sin s\right)\right)$$

hence, since φ_1 and φ_2 are continuous in U, they are also bounded in the compact set U_0 and there exists $r_0 \leqslant R/2$ such that

$$1 + r\left(\varphi_1(r, s)\cos s + \varphi_2(r, s)\sin s\right) \neq 0$$

for all $r \in \left(0, r_0\right]$, and in this case, $\frac{D\mu_r'}{ds}$ and μ_r' are linearly independent.

At this moment, we have seen that μ_r is a twisted null curve passing through $\gamma\left(t_0\right)$ for $0 < r \leqslant r_0$, and it remains to show that there exists $\delta > 0$ such that μ_r also passes through $\gamma(t)$ for every $t \in \left(t_0, t_0 + \delta\right]$.

Now, we want to prove that for every $r \in \left(0, r_0\right]$ there exists $s_r > 0$ such that $f_r(s_r) = \bar{t}$. Given $r \in \left(0, r_0\right]$, we define $\omega_r = \sup\left\{s: f_r(s)\text{exists}\right\}$. Let us assume that $\lim_{s\mapsto\omega_r} f_r(s) = c \leqslant \bar{t}$. In case of $\omega_r < +\infty$, the solution $\overline{f_r}$ of equation (3.3) verifying the initial condition $\overline{f_r}(\omega_r) = c$ would coincide with $f_r = f_r(s)$ for $s < \omega_r$ contradicting the maximality of f_r up to ω_r because in that case f_r could be extended beyond $s = \omega_r$. On the other hand, if $\omega_r = +\infty$, the derivability of f_r would imply that $\lim_{s\mapsto+\infty} f_r'(s) = 0$ and hence the curve solution μ_r would approximate to the curve β_r verifying

$$\beta_r(s) = (c, r(1 - \cos s), r\sin s) \in U_0$$

in TM, i.e. for every $s_0 \in \mathbb{R}$ the sequence $\left\{s_n = s_0 + 2\pi n\right\}_{n\in\mathbb{N}}$ would verify

$$\lim_{s\mapsto+\infty}\mu_r(s_n) = \beta_r(s_0)\text{and} \lim_{s\mapsto+\infty}\mu_r'(s_n) = \beta_r'(s_0).$$

By the continuity of the metric $\mathbf{g}$ then we have

$$\lim_{s\mapsto+\infty}\mathbf{g}\left(\mu_r'(s_n), \mu_r'(s_n)\right) = \mathbf{g}\left(\beta_r'(s_0), \beta_r'(s_0)\right) \neq 0$$

since β_r is contained in the space-like hypersurface $\{t = c\}$, but this contradicts that $\mathbf{g}\left(\mu_r', \mu_r'\right) = 0$. Therefore, independently from ω_r, for every $r \in \left(0, r_0\right]$ we have that $\lim_{s\mapsto\omega_r} f_r(s) > \bar{t}$ and hence, for every $r \in \left(0, r_0\right]$ there exists $s_r \in \left(0, \omega_r\right)$ such that $f_r(s_r) = \bar{t}$.

Since the functions g_{ij} are continuous in U for $i, j = 1, 2$, then their restrictions to the compact set U_0 reach their maximum, therefore there exists $M_{ij} > 0$ such that $\left|g_{ij}(t, x, y)\right| \leqslant M_{ij}$ for $(t, x, y) \in U_0$. Then,

$$\begin{aligned} 0 < f_r'(s) = r\sqrt{g_{11}\sin^2 s + 2g_{12}\sin s\cos s + g_{22}\cos^2 s} \leqslant \\ \leqslant r\sqrt{\left|g_{11}\sin^2 s\right| + 2\left|g_{12}\sin s\cos s\right| + \left|g_{22}\cos^2 s\right|} \leqslant \\ \leqslant r\sqrt{M_{11} + 2M_{12} + M_{22}} = rM, \end{aligned}$$

where $M = \sqrt{M_{11} + 2M_{12} + M_{22}} \in \mathbb{R}$ is independent from r and s. So integrating, we have that $t_0 \leqslant f_r(s) \leqslant rMs + t_0$ and therefore

$$\bar{t} = f_r(s_r) \leqslant rMs_r + t_0 \Rightarrow \frac{\bar{t} - t_0}{rM} \leqslant s_r$$

then there exists $\rho \in \left(0, r_0\right]$ small enough such that $s_r \geqslant 2\pi$ for all $r \in (0, \rho]$ and hence the parameter s of f_r can be extended beyond $s = 2\pi$. Since $f_\rho'(s) > 0$ then $f_\rho(s) > t_0$ for all $s > 0$, therefore there exists $\delta > 0$ such that $f_\rho(2\pi) = t_0 + \delta$. So, by the inequality $t_0 \leqslant f_r(2\pi) \leqslant 2\pi rM + t_0$ we have that $\lim_{r\mapsto 0} f_r(2\pi) = t_0$ and for every $t \in \left(t_0, t_0 + \delta\right]$ there exists $r \in (0, \rho]$ such that

$$\mu_r(0) = \left(t_0, 0, 0\right) = \phi\left(\gamma(t_0)\right)$$

$$\mu_{\tilde{r}}(2\pi) = \left(f_r(2\pi), 0, 0\right) = (t, 0, 0) = \phi(\gamma(t))$$

therefore we have shown that there exists $\delta > 0$ such that for every $t \in \left(t_0, t_0 + \delta\right]$ the points $\gamma\left(t_0\right)$ and $\gamma(t)$ can be connected by some future-directed twisted null curve $\mu_{\tilde{r}}$. Analogously, this construction can be done to obtain a future-directed twisted null curve joining $\gamma(t)$ to $\gamma\left(t_0\right)$ for all $t \in \left[t_0 - \delta, t_0\right)$. □

Lemma 11. *The statement of lemma 10 is true in a m-dimensional spacetime M.*

Proof. We can find a synchronous coordinate system (U, ϕ) with $\phi = \left(t, x_1, \ldots, x_{m-1}\right)$ (as done previously) such that the expression of the geodesic γ in these coordinates is $\phi(\gamma(s)) = (s, 0, \ldots, 0) \in \mathbb{R}^m$, so this chart is adapted to γ. Consider the restriction

$$V = \left\{(t, x_1, \ldots, x_{m-1}) : x_i = 0, i = 3, \ldots, m-1\right\} \subset \phi(U)$$

then $N = \phi^{-1}(V) \subset M$ is a 3-dimensional manifold embedded in M. Moreover, by [On83, lemma 4.3] we have that Levi-Civita connection in N coincides with the orthogonal projection over N of the Levi-Civita connection in M, hence we have $\frac{D^N}{ds} = \tan\left(\frac{D}{ds}\right)$ where $\frac{D^N}{ds}$ and $\frac{D}{ds}$ denote the covariant derivatives in N and M respectively. So the geodesics in M contained in N are also geodesics in N and the restriction $\left(N, \phi|_N = (t, x_1, x_2)\right)$ of the synchronous coordinate system is still a synchronous coordinate system for N. Then, since γ is a geodesic contained in N, by step 10, there exists $\delta > 0$ and a future-directed twisted null curve $\mu \subset N$ such that μ joins $\gamma\left(t_0\right)$ to $\gamma\left(t_0 + \delta\right)$. Since the metric in N is the restriction of the metric in M, then μ as curve in M is also null. Finally, since μ' and $\frac{D^N\mu'}{ds} = \tan\left(\frac{D\mu'}{ds}\right)$ are lineally independent in $T_{\mu(s)}N$ then is animmediate consequence that μ' and $\frac{D\mu'}{ds}$ are lineally independent in $T_{\mu(s)}M$. Therefore, we have shown that there exists $\delta > 0$ and μ a future-directed twisted null curve in M joining $\gamma\left(t_0\right)$ to $\gamma\left(t_0 + \delta\right)$.

We can now prove as a direct consequence of lemmas 10 and 11 the following:

Proposition 12. *Let $\gamma: I \to M$ be a future-directed timelike geodesic. Then, for any $t_0, t_1 \in I$, there exists a future-directed piecewise twisted null curve μ joining $\gamma\left(t_0\right)$ to $\gamma(t_1)$.*

Proof. By lemma 11, for all $t \in \left[t_0, t_1\right]$ there exists an open interval $I_t = \left[t - \delta_t, t + \delta_t\right] \subset \left[t_0, t_1\right]$ relative to $\left[t_0, t_1\right]$ such that $\gamma(t)$ can be joined to $\gamma(u)$ with $u \in I_t$ by means of a piecewise twisted null curve. By the compactness of $\left[t_0, t_1\right]$, we can extract a finite covering $\left\{I_n\right\}_{n=1,\ldots,N}$ such that, with no lack of generality, verifies $I_i \cap I_k \neq \varnothing \Leftrightarrow k = i \pm 1$. We can choose a partition

$$\left\{t_0 = a_1 < b_1 < \cdots < a_{N-1} < b_{N-1} < a_N = t_1\right\}$$

such that $a_i \in I_i$ and $b_i \in I_i \cap I_{i+1}$ and therefore there exists future-directed twisted null curves joining $\gamma\left(a_i\right)$ to $\gamma\left(b_i\right)$ and $\gamma\left(b_i\right)$ to $\gamma\left(a_{i+1}\right)$ for $i = 1, \ldots, N-1$. The union of these curves forms a future-directed piecewise twisted null curve connecting $\gamma\left(t_0\right)$ to $\gamma(t_1)$. □

Now we can proceed with the proof of theorem 9.

Proof. Theorem 9: consider $p, q \in M$ such that $q \in I^+(p)$, then there exists a continuous future-directed time-like curve λ connecting p and q. By compactness of λ between p and q, there exists a finite covering $\{W_k\}_{k=1,\ldots,K}$ of globally hyperbolic and causally convex open sets, then it is possible to built a continuous curve γ joining p and q formed by segments $\gamma_k \subset W_k$ of future-directed time-like geodesics with endpoints at λ. So γ becomes a future-directed piecewise time-like geodesic.

By proposition 12, the endpoints of the time-like geodesic segments γ_k of γ can be connected by a future-directed piecewise twisted null curve μ_k. Since γ is continuous, we can glue together all μ_k to obtain another piecewise twisted null curve μ joining p and q. □

Notice that if M is globally hyperbolic, the proof simplifies greatly. In fact if M is globally hyperbolic, then it is also causally convex and the curve γ can be considered as a time-like geodesic and applying proposition 12, the result follows.

4. The smooth structure of the space of skies and the non-refocussing property

4.1. Regular sets

The smooth structure on the space of skies will be obtained by selecting a family of neighbourhoods possessing the properties that will make obvious the construction of an atlas on Σ. We will call such neighbourhoods *regular neighbourhoods* and they refine the notion of regular set already introduced in [Ba14, definiton 3].

From now on, let M be a strongly causal, null pseudo-convex, sky-separating space-time. Let $W \subset \Sigma$ be a non-empty set satisfying the conditions:

(i) $\hat{T}X \cap \hat{T}Y = \varnothing$ for all $X \neq Y \in W$.

(ii) The union

$$\widehat{W} = \bigcup_{X \in W} \hat{T}X \subset \hat{T}\mathcal{N}$$

is a regular $(3m - 4)$–dimensional submanifold of $\hat{T}\mathcal{N}$.

(iii) Let $\widehat{\mathcal{D}}$ be the distribution in $\widehat{W}$ whose leaves are $\widetilde{X} = \hat{T}X$. Then the space of leaves $\widetilde{W} = \left\{\widetilde{X}: X \in W\right\} = \widehat{W}/\widehat{\mathcal{D}}$ is a differentiable quotient manifold.

It is clear that in this case, $\widetilde{W}$ can be identified with W via the bijective map

$$\begin{aligned} \Theta: W &\to \widetilde{W} \\ X &\mapsto \widetilde{X} \end{aligned} \tag{4.1}$$

and hence W inherits the quotient topology such that $U \subset W$ is open $\Leftrightarrow \widehat{U} = \bigcup_{X \in U} \hat{T}X \subset \widehat{W}$ is open, and also a differentiable structure from $\widetilde{W}$. So, we will denote W equipped with the previous structure as $W^{(\sim)} \simeq \widetilde{W}$.

(iv) For every $X_0 \in W$ and every celestial curve $\Gamma: I_\epsilon \to \mathcal{N}$ such that $\Gamma'(0) \in \hat{T}X_0$,
   (a) there exists $0 < \delta \in I_\epsilon$ such that $\Gamma': I_\delta \to \widehat{W}$ with $I_\delta = (-\delta, \delta)$,
   (b) the curve $\chi^{\Gamma}_{X_0}: I_\delta \to W^{(\sim)}$ defined in lemma 8 is differentiable.

(v) Given $\widetilde{X}, \widetilde{Y} \in \widetilde{W}$, for any causal curve $\chi: [a, b] \to \Sigma$, joining X and Y, then $\chi(s) \in W$ for all $s \in [a, b]$.

Now we are ready to state the next definition.

Definition 13. A not–empty subset $W \subset \Sigma$ is said to be a *regular* subset, and denoted as $W \subset_{\mathrm{reg}} \Sigma$, if it verifies conditions (i)–(v) above.

Observe that both, the definition of regular subset and the differentiable structure of $W^{(\sim)} \simeq \widetilde{W}$, depend only on $\mathcal{N}$ and Σ.

4.2. The topology of the space of skies and regular sets

We will show next that the class of regular subsets is not empty.

We will say that $V \subset M$ is an open normal set if V is globally hyperbolic, causally convex, relatively compact, open set of M. A classical theorem due to Whitehead guarantees the existence of convex normal neighbourhoods V at any point $x \in M$, (see [On83, chapter 5] and [Mi08, theorem 2.1 and definition 3.22] for a treatment of this result in Lorentz manifolds). Thus for a strongly causal space-time M there exists a basis of neighbourhoods at any $p \in M$ formed by normal open sets.

Proposition 14. *Let $V \subset M$ be a normal open set, then $U = S(V) \subset_{\mathrm{reg}} \Sigma$ is regular. Moreover, $S: V \to U^{(\sim)}$ is a diffeomorphism.*

Proof. Let $V \subset M$ be a normal open set, then condition (i) is verified since V is causally convex. By [Ba14, theorem 1], condition (ii) is verified. The condition (iii) and the fact of $S: V \to U^{(\sim)}$ being a diffeomorphism are consequences of [Ba14, theorem 2]. Lemma 8 trivially implies ((iv)a) and permits to construct the curve $\chi^{\Gamma}_{X_0}$ as the following composition of differentiable maps

$$\begin{array}{ccccccc} I_\delta & \xrightarrow{\Gamma} & \widehat{U} & \xrightarrow{\pi} & \widetilde{U} & \xrightarrow{\Theta^{-1}} & U^{(\sim)} \\ s & \mapsto & \Gamma'(s) & \mapsto & \widehat{T}\chi^{\Gamma}_{X_0}(s) & \mapsto & \chi^{\Gamma}_{X_0}(s) \end{array}$$

then ((iv)b) is verified. Finally, in order to verify (v), we know that $\Gamma'(a) \in \widehat{T}X, \Gamma'(b) \in \widehat{T}Y$ and $X, Y \in U$, by lemma 8, there exists a piecewise twisted null curve $\mu: [a, b] \to M$ such that $\mu(a) = x \in V$ and $\mu(b) = y \in V$. Since V is causally convex, then μ is fully contained in V and therefore $\chi = S \circ \mu$ is fully contained in $U = S(V)$. So, we conclude that $U \subset_{\mathrm{reg}} \Sigma$.

We may call the regular sets $U = S(V)$ with V open normal, elementary regular sets in Σ.

Using now the technical lemma:

Lemma 15. *Given $W \subset_{\mathrm{reg}} \Sigma$ a regular set and $X_0 = S(x_0) \in W$, then for any twisted null curve $\mu: I_\epsilon \to M$ such that $\mu(0) = x_0$ there exists $\delta > 0$ verifying that $\mu((-\delta, \delta)) \subset S^{-1}(W)$.*

Proof. Consider $X_0 = S(x_0) \in W \subset_{\mathrm{reg}} \Sigma$, then by lemma 8, there exists a celestial curve $\Gamma: I_\epsilon \to \mathcal{N}$ and a continuous curve $\chi^{\Gamma}_{X_0}: I_\epsilon \to \Sigma$ such that $\chi^{\Gamma}_{X_0} = S \circ \mu$. Since W is regular, then there exists $\delta > 0$ such that $\chi^{\Gamma}_{X_0}: (-\delta, \delta) \subset I_\epsilon \to W^{(\sim)}$ is differentiable. Then we have

$$\mu((-\delta, \delta)) = S^{-1} \circ \chi^{\Gamma}_{X_0}((-\delta, \delta)) \subset S^{-1}\left(W^{(\sim)}\right) = S^{-1}(W).$$

$\square$

It is easy to prove the following:

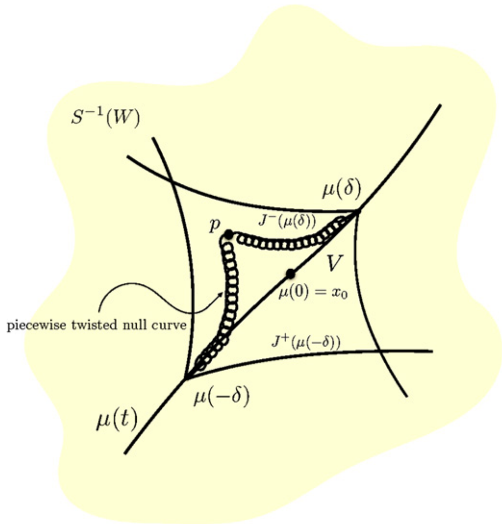

Figure 1. For any point $x_0 \in S^{-1}(W)$ there is a diamond-shaped set V such that $x_0 \in V \subset S^{-1}(W)$ (any point $p \in V$ can be joined to its end-points by piecewise twisted null curves), hence $S^{-1}(W)$ is open.

Theorem 16. *Let $W \subset_{\text{reg}} \Sigma$ be a regular set, then $S^{-1}(W)$ is open in M.*

Proof. Given $W \subset_{\text{reg}} \Sigma$ and consider $X_0 \in W$ such that $x_0 = S^{-1}\left(X_0\right) \in M$. Take a future-directed twisted null curve $\mu: I_\epsilon \to M$ with $\mu(0) = x_0$, then by lemma 15, there exists $\delta > 0$ verifying that $\mu((-\delta, \delta)) \subset S^{-1}(W)$. Without any lack of generality, we can assume that δ is small enough for $V = I^+(\mu(-\delta)) \cap I^-(\mu(\delta))$ being globally hyperbolic and causally convex. Observe that $x_0 \in V$ and for any $p \in V$, we have that $p \in I^+(\mu(-\delta))$, then by theorem 9, for any $p \in V$ there exists a future-directed piecewise twisted null curve μ_p connecting $\mu(-\delta)$ and $\mu(\delta)$ passing through p (see figure 1). Now, since W is regular, then by property (v), the curve $\chi_p = S \circ \mu_p$ is fully contained in W, therefore $p \in S^{-1}(W)$ and hence $V \subset S^{-1}(W)$ and $S^{-1}(W)$ is open in M. □

It is interesting to point out that whenever M is globally hyperbolic, then any non-empty $V = I^+(\mu(-\delta)) \cap I^-(\mu(\delta))$ is automatically globally hyperbolic and the conclusion of the theorem is reached easily without referring to the previous lemmas.

In virtue of proposition 14 and theorem 16, since the sky map S is an homeomorphism with the reconstructive topology in Σ, it is clear that this topology coincides with the topology generated by regular sets in Σ. So, by [Ba14, corollary 1,theorem 2 and corollary 2], we get the following corollary.

Corollary 17. *The family of regular sets $\left\{ W \,|\, W \subset_{\text{reg}} \Sigma \right\}$ is a basis for the reconstructive topology of Σ. Moreover, there exists a unique differentiable structure in Σ compatible with the manifolds $W^{(\sim)} \subset \Sigma$ that makes of $S: M \to \Sigma$ a diffeomorphism.*

The following lemma corroborates the relation between neighbourhood basis of M and its space of skies Σ and will be used to establish the conclusion that for strongly causal and null pseudo-convex space-times, sky-separating implies non-refocussing.

Lemma 18. *Let $\mathcal{B}(x)$ be a neighbourhood basis consisting on globally hyperbolic, normal and causally convex open sets. For any $U \in \mathcal{B}(x)$, denote by $\mathcal{U} = \{\gamma \in \mathcal{N} | \gamma \cap U \neq \varnothing\}$. Then $\{\Sigma(\mathcal{U}) | U \in \mathcal{B}(x)\}$ is a neighbourhood basis of $S(x) \in \Sigma$.*

Proof. Because the bundle $\mathbb{PN}(M) \to M$ is locally trivial, let us take a neighbourhood $V \subset M$ of $x \in M$ such that there is a diffeomorphism $\varphi: V \times \mathbb{S}^{m-2} \to \mathbb{PN}(V)$ with $\varphi\left(\{y\} \times \mathbb{S}^{m-2}\right) = \mathbb{PN}_y$ for all $y \in V$.

Consider the map $\sigma: \mathbb{PN}(V) \to \mathcal{V} \subset \mathcal{N}$ defined by $\sigma([v]) = \gamma_{[v]}$. It is clear that σ is continuous and hence $\overline{\sigma} = \sigma \circ \varphi: V \times \mathbb{S}^{m-2} \to \mathcal{V}$ is also so. Observe that

$$S(x) = \overline{\sigma}\left(\{x\} \times \mathbb{S}^{m-2}\right),$$

and $\overline{\sigma}(V \times \mathbb{S}^{m-2}) = \mathcal{V}$.

Now, take any open $\mathcal{W} \subset \mathcal{V}$ containing the sky $S(x)$, then

$$\{x\} \times \mathbb{S}^{m-2} \subset \overline{\sigma}^{-1}(S(x)) \subset \overline{\sigma}^{-1}(\mathcal{W})$$

Since $\overline{\sigma}$ is continuous then $\overline{\sigma}^{-1}(\mathcal{W})$ is open in $V \times \mathbb{S}^{m-2}$.

For any $(y, q) \in V \times \mathbb{S}^{m-2}$ there exists a neighbourhood basis whose elements are $U^{(y,q)} = K^y \times H^q$ where $K^y \subset V$ and $H^q \subset \mathbb{S}^{m-2}$ are open neighbourhoods of $y \in V$ and $q \in \mathbb{S}^{m-2}$ respectively. Then for any $(x, q) \in \{x\} \times \mathbb{S}^{m-2}$, there exist $U^{(y,q)}$ with $(x, q) \in U^{(y,q)} \subset \overline{\sigma}^{-1}(W)$. Since $\{x\} \times \mathbb{S}^{m-2}$ is compact, then there exists a finite sub-covering $\left\{U_j = K_j \times H_j\right\}_{j=1,\ldots,n} \subset \overline{\sigma}^{-1}(\mathcal{W})$. Then

$$\{x\} \times \mathbb{S}^{m-2} \subset \bigcup_{j=1}^{n} U_j \subset \overline{\sigma}^{-1}(\mathcal{W}).$$

Observe that $K_0 = \bigcap_{j=1}^{n} K_j$ is an open neighbourhood of x and $\bigcup_{j=1}^{n} H_j = \mathbb{S}^{m-2}$.

Since $\mathcal{B}(x)$ is a neighbourhood basis of $x \in M$, there exists $U \in \mathcal{B}(x)$ such that $U \subset K_0$.

For any $(y, q) \in U \times \mathbb{S}^{m-2}$, we have that

$$(y, q) \in U \times \bigcup_{j=1}^{n} H_j$$

therefore there exists j such that $q \in H_j$ and since $y \in K_0 \subset K_j$, then $(y, q) \in U_j \subset \overline{\sigma}^{-1}(W)$. This implies that

$$\{x\} \times \mathbb{S}^{m-2} \subset U \times \mathbb{S}^{m-2} \subset \overline{\sigma}^{-1}(\mathcal{W}).$$

and hence

$$S(x) \subset \overline{\sigma}\left(U \times \mathbb{S}^{m-2}\right) \subset \mathcal{W}$$

and since $\mathcal{U} = \overline{\sigma}\left(U \times \mathbb{S}^{m-2}\right)$ then

$$S(x) \in \Sigma(\mathcal{U}) \subset \Sigma(\mathcal{W})$$

is verified. Then $\{\Sigma(\mathcal{U}): U \in \mathcal{B}(x)\}$ is a neighbourhood basis of $S(x) \in \Sigma$ as we claimed. □

A direct consequence of the previous results is the following:

Theorem 19. *Let M be a strongly causal null pseudo-convex, space-time separating skies such that it is refocussing at x, then the sky map $S: M \to \Sigma$ is not open.*

Proof. We will show that there exists a sequence $\{x_n\}$ in M that does not converge to x and such that $S(x_n)$ converges to $S(x)$ in Σ does contradicting the statement that S is open.

Because M is refocussing at x there exists an open neighbourhood $W \subset M$ of x such that for every open neighbourhood $V \subset W$ of x there is $y \notin W$ such that every light ray passing through y enters V. Let us choose a sequence of globally hyperbolic neighbourhoods $V_n^x \subset W$ of x such that $\bigcap_n V_n^x = \{x\}$. More specifically, let $\sigma(t)$ be a time-like curve contained on a causally convex, globally hyperbolic neighbourhood $U \subset W$ of x and let a_n (respect. b_n) be a sequence of points on σ, in the past (future) of x, such that $a_n \to x$ (respect. $b_n \to x$). Now we choose the sequence of open neighbourhoods as $V_n^x = I^+(a_n) \cap I^-(b_n)$.

Then for any V_n^x in the previous sequence there exists $x_n \notin W$ such that $\gamma \cap V_n^x \neq \varnothing$ and $x_n \in \gamma \in \mathcal{N}$. Hence, since $x_n \notin W$ for all n, then x_n cannot converge to x.

On the other hand, considering the open subsets $\mathcal{U}_n = \{\gamma \in \mathcal{N} | \gamma \cap V_n^x \neq \varnothing\}$, and because of lemma 18, it is clear that $\Sigma(\mathcal{U}_n)$ define a neighbourhood basis at $S(x)$ in Σ, and because $S(x_n) \in \Sigma(\mathcal{U}_n)$ then we conclude that $S(x_n) \to S(x)$. □

Then we get as a corollary of theorem 19:

Corollary 20. *If M is a strongly causal, null pseudo-convex, space-time such that the skies of M separate events, then M is non-refocussing.*

Remark 21. As it was indicated before, section 2.3, if M is globally hyperbolic, the space of light rays $\mathcal{N}$ can be identified with the bundle of spheres over C where C is a space-like smooth Cauchy surface [Be05]. Now, lemma 18 can be slightly reformulated saying that if $\mathcal{B}_C(x)$ is any neighbourhood basis at x in C, then $\{\Sigma(S(TU)) | U \in \mathcal{B}_C(x)\}$ is a neighbourhood basis of $S(x) \in \Sigma$, and theorem 19 follows again from it.

5. Conclusions and discussion

We have reached the main conclusion that the topological, differentiable and causal structures of sky-separating strongly causal space-times can be reconstructed from the corresponding ones in their spaces of light rays and skies provided that they are null pseudo-convex. It is also important to point out that because of lemma 18 any strongly causal space-time is locally sky-separating, thus the property of being sky-separating has a global character. Moreover under being sky-separating implies that the space-time is non-refocussing.

The description of the causal structure of a space-time in terms of the partial order defined in the space of skies by non-negative Legendrian isotopies, provides a new interpretation to the Malament–Hawking theorem, [Ha76, Ma77], in the sense that such partial order on the space of skies characterizes the conformal structure of the original space-time. Actually, suppose that $\Phi: \mathcal{N}_1 \to \mathcal{N}_2$ is a sky preserving diffeomorphism between the spaces of light rays of two strongly causal sky-separating space-times M_1 and M_2. If the map Φ preserves the partial orders $\prec_a$, $a = 1, 2$ defined in the spaces of skies Σ_1 and Σ, i.e., if $X \prec_1 Y$ then $\Phi(X) \prec_2 \Phi(Y)$, for any $X, Y \in \Sigma_1$, then because of corollary 3, we have that Φ induces a causal diffeomorphism $\varphi: M_1 \to M_2$, hence a conformal diffeomorphism.

The characterization of causal relations in terms of sky isotopies opens a new direction in the foundations of the causal sets programme of quantum gravity [Br91, Ri00], as it shows that the contact structure in the space of light rays is needed for the description of causal structures.

It is also worth pointing out here that the causal completion of a given spacetime is just continuous and often fails to be smooth space. According to the reconstruction theorems discussed in this paper, a similar analysis could be performed directly on the space of light rays and skies. In this setting a concrete proposal of a new causal boundary construction was proposed by R Low [Lo06] but has not been discussed in detail so far.

A particularly interesting situation happens for three dimensional space-times that will be discussed in a forthcoming paper. In such case the space of light rays happens to be three dimensional too as well as the space of skies. Low's causal boundary can be constructed explicitly and their topology can then be compared with that of the original space-time.

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