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MAXIMAL GEODESICS IN COMPACT MANIFOLDS (*)

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ABSTRACT:
We prove that the right or left limit points set of a geodesic γ in a compact manifold is union of maximal geodesics (which are maximal null geodesics in the semi-Riemannian case). If such set is just a unique geodesic σ, then this is necessarily smoothly closed, and we establish some relation. between the completeness of σ and γ.
Next we obtain stronger results in the particular case of Lorentz orientable surfaces.
Finally we prove that null-completeness implies global completeness for certain classes of Lorentzian manifolds.

§0 INTRODUCTION

This paper is devoted to the study of general geometrical behaviour of the geodesics of an arbitrary linear connection on a compact manifold, with additional analysis of some special cases.

Roughly speaking, our results in §1 and §2 can be summed up by saying that geodesics rays in a compact manifold, wind up approaching a limit set which is a union of maximal geodesics. These geodesics are null geodesics if our connection is compatible with a Lorentz metric, and the ray is incomplete
We have also discovered a certain dependence between the completeness of the ray and that of the limit geodesics. This dependence is shown in the theorem 4.5 which proves that in case this limit set has a unique incomplete geodesic, then the ray is also incomplete. In §6 we analyze this dependence more thoroughly, determining the degre of incompleteness of the initial ray, if some geodesic in the limit set is incomplete.
The intuitive idea of 6.2 and 6.3 is that the ray is more incomplete than any closed incomplete geodesic. In fact a general aim of §6 is to make precise this sort of statement.

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We specialize in §5 to Lorentz surfaces using the Poincaré-Bendixson theory as a tool. We get that in most cases any incomplete ray is either dense or has limit set a null smoothly closed geodesic.

Finally we point out 3.5 as a useful tool. This theorem states that every maximal compact geodesic is smoothly closed.

In this paper "Lorentz manifold" can be substituted by "semi-Riemannian manifold which is not Riemannian".

§1 INCOMPLETENESS FROM RIEMANNIAN VIEW POINT.

We describe here some geometrical tools and basic notations that we shall use through this paper.

Henceforth M denotes a differential manifold endowed by a linear symmetric connection Γ, and an auxiliary Riemannian complete metric.

If $u \in T_pM$ let $\|u\|$ denote the Riemannian norm. Also ∇ is the Γ-covariant derivate.

Let α be a curve on M (i.e a differentiable curve) such that zero belongs to its parameterization interval and $\alpha'(t) \neq 0$ for all t. We denote by the bold character $\underline{\alpha}$ the unique riemann arc length parameterized curve with the same image as α, such that $\underline{\alpha}(0)=\alpha(0)$.

If $v \in T_pM$, γ_v is the maximal Γ-geodesic such that $\gamma_v(0)=p$ and $\gamma_v'(0)=v$. Finally, if v is a non null vector, $\underline{\gamma}_v$ is the unique Γ-pregeodesic parameterized proportionally to the riemann arc lenght, with the same image than γ_v and such that $\underline{\gamma}_v(0)=p$ and $\underline{\gamma}_v'(0)=v$ (note that v is not a bold character in $\underline{\gamma}_v$).

We show next some technical results which connect the two structures defined on M.

PROPOSITION 1.1

There is a unique symmetric connection $\underline{\Gamma}$ such that curves $\underline{\gamma}_v$ are the maximal geodesics. Moreover, this connection is geodesically complete.

Proof:

Let us define a second order differential equation Ξ on M by:

$$\Xi: TM \ni v \longrightarrow \underline{\gamma}_v''(0) \in TTM$$

Then for each $v \in TM$, $\underline{\gamma}_v$ is an integral curve of Ξ. In fact, if $\underline{\gamma}_v'(s_0)=v_0$ then $\underline{\gamma}_v(s+s_0)=\underline{\gamma}_{v_0}(s)$ for sufficiently small s, hence

$$\underline{\gamma}_v''(s_0)= \underline{\gamma}_{v_0}''(0)= \Xi\ (\underline{\gamma}_{v_0}'(0))= \Xi\ (v_0)= \Xi\ (\underline{\gamma}_v'(s_0))$$

Moreover Ξ is a spray, thus for all $r \in \mathbb{R}$ if $\underline{\gamma}_v(rs)$ and if $\underline{\gamma}_{rv}(s)$ are defined, then it is trivial to see that $\underline{\gamma}_v(rs)=\underline{\gamma}_{rv}(s)$. Finally, if $\underline{\gamma}: (a,b) \longrightarrow M$ is an integral curve of Ξ with $b<\infty$, since $\underline{\gamma}$ is parameterized proportionally to Riemann arc-length , we conclude that $\lim_{s \to \infty} \underline{\gamma}(s)$ exists, and $\underline{\gamma}$ is right-extendible to a longest geodesic. Hence the unique symmetric connection $\underline{\Gamma}$ induced to Ξ is complete.∎

The exponential map associate to $\underline{\Gamma}$ is denoted by $\underline{\exp}: TM \longrightarrow M$. Since the connections Γ and $\underline{\Gamma}$ are projectively equivalent, there is a 1-form λ on M such that

$$\nabla_X Y - \underline{\nabla}_X Y = \frac{1}{2}(\lambda(X)Y + \lambda(Y)X) \ ,$$

where $\underline{\nabla}$ is the covariant derivate of $\underline{\Gamma}$. The 1-form λ has the following geometric meaning.

PROPOSITION 1.2

Let $\gamma: [0,b) \longrightarrow M$ be a right-inextendible Γ-geodesic. For $t \in [0,b)$ let $s_\gamma(t)$ be the Riemannian length of γ between 0 and t, i.e:

$$[0,b) \ni t \longrightarrow s_\gamma(t) = \int_0^t \|\gamma'(\eta)\|\, d\eta \in [0,\infty)$$

If $t_\gamma: [0,\infty) \longrightarrow [0,b)$ denotes the inverse function of s , and

$$\Lambda_\gamma(s) = \int_0^s \lambda(\underline{\gamma}'(\xi))d\xi \text{ for all } s \geq 0$$

then we have that $t_\gamma(s) = \dfrac{1}{\|\gamma'(0)\|} \displaystyle\int_0^s e^{\Lambda_\gamma(\xi)} d\xi$

Proof:

Since $\dfrac{\nabla}{d\,t}\left(\dfrac{d\gamma}{dt}\right)=0$ and $\dfrac{\underline{\nabla}}{d\,s}\left(\dfrac{d\underline{\gamma}}{ds}\right)=0$, we have

$$\frac{\nabla}{ds}\left(\frac{d\gamma}{ds}\right)=\frac{\nabla}{ds}\left(\frac{d\gamma}{ds}\right)-\frac{\nabla}{ds}\left(\frac{d\gamma}{ds}\right)=\frac{1}{2}\,2\lambda\left(\frac{d\gamma}{ds}\right)\frac{d\gamma}{ds}=\lambda\left(\frac{d\gamma}{ds}\right)\frac{d\gamma}{ds}$$

Moreover

$$\frac{\nabla}{ds}\left(\frac{d\gamma}{ds}\right)=\frac{dt_\gamma}{ds}\frac{\nabla}{dt}\left(\frac{dt_\gamma}{ds}\frac{d\gamma}{dt}\right)=\frac{dt_\gamma}{ds}\frac{d}{dt}\left(\frac{dt_\gamma}{ds}\right)\frac{d\gamma}{dt}=\frac{d^2t_\gamma/ds^2}{dt_\gamma/ds}\frac{d\gamma}{ds}$$

Thus we have $\lambda\left(\frac{d\gamma}{ds}\right)=\frac{dh/ds}{h}$, where $h(s)=\frac{dt_\gamma}{ds}$. Hence,

$$\text{Log } h(s)=\int_0^s \lambda(\gamma'(\xi))d\xi+\text{Log } h(0),\quad \text{since}\quad h(0)=\frac{dt_\gamma}{ds}\Big|_{s=0}=\frac{1}{\|\gamma'(0)\|}$$

Finally we have: $h(s)=\frac{1}{\|\gamma'(0)\|}e^{\Lambda_\gamma(s)}$ ∎

COROLLARY 1.3

With the same hypothesis of 1.2, suppose that $b<\infty$, then we have that $\lim_{t\to b}\|\gamma'(t)\|=\infty$.

Proof:

Since $b=\int_0^\infty e^{\Lambda_\gamma(\xi)}d\xi<\infty$, we have that $\lim_{s\to\infty}e^{\Lambda_\gamma(s)}=\lim_{s\to\infty}\frac{dt_\gamma}{ds}=0$

and $\lim_{s\to\infty}e^{-\Lambda_\gamma(s)}=\lim_{t\to b}\frac{ds_\gamma}{dt}=\lim_{t\to b}\|\gamma'(t)\|=0$

§2 STRUCTURE OF THE LIMIT POINT SET OF A INCOMPLETE GEODESIC.

First we establish the following preliminary concept:

DEFINITION 2.1

Let $\alpha:(a,b)\longrightarrow M$ be a curve. We will say that $p\in M$ is a right-limit point of α if there is a sequence (t_i) into (a,b) such that: $\lim_{i\to\infty}t_i=b$ and $\lim_{i\to\infty}\gamma(t_i)=p$. We denote by $\lim^+\alpha$ the set of such points. Analogously we can define left-limit points and $\lim^-\alpha$. Henceforth we will consider only right-limit points. All the results can be easily adapted to the other case.

We prove here that the right-limit point set of an incomplete maximal Γ-geodesic is the union of maximal geodesics, which are null geodesics if Γ is Lorentzian. Note that if M is non compact, such union can eventually be empty.

THEOREM 2.2

Let $\gamma: [0,b) \longrightarrow M$ be a right-inextendible Γ-geodesic. If $p \in \lim^+\gamma$ then for all sequence (t_i) into $[0,b)$ which converges to b, there are a subsequence (t'_i) and $u \in T_p(M)-\{0\}$, such that

$$\lim_{i\to\infty} \frac{\gamma'(t'_i)}{\|\gamma'(t'_i)\|} = u.$$

Moreover the Γ-maximal geodesic γ_u is made of right-limit points of γ.

Proof:

Taking a compact set K such that $p \in \mathring{K}$, there is an integer N with $\gamma(t_i) \in \mathring{K}$ for $i \geq N$. If $x \in M$ let $S_x(M)=\{v \in T_xM : \|v\|=1\}$ be the unit sphere in T_xM. Then $S(K)= \bigcup_{x\in K} S_x(M)$ is a compact set which contains $\dfrac{\gamma'(t_i)}{\|\gamma'(t_i)\|}$ for $i \geq N$. Hence there is a subsequence (t'_i) such that

$u= \lim_{i\to\infty} \dfrac{\gamma'(t'_i)}{\|\gamma'(t'_i)\|}$ exists, and by continuity u belongs to T_pM.

Finally, if $s_i=s_\gamma(t'_i)$, and $s \in \mathbb{R}$ then $\underline{\exp}\left[s \dfrac{\gamma'(t'_i)}{\|\gamma'(t'_i)\|} \right] =$

$= \underline{\exp}\ [s\ \underline{\gamma}'(s_i)] = \underline{\gamma}(s+s_i)$ belongs to $\underline{\gamma}[0,\infty)=\gamma[0,b)$ for i sufficiently large, since $\lim s_i=\infty$. Using continuity we have

$$\lim_{i\to\infty} \underline{\exp}\left(s \frac{\gamma'(t_i)}{\|\gamma'(t_i)\|} \right) = \underline{\exp}\ (s\ u) = \underline{\gamma}_u(s)$$

This proves that $\text{im}\ \gamma_u \subseteq \lim^+\gamma$. ■

We analyze now the semi-Riemannian case:

COROLLARY 2.3

With the same hypothesis as in Theorem 2.2, we suppose now that Γ is the Levi-Civita connection of a Lerentzian metric on M.

Then, if $b<\infty$ the geodesic γ_u constructed in 2.2 is a null geodesic.

Proof:

Let "$|\ |$" the Lorentzian norm. Since γ is Γ-geodesic then $c=|\gamma'(t)|$ $(t\in[0,b))$ is a real constant. Using 1.3 we have:

$$|u|=\left|\lim_{i\to b}\frac{\gamma'(t_i)}{\|\gamma'(t_i)\|}\right|=\lim_{i\to b}\frac{|\gamma'(t_i)|}{\|\gamma'(t_i)\|}=c\lim_{i\to b}\frac{1}{\|\gamma'(t_i)\|}=0$$

This proves that γ_u is null geodesic ∎

§3 MAXIMAL CLOSED GEODESICS.

Using the Baire theorem and the results of §2 we prove that the conditions of topological and smooth closure are equivalents for maximal Γ-geodesics in compact manifolds. First we recall some preliminary concepts

DEFINITION 3.1

A curve segment $\alpha:[a,b]\longrightarrow M$ is smoothly closed provided $\alpha(a)=\alpha(b)$ and $\alpha'(a)=c\alpha'(b)$ for some $c>0$. A smoothly closed geodesic is a geodesic $\gamma:(a,b)\longrightarrow M$ which has a segment $\gamma/[a_1,b_1]$ smoothly closed. (Therefore by uniqueness of geodesic im $\gamma=\gamma([a_1,b_1])$).

A point $\gamma(t_1)$ of im γ is called an autointersection point of γ, if there is $t_2\in(a,b)$ such that $\gamma(t_1)=\gamma(t_2)$ and $\gamma'(t_1)$, $\gamma'(t_2)$ are not proportional.

We say that a curve is topologically closed, if its image is a closed set. This is equivalent to saying that $\lim^+\gamma$ and $\lim^-\gamma$ are contained into im γ.

REMARK 3.2

The set of autointersection points of a non constant geodesic $\gamma:(a,b)\longrightarrow M$ is obviously numerable. In fact, for any compact segment γ_1 of γ such that the autointersection set of γ_1 is empty, we have that the set defined by:

$\{t\in(a,b)-[a_1,b_1]:\ \gamma(t)\in \text{im}\,\gamma_1$ and $\gamma'(t)$ is transversal to $\gamma_1\}$

is discrete.

PROPOSITION 3.3

Let $\sigma:[0,b)\longrightarrow M$ be a right-inextendible Γ-geodesic. Suppose that $\lim^+\sigma\neq\emptyset$. If σ is topologically closed then, his geodesic maximal extension $\bar{\sigma}$ is smoothly closed.

Proof:

If $p \in \lim^{+}\sigma$ and $u \in T_p(M)$ are as in 2.2, using $\lim^{+}\sigma \subseteq \overline{\text{im }\sigma} = \text{im }\sigma$, we have that $p \in \text{im }\sigma$ and, by 2.2, we also have $\text{im }\gamma_u \subseteq \lim^{+}\sigma \subseteq \text{im }\sigma$. Hence, $\text{im }\bar{\sigma} = \text{im }\gamma_u \subseteq \text{im }\sigma$, and $\text{im }\bar{\sigma} = \text{im }\sigma$ by maximality. Therefore for all $\varepsilon > 0$ there is a $t_\varepsilon \in [0,b)$ such that $\bar{\sigma}(-\varepsilon) = \sigma(t_\varepsilon)$. Since the set of autointersection points of $\bar{\sigma}$ is numerable, there is $\varepsilon > 0$ with $\bar{\sigma}'(-\varepsilon)$ proportional to $\sigma'(t_\varepsilon)$. Hence $\bar{\sigma}$ is smoothly closed. ■

This result is completed with theorem 3.5 which requires the following preliminary classical result.

THEOREM 3.4 (Baire Theorem)

There is no complete metric space which is a numerable union of closed sets with empty interior.

THEOREM 3.5

Let σ: $(a,b) \longrightarrow M$ be a inextendible topologically closed non constant Γ-geodesic. Then, if M is compact, σ is smoothly closed.

Proof:

For $r>0$ and $t \in (a,b)$ consider $S^{\sigma}_{r}(t) = \{\underline{\exp}(v) : \langle v, \sigma'(t)\rangle = 0, \|v\| < r\}$ and $K^{\sigma}_{r}(t) = S^{\sigma}_{r}(t) \cap \text{im }\sigma$. We claim that there is $t_0 \in (a,b)$ and $r_0 > 0$ such that:

(1) $K^{\sigma}_{2r_0}(t_0)$ has not auto intersection points of σ

(2) σ allways intersect transversally $S^{\sigma}_{2r_0}(t_0)$

In order to prove the claim (1) note that there are $\varepsilon>0$, $r_1>0$ such that if $0<r\leq r_1$ then $\bigcup_{-r<t<r} S^{\sigma}_{r}(t)$ is open and $S^{\sigma}_{r}(t_1) \cap S^{\sigma}_{r}(t_2) = \emptyset$ for $t_1 \neq t_2$. Hence there is t_0 with $|t_0| < \varepsilon$ such that $S^{\sigma}_{r}(t_0)$ contains no aut o intersection points of σ. In other case such autointersection set should be not numerable.

Moreover if we suppose that (2) is false for all r_0 such that $0<r_0<r$, using 2.2, we have a sequence $(t_i) \subset (a,b)$ which converges to a or b, such that $\sigma(t_i) \in S^{\sigma}_{r_1}(t_0)$, $\sigma'(t_i)$ is tangent to $S^{\sigma}_{r_1}$ and $\lim \frac{\sigma'(t_i)}{\|\sigma'(t_i)\|} = u \in T_{\sigma(t_0)}M$.

Obviously u is non zero and tangent to $S^{\sigma}_{r_1}(t_0)$, hence it is orthogonal to $\sigma'(t_0)$. Again by 2.2, the maximal Γ-geodesic γ_u is contained in $\overline{\text{im}\ \sigma}$= im σ. Thus im γ_u= im σ. This contradicts that $\gamma(t_0)$ is not an autointersection point of γ, and the claim is proved.

Suppose now that σ is not smoothly closed and let r_0 and t_0 be the numbers obtained in the claim. In order to get a contradiction using the Baire Theorem, it is sufficient to prove the following two statements relatives to the closure C of $K_{r_0}(t_0)$:

a) For all $x\in C$ and all neighborhood V of x, we have $V\cap(C-\{x\})\neq\emptyset$.

b) C is a numerable compact set.

Note first that by compactness of M, the set $\lim^{+}\sigma$ is not empty and it is contained into im σ. By 2.2 we have that $\lim^{+}\sigma$=im σ.

Let $\underline{\sigma}:\mathbb{R}\longrightarrow M$, the arc length riemann parameterization of σ, such that $\underline{\sigma}(0)=\sigma(0)$ and $\|\underline{\sigma}'(0)\|=1$. If $x=\underline{\sigma}(s)\in K_{r_0}(t_0)$, using again 2.2 we obtain a sequence $(s_i)\subset\mathbb{R}$ such that

$$\lim s_i=+\infty,\ \lim \underline{\sigma}(s_i)=\underline{\sigma}(s),\ \text{and}\ \lim \underline{\sigma}'(s_i)=\underline{\sigma}'(s)$$

For some $\eta>0$ and i sufficiently large the segment $\underline{\sigma}(s_i-\eta,s_i+\eta)$has transversal intersection with $S^{\sigma}_{r_0}(t_0)$ in a point x_i, and $\lim x_i=x$. By the hypothesis and the property (1) of the claim we get that $x_i\neq x_j$ for $i\neq j$. This proves a).

In order to prove b), note that by thetransversalily property (2) $K^{\sigma}_r(t)$ is numerable for $0<r<2r_0$.Since σ is topologically closed we obtain also that $C\subset$ im $\sigma\cap S_{2r_0}(t_0)$. Hence C is a numerable and compact set.∎

COROLLARY 3.7

Suppose M compact, and let γ be a right-inextendible Γ-geodesic. If $\lim^{+}\gamma$ is the image of a geodesic σ, then σ is smoothly closed.

Proof:

Obviously $\lim^{+}\gamma$ is a closed set , thus im σ also is closed. Use now 3.6.

§4 CLOSED GEODESICS INTO A LIMIT POINT SET.

In 2.2 we have proved that the right limit points set of a Γ-geodesic γ is made of a union of maximal ones. If this union is just a unique geodesic, then by 3.7 this geodesic is smoothly closed. As a partial converse, we prove here that if such limit points set contains a closed isolated geodesic σ, then this set coincides with im σ. Moreover if the initial geodesic γ is right complete, then σ is right or left complete.

First we prove the following technical lemma:

LEMMA 4.1

Let $\gamma:[0,\beta]\longrightarrow M$ be a right inextendible non closed Γ-geodesic and let $\sigma:(a,b)\longrightarrow M$ be a closed maximal Γ-geodesic with Riemannian length L such that:

(1) im $\sigma \subset \lim^{+}\gamma$.

(2) There is $p\in$ im σ and U neighborhood of p in M such that $(U-\text{im }\sigma)\cap \lim^{+}\gamma=\emptyset$.

Then there are a sequence $(s_i)\subset\mathbb{R}^{+}$ and $s^{*}\in[0,L]$ such that:

$\lim s_i=+\infty$, $\lim \underline{\gamma}(s_i)=\underline{\sigma}(s^{*})$, $\lim \underline{\gamma}'(s_i)\in \pm\underline{\sigma}'(s^{*})$, $\lim (s_{i+1}-s_i)=L$

Proof:

Using the same sort of argument as in 3.6 and the hypothesis, we conclude that there is a point $s^{*}\in\mathbb{R}$ and $r>0$ such that

(a) $\sigma(s^{*})\in U$ is not auto intersection point of σ

(b) $S^{\sigma}_{2r}(s^{*})\cap \lim^{+}\gamma=\{\sigma(s^{*})\}$

(c) γ has always transversal intersection with $S^{\sigma}_{2r}(s^{*})$

Hence, the set $\{\xi>0:\ \underline{\gamma}(\xi)\in S^{\sigma}_{r}(s^{*})\}$ is infinite numerable and can be indexes by a sequence (ξ_k) with $\xi_k<\xi_{k+1}$ for $k\in\mathbb{N}$. Note that $\lim \xi_k=+\infty$.

Since $(\underline{\gamma}(\xi_k))$ is contained in a compact set, we see by (b) that $\underline{\sigma}(s^{*})$ is the only accumulation point. Therefore $\lim \underline{\gamma}(\xi_k)=\underline{\sigma}(s^{*})$. Moreover, $\pm\underline{\sigma}'(s^{*})$ are the only possible accumulation points of $\underline{\gamma}'(\xi_k)$, since in other case there is a subsequence (ξ'_i) of (ξ_i) such that $\lim \underline{\gamma}'(\xi'_i)=u\in T_{\sigma(s^{*})}M$, which is not in the direction of $\underline{\sigma}'(s^{*})$. Thus, we obtain by 2.2 that the maximal geodesic γ_u has points into $(U-\text{im }\sigma)\cap\lim^{+}\gamma$. This contradicts the hypothesis.

Let D be a oriented domain of M which contains S^{σ}_{r}. The vector

$\sigma'(s^*)$ defines an orientation in S_r^σ. Reversing if necessary the parameterization of σ, we can suppose that there are a infinite ξ_i such that $\gamma'(\xi_i)$ defines the same orientation in S_r^σ. Let (s_k) be the increasing sequence of such ξ_i. Thus we have that
$\lim \gamma'(s_k)=\sigma'(s^*)$.
Let $\rho:\mathbb{R}\longrightarrow\mathbb{R}$ be the function defined by $\rho(s)=\mathrm{Min}\{|s-s_k|:k\in\mathbb{N}\}$. We claim that $\lim \rho(s_k+L)=0$.
First note that for all h>0 we have:

$$\lim \underline{\gamma}(s_k+h)= \lim \underline{\exp}\ (h\ \gamma'(s_k))= \underline{\exp}\ (h\ \sigma'(s^*)=\sigma(s^*+h) \quad (*)$$

Let $\varepsilon>0$ be sufficiently small and V Γ-convex neighborhood of $\sigma(s^*)$ such that $\sigma((s^*-\varepsilon,s^*+\varepsilon))\subset V$, and $S_r^\sigma(s^*)$ divides U in two connected components V^- and V^+.

Taking in (*) $h= L+\varepsilon$, and $h=L-\varepsilon$
we see that $\lim \underline{\gamma}(s_k+L+\varepsilon)=\underline{\sigma}(s^*+\varepsilon)$ and $\lim \underline{\gamma}(s_k+L-\varepsilon)=\underline{\sigma}(s^*-\varepsilon)$, thus there is $N\in\mathbb{N}$ such that for k>N we have that $\underline{\gamma}(s_k+L-\varepsilon)$ and $\underline{\gamma}(s_k+L+\varepsilon)$ belongs to V^- and V^+ respectively.
By continuity we can suppose that $\underline{\gamma}((s_k+L-\varepsilon,s_k+L+\varepsilon))\subset V$. Hence this geodesic segment intersect $S_{r_1}^\sigma(s^*)$, transversaly in a first point $\underline{\gamma}(\xi_j)$ and by construction $\gamma'(\xi_j)$ defines the same orientation in $S_r^\sigma(s^*)$ than $\sigma'(s^*)$. Thus there is $s_i=\xi_j$ belonging to $(s_k+L-\varepsilon,s_k+L+\varepsilon)$. This proves the claim.
Taking necessary a subsequence (but it is not necessary!) we can suppose that $\lim (s_{k+1}-s_k)=L$. Again $\lim \underline{\gamma}(s_k)=\underline{\sigma}(s)$ and
$\lim \gamma'(s_k)= \underline{\sigma}'(s^*)$.

REMARK 4.2
Note that if (s_k) is the sequence of 4.1, then we can take an orientation for the parameterization of σ, such that $\lim \underline{\gamma}'(s_i)=$ $=\underline{\sigma}'(s^+)$. Moreover for sufficiently small $\delta>0$, we have that $(s_k+\delta)$ verifies the same conditions required for (s_k).

DEFINITION 4.3
We say that σ has the same orientation of γ, if σ has the orientation explained in 4.2.

The following corollary show the consistency of the previous definition, and proves that $\lim^+\gamma$ coincides with im σ.

COROLLARY 4.4

With the same hypothesis of 4.1, and assuming that σ has the orientation explained in 4.2, let (ξ_k) be a sequence such that $\lim \xi_k=+\infty$, and $\lim \underline{\gamma}(\xi_k)$ exists. Then there is $\theta\in[0,1]$ such that $\lim \underline{\gamma}(\xi_k)=\underline{\sigma}(L\theta)$. Moreover $\lim \underline{\gamma}'(\xi_k)=\underline{\sigma}'(L\theta)$.
In particular we have that $\lim^{+}\gamma = \text{im}\,\sigma$.

Proof:

Let $(\xi_k) \longrightarrow +\infty$ be such that $\underline{\gamma}(\xi_k)$ is convergent. Let (s_k) be the sequence obtained in 4.1. We define:

$$\theta_k = \frac{\xi_k - s_i}{\Delta_i} \quad \text{where } \xi_k\in [s_i, s_{i+1}) \text{ and } \Delta_i=s_{i+1}-s_i$$

Since $\theta_k\in [0,1)$ there is a convergent subsequence. Hence we can suppose that $\theta=\lim \theta_k\in [0,1]$ exists. We have then therefore,

$$\lim \underline{\gamma}(\xi_k)=\lim \underline{\gamma}(s_{j_k} + \Delta_{j_k}\theta_k)= \lim \underline{\exp}\, (\Delta_{j_k}\theta_k\, \underline{\gamma}'(s_{j_k}))=$$
$$= \underline{\exp}\, (L\, \theta\, \underline{\sigma}'(s^*)) =\sigma(L\theta)\in \text{im}\,\sigma .$$

We prove now that $\lim \underline{\gamma}'(\xi_k)=\underline{\sigma}'(L\theta)$

Using the remark to change (s_k) if it is necessary , we can suppose that θ belongs to $(0,1)$, and we can take $\varepsilon>0$ and $N\in\mathbb{N}$ such that for $k>N$, the points ξ_k and $\bar{\xi}_k=\xi_k+\varepsilon$ belongs to same interval $[s_i,s_{i+1})$.
Then

$$\bar{\theta}_k=\frac{\xi_k+\varepsilon-s_i}{\Delta_i}= \theta_k+ \frac{\varepsilon}{\Delta_i} \text{ and } \lim \bar{\theta}_k = \theta +\frac{\varepsilon}{L}.$$ Using the before argument we see that $\lim \underline{\gamma}(\xi_k+\varepsilon)= \underline{\sigma}(L\theta+\varepsilon)$.

Moreover if we suppose that there is a subsequence (ξ_{k_j}) such that $\lim \underline{\gamma}(\xi_{k_j})= -\underline{\sigma}'(L\theta)$, then :

$$\lim \underline{\gamma}(\xi_{k_j}+\varepsilon)=\lim \underline{\exp}(\varepsilon\, \underline{\gamma}'(\xi_{k_j}))=\underline{\exp}(-\varepsilon\, \underline{\sigma}'(L\theta))=\underline{\sigma}(L\theta-\varepsilon)$$

which contradicts the statement above.∎

We prove now the following main theorem

THEOREM 4.5

Let γ: $[0,b)\longrightarrow M$ be a right-inextendible Γ-geodesic (b can be eventually infinite), such that $\lim^{+}\gamma$ is the image of a closed right-incomplete Γ-geodesic σ with the same orientation as γ.

Then γ is right-incomplete.

The proof of 4.5 requires some technical preliminary lemmas

The hypothesis of theorem 4.5 are assumed in lemmas 4.6 to 4.10

We can suppose without lossing generality that γ is not closed, and that the riemann length of σ is L=1. Let (s_k) be the sequence obtained in 4.1, i.e. (s_k) verifies $\lim s_k=+\infty$, $\lim \underline{\gamma}(s_k)=\underline{\sigma}(0)$, $\lim \underline{\gamma}'(s_k)=\underline{\sigma}'0)$, $\Delta_k=s_{k+1}-s_k>0$, and $\lim \Delta_k=1$

We define for $\xi\in\mathbb{R}$, $\Theta(\xi)=\dfrac{\xi-s_k}{\Delta_k}$ when $\xi\in[s_k,s_{k+1})$. Note that $\Theta(\xi)$ belongs to the interval $[0,1)$.

Recall that λ is the 1-form introduced in §1 which relate the connection Γ and $\underline{\Gamma}$. We consider $\lambda_\gamma=\lambda.\underline{\gamma}:[0,+\infty)\longrightarrow\mathbb{R}$, and $\lambda_\sigma=\lambda.\sigma':[0,1]\longrightarrow\mathbb{R}$.

LEMMA 4.6

Let (ξ_j) be a sequence such that $\lim \xi_j=+\infty$ and $(\underline{\gamma}(\xi_j))$ is a convergent sequence. If (ξ'_j) verifies $\lim |\xi_j-\xi'_j|=0$, then $\lim \underline{\gamma}(\xi'_j)=\lim \underline{\gamma}(\xi_j)$.

Proof:

If $\theta_j=\Theta(\xi_j)$, then (θ_k) has subsequence which converges to some $\theta\in[0,1]$. Using the argument of 4.3 and the convergence of $(\underline{\gamma}(\xi_j))$we see that $\lim \underline{\gamma}(\xi_j)=\underline{\sigma}(\theta)$. As in4.3 we can suppose that $\theta\in(0,1)$. If $\theta'_j=\Theta(\xi'_j)$, using that $\lim|\xi_j-\xi'_j|=0$ and $\lim \Delta_j=1$ we see that $\lim|\theta_j-\theta'_j|=0$, hence $\lim \theta'_j=\lim \theta_j=\theta$. Using again the argument of 4.2 we have $\lim \underline{\gamma}(\xi'_j)=\underline{\sigma}(\theta)$.

LEMMA 4.7

The function $\lambda_\gamma=\lambda.\underline{\gamma}':[0,+\infty)\longrightarrow\mathbb{R}$ is uniformly continuous.

Proof:

If λ_γ were not uniformly continuous then there would be $\varepsilon>0$, ξ_k and ξ'_ksuch that $\lim |\xi_k-\xi'_k|=0$ and $|\lambda_\gamma(\xi_k)-\lambda_\gamma(\xi'_k)|>\varepsilon$. Since λ_γ is

continuous , $\lim \xi_k = +\infty$. Taking if it is necessary a subsequence we can suppose that $\lim \underline{\gamma}(\xi_k) = \underline{\sigma}(\theta)$ exists, where $\theta \in [0,1]$. By 4.3 and 4.6 we have that $\lim \underline{\gamma}(\xi'_k) = \underline{\sigma}(\theta)$, and $\lim \underline{\gamma}'(\xi_k) = \underline{\sigma}'(\theta) = \lim \underline{\gamma}'(\xi'_k) = \underline{\sigma}'(\theta)$. Using that λ is continuous we get:
$\lim \lambda_\gamma(\xi_k) = \lim \lambda(\underline{\gamma}'(\xi_k)) = \lambda(\sigma'(\theta)) = \lim \lambda(\underline{\gamma}'(\xi'_k)) = \lim \lambda_\gamma(\xi'_k)$.
This contradicts that $|\lambda_\gamma(\xi_k) - \lambda_\gamma(\xi'_k)| > \varepsilon$.

LEMMA 4.8

Let $\lambda_j : [0,1] \longrightarrow \mathbb{R}$ be such that $\lambda_j(\theta) = \lambda_\gamma(s_j + \theta\ \Delta_j)$. Then (λ_j) converges uniformly to $\lambda_\sigma = \lambda.\underline{\sigma}' : [0,1] \longrightarrow \mathbb{R}$.

Proof:

Obviously (λ_j) converges pointwise to λ_σ. Hence it is sufficient to prove that (λ_j) is an equicontinuous family of functions.
By 4.7 , for all $\varepsilon > 0$ there is $\delta > 0$ such that if $|\xi - \xi'| < \delta$ then $|\lambda_\gamma(\xi) - \lambda_\gamma(\xi')| < \varepsilon$. Since $\lim \Delta_j = 0$, there is also $\rho > 0$ such that if $|\theta - \theta'| < \rho$ then $|s_j + \theta\ \Delta_j - (s_j + \theta'\Delta_j)| = \Delta_j |\theta - \theta'| < \delta$ for all j. Thus for $|\theta - \theta'| < \rho$ we have $|\lambda_j(\theta) - \lambda_j(\theta')| < \varepsilon$. ∎

LEMMA 4.9

For all $\varepsilon > 0$ there is $N \in \mathbb{N}$ such that if $j \geq N$ and $s \in [s_j, s_{j+1}]$ we have

$$\left| \int_0^{\Theta(s)} \lambda_\sigma(\theta) d\theta - \int_{s_j}^{s} \lambda_\gamma(\xi) d\xi \right| < \varepsilon$$

Proof:

We define $\Lambda_j(\tau) = \int_0^\tau \lambda_j(\theta) d\theta$ and $\Lambda_\sigma(\tau) = \int_0^\tau \lambda_\sigma(\theta) d\theta$ for $\tau \in [0,1]$. By 4.8, (λ_j) converges uniformly to λ_σ, thus (Λ_j) converges uniformly to Λ_σ. In particular, there is $K > 0$ such that $|\Lambda_j(\theta)| < K$ for $\theta \in [0,1]$ and $j \in \mathbb{N}$. Making the change of variables $\xi = s_j + \theta\ \Delta_j$, we have:

$$\Lambda_j(\Theta(s)) = \frac{1}{\Delta_j} \int_{s_j}^{s} \lambda_\gamma(\xi) d\xi \text{ for } s \in [s_j, s_{j+1}]$$

Fixing $\varepsilon > 0$, we pick up $N \in \mathbb{N}$ such that $j \geq N$ implies $|\Delta_j - 1|\ K < \varepsilon/2$, and $|\Lambda_j(\tau) - \Lambda_\sigma(\tau)| < \varepsilon/2$ for all $\tau \in [0,1]$. Hence for $s \in [s_j, s_{j+1}]$ we have:

$$\left| \int_{s_j}^{s} \lambda_\gamma(\xi) d\xi - \int_0^{\Theta(s)} \lambda_\sigma(\theta) d\theta \right| = |\Delta_j\ \Lambda_j(\theta(s)) - \Lambda_\sigma(\theta(s))| =$$

$$= |(\Delta_j - 1)\Lambda_j(\theta(s)) + \Lambda_j(\theta(s)) - \Lambda_\sigma(\theta(s)| \leq |\Delta_j - 1|\ K + |\Delta_j(\theta(s)) - \Lambda_\sigma(\theta(s)| \leq \varepsilon \ \blacksquare$$

PROOF OF THEOREM 4.5

Since σ is right-incomplete, by 1.2 we get $\int_0^\infty e^{\Lambda_\sigma(\xi)}d\xi <+\infty$, thus $\lim_{s\to\infty} \Lambda_\sigma(s)=-\infty$. Since λ_σ is periodic we conclude that for $k\in\mathbb{N}$

$$\Lambda_\sigma(k)=\sum_{j=1}^{K}\int_{j-1}^{j}\lambda_\sigma(\theta)d\theta = k\,\Lambda_\sigma(1).$$ In particular we get $\Lambda_\sigma(1)<0$.

For $s\in[s_k,s_{k+1}]$ we write:

$$\delta_k(s)=\int_{s_k}^{s}\lambda_\gamma(\xi)d\xi-\int_0^{\Theta(s)}\lambda_\sigma(\theta)d\theta\ ,\quad \Lambda_\gamma^k=\int_{s_k}^{s_{k+1}}\lambda_\gamma(\xi)d\xi \quad\text{and}\quad \varepsilon_k=\Lambda_\gamma^k-\Lambda_\sigma(1)$$

Note that $\varepsilon_k=\delta_k(s_{k+1})$.

Using 4.9 we get a number $N\in\mathbb{N}$ such that if $k\geq N$ then

$$|\varepsilon_k|<\frac{|\Lambda_\sigma(1)|}{4}\quad\text{and}\quad|\delta_k(s)|<\frac{|\Lambda_\sigma(1)|}{4}\quad\text{for all } s\in[s_k,s_{k+1}] \tag{4.5.1}$$

Translating if necessary the origin in γ we can suppose for simplicity that $N=0$ and $s_0=0$. If $s\in[s_k,s_{k+1})$ we get:

$$\Lambda_\gamma(s)=\sum_{j=0}^{k-1}\Lambda_\gamma^j+\int_{s_k}^{s}\lambda_\gamma(\xi)d\xi=\sum_{j=0}^{k-1}(\Lambda_\sigma(1)+\varepsilon_j)+\delta_k(s)+\int_0^{\Theta(s)}\lambda_\sigma(\theta)d\theta$$

thus we have
$$\Lambda_\gamma(s)=A_k(s)+\int_0^{\Theta(s)}\lambda_\sigma(\theta)d\theta \tag{4.5.2}$$

where $A_k(s)=\sum_{j=0}^{k-1}(\Lambda_\sigma(1)+\varepsilon_j)+\delta_k(s)$. Using (4.5.1) and taking $H=\frac{\Lambda_\sigma(1)}{2}$ we obtain that $A_k(s)<k\,H<0$. Using now (4.4.2) we get

$$\Lambda_\gamma(s)<k\,H+\int_0^{\Theta(s)}\lambda_\sigma(\theta)d\theta,\quad e^{\Lambda_\gamma(s)}<e^{k\,H}\,e^{\Lambda_\sigma(\Theta(s))}\quad\text{for } s\in[s_k,s_{k+1})\text{ and}$$

$$\int_{s_k}^{s_{k+1}}e^{\Lambda_\gamma(s)}ds<e^{k\,H}\int_{s_k}^{s_{k+1}}e^{\Lambda_\sigma(\Theta(s))}ds=e^{k\,H}\,\Delta_k\int_0^1 e^{\Lambda_\sigma(\theta)}d\theta$$

(the last equality has been obtained making the change of variable:

$\theta=\dfrac{s-s_k}{\Delta_k}=\Theta(s))$.

Defining $J=\int_0^1 e^{\Lambda_\sigma(\theta)}d\theta$, using the convergence of the geometric series $\sum e^{k\,H}$ for H<0 and $\lim \Delta_k=1$, we have:

$\sum_{k=0}^{\infty}\dfrac{e^{kH}}{\Delta_k}=B\in\mathbb{R}^+$, thus for all $n\in\mathbb{N}$ we get:

$$\int_0^{s_n} e^{\Lambda_\gamma(s)}ds=\sum_{k=0}\int_{s_k}^{s_{k+1}} e^{\Lambda_\gamma(s)}ds \le J\sum_{k=0}^{n-1}\frac{e^{k\,H}}{\Delta_k} < J\,B$$

By 1.2 we conclude that γ is right-incomplete.■

We have proved that if the closed geodesic σ is right-incomplete then $\Lambda_\sigma(1)=\int_0^1\lambda_\sigma(\theta)d\theta$ is negative. Reciprocally it is easy to prove that if such integral is negative then σ is right-incomp lete. Therefore, the sign of $\Lambda_\sigma(1)$ must have a geometric meaning which is independent of the Riemannian auxiliary metric. In fact we have the following result.

PROPOSITION 4.10

Let σ be a smoothly closed Γ-geodesic with Riemannian length L>0, and let β be the first $t\in\mathbb{R}^+$ such that $\sigma'(t)$ is proportional to $\sigma'(0)$. If $\sigma'(\beta)=c\sigma'(0)$, then $c=e^{-\Lambda_\sigma(L)}$. In particular σ is complete iff $\Lambda_\sigma(L)=0$.

Proof:

Let us consider the function $\nu:[0,\beta)t\ni\!\longrightarrow\in\|\sigma'(t)\|\in\mathbb{R}^+$. Then $\nu'(t)=2<\dfrac{\underline{\nabla}\,\sigma'}{dt}, \sigma'(t)>$. Changing the roles of Γ and $\underline{\Gamma}$ in 1.1 we get: $\dfrac{\underline{\nabla}\,\sigma'}{dt}=-\lambda(\sigma'(t)\ \nu(t)$, thus $\nu'(t)=-2\ \lambda(\sigma'(t))\ \nu(t)$.

If we suposse for simplicity that $\|\sigma'(0)\|=1$, we can write:

$J_\sigma(t)=\int_0^t\lambda(\sigma'(t))dt$, and we get $\nu(\beta)=e^{-2\ J_\sigma(\beta}$.

Since $\sigma'(\beta)=+\sqrt{\nu(\beta)}\sigma'(0)$ we have that $\sigma'(\beta)=e^{-J_\sigma(\beta)}\sigma'(0)$. But making $t=t_\sigma(s)$ we get:

$$J_\sigma(\beta)=\int_0^\beta \lambda(\sigma'(t_\sigma(s))\frac{dt_\sigma}{ds}ds=\int_0^\beta \lambda(\sigma'(t_\sigma(s)\frac{dt_\sigma}{ds})ds=\int_0^L \lambda_\sigma(s)ds=\Lambda_\sigma(L)\blacksquare$$

We have finally the following main result:

THEOREM 4.11

Let $\gamma:[0,b)\longrightarrow M$ be a right inextendible Γ-geodesic such that $\lim^+\gamma$ is the image of a smoothly closed Γ-geodesic σ. Let $\beta>0$ and $c>0$ such that $\sigma'(\beta)=c\sigma'(0)$. Suppose $c<1$, then:

if γ has the same orientation as σ, γ is right-complete.

Otherwise γ is right-incomplete.

Proof:

Using 4.10 we see that the last assert is a reformulation of 4.5.

In order to prove the first assertion, note that we can use the same argument as in the proof of 4.5 with $H=\frac{\Lambda_\sigma(1)}{2}>0$, and we get:

$$\Lambda_\gamma(s)=A_k(s)+\int_0^{\Theta(s)}\lambda_\sigma(\theta)d\theta \text{ where } A_k(s)=\sum_{j=0}^{k-1}(\Lambda_\sigma(1)+\varepsilon_j)+\delta_k(s)$$

and $A_k(s)>k\,H>0$ for $k\in\mathbb{N}$ and $s\in[s_k,s_{k+1})$, obtaining finally:

$$\int_0^{s_n} e^{\Lambda_\gamma(s)}ds \geq J\sum_{k=0}^{n-1}\frac{e^{k\,H}}{\Delta_k}\xrightarrow[n\to\infty]{}+\infty .$$

By 1.2 we conclude that γ is right-complete.$\blacksquare$

§5 MAXIMAL GEODESICS IN COMPACT ORIENTABLE LORENTZ SURFACES

We suppose now that M is a compact Lorentz surface which is time-orientable. Thus M has vanishing Euler characteristic and is diffeomorphic to the 2-torus or the Klein- bottle depending on whether it is topologically orientable or not.

We will prove here that escept for some specific cases (that we do not analize in this paper), the right-limit point set of a right-incomplete non-null geodesic is either a closed null geodesic or the whole surface M.

We can take null differentiable vector fields X and Y on M which are linearly independent, and defining the same time-cone in each point of M. We choose the auxiliary Riemannian metric such that (X,Y) determine a global ortonormal parallelitation. The null Γ-pregeodesics parameterized by the Riemann arc length are exactly the orbits of ±X or ±Y. In particular these Γ-geodesics have no auto intersection points. We recall now the following classic theorem for differentiable fields without critical points in compact surfaces.

THEOREM 5.1 (See [7])
Let V be a differentiable field without critical points on a compact surface S, and let α be a orbit of V. then either $\lim^{+}\alpha$ is periodic or $\lim^{+}\alpha$ is the whole S. In this case S is the 2-torus, and the right (and left) limit point set of each orbit of V is the whole S.

Using this result, and theorems 2.2 and 4.5 we obtain immediately the following properties of the null geodesics in compact Lorentz surfaces:

COROLLARY 5.2
I) If there is a null closed geodesic in the direction of X, then:
   a) The right-limit point set of each null geodesic in the direction of X , is a closed null geodesic.
   b) If each closed null geodesic on the direction of X is incomplete, then each null geodesic in the direction of X is incomplete.

II) If M has no closed null geodesics, then M is a 2-torus, and each null geodesic is dense in M.∎

We are interested now in the study of the right limit point set of a right incomplete non-null geodesic. We start with the following auxiliary result:

LEMMA 5.3
Let $\gamma\colon [0,b)\longrightarrow M$ be a right-inextendible and right-incomplete non null Γ-geodesic. Let p be a point belonging to $\lim^{+}\gamma$, and (s_i) a sequence in $[0,+\infty)$ such that $\lim_{i\to\infty} s_i=+\infty$, $\lim_{i\to\infty} \underline{\gamma}(s_i)=p$ and

$\lim_{i\to\infty} \underline{\gamma}'(s_i)=X(p)$. Then $\lim_{s\to\infty} \| \underline{\gamma}'(s)-X(\underline{\gamma}(s))\|=0$.
In particular, $\lim^{+}\gamma$ is a union of orbits of X.

Proof:
Let (ξ_j) be a sequence of $[0,\infty)$ such that $\lim_{j\to\infty} \xi_j=+\infty$, and that $\underline{\gamma}(\xi_j)$ converges to a point $m\in M$. We claim that $(\underline{\gamma}'(\xi_j))$ converges to a null vector $u\in T_mM$. In fact if there are subsequences (ξ'_j) and (ξ''_j) of (ξ_j) such that $\lim_{j\to\infty} \underline{\gamma}'(\xi'_j)=u'\neq\lim_{j\to\infty} \underline{\gamma}'(\xi''_j)=u''$, using 5.5 we conclude that u' and u'' are null linearly independent vectors in T_mM which define the same time-cone than (X,Y), and $\|u'\|=\|u''\|=1$. Thus, we can suppose that $u'=X(m)$ and $u''=Y(m)$, and taking necessary subsequences we also get

$\xi'_j<\xi''_j<\xi_{j+1}$, $\|\underline{\gamma}'(\xi'_j)-X(\underline{\gamma}\ (\xi'_j)\|<\frac{1}{2}$ and $\|\underline{\gamma}'(\xi'_j)-X(\underline{\gamma}\ (\xi'_j)\|<\frac{1}{2}$

By continuity we can take a sequence (ζ_j) such that:

$\xi'_j<\zeta_j<\xi''_j$ and $\|\underline{\gamma}'(\zeta_j)-X(\underline{\gamma}(\zeta_j)\|=\frac{\sqrt{2}}{2}$, and there is not a subsequence (ζ'_j) of (ζ_j) such that $(\underline{\gamma}'(\zeta'_j))$ converges to a null vector of T_mM. This contradicts 2.3 and the claim is proved.
Using the same sort of argument comparing (s_j) and (ξ_j) we can prove that $\lim_{j\to\infty} \gamma'(\xi_j)=X(m)$.∎

REMARK 5.4
Note that with the hypothesis of 5.3, if (ξ_j) is a sequence in $[0,+\infty)$ such that $(\underline{\gamma}(\xi_j))$ converges to a point $m\in M$, then $(\underline{\gamma}'(\xi_j))$ converges to X(m).

REMARK 5.5
Suppose that there is a dense orbit of X. Then by 5.1 , M is a 2-torus and all the null geodesics are dense. Thus with the hypothesis of 5.3 we see easily using 2.2 and 2.3 that $\lim^{+}\gamma$ is the whole M.

The rest of this section is devoted to proving the following main result:

THEOREM 5.6
Suppose that the Lorentzian surface M is a 2-torus which has all its null geodesics closed, and let $\gamma:[0,b)\longrightarrow M$ be a right-

inextendible and right-incomplete non null Γ-geodesic. Then either $\lim^{+}\gamma$ is the image of a smoothly closed null geodesic, or im γ is dense in M. In the last case, the autointersection point set of γ is dense in im γ.

In the rest of §5 the hypothesis of 5.3 and 5.6 are implicitly assumed

The main idea to prove the theorem is to show that either γ cuts infinitely many times each orbit of X, or there is an orbit β of X which is the frontier of the union of all orbits that cut γ.
In the first case, by 5.4 and 2.2 we have that $\lim^{+}\gamma=M$. In the second case, $\lim^{+}\gamma=\mathrm{im}\beta$.
We introduce now some basics tools.

Preliminaries 5.7

5.7.1

Since M is a 2-torus it can be represented by a quotient $M=\mathbb{R}^2/\mathbb{Z}^2$, where each element $(p,q)\in\mathbb{Z}^2$ is identified with the integer translation: $\tau_{pq}:\mathbb{R}^2\ni(x,y)\longrightarrow(x+p,y+q)\in\mathbb{R}^2$.
Note that $\mathbb{R}^2$ is canonically endowed with Lorentzian and Riemannian structures such that the canonical covering $\pi:\mathbb{R}^2\longrightarrow M$ is a local isometry. We denote by $\tilde{\Gamma}$ the Levi-Civita Lorentzian connection in $\mathbb{R}^2$. Let $\underline{\tilde{\Gamma}}$ be the connection on $\mathbb{R}^2$ projectively related with $\tilde{\Gamma}$ as $\underline{\Gamma}$ is related with Γ (see 1.1). Thus $\underline{\tilde{\Gamma}}$-geodesics are $\tilde{\Gamma}$-pregeodesics parameterized proportionally to the Riemann arc length, and their π-projections are $\underline{\Gamma}$-geodesics.
Let us denote by $\tilde{X}$ (respectively $\tilde{Y}$) the unique vector field on $\mathbb{R}^2$such that $\pi_*(\tilde{X})=X$ $(\pi_*(\tilde{Y})=Y)$. Observe that $\tilde{X}$ and $\tilde{Y}$ are complete vector fields which are invariant by $\mathbb{Z}^2$. Moreover the orbits of $\tilde{X}$ and $\tilde{Y}$ are null $\tilde{\Gamma}$-pregeodesics parameterized by Riemann arc length.

5.7.2

By topological reasons we know that two non isotopic knots in the 2-torus have non-empty intersection. Thus the orbits of X are all knots in the same isotopy class. We can translate this property to $\tilde{X}$ in the following way

LEMMA

There is $(k,\ell)\in\mathbb{Z}\times\mathbb{Z}$ with $k\geq 0$, $\ell\geq 0$ $(k,\ell)\neq(0,0)$ $\mathrm{mcd}(k,\ell)=1$ such that for each $\alpha:\mathbb{R}\longrightarrow\mathbb{R}^2$ orbit of $\tilde{X}$ we have $\tau_{k\ell}(\alpha(s))\in\mathrm{im}\ \alpha$ for all $s\in\mathbb{R}$. Moreover, if $(k',\ell')\in\mathbb{Z}^2-\{(0,0)\}$ verifies that $\tau_{k'\ell'}(\alpha(s))\in\mathrm{im}\ \alpha$ for some orbit α of $\tilde{X}$ and some $s\in\mathbb{R}$, then $k'\in\mathbb{Z}k$ and $\ell'\in\mathbb{Z}\ell$.

5.7.3

Using the lemma in 5.7.2, it is easy to prove that if α is an orbit of $\tilde{X}$ then $\mathbb{R}^2$-im α has just two connected components, D_α^+ and D_α^- which are open, non bounded and invariant by the flow of $\tilde{X}$, where D_α^+ denotes the component defined by the vector field $\tilde{Y}$ in the following way

If $p\in\mathrm{im}\ \alpha$ and β is the orbit of $\tilde{Y}$ through p (i.e $\beta(0)=p$) then $\beta(s)\in D_\alpha^+$ for $s>0$.

We obtain easily the following technical result

LEMMA

For each orbit σ of X, there are infinitely many orbits $\tilde{\sigma}$ of $\tilde{X}$ in D_α^+ which are liftings of σ. Moreover, if (α_k) is a sequence of liftings of σ with im $\alpha_j\neq$im α_k for $j\neq k$, then if $p_k\in$ im α_k, (p_k) has no accumulation points.

5.7.4

Without lossing generality we can suppose that $\pi(0,0)=\underline{\gamma}(0)$. Let $\psi:[0,+\infty)\longrightarrow\mathbb{R}^2$ be the lifting of $\underline{\gamma}$ through $(0,0)$, and α_0 the orbit of $\tilde{X}$ such that $\alpha_0(0)=(0,0)$. By 5.3, since $\underline{\gamma}'(s)$ belongs to the time-cone defined by $(X(\underline{\gamma}(s)),Y(\underline{\gamma}(s)))$ for all $s\in\mathbb{R}$, we conclude that this happens for the respective elevations and we have:

LEMMA:

There is a differentiable function $\vartheta:\mathbb{R}\longrightarrow(0,\pi/2)$ such that for all $s\in[0,+\infty)$:

$$\psi'(s)=\cos\vartheta(s)\ \tilde{X}(\psi(s))+\sin\vartheta(s)\ \tilde{Y}(\psi(s))$$

Moreover $\lim_{s\to\infty}\vartheta(s)=0.$ ■

5.7.5

Let α be an orbit of $\tilde{X}$ (im $\alpha \neq$ im α_0) intersecting to ψ in a point $\psi(\xi)$. By 5.7.4 we have $<\psi'(\xi),\tilde{Y}(\psi(\xi))>= \sin \vartheta(\xi)>0$. Using the definition of D^+_α we see that there is $\varepsilon>0$ such that for $0<s<\varepsilon$ we have: $\psi(\xi+s)\in D^+_\alpha$ and $\psi(\xi-s)\in D^-_\alpha$. Moreover, using this fact it is straightforward to show that this is the only cut point between α and ψ.. Thus we have prove the following

LEMMA

If α is a orbit of $\tilde{X}$and $\psi(\xi)\in$ im α then $\psi((\xi,+\infty])\subset D^+_\alpha$ and $\psi(([0,\xi))\subset D^-_\alpha$.■

5.7.6

We denote by C_ψ the union of all im α such that α is an orbit of $\tilde{X}$ which cuts ψ and im $\alpha\neq$im α_0. We have then:

LEMMA

If α is an orbit of $\tilde{X}$ such that im $\alpha \subset D^+_{\alpha_0}$ and $D^+_\alpha \cap C_\psi\neq\emptyset$ then α cuts ψ (thus im $\alpha \subset C_\psi$).

Proof:

Let p be a point belonging to $D^+_\alpha \cap C_\psi$ and let β be the orbit of $\tilde{X}$ through p. Thus im $\beta \subset D^+_\alpha \cap C_\psi$ and there is $\bar{s}>0$ such that $\psi(\bar{s})\in$imβ. In particular, ψ connects $\psi(0)=(0,0)\in D^-_\alpha$ with $\psi(\bar{s})\in D^+_\alpha$. Therefore ψ must cut α, and im $\alpha \subset C_\psi$.■

We conclude the preliminaries to the proof of our main theorem with the following technical result. The hypothesis remarks and lemmas of 5.7 are implicitly assumed:

PROPOSITION 5.8

C_ψ is a non empty open set contained in $D_{\alpha_0}^{\ +}$. Moreover, if $C_\psi\neq D_{\alpha_0}^{\ +}$ then the topological frontier ∂C_ψ of C_ψ is the union of im α_0 and im β where β is orbit of $\tilde{X}$ contained into $D_{\alpha_0}^{\ +}$, and if d denote the Riemannian distance in $\mathbb{R}^2$we have: $\lim_{s\to+\infty} d(\psi(s),\text{im } \beta)=0$

Proof:

In order to prove that C_ψ is open, take $p\in C_\psi$ and let α be the orbit of $\tilde{X}$ through p . This orbit intersects ψ in a point $\psi(\xi)\in\mathrm{im}\,\psi \cap \mathrm{im}\,\alpha$. Let $\tilde{F}_s$ be the flow of $\tilde{X}$. Using the transversality property of 5.7.4 we see that for $\varepsilon>0$ sufficiently small and $K>0$ sufficiently large, the set $E^{\varepsilon}_{\xi}=\{\tilde{F}_s(\psi(\xi+\zeta)): |\zeta|<\varepsilon,\ |s|<K\}$ is an open neighbourhood of p. Since $\mathrm{im}\,\psi \subset C_\psi$ and C_ψ is invariant by $\tilde{F}_s$, we conclude that E^{ε}_{ξ} is contained in C_ψ , and p is an interior point of C_ψ.

We prove now the second statement:

Obviously $\mathrm{im}\,\alpha \subset \partial\, C_\psi$ and by 5.7.5 $C_\psi \subset D^+_{\alpha_0}$. If $\partial\, C_\psi \cap D^+_{\alpha_0}=\emptyset$ then C_ψ is open and closed in $D^+_{\alpha_0}$. Since $D^+_{\alpha_0}$ is connected, we get $C_\psi=D^+_{\alpha_0}$.

Suppose now that $\partial\, C_\psi \cap D^+_{\alpha_0}$ is non empty. Thus there is a point p belonging to $\partial\, C_\psi \cap D^+_{\alpha_0}$. Let (p_k) be a sequence into C_ψ such that $\lim p_k= p$. Let β_k and β be the orbits of $\tilde{X}$ through p_k and p respectively. Using the flow $\tilde{F}_s$ of $\tilde{X}$ we see that for all $s>0$, $\tilde{F}_s(p_k)=\beta_k(s)\in C_\psi$ and $\lim_{k\to+\infty} \tilde{F}_s(p_k)=\tilde{F}_s(p)=\beta(s)$. Thus $\mathrm{im}\,\beta\subset\partial C_\psi\cap D^+_{\alpha_0}$.

In order to prove that $\partial C_\psi\cap D^+_{\alpha_0}=\mathrm{im}\,\beta$, pick now other $\bar{p}\in\partial C_\psi\cap D^+_{\alpha_0}$, and let $(\bar{p}_k) \subset C_\psi$ be such that $\lim p_k= p$, .Let $\bar{\beta}_k$ and $\bar{\beta}$ be the orbits of $\tilde{X}$ through $\bar{p}_k$ and $\bar{p}$ respectively. We prove that $\mathrm{im}\,\bar{\beta} =\mathrm{im}\,\beta$, showing that it is not possible that $\mathrm{im}\,\bar{\beta} \subset D^+_\beta$ or $\mathrm{im}\,\bar{\beta} \subset D^-_\beta$.

In fact if $\mathrm{im}\,\bar{\beta}\subset D^+_\beta$, since D^+_β is open and $\bar{p}\in D^+_\beta$ there is k such that $\bar{p}_k\in D^+_\beta$, and $\mathrm{im}\,\bar{\beta}_k \subset D^+_\beta\cap C_\psi$ .We conclude by 5.7.6 that $\mathrm{im}\,\beta\subset C_\psi$. Since C_ψ is open, this contradicts $\mathrm{im}\,\beta \subset \partial\, C_\psi$.

If $\mathrm{im}\,\bar{\beta} \subset D^-_\beta$ then $\mathrm{im}\,\beta \subset D^+_{\bar{\beta}}$ and by symmetry this is also contradictory.

Finally to prove that $\lim_{s\to+\infty} d(\psi(s),\mathrm{im}\,\beta)=0$, it is sufficient to show that if (s_k) is a sequence in $[0,+\infty)$ such that $\lim_{k\to\infty} s_k= +\infty$ and $s_k<s_{k+1}$, then there is a subsequence (s'_k) with $\lim_{k\to\infty} d(\psi(s'_k),\mathrm{im}\beta)=0$

Let (s_k) be such a sequence. If β_k is the orbit of $\tilde{X}$ through $\psi(s_k)$, then for all α orbit of $\tilde{X}$ with $\mathrm{im}\,\alpha \subset C_\psi$ there is N such that $\mathrm{im}\beta_k\subset D^+_\alpha$ for $k\geq N$ (by 5.7.5 and 5.7.3 it is is sufficient to take N such that $s_N>s$, where $\psi(s)$ is the cut point of ψ and α). Using this fact we see easily that if Σ_0 is the straight line in $\mathbb{R}^2$ orthogonal to β through $p=\beta(0)$, the orbits β_k cut Σ_0 transversally for k sufficiently large. Also if p_k is such intersection then $\lim p_k=p$. We can suppose (taking a suitable reparameterization of orbits β_k), that $\beta_k(0)=p_k$ and $\beta_k(\zeta_k)=\psi(s_k)$for some $\zeta_k>0$.

Let k and ℓ be the integers obtained in 5.7.2, τ_n the translation in $\mathbb{R}^2$ defined by the vector $(nk,n\ell)$ for $n\in\mathbb{Z}$ and $\Sigma_1=\tau_1(\Sigma_0)$. Since $\tau_1(p_k)=\tau_1(\beta_k(0))\in\text{im }\beta_k$ and $\tau_1(p)\in\text{im }\beta$ there are $L_k>0$ and $L>0$ such that $\tau_1(p_k)=\beta_k(L_k)$and $\tau_1(p)=\beta(L)$. It is straightforward to see that $\lim_{k\to\infty} L_k=L$. Using again 5.7.2 we can take $\xi_k\in[0,L_k]$ such that $\tau_n(\beta_k(\xi_k))=\beta_k(\zeta_k)$ for some integer n, and we have that $d(\beta_k(\xi_k),\text{im }\beta)=d(\beta_k(\zeta_k),\text{im }\beta)=d(\psi(s_k),\text{ im }\beta)$. Finally since $\lim_{k\to\infty} L_k=L<+\infty$, there is a subsequence (ξ_{k_j}) of (ξ_k) such that

$\lim_{j\to\infty} \xi_{k_j}=\xi<+\infty$. Hence we have $\lim_{j\to\infty} \beta_{k_j}(\xi_{k_j})=\lim_{j\to\infty} \underline{\exp}\ (\xi_{k_j}\beta'_{k_j}(0))=$

$=\underline{\exp}\ (\xi\beta'(0))=\beta(\xi)$, and $\lim_{j\to\infty} d(\psi(s_{k_j}),\text{im }\beta)=d(\beta(\xi),\text{im }\beta)=0$. ∎

PROOF OF THEOREM 5.6

Following the argument of 5.8, there are two possibilities:

a) If $C_\psi\neq D^+_{\alpha_0}$, let β be the orbit of $\tilde{X}$ obtained in 5.8, such that $\lim_{s\to+\infty} d(\psi(s),\text{im }\beta)=0$. Denote $\underline{\sigma}=\pi.\beta$. To prove that $\lim^+\gamma=\text{im }\underline{\sigma}$ it is sufficient (by 2.2) to show that $\lim^+\gamma\subset\text{im }\underline{\sigma}$:
Suppose $p\in\lim^+\gamma$, and let (s_k) be a sequence in $[0,+\infty)$ such that $\lim \underline{\gamma}(s_k)=p$. By the construction of 5.7.1, we see that there is a positive constant $\varepsilon_0>0$, such that π induces isometry from Riemannian ball $\tilde{B}(x,\varepsilon)$ in $\mathbb{R}^2$ onto the Riemannian ball $B(\pi(x),\varepsilon)$ in M, for all $x\in\mathbb{R}^2$ and all ε such that $0<\varepsilon<\varepsilon_0$.
Since $\lim_{k\to\infty} d(\psi(s_k),\text{im }\beta)=0$, there is N a positive integer such that $d(\psi(s_k),\text{im }\beta)<\varepsilon_0$ for k>N. Hence for k>N we have $d(\psi(s_k),\text{im}.\beta)=d(\underline{\gamma}(s_k),\text{im }\sigma)<\varepsilon_0$, and taking limits we get $0=\lim d(\underline{\gamma}(s_k),\text{im }\sigma)=\lim d(p,\text{im }\sigma)$. Since im σ is compact we conclude that $p\in\text{im }\sigma$.

b) If $C_\psi=D^+_{\alpha_0}$ then (by Lemma of 5.7.5) ψ intersects each orbit of $\tilde{X}$

which is contained into $D^+_{\alpha_0}$ only once. Moreover, by 5.7.3 if $\underline{\sigma}$ is a orbit of X there are infinitely many liftings $\tilde{\underline{\sigma}}$ of $\underline{\sigma}$ such that im $\tilde{\underline{\sigma}} \subset D^+_{\alpha_0}$, and we can find a sequence $(s_k)\subset[0,+\infty)$ with $s_k<s_{k+1}$ and $\psi(s_k)$ belongs to a lifting α_k of $\underline{\sigma}$ where im $\alpha_k\neq$im α_j if $k\neq j$.
We claim that lim $s_k=+\infty$. In fact, if the opposite happened, there would be s=lim s_k and points $p_k=\psi(s_k)\in$im α_k giving a sequence (p_k) having $\psi(s)$ as accumulation point. This would contradicts 5.7.3.
Projecting through π we see that $\underline{\gamma}(s_k)=\pi.\psi(s_k)\in\pi(\text{im } \alpha_k)=\text{im } \underline{\sigma}$.
Since im $\underline{\sigma}$ is closed, taking if necessary a subsequence, we can suppose that there is $\lim_{k\to\infty} \underline{\gamma}(s_k)=p\in\text{im } \underline{\sigma}$ and $\lim_{k\to\infty} \underline{\gamma}(s_k)=u\in T_pM$.
By the claim $p\in\lim^+\underline{\gamma}$, and by 5.3 u=X(p) is tangent to $\underline{\sigma}$.
Using now 2.2 we have that im $\underline{\sigma}$ is contained in $\lim^+\underline{\gamma}$. Since this occurs for each orbit $\underline{\sigma}$ of X we conclude that $\lim^+\underline{\gamma}=M$.
Moreover the auto intersection points set of γ is dense in im γ.
In fact for $\xi>0$ the orbit σ of X through $\underline{\gamma}(\xi)$ is transversal to $\underline{\gamma}$, using now the above argument we see that there is $(s_k)\longrightarrow+\infty$ with $\lim_{k\to\infty} \underline{\gamma}(s_k)=\underline{\gamma}(\xi)$ and $\lim_{k\to\infty} \underline{\gamma}'(s_k)=\underline{\sigma}'(0)$. Thus we can find $\eta>0$ such that $\underline{\gamma}((s_k-\eta,s_k+\eta))$ cuts $\underline{\gamma}$ transversally for k sufficiently large, and such intersection points converge to $\underline{\gamma}(\xi)$.■

§ 6 GRADED COMPLETENESS

We end this paper explaining some thinking on a possible "graduation" of the completeness concept, which put our results into a new perspective.
The μ-completeness concept has been partially motivated in order to give an adequate formulation of a completeness result (see 6.3 below). This result arises from the analysis of a natural reformulation of the following solved question:
Are there some logical relation between null completeness and timelike or spacelike completeness in a Lorentzian manifold ?

The negative answer to this question comes from counter-examples given by Kundt [3], Geroch [2] and Beem [1]. However, in [4] the author has proved that for locally symmetric Lorentz manifolds, they are logically equivalent. This suggests the following alternative vague question:
Which sort of geometrical conditions must be imposed to the class

of Lorentzian manifolds, inducing some logical relations between the three sorts of geodesics completeness ?

The main theorem 4.5 shows that under certain geometrical conditions there are some logical relations between the completeness of two given geodesics. We exploit next the argument to give a geodesic completeness theorem (which relates null and non-null completeness) for a certain class of Lorentzian manifolds. The hypothesis and general notations of §1 and §2 are implicitly assumed.

PROPOSITION 6.1

Let $\gamma:[0,b)\longrightarrow M$ be a right-inextendible Γ-geodesic, and let (s_i) be a sequence into $[0,+\infty)$ such that $\lim_{i\to\infty} s_i=+\infty$, $\lim_{i\to\infty} \underline{\gamma}(s_i)=p$ and $\lim_{i\to\infty} \underline{\gamma}'(s_i)=u$. We write $\sigma=\gamma_u$, $u_i=\underline{\gamma}'(s_i)$ $\underline{\gamma}_i(\xi)=\underline{\gamma}(s_i+\xi)$ (for $\xi>0$) Given $L>0$ we define for $\xi\in[0,L]$ $\lambda_\sigma(\xi)=\lambda(\underline{\sigma}'(\xi))$ $\lambda_i(\xi)=\lambda(\underline{\gamma}_i'(\xi))$ and $\Lambda_i(\xi)=\int_0^\xi \lambda_i(s)ds$ (Λ_σ and Λ_γ are as in 1.2). Then we have that (λ_i) converges uniformly to λ_σ in $[0,L]$. In particular (Λ_i) converges uniformly to Λ_σ and $\lim_{i\to\infty} \int_0^L e^{\Lambda_i(s)}ds = \int_0^L e^{\Lambda_\sigma(s)}ds$

Proof:

Note first that for each $\xi\in[0,L]$ we have $\underline{\gamma}_i(\xi)=\underline{\exp}(\xi u_i)$ and $\underline{\gamma}_i'(\xi)=\frac{d}{ds}\big|_{s=\xi}\ \underline{\exp}(su_i)$.

In order to prove the uniform convergence of (λ_i) we denote by $K_i=\sup\ \{|\lambda_\sigma(\xi)-\lambda_i(\xi)|:\xi\in[0,1]\}$. Let $\xi_i\in[0,L]$ be such that $|\lambda_\sigma(\xi)-\lambda_i(\xi)|\ =K_i$. By simplicity we suppose that (ξ_i) is a convergent sequence, and let ξ be the limit of (ξ_i) (otherwise we can take a suitable subsequence). Thus $\lim_{i\to\infty} \xi_i u_i=\xi u$ and we have

$$\lim_{i\to\infty} \underline{\gamma}_i(\xi_i)=\lim_{i\to\infty} \underline{\exp}(\xi_i u_i)=\underline{\exp}\ (\xi u)=\underline{\sigma}(\xi)$$

$$\lim_{i\to\infty} \underline{\gamma}_i'(\xi_i)=\lim_{i\to\infty} \frac{d}{ds}\Big|_{s=\xi_i} \underline{\exp}(su_i)= \frac{d}{ds}\Big|_{s=\xi} \underline{\exp}(su)=\underline{\sigma}'(\xi)$$

Since λ is continuous we get $\lim \lambda_i(\xi_i)=\lambda_\sigma(\xi)$, and we conclude

$$K_i=|\lambda_\sigma(\xi_i)-\lambda_i(\xi_i)|\le |\lambda_\sigma(\xi_i)-\lambda_\sigma(\xi)| + |\lambda_\sigma(\xi)-\lambda_i(\xi_i)| \xrightarrow[i\to\infty]{} 0$$

Using a slight modification of preceding argument, we can prove that for each subsequence (K_i') of (K_i) there is a subsequence of (K_i') which converges to zero. Thus $\lim_{i\to\infty} K_i=0$, and (λ_i) converges uniformly to λ_σ in $[0,L]$. The other statements are now obvious.■

COROLLARY 6.2

With the same conditions as in 6.1, assuming $b<+\infty$ if σ is right-complete then we have $\lim (b-t_i)\,\|\gamma'(t_i)\|=0$, where $t_i=$ $=t_\gamma(s_i)$ (See 1.2).

Proof:

Let us consider $\gamma_i=[0,b-t_i)\longrightarrow M$ such that $\gamma_i(t)=\gamma(t+t_i)$. Thus γ_i is a right-incomplete Γ-geodesic, and we have $\Lambda_{\gamma_i}(\xi)=\int_0^\xi \lambda_i(s)ds$.

Taking $b_i(\xi)=\int_0^\xi e^{\Lambda_i(s)}ds$ and using 1.2, since $\gamma_i'(0)=\gamma'(t_i)$ we have

$t_{\gamma_i}(\xi)=\frac{1}{\|\gamma'(t_i)\|}\,b_i(\xi)$, and $\lim_{\xi\to+\infty} t_{\gamma_i}(\xi)=b-t_i$. Thus if $b_i=(b-t_i)\|\gamma'(t_i)\|$, we get $\lim_{\xi\to+\infty} b_i(\xi)=b_i$.

To end the proof it is sufficient to prove that there is no a bounded subsequence of (b_i). Suppose that there where (b_{i_k}) and $H>0$ such that $b_{i_k}\le H$ for all k. Since σ is right-complete, by 1.2 we would have $\int_0^{+\infty} e^{\Lambda_\sigma(s)}ds=+\infty$ and $L>0$ would exist such that $\int_0^L e^{\Lambda_\sigma(s)}ds>H$

Using now 6.1 we get for k sufficiently large $b_{i_k}>b_{i_k}(L)>H$, and this is a contradiction.■

COROLLARY 6.3

Suppose that M is a Lorentzian compact manifold which has all their null geodesics completes (i.e M is null-complete).

If $\gamma:[0,b)\longrightarrow M$ is a non null right-inextendible and right-incomplete geodesic then $\underline{\lim}_{t\to b}(b-t)\|\gamma'(t)\|=+\infty$

Proof

Let (t_i) be a sequence in $[0,b)$ such that $\lim_{i\to\infty} t_i=b$, and let s_i be equal to $s_\gamma(t_i)$. Using 2.2 and 2.3 we obtain a subsequence (s_{i_k}) such that $u=\lim_{k\to\infty}\underline{\gamma}'(s_{i_k})$ is a null vector and $\sigma=\gamma_u$ is a null geodesic. Since σ is complete by hypothesis, using 6.2 we conclude that $\lim_{k\to\infty} (b-t_{i_k})\|\gamma'(t_{i_k})\|=+\infty$. Since this happens for each sequence (t_i) into $[0,b)$ such that $\lim_{i\to\infty} t_i=b$, we have $\lim_{t\to b} (b-t)\|\gamma'(t)\|=+\infty$ ∎

The property $\lim_{t\to b} (b-t)\|\gamma'(t)\|=+\infty$ used in the hypothesis of the preceding corollary depends initially on the auxiliary Riemannian metric. Next we establish a topological reformulation of this property in order to find the graded completeness concept, and give consistency to the result above.
We recall previously the following general definition

DEFINITION 6.4
Given X and Y topological spaces, we consider D a subset of X, and $a\in X$ an accumulation point of D. If $\psi: D\longrightarrow Y$ is a continuous function, we say that $\lim_{x\to a} \psi(x)=\infty$, if for every compact set K of Y, there is a neighborhood V of a, such that $\psi(x)\notin K$ for all $x\in V\cap D$. Otherwise we say that $\lim_{x\to a} \psi(x)\neq\infty$.
Since TM has a standard topological structure, the following definition is independent of the auxiliary Riemannian metric:

DEFINITION 6.5 : Graded Completeness
Let μ be a real number such that $0\le\mu\le1$, and let γ be a Γ-geodesic.

a) The geodesic γ is called right μ-complete provided that for every bounded interval (a,b) where γ is defined, we have either, $\lim_{t\to b} (b-t)^{-\log\mu}\ \gamma'(t)\neq\infty$ if $\mu\neq0$, or $\lim_{t\to b} \gamma(t)\neq\infty$ if $\mu=0$.

   Analogously we can define the left μ-completeness property for γ.

b) The geodesic γ is called μ-complete if it is right and left μ-complete.

c) Finally we say that the connection Γ is μ-complete if all their geodesics are μ-completes.

REMARKS 6.6
1.- Completeness is equivalent to 1-completeness. In fact if $\gamma: [0,b) \longrightarrow M$ is a right inextensible and incomplete Γ-geodesic, then by 1.3 $\lim \|\gamma'(t)\|=+\infty$ and γ is not right 1-complete. The other implication is trivial.

2.- Ovbiously, μ'-completeness implies μ-completeness, if $0\le\mu<\mu'\le 1$ This occurs in particular when $\mu=0$, and suggests that the definition of 0-completeness is natural.

3.-In Riemannian manifolds 0-completeness and global completeness are equivalent.

4.- Note that if M is compact, every connection Γ in M is 0-complete. In fact the compact property for M could be replaced through the paper, by the 0-completeness property.

5.- If $\gamma:[0,b)\longrightarrow M$ is a right inextendible Γ-geodesic, and $\mu\in(0,1]$, the following statements are equivalents:

a) γ is right μ-complete.

b) γ is right 0-complete and $\lim_{t\to b}(b-t)^{-\log\mu}\|\gamma'(t)\|\neq\infty$.

Using the last two remarks and corollary 6.3 we have the following completeness theorem

THEOREM 6.7

Suppose that M is a Lorentzian e^{-1}-complete manifold. Then, null completeness implies global geodesic completeness.■

The next result shows that the e^{-1}-completeness is not a very strange property for incomplete geodesics of a 0-complete connection

PROPOSITION 6.8

Let $\gamma:[0,b)\longrightarrow M$ be a right-inextendible Γ-geodesic with $b<+\infty$. Then γ is right e^{-1}-complete iff a sequence (s_i) exists into $[0,+\infty)$ such that $\lim s_i=+\infty$ and there is $\lim_{i\to\infty}\dfrac{\int_{s_i}^{+\infty}e^{\Lambda_\gamma(s)}ds}{e^{\Lambda_\gamma(s_i)}}<+\infty$.

In particular γ is e^{-1}-complete, if $\lim_{s\to\infty}\lambda_\gamma(s)\neq 0$ exists.

Moreover, if γ is smoothly closed then γ is right 1-complete, but it is not μ-complete for $\mu\in(e^{-1},1]$.

Proof

The geodesic γ is right e^{-1}-complete iff a sequence (t_i) into $[0,b)$ exists, such that $\lim t_i = b$, and $\lim (b-t_i)\|\gamma'(t_i)\| \neq \infty$.

Using 1.2 we get for $t\in[0,b)$, $(b-t)\|\gamma'(t)\| = \dfrac{\displaystyle\int_{s_\gamma(t)}^{+\infty} e^{\Lambda_\gamma(s)}\,ds}{e^{\Lambda_\gamma(s_\gamma(t))}}$

We can be prove the first statement taking $s_i = s_\gamma(t_i)$.

Moreover if there is $\lim \lambda_\gamma(s) = \lambda_\infty \neq 0$, then by l'Hopital rule we have

$$\lim_{s\to\infty} \frac{\displaystyle\int_s^{+\infty} e^{\Lambda_\gamma(\xi)}\,d\xi}{e^{\Lambda_\gamma(s)}} = \lim_{s\to\infty} \frac{-\,e^{\Lambda_\gamma(s)}}{\lambda_\gamma(s)\,e^{\Lambda_\gamma(s)}} = \frac{-1}{\lambda_\infty} \neq \infty .$$

Finally, if the geodesic γ is smoothly closed, let L be the Riemannian length of γ. We consider

$$s_i = i.L \ , \quad J = \int_0^L e^{\Lambda_\gamma(s)}\,ds \ \text{ and } \ c = e^{-\Lambda_\gamma(L)} > 1 \ \text{(see proposition 4.10).}$$

Since λ_γ is periodic with periode L we get $\Lambda_\gamma(s_i) = i\,\Lambda_\gamma(L)$, and if $s\in[s_k, s_{k+1}]$ then $\Lambda_\gamma(s) = k\Lambda_\gamma(L) + \Lambda_\gamma(s-s_k)$, therefore

$$\int_{s_k}^{s_{k+1}} e^{\Lambda_\gamma(s)}\,ds = e^{k\Lambda_\gamma(L)} \int_{s_k}^{s_{k+1}} e^{\Lambda_\gamma(s-kL)}\,ds = e^{k\Lambda_\gamma(L)} \int_0^L e^{\Lambda_\gamma(s)}\,ds = \frac{J}{c^k}$$

and

$$\lim_{i\to\infty} \frac{\displaystyle\int_{s_i}^{+\infty} e^{\Lambda_\gamma(s)}\,ds}{e^{\Lambda_\gamma(s_i)}} = \lim_{i\to\infty} c^i \sum_{k=i}^{\infty} \int_{s_k}^{s_{k+1}} e^{\Lambda_\gamma(s)}\,ds =$$

$= \lim_{i\to\infty} Jc^i \sum_{k=i}^{\infty} \frac{1}{c^k} = J\,\frac{c}{c-1}$. This proves that γ is right e^{-1}-complete

Since $\lim_{i\to\infty} Jc^i \left(\sum_{k=i}^{\infty} \frac{1}{c^k} \right)^{\nu} = \frac{c-1}{c^\nu} \lim_{i\to\infty} c^{i(1-\nu)} = \infty$ if $\nu<1$, a slight

modification the argument before proves that the smoothly closed geodesic γ is not μ-complete for $e^{-1}<\mu\leq 1$. ∎

FINAL REMARKS AND OPEN CUESTIONS

Theorem 6.7 gives a criteria for non null completeness in a Lorentz manifold in the following way

A Lorentz manifold is non-null complete if there is a non-null geodesic which is not e^{-1}-complete. This sugest the following open question

QUESTION 1

Could e^{-1}-complete condition be replaced by the 0-complete condition in Theorem 6.7 ?.

An affirmative answer to this question mean that null-completeness implies global completeness in compact Lorentzian manifolds.

On the other hand it is natural to ask if the μ-completeness concept, $\mu\in[0,1]$ is available for every μ.

QUESTION 2

Pick up a number μ such that $0\leq\mu<1$, are there some Lorentzian manifolds which are μ-complete, but which are not μ'-complete for $\mu<\mu'<1$?

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