UNIVERSIDAD
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MADRID

FACULTAD DE CIENCIAS MATEMÁTICAS
Departamento de Geometría y Topología

GEOMETRIC STRUCTURES AND CAUSALITY IN THE SPACE OF LIGHT RAYS OF A SPACETIME

(Estructuras geométricas y causalidad en el espacio de rayos de luz de un espacio–tiempo)

Memoria para optar al grado de Doctor presentada por:

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Madrid, Enero de 2016

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PhD THESIS

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A Gemma

A Jaime

Broadly speaking, "geometry", after all, means any branch of mathematics in which pictorial representations provide powerful aids to one's mathematical intuition.

(Roger Penrose, The complex geometry of the natural world)

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Resumen

Título

Estructuras geométricas y causalidad en el espacio de rayos de luz de un espacio–tiempo.

Introducción

Inspirado por algunos de los más grandes matemáticos del siglo XIX y principios del XX como Felix Klein, Julius Plücker, Arthur Cayley and Sophus Lie entre otros, R. Penrose desarrolló, en las décadas de 1960 y 1970, el *programa twistor* [56], [58]. Esta teoría está motivada en la obtención de un formalismo que permita unir la Relatividad general con la Física cuántica. Los espacios Twistor son estructuras complejas que contienen información del espacio–tiempo de Minkowski 4–dimensional de modo que las geodésicas luz pueden verse como elementos básicos (puntos) de esta geometría compleja. Así, a partir de este nuevo punto de vista, surje la siguiente idea: los conjuntos de todas las geodésicas luz que pasan por diferentes puntos, son distintos, o de forma equivalente, si dos observadores contemplan exactamente el mismo cielo, entonces están en el mismo punto del espacio–tiempo. Por tanto, en el espacio–tiempo de Minkowski, el conjunto de geodésicas luz que pasan por un determinado punto, caracteriza dicho punto.

A finales de la década de 1980, R. Low comenzó a trabajar en esta idea aplicándola más tarde a espacio–tiempos generales, no necesariamente minkowskianos, y sobre una variedad diferenciable real. En su trabajo [39], [41], [40], [42], [44], [45] el autor estudia la topología y la geometría del espacio de geodésicas luz (desparametrizadas) y ofrece condiciones para que éstas tengan buenas propiedades. Además, apunta la existencia de una estructura de contacto en el espacio de geodésicas luz y observa que la estructura causal del espacio–tiempo también se halla codificada en dicho espacio $\mathcal{N}$ de geodésicas luz, o rayos de luz como los llamaremos de ahora en adelante. Una structura importante contenida en $\mathcal{N}$ es la familia de cielos: el conjunto de todos los rayos de luz que pasan por x se denomina cielo de x y se denota como X. En este trabajo, llamaremos Σ al conjunto de cielos. De acuerdo con el teorema de Malament-Hawking [26], [46], teniendo en cuenta los progresos realizados en [36] y [54], la estructura causal está relacionada con las estructuras topológica, diferenciable y métrica, así como con la dimensión: si hay una biyección causal f entre dos espacio–tiempos de dimensiones $n_1, n_2 > 2$ que verifican las condiciones de distinción de futuro y de pasado, entonces $n_1 = n_2$ y los espacio–tiempos son conformemente isométricos. Así, la búsqueda de la estructura causal de M escondida en el espacio de rayos de luz $\mathcal{N}$ podría ser importante para determinar la geometría y la topología de la correspondiente clase conforme de espacio–tiempos. De nuevo, inspirado por la geometría de twistors, Low en [41], [43], [44] seguido de Chernov–Rudyak en [17], Chernov–Kinlaw–Sadykov en [14], Chernov–Nemirovski en [15], [16], Natario en [50] y Natario–Tod en [51], entre otros, han estudiado las relaciones causales mediante un tipo de enlazamiento entre cielos de la variedad de contacto $\mathcal{N}$.

Otro tema es la reconstrucción de la variedad Lorentz conforme M mediante la información contenida en $\mathcal{N}$. En [42], Low describe cómo recuperar M en el caso globalmente hyperbólico: la intersección entre el cono de luz en x y una superficie de Cauchy puede identificarse con el cielo X y, por lo tanto, también con el suceso $x \in M$. Pero aparece un problema cuando existe un par de puntos $p, q \in M$ verificando que todos los rayos de luz que pasan por p también pasan por q. En este caso, se dice que M no separa cielos. Cuando aparece esta propiedad, ésta no permite hacer una identificación adecuada entre M y Σ, por lo que se supone que M separa cielos. E incluso más, también hay dificultades cuando existe un entorno abierto V de p tal que para todo abierto U con $p \in U \subset V$, existe un suceso $q \notin V$ tal que todos los rayos de luz que pasan por q entran en U. A esto se le conoce como la propiedad de reenfocamiento en p y que ha sido ampliamente estudiada por Kinlaw en [31].

Objetivos y resultados

En este punto, los dos objetivos principales de este trabajo son:

- caracterizar la estructura causal de las variedades Lorentz conforme M en términos de sus espacios de rayos de luz $\mathcal{N}$, y
- establecer si es posible la reconstrucción de la variedad Lorentz conforme M a partir de sus correspondientes espacios de rayos de luz $\mathcal{N}$.

Para conseguir estos propósitos, buscaremos otros objetivos secundarios tales como:

- determinar si la hipótesis de no–reenfocamiento es necesaria para separar cielos,
- recopilar y ordenar los resultados de la literatura sobre la construcción de los espacios de rayos de luz y sus estructuras geométricas, añadiendo demostraciones detalladas para hacer que esta memoria sea autocontenida,
- contribuir con algún avance a la construcción de la frontera propuesta por Low, e
- ilustrar los resultados teóricos con ejemplos de espacio–tiempos particulares.

Para alcanzar esta meta, procedemos de la siguiente manera. En primer lugar, en el capítulo 1, exponemos los antecedentes necesarios que se utilizarán en el resto de este trabajo, es decir, definiciones básicas sobre geometría diferencial en la sección 1.1 y una breve introducción sobre causalidad en la sección 1.2.

Posteriormente, el capítulo 2 se dedica a las estructuras topológica y diferenciable del espacio de rayos de luz $\mathcal{N}$, además del estudio de su fibrado tangente $T\mathcal{N}$. La mayor parte de los resultados de las secciones 2.1 y 2.2 ya son conocidos aunque están dispersos en la literatura. Los formalizamos y ordenamos comenzando con la definición de espacio de rayos de luz $\mathcal{N}$ de una variedad (Lorentz) conforme M hasta la descripción de su estructura topológica y diferenciable. Además, se construyen sistemas coordenados en $\mathcal{N}$ a partir de la restricción a conjuntos adecuados de coordenadas en TM, y la proposición 2.2.14 aporta condiciones en M para que $\mathcal{N}$ sea Hausdorff, de hecho, establece que si M es fuertemente causal y pseudoconvexo para los rayos de luz, entonces $\mathcal{N}$ es Hausdorff. De esta manera, supondremos que M verifica tales condiciones. En la sección 2.3, caracterizamos los vectores tangentes de $T_\gamma\mathcal{N}$ como campos de Jacobi de variaciones infinitesimales compuestas de rayos de luz para un rayo de luz dado $\gamma \in \mathcal{N}$. Esto permite interpretar $v \in T_\gamma\mathcal{N}$ en términos de elementos de M de manera que nos será útil. De nuevo, ofrecemos los detalles omitidos en la literatura necesarios para obtener dicha caracterización, siendo la proposición 2.3.15, el resultado principal de esta sección.

La estructura de contacto $\mathcal{H} \subset T\mathcal{N}$ de $\mathcal{N}$ se construye en las secciones 2.4 y 2.5 mediante tres formas distintas. Primero, en la sección 2.4.2, construimos $\mathcal{H}$ pasando el núcleo de la 1–forma canónica $\theta \in \mathfrak{X}^*(T^*M)$ de la variedad simpléctica T^*M al fibrado tangente TM mediante la transformación de Legendre. Después, restringimos la distribución de hiperplanos resultante, como se hizo previamente para obtener las coordenadas en $\mathcal{N}$, dando lugar a una estructura de contacto $\mathcal{N}$, como se muestra en la proposición 2.4.13. En la sección 2.5.1, llevamos a cabo un procedimiento más elegante, pero equivalente al anterior: la reducción coisotrópica. El teorema 2.5.5 determina el mecanismo de reducción coisotrópica que se utiliza en los teoremas 2.5.6 y 2.5.7 para obtener $\mathcal{H}$. El tercer procedimiento es la reducción de Marsden–Weinstein. Se puede observar que no es la manera más sencilla de construir $\mathcal{H}$ ya que es necesario utilizar resultados potentes que necesitan verificar un número mayor de hipótesis, pero se incluye como sección 2.5.3 aportando así los detalles que faltan en [30] sobre este tema.

En el capítulo 3, nos ocupamos del estudiar el espacio de cielos Σ. El cielo $X \in \Sigma$ de un punto $x \in M$ es el conjunto de todos los rayos de luz que pasan por x, dando lugar a una subvariedad legendriana de $\mathcal{N}$. De hecho, esta propiedad para todo X caracteriza la estructura de contacto de $\mathcal{N}$, y se demuestra que depende únicamente de la estructura conforme de M. La aplicación cielo $S : M \to \Sigma$ se puede definir como $S(x) = X$, y a lo largo de todo este trabajo supondremos que S es inyectiva, o en otras palabras, M separa cielos. Esta condición es necesaria a la hora de identificar sin ambigüedades M y Σ. Después de dar, en la sección 3.1, un tipo espacial de coordenadas no canónicas en $T\mathcal{N}$, definimos la topología de Low en Σ. Con esta topología, la aplicación cielo S es continua (proposición 3.2.4) y, suponiendo que no hay reenfocamiento en M, también es abierta (proposición 3.2.5). Por lo tanto, como se enuncia en el corolario 3.2.6, S es un homeomorfismo con la hipótesis adicional de ausencia de reenfocamiento. Un resultado importante de esta sección 3.2 es el teorema 3.2.8: el subconjunto $\widehat{\Sigma} \subset T\mathcal{N}$ de vectores celestiales, esto es vectores tangentes a cielos de Σ, es localmente una subvariedad regular de $T\mathcal{N}$. Este teorema será fundamental para demostrar otros resultados posteriores.

En la sección 3.3 estudiamos dos tipos de curvas: curvas celestiales en $\mathcal{N}$ y curvas luz retorcidas en M. Las curvas celestiales son curvas en $\mathcal{N}$ tales que su vector tangente es un vector celestial. La proposición 3.3.2 muestra que para cualquier curva celestial $\Gamma \subset \mathcal{N}$ existe una curva $\mu \subset M$, llamada estela de Γ, tal que $\Gamma'(s) \in T_{\Gamma(s)}S(\mu(s))$ y $\mu'(s)$ es proporcional al vector tangente del rayo de luz $\Gamma(s)$ en el punto $\mu(s)$ siempre que μ sea regular. El lema 3.3.7 nos dice que la estela μ de una curva celestial Γ es una curva luz retorcida a trozos, es decir, una curva que no es geodésica en ningún punto y μ' es luz en los puntos en los que μ es regular, y recíprocamente, cualquier curva luz retorcida μ define una curva celestial Γ tal que μ es su estela. También, como herramienta que utilizaremos más adelante, en el corolario 3.3.12 de la sección 3.3.2, se demuestra que cualquier par de puntos relacionados temporalmente en M se pueden conectar mediante una curva luz retorcida a trozos orientada en el tiempo.

Se caracteriza la estructura causal de M en términos de la geometría de $\mathcal{N}$ en la sección 3.4. En una variedad de contacto $(Y, \mathcal{H})$ con estructura de contacto $\mathcal{H} = \ker\alpha$ donde $\alpha \in T^*Y$, una familia diferenciable $\{\Lambda_s\}_{s\in[0,1]}$ de subvariedades legendrianas se denomina isotopía legendriana. Puede ser descrita mediante una parametrización $F : \Lambda_0 \times [0,1] \to Y$ que verifique $F(\Lambda_0 \times \{s\}) = \Lambda_s \subset Y$ donde $s \in [0,1]$. Se dice que una parametrización F de una isotopía legendriana es no negativa si $(F^*\alpha)\left(\frac{\partial}{\partial s}\right) \geq 0$. El lema 3.4.3 asegura que la no negatividad de una isotopía legendriana es independiente de su parametrización. Además, el corolario 3.4.8 nos da una relación entre la estructura causal de M y las isotopías legendrianas con signo en $\mathcal{N}$: una isotopía legendriana de cielos $\{S(\mu(s))\}_{s\in[0,1]}$ es no negativa si y sólo si la curva $\mu : [0,1] \to M$ es causal dirigida hacia el pasado. Así, obtenemos una descripción de la estructura causal de M en función de la geometría de $\mathcal{N}$, luego la información causal está codificada en la existencia de isotopías legendrianas de cielos que unen dos cielos dados.

La estructura diferenciable de Σ se estudia en la sección 3.5. Después de dar la definición de una nueva topología en Σ, llamada la topología de los conjuntos regulares, en el corolario 3.5.5 demostramos que, en este caso, la aplicación cielo $S : M \to \Sigma$ es un homeomorfismo. La estructura diferenciable de Σ se determina en el corolario 3.5.6: los conjuntos regulares constituyen una base de la topología de Low de Σ, y existe una única estructura diferenciable en Σ compatible con la topología de los abiertos regulares que hace de $S : M \to \Sigma$ un difeomorfismo. Los resultados anteriores se obtienen sin asumir la hipótesis de no reenfocamiento en M, y de acuerdo con el teorema 3.5.8 y el corolario 3.5.9, si M es una variedad Lorentz conforme fuertemente causal, pseudoconvexa para los rayos de luz, que separa cielos y Σ está dotada de la topología de Low, entonces la aplicación cielo $S : M \to \Sigma$ es un homeomorfismo y no hay reenfocamiento en M.

En las secciones 3.6 y 3.7 establecemos condiciones para la reconstrucción de las variedades Lorentz conforme $(M, \mathcal{C})$ a partir de sus espacios de rayos de luz $\mathcal{N}$. Dada una variedad conforme fuertemente causal $(M, \mathcal{C})$ tal que $(\mathcal{N}, \Sigma)$ es el correspondiente par de espacios de rayos de luz y de cielos, decimos que $(M, \mathcal{C})$ es recuperable si para cada par $\left(\overline{\mathcal{N}}, \overline{\Sigma}\right)$ correspondiente a otra variedad fuertemente causal $\left(\overline{M}, \overline{\mathcal{C}}\right)$, con un difeomorfismo $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ tal que $\phi(X) \in \overline{\Sigma}$ para cualquier $X \in \Sigma$, entonces la aplicación $\varphi = \overline{S}^{-1} \circ \phi \circ S : M \to \overline{M}$ es un difeomorfismo conforme sobre su imagen. Demostramos, en el teorema 3.6.3, que una variedad conforme M fuertemente causal, pseudoconvexa para rayos de luz y que separa cielos es recuperable. En la sección 3.7 determinamos condiciones de equivalencia entre dos espacios de rayos de luz $\mathcal{N}_1$ y $\mathcal{N}_2$ tales que reconstruyen la misma variedad Lorentz conforme M. Esta equivalencia se da en términos de la imagen de curvas celestiales y curvas cielo de $\mathcal{N}_1$ mediante el difeomorfismo (que preserva cielos) $\phi : \mathcal{N}_1 \to \mathcal{N}_2$. El teorema 3.7.8 establece que $\mathcal{N}_1$ y $\mathcal{N}_2$ recuperan el mismo M si y sólo si ϕ conserva las curvas celestiales causales y también si y sólo si ϕ conserva las curvas cielo.

Finalmente, el capítulo 4 está concebido como cajón de sastre de diferentes temas. En la sección 4.1, afrontamos la frontera de M propuesta por Low en [45]. Estudiamos la construcción de la frontera de Low en el caso 3–dimensional. Construimos una subvariedad $\widetilde{\mathcal{N}}$ con frontera en $\mathbb{P}(\mathcal{H})$ tal que está foliada por las hojas de una distribución regular $\widetilde{\mathcal{D}}$. La variedad cociente $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$ es difeomorfa a M y la frontera $\partial\widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$ también está foliada, bajo ciertas condiciones, por las órbitas de campos de direcciones $\ominus$ y $\oplus$ definidas respectivamente por los límites pasado y futuro de las curvas $\widetilde{\gamma}(s) = T_\gamma S(\gamma(s))$ con $\gamma \in \mathcal{N}$. Estas órbitas se identifican con puntos de la frontera de $\widetilde{\mathcal{N}}$ en $\mathbb{P}(\mathcal{H})$ y entonces, esta frontera se propaga a M mediante una extensión del difeomorfismo $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}} \simeq M$. Posteriormente, comprobamos si la frontera de Low se puede comparar con la c–frontera utilizando condiciones muy sencillas y sin generalidad. Se observa que no son la misma frontera pero tienen características comunes.

En la sección 4.2, mostramos como pueden describirse los espacios de rayos de luz de algunos espacio–tiempos de Minkowski y de–Sitter. Se describen coordenadas del correspondiente $\mathcal{N}$, sus estructuras de contacto $\mathcal{H}$ y también la frontera de Low, así como otras de sus estructuras.

Concluimos este trabajo con la sección 4.3, en la que se enumera una lista de cuestiones pendientes y de líneas de investigación que pueden seguirse en el futuro.

Conclusiones

Se han alcanzado todos los objetivos principales. Hemos caracterizado la estructura causal de M fuertemente causal, pseudoconvexo para rayos de luz y que separa cielos en términos de la existencia de isotopías legendrianas con signo en su espacio de rayos de luz $\mathcal{N}$. También hemos obtenido resultados secundarios sobre la causalidad de M como el lema 3.4.6 y el corolario 3.3.12. La reconstrucción de M a partir de $(\mathcal{N}, \Sigma)$ es posible y hemos encontrado condiciones para determinar cuándo dos diferentes $(\mathcal{N}_1, \Sigma_1)$ y $(\mathcal{N}_2, \Sigma_2)$ reconstruyen la misma variedad conforme $(M, \mathcal{C})$. Hemos probado que la ausencia de reenfocamiento es equivalente a la separación de cielos en el caso fuertemente causal. En estos aspectos, y para variedades conformes fuertemente causales, este trabajo resulta bastante completo y autocontenido, pero todavía quedan muchas preguntas sin respuesta cuando se debilitan las hipótesis asumidas. Además, el estado en el que queda en este punto el tema tratado, permite el estudio de las variedades Lorentz conformes desde la perspectiva de la geometría de contacto, de manera que se puede recorrer este nuevo camino en paralelo con el de la geometría clásica de espacio–tiempos.

Summary

Title

Geometric structures and causality in the space of light rays of a spacetime.

Introduction

Inspired by some of the greatest mathematicians in 19th century and beginning of 20th such as Felix Klein, Julius Plücker, Arthur Cayley and Sophus Lie among others, R. Penrose in 1960–70s developed the *twistor programme* [56], [58]. This theory is motivated to set a formalism in order to merge general relativity and quantum physics. Twistor spaces are complex structures containing information of 4–dimensional Minkowski spacetime in such a way null geodesics can be seen as basic elements (points) in this complex geometry. Then, an idea emerges from this new point of view: the sets of all null geodesics passing through different events in the spacetime are different, or equivalently, if two observers watch exactly the same sky, then they are at the same point of the spacetime. So, in Minkowski spacetime, all null geodesics passing through a specific point characterizes said point.

In late 1980s, R. Low started to work out this idea and he applied it later for a general spacetime, not necessarily Minkowskian, and in a real differential manifold. In his work [39], [41], [40], [42], [44], [45] the author studies the topology and geometry of the space of (unparametrized) null geodesics and offers conditions for having good properties. He points out the existence of a contact structure in the space of null geodesics and observes that the causal structure of the spacetime is also encoded in said space $\mathcal{N}$ of null geodesics, or light rays as we will name them from now on. An important structure contained in $\mathcal{N}$ is the family of skies: the set of all light rays passing through x is called the sky of x and is denoted by X. In the present work, we will name the set of all skies by Σ. According to Malament-Hawking theorem [26], [46], taking account of the improvements in [36] and [54], the causal structure is related to the topological, differentiable and metric structures, as well as the dimension: if a causal bijection f exists between two spacetimes of dimensions $n_1, n_2 > 2$ which are both future and past distinguishing, then $n_1 = n_2$ and the spacetimes are conformally isometric. So, the quest of the causal structure of M hidden in the space of light rays $\mathcal{N}$ could be important to determine the geometry and topology of the conformal class of spacetimes. Again, inspired by twistor geometry, Low in [41], [43], [44] followed by Chernov–Rudyak in [17], Chernov–Kinlaw–Sadykov in [14], Chernov–Nemirovski in [15], [16], Natario in [50] and Natario–Tod in [51], among others, have studied causal relations by some kind of linking between skies in the contact manifold $\mathcal{N}$.

Another topic is the reconstruction of the conformal Lorentz manifold M by the information contained in $\mathcal{N}$. In [42], Low describes how to recover M in a globally hyperbolic case: the intersection between the lightcone at x and a Cauchy surface can be identified with the sky X and therefore also with the event $x \in M$. But a problem arises when there exists a pair of points $p, q \in M$ verifying that all light rays passing through p also pass through q. In this case, it is said that M is not sky–separating. When this property appears, it does not permit to do an adequate identification between M and Σ, so M is assumed to be sky–separating. And even more, there are also difficulties when there exists an open neighbourhood V of p such that for all open U with $p \in U \subset V$, there exists an event $q \notin V$ such that all light rays through q enter U. This is known as the property of refocusing at p and it has been widely studied by Kinlaw in [31].

Objectives and results

At this point, the two main objectives of this work are:

- to characterize the causal structure of conformal Lorentz manifolds M in terms of their spaces of light rays $\mathcal{N}$, and
- to establish if the reconstruction of conformal Lorentz manifolds M is possible from their corresponding spaces of light rays $\mathcal{N}$.

In order to achieve these aims, we will find some other secondary objectives such as:

- to determine if non–refocusing hypothesis is needed for sky–separating,
- to collect and order the results in the literature about the construction of the space of light rays and its geometrical structures, adding detailed proofs to make of this work a self–contained report,
- to contribute with some breakthrough in the construction of boundary proposed by Low, and
- to illustrate the theoretical results with examples of specific spacetimes.

To get this goal, we proceed in the following way. Firstly, in chapter 1, we expound the background needed in the rest of this work, that is, basic definitions on differential geometry in section 1.1 and a brief introduction on causality in section 1.2.

Then, chapter 2 is devoted to the topological and differentiable structures of the space of light rays $\mathcal{N}$, as well as the study of its tangent bundle $T\mathcal{N}$. The most of the results in sections 2.1 and 2.2 are already known even though they are dispersed in the literature. We formalize and order them starting with the definition of space of light rays $\mathcal{N}$ of a conformal (Lorentz) manifold M up to the description of its differentiable and topological structure. Moreover, coordinate charts are built in $\mathcal{N}$ from the restriction of coordinates in TM to adequate sets, and proposition 2.2.14 gives conditions in M for hausdorffness in $\mathcal{N}$, in fact, it states that if M is strongly causal and null pseudo–convex, then $\mathcal{N}$ is Hausdorff. So, we will assume M verifies these conditions. In section 2.3, we characterize tangent vectors in $T_\gamma\mathcal{N}$ as Jacobi fields of infinitesimal variations by light rays of a given light ray $\gamma \in \mathcal{N}$. This permits us to interpret $v \in T_\gamma\mathcal{N}$ in a useful way in terms of elements of M. Again, we offer details omitted in the literature to obtain said characterization, being proposition 2.3.15, the main result of this section.

The contact structure $\mathcal{H} \subset T\mathcal{N}$ of $\mathcal{N}$ is built in sections 2.4 and 2.5 by using three different procedures. First, carried out in section 2.4.2, we construct $\mathcal{H}$ passing the kernel of the canonical 1–form $\theta \in \mathfrak{X}^*(T^*M)$ of the symplectic manifold T^*M to the tangent bundle TM by means of the Legendre transform. Then, we restrict the resulting distribution of hyperplanes, in the same way we previously did to obtain coordinates in $\mathcal{N}$, becoming a contact structure in $\mathcal{N}$, as proposition 2.4.13 shows. In section 2.5.1, we implement a more elegant procedure, but equivalent to the previous one: coisotropic reduction. Theorem 2.5.5 determines the coisotropic reduction mechanism used in theorems 2.5.6 and 2.5.7 to get $\mathcal{H}$. The third procedure is the Marsden–Weinstein reduction. It can be seen that this is not the easiest way to construct $\mathcal{H}$ because it is necessary to use powerful results needing to verify more hypotheses, but it is included as section 2.5.3 providing the missing details of [30] about this topic.

In chapter 3, we deal with the space of skies Σ. The sky $X \in \Sigma$ of a point $x \in M$ is the set of all light rays passing through x and it becomes a legendrian submanifold of $\mathcal{N}$. In fact, this property for all sky X characterizes the contact structure of $\mathcal{N}$, and we show that it only depends on the conformal structure of M. The sky map $S : M \to \Sigma$ can be defined by $S(x) = X$ and, throughout this work, we are assuming that S is injective, or in other words, M is sky–separating. This condition is necessary in order to identify unambiguously M and Σ. After giving, in section 3.1, a special type of non–canonical coordinates in $T\mathcal{N}$, we define the Low's topology in Σ. Equipped with this topology, the sky map S is continuous (proposition 3.2.4) and, assuming that M is non–refocusing, it is also open (proposition 3.2.5). Therefore, as enunciated in corollary 3.2.6, S is a homeomorphism with the further hypothesis of non–refocusing. An important result in section 3.2 is theorem 3.2.8: the subset $\widehat{\Sigma} \subset T\mathcal{N}$ of celestial vectors, that is, vectors tangent to skies in Σ, is locally embedded in $T\mathcal{N}$. This theorem will be fundamental in some proofs of subsequent sections.

In section 3.3 we study two kinds of curves: celestial curves in $\mathcal{N}$ and twisted null curves in M. Celestial curves are curves in $\mathcal{N}$ such that their tangent vector are celestial. Proposition 3.3.2 shows that for any celestial curve $\Gamma \subset \mathcal{N}$ there exists a curve $\mu \subset M$, called the dust of Γ, such that $\Gamma'(s) \in T_{\Gamma(s)}S(\mu(s))$ and $\mu'(s)$ is proportional to the tangent vector of the light ray $\Gamma(s)$ at the point $\mu(s)$ wherever μ is regular. Lemma 3.3.7 says that the dust μ of a celestial curve Γ is a piecewise twisted null curve, that is, a curve non–geodesic at any point and μ' is null wherever μ is regular, and conversely, any such twisted null curve μ defines a celestial curve Γ such that μ is its dust. Also, as a tool that we will use later, in corollary 3.3.12 of section 3.3.2, we show that any pair of timelike related points in M can be connected by a time–oriented piecewise twisted null curve.

We characterize the causal structure of M in terms of the geometry of $\mathcal{N}$ in section 3.4. In a contact manifold $(Y, \mathcal{H})$ with contact structure $\mathcal{H} = \ker \alpha$ where $\alpha \in T^*Y$, a differentiable family $\{\Lambda_s\}_{s\in[0,1]}$ of legendrian submanifolds is called a legendrian isotopy. It can be described by a parametrization $F : \Lambda_0 \times [0,1] \to Y$ verifying $F(\Lambda_0 \times \{s\}) = \Lambda_s \subset Y$ where $s \in [0,1]$. A parametrization F of a legendrian isotopy is said to be non–negative if $(F^*\alpha)\left(\frac{\partial}{\partial s}\right) \geq 0$. Lemma 3.4.3 ensures that non–negativity of a legendrian isotopy is independent of the used parametrization. Moreover corollary 3.4.8 gives us a relation between causal structure in M and signed legendrian isotopies in $\mathcal{N}$: a legendrian isotopy of skies $\{S(\mu(s))\}_{s\in[0,1]}$ is non-negative if and only if the curve $\mu : [0,1] \to M$ is causal past–directed. So, we obtain a description of the causal structure of M in terms of the geometry of $\mathcal{N}$, then the causal information is encoded in the existence of signed legendrian isotopies of skies connecting two given skies.

The differentiable structure of Σ is studied in section 3.5. After the definition of a new topology in Σ, called the topology of regular sets, we show in corollary 3.5.5 that, in this case, the sky map $S : M \to \Sigma$ is a homeomorphism. The differentiable structure of Σ is determined in corollary 3.5.6: regular sets constitute a basis for the Low's topology of Σ, and there exists a unique differentiable structure in Σ compatible with topology of regular sets that makes of $S : M \to \Sigma$ a diffeomorphism. The previous results have been obtained without the assumption of non–refocusing in M, and according to theorem 3.5.8 and corollary 3.5.9, if M is a strongly causal, null pseudo–convex, sky–separating conformal manifold and the Low's topology is provided in Σ, then the sky map $S : M \to \Sigma$ is a homeomorphism and M is non–refocusing.

In sections 3.6 and 3.7 we set conditions of reconstruction of the conformal Lorentz manifold $(M, \mathcal{C})$ from its spaces of light rays $\mathcal{N}$. Given $(M, \mathcal{C})$ a strongly causal manifold such that $(\mathcal{N}, \Sigma)$ is the corresponding pair of spaces of light rays and skies, we say that $(M, \mathcal{C})$ is recoverable if for any pair $\left(\overline{\mathcal{N}}, \overline{\Sigma}\right)$ corresponding to another strongly causal conformal manifold $\left(\overline{M}, \overline{\mathcal{C}}\right)$, with a diffeomorphism $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ such that $\phi(X) \in \overline{\Sigma}$ for any $X \in \Sigma$, then the map $\varphi = \overline{S}^{-1} \circ \phi \circ S : M \to \overline{M}$ is a conformal diffeomorphism on its image. We show, in theorem 3.6.3, that a strongly causal, null pseudo–convex and sky–separating conformal manifold M is recoverable. In section 3.7 we determine conditions of equivalence between two spaces of light rays $\mathcal{N}_1$ and $\mathcal{N}_2$ such that they reconstruct the same conformal Lorentz manifold M. Said equivalence is given in terms of the image of celestial and sky curves in $\mathcal{N}_1$ by the (preserving skies) diffeomorphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$. Theorem 3.7.8 states that $\mathcal{N}_1$ and $\mathcal{N}_2$ recover the same M if and only if ϕ maps causal celestial curves into causal celestial curves, and also if and only if ϕ maps sky curves into sky curves.

Finally, chapter 4 is conceived as a mixture of different topics. In section 4.1, we deal with the boundary of M proposed by Low in [45]. We study the construction of Low's boundary in 3–dimensional cases. We construct a submanifold $\widetilde{\mathcal{N}}$ with boundary in $\mathbb{P}(\mathcal{H})$ such that it is foliated by leaves of a regular distribution $\widetilde{\mathcal{D}}$. The quotient manifold $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$ is diffeomorphic to M and the boundary $\partial\widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$ is also foliated, under some conditions, by orbits of fields of directions $\ominus$ and $\oplus$ defined respectively by the past and future limits of all curves $\widetilde{\gamma}(s) = T_\gamma S(\gamma(s))$ with $\gamma \in \mathcal{N}$. These orbits are identified to points at the boundary of $\widetilde{\mathcal{N}}$ in $\mathbb{P}(\mathcal{H})$ and then, this boundary is propagated to M by an extension of the diffeomorphism $\widetilde{\mathcal{N}}/\widetilde{\mathcal{D}} \simeq M$. Later, we check if Low's boundary can be compared with GKP c–boundary using very simple and not general conditions. We observe they are not the same boundary but both have common features.

In section 4.2, we show how spaces of light rays can be described for some Minkowski and de Sitter spacetimes. We describe coordinates of corresponding $\mathcal{N}$, their contact structures $\mathcal{H}$ and also Low's boundary as well as other structures therein.

We conclude this work with section 4.3, in which we list some pending questions and lines of research that can be followed in the future.

Conclusions

All the proposed main objectives have been reached. We have characterized the causal structure of strongly causal, null pseudo–convex and sky–separating M in terms of the existence of signed legendrian isotopies in its space of light rays $\mathcal{N}$. We also have obtained side results on causality on M as lemma 3.4.6 and corollary 3.3.12. The reconstruction of M from the pair $(\mathcal{N}, \Sigma)$ is possible and we have found conditions to determine when two different $(\mathcal{N}_1, \Sigma_1)$ and $(\mathcal{N}_2, \Sigma_2)$ recover the same conformal manifold $(M, \mathcal{C})$. It has been proven that non–refocusing is equivalent to sky–separating condition in a strongly causal spacetime. In these aspects, and for strongly causal conformal manifolds, this work is nearly complete and self–contained, but there are still many questions without answer when we weaken the assumed hypothesis. Moreover, the state in which the covered matter is at this point, permits the study of conformal Lorentz manifolds from the view of contact geometry, so that, this new path can be walked in parallel with the classical spacetime geometry.

Chapter 1

Background

In this chapter we will compile the basic definitions and results forming the background for the target of our study and it pretends to be a working basis for a good understanding of following chapters. Section 1.1 is devoted to introduce some of the notation and basic concepts in the scope of differential geometry. We will also offer, in section 1.2, some elementary results on causality theory we will need later.

First, we need to fix the notation of elementary concepts. We will use the word *smooth* as well as *differentiable* to name C^∞ objects. If M is a differentiable manifold and $p \in M$ is any of its points, then we will denote by TM the *tangent bundle* of M, then T_pM will be the *tangent space* of M at p and $\mathbf{0}_p$ its zero vector. The ring of differentiable real–valued functions over M will be $\mathfrak{F}(M)$, by $\mathfrak{X}(M)$ we will denote the set of *differentiable vector fields* in M, the set of *1–form* in M will be denoted by $\mathfrak{X}^*(M)$. In general, the set of *p–forms* in M will be denoted by $\Lambda^p(M)$ and $\mathfrak{T}^p_q(M)$ will be the bundle of *differentiable tensors* of type (p,q) over M.

If P is a fibre bundle over M then the canonical projection will be usually denoted by $\pi^P_M : P \to M$.

Section 1.1

Differential geometry

Now we will do a brief summary of some important topics of differential geometry to avoid possible confusion because there exists non–equivalent definitions of some geometric objects in the literature.

Let M and N be two differentiable manifolds and $f : M \to N$ a differentiable map, the *push–forward* of a vector field $X \in \mathfrak{X}(M)$ can be defined pointwise by

$$(f_*X)_p = (df)_p \, X_p \in T_{f(p)}N$$

where $(df)_p$ is the differential of f at $p \in M$ and $X_p := X(p)$. In general, (f_*X) is not a vector field in N (see [35, p. 87]).

Then the *pull–back* of a k–covariant tensor field $T \in \mathfrak{T}_k^0(N)$ by f is defined by

$$(f^*T)(X_1, \ldots, X_k) = T(f_*X_1, \ldots, f_*X_2)$$

for $X_i \in \mathfrak{X}(M)$ with $i = 1, \ldots, k$. In this case, according to [35, Prop. 11.9], $f^*T \in \mathfrak{T}_k^0(M)$ is a k–covariant tensor in M.

If $f : M \to N$ is a diffeomorphism then it is possible to establish the following definition of push–forward and pull–back of tensor and vector fields. So, given $T \in \mathfrak{T}_k^n(M)$, $\alpha_i \in \mathfrak{X}^*(M)$ with $i = 1, \ldots, n$ and $X_j \in \mathfrak{X}(M)$ with $j = 1, \ldots, k$ we can define $f_*T \in \mathfrak{T}_k^n(N)$ by

$$(f_*T)(\alpha_1, \ldots, \alpha_n, X_1, \ldots, X_k) = T\left(f^*\alpha_1, \ldots, f^*\alpha_n, \left(f^{-1}\right)_* X_1, \ldots, \left(f^{-1}\right)_* X_k\right)$$

and for $T \in \mathfrak{T}_k^n(N)$ we have that

$$f^*T = \left(f^{-1}\right)_* T \in \mathfrak{T}_k^n(M)$$

For $Y \in \mathfrak{X}(N)$ we have that the pull–back f^*Y of Y by f is

$$(f^*Y)(p) = \left(d\left(f^{-1}\right)\right)_{f(p)} Y(f(p)) \in T_pM. \tag{1.1.1}$$

and the push–forward f_*X of $X \in \mathfrak{X}(M)$ by f is

$$(f_*X)(q) = (df)_q X\left(f^{-1}(q)\right) \in T_qN. \tag{1.1.2}$$

Given a vector field $X \in \mathfrak{X}(M)$, we will denote by $\mathcal{L}_X$ the *Lie derivative* along X. For a geometric object A, it can be defined by

$$\mathcal{L}_X A = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_t^X\right)^* A \tag{1.1.3}$$

where $\Phi_t^X : M \to M$ denotes the flow of X.

So, for a differentiable function $f \in \mathfrak{F}(M)$ it is known that

$$\mathcal{L}_X f = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_t^X\right)^* f = X(f) \in \mathfrak{F}(M).$$

In case of a vector field $Y \in \mathfrak{X}(M)$ we have that

$$\mathcal{L}_X Y = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_t^X\right)^* Y = [X, Y] \in \mathfrak{X}(M)$$

For a differentiable p–form $\omega \in \Lambda^p(M)$ we have the *Cartan's formula*

$$\mathcal{L}_X \omega = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_t^X\right)^* \omega = i_X d\omega + d(i_X\omega) \in \Lambda^p(M) \tag{1.1.4}$$

where $i_X\theta \in \Lambda^{p-1}(M)$ denotes the *inner product* of $X \in \mathfrak{X}(M)$ and $\theta \in \Lambda^p(M)$, that it is defined by

$$i_X\theta(Y_1, \ldots, Y_{p-1}) = \theta(X, Y_1, \ldots, Y_{p-1}).$$

and it is the contraction of θ in the first variable.

By expression 1.1.1, if Φ_t^X denotes the flow of X, then

$$[X,Y] = \left.\frac{d}{dt}\right|_{t=0} \left(\Phi_t^X\right)^* Y = \left.\frac{d}{dt}\right|_{t=0} \left(d\Phi_{-t}^X\right) Y \circ \Phi_t^X \tag{1.1.5}$$

See, for example, [2] for further details.

Definition 1.1.1. *Let M be a differentiable manifold such that* $\dim M \geq 2$ *and* $\mathbf{g} \in \mathfrak{T}_2^0(M)$ *a symmetric tensor. Then $(M,\mathbf{g})$ is said to be:*

1. *a* riemannian manifold *if* $\mathbf{g}$ *is positive definite at every $p \in M$.*
2. *a* semi–riemannian manifold *if* $\mathbf{g}$ *is non-degenerated at every $p \in M$.*
3. *a* Lorentzian manifold *if $(M,\mathbf{g})$ is a semi–riemannian manifold and for every $p \in M$ there exists a basis at T_pM in which $\mathbf{g}_p = diag(-1,+1,...,+1)$.*

Equivalently, we will say that M is riemannian, semi–riemannian or Lorentzian when $(M,\mathbf{g})$ is so and the metric $\mathbf{g}$ is not necessary to be specified.

For any given smooth function $\sigma \in \mathfrak{F}(M)$, let us define a *conformal metric* in M equivalent to $\mathbf{g}$ by

$$\mathcal{C}_{\mathbf{g}} = \left\{\overline{\mathbf{g}} \in \mathfrak{T}_0^2(M) : \overline{\mathbf{g}} = e^{2\sigma}\mathbf{g},\ \ \sigma \in \mathfrak{F}(M)\right\}$$

and a *conformal manifold* equivalent to $(M,\mathbf{g})$ by the pair $(M,\mathcal{C}_{\mathbf{g}})$. We can talk about *conformal Lorentz manifold*, *conformal semi-riemannian manifold*,... when the related metric is Lorentz, semi-riemannian,... but since the only sort of metric we work with is Lorentzian one, for brevity, we will call them just *conformal manifold*.

By ∇, we will denote the *Levi–Civita connection* of M, that is the unique connection verifying

$$[X,Y] = \nabla_X Y - \nabla_Y X$$

and

$$X(\mathbf{g}(Y,Z)) = \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z)$$

where $X,Y,Z \in \mathfrak{X}(M)$ and $[X,Y] = XY - YX$ is the *Lie bracket* of X and Y.

Considering a curve $\lambda = \lambda(t)$ in M, we will denote by

$$\frac{D}{dt} : \mathfrak{X}_\lambda \to \mathfrak{X}_\lambda$$

the *covariant derivative along* λ, where $\mathfrak{X}_\lambda$ denotes the set of all smooth vector fields on the curve λ.

The *curvature* or *Riemann tensor* is defined by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Given a function $f \in \mathfrak{F}(M)$, we will say that $\operatorname{grad} f$ is the *gradient* of f and it describes the vector field metrically equivalent to the 1–form df, that is, for any $X \in \mathfrak{X}(M)$

$$\mathbf{g}(\operatorname{grad} f, X) = df(X) = X(f) \in \mathfrak{F}(M)$$

A detailed exposition of the properties of the previous geometrical objects can be found in, for example, [2], [8], [25] and [53].

Section 1.2

Causality in spacetimes

From now on, we will consider that $(M, \mathbf{g})$ is a Lorentzian manifold. In such a manifold we can classify the tangent vectors depending on their *causal character*, that is, we will say that a vector $v \in T_pM$ such that $v \neq \mathbf{0}_p$ is:

spacelike	if $\mathbf{g}_p(v,v) > 0$
null or *lightlike*	if $\mathbf{g}_p(v,v) = 0$
timelike	if $\mathbf{g}_p(v,v) < 0$

It is trivial to notice that any metric $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ with $\sigma \in \mathfrak{F}(M)$ defines the same causal character for every $v \in T_pM$ since $e^{2\sigma} > 0$. Therefore, clearly the causality is well defined in the conformal Lorentz manifold $(M, \mathcal{C}_{\mathbf{g}})$.

We get from [53, p. 145] the following definition. Let τ be a continuous function on M assigning to every $p \in M$ a connected component τ_p of the set of causal vectors in T_pM. A such function τ will be called *time–orientation* of M. We will say that (M,g) is *time–orientable* if M admits a time–orientation. If a time–orientation τ is provided at (M,g) then we will say that (M,g) is *time–oriented*.

Time–orientability is equivalent to the existence of a *timelike vector field* X (see [53, Lem. 5.32] for details), that is for all $p \in M$ the tangent vector $X_p \in T_pM$ is timelike. In fact, if X exists, then it is possible to assign to every $p \in M$ the connected component of T_pM containing X_p and so we get a time–orientation. On the other hand, if M is furnished of a time–orientation τ then for every $p \in M$ there exists a neighbourhood U_p where a timelike vector field X_{U_p} is defined, and its image for any $q \in U_p$ is in τ_q. Using partitions of unity a global timelike vector field X can be constructed in M, see [53, Lem. 5.32] for more details.

In a time–oriented Lorentzian manifold (M,g) we can distinguish both connected component of the set of causal vectors calling *future causal cone* of p to the τ component and *past causal cone* of p to the $-\tau$ one. So we will say that a causal vector $v \in T_pM$ is *future* (respectively *past*) if $v \in \tau_p$ (respectively $-v \in \tau_p$).

In what follows, we will consider time–oriented Lorentzian manifold.

Definition 1.2.1. *A time–oriented Lorentzian manifold $(M, \mathbf{g})$ of dimension $m \geq 3$ will be called a* spacetime*.*

Let us consider the tangent bundle TM. If (φ_M, U) is a coordinate chart in M such that $\varphi_M = \left(x^1, \dots, x^m\right)$ in which a tangent vector $v \in TU$ can be written as $v = v^k \frac{\partial}{\partial x^k}$, then (φ, TU) such that $\varphi = \left(x^k, v^k\right)$ is a coordinate chart in TM. We can express the metric $\mathbf{g}$ in this coordinates as $\mathbf{g}(u,v) = g_{ij}u^iv^j$.

Notation 1.2.2. *If N is a differentiable manifold, the notation $\widehat{T}N$ will be used to make reference to the bundle resulting of eliminating the zero section of TN, that is*

$$\widehat{T}N = \{v \in TN : v \neq \mathbf{0}\}.$$

Let us consider the restriction $\mathbb{N}$ of $\widehat{T}M$ defined by

$$\mathbb{N} = \left\{v \in \widehat{T}M : \mathbf{g}(v,v) = 0\right\}$$

that is the set of null vectors in M.

Given $L : \widehat{T}M \to \mathbb{R}$ the differentiable function defined by

$$L(v) = \frac{1}{2}\mathbf{g}(v,v) \tag{1.2.1}$$

that can be written as $L(x^k, v^k) = \frac{1}{2}g_{ij}v^iv^j$ in the local natural bundle coordinates (x^k, v^k). By definition of $\mathbb{N}$, it is trivial to see that $\mathbb{N} = L^{-1}(0) \subset \widehat{T}M$. The differential of L in $\varphi(v)$ is

$$dL = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}v^iv^jdx^k + g_{ik}v^idv^k. \tag{1.2.2}$$

Since $\mathbf{g}$ is a non–degenerate metric, then for every $v \in \widehat{T}M$ there exists $u \in \widehat{T}M$ such that $\mathbf{g}(v,u) \neq 0$. This implies that some $g_{ik}v^i$ with $k = 1, \ldots, n$ is not zero, then the rank of $dL_{(x^k,v^k)}$ is 1 and therefore $0 \in \mathbb{R}$ is a regular value of the function L. By [10, Cor. II.7.4], since $\mathbb{N} = L^{-1}(0)$ is the inverse image of a regular value, then it is a regular submanifold of $\widehat{T}M$ and, by restriction, it inherits the structure of bundle of $\widehat{T}M$ over M. So $\mathbb{N}$ is a bundle over M and we will denote by $\pi_M^{\mathbb{N}} : \mathbb{N} \to M$ its canonical projection and by $\mathbb{N}_p$ its fibre at $p \in M$.

The zero section of TM separates both connected components of $\mathbb{N}$ denoted by

$$\mathbb{N}^+ = \{v \in \mathbb{N} : v \text{ future }\}$$

$$\mathbb{N}^- = \{v \in \mathbb{N} : v \text{ past }\}$$

We will call the fibres $\mathbb{N}_p$, $\mathbb{N}_p^+$ and $\mathbb{N}_p^-$ *lightcone*, *future lightcone* and *past lightcone* at $p \in M$ respectively.

By the previous classification of tangent vectors, we will say that a curve γ is *timelike* (respectively *null*, *spacelike*, *causal*) if its tangent vector is timelike (respectively null, spacelike, causal) at every of its points. We will say that a causal curve is *future–directed* (respectively *past–directed*) if it is equipped with future tangent vectors (respectively past) at any of its point.

Definition 1.2.3. *Let S be a subset of M.*

1. *The* chronological future of S *is the set of all points in M that can be connected to S by a future–directed timelike curve. It will be denoted by $I^+(S)$. Analogously, it is possible to define the* chronological past of S *denoted by $I^-(S)$.*

2. *The* causal future of S *is the union of S and the set of all points in M that can be connected to S by a future–directed causal curve. It will be denoted by $J^+(S)$. In the same way, we can define the* causal past of S *denoted by $J^-(S)$.*

3. *A subset $S \subset M$ is* achronal *if any $p \in S$ verifies $I^+(p) \cap S = \varnothing$.*

4. *Let S be an achronal set, we will name* future *(*past*)* Cauchy development of S *to the set of points $p \in M$ such that any causal curve inextensible to the past (future) passing through p intersects S. We will denote it by $D^+(S)$ $(D^-(S))$. And we will say that $D(S) = D^+(S) \cup D^-(S)$ is the* Cauchy development *of S.*

In a equivalent way, we will use the notation

$$p \prec q$$

to indicate $q \in I^+(p)$. Also

$$p < q$$

can be used to denote the existence of a future–directed causal curve from p to q. The notation

$$p \leq q$$

is used to indicate both $p < q$ or $p = q$, that is $q \in J^+(p)$.

Next theorem is a basic result to study the causal structure of spacetimes and it can be found in [53, Prop. 10.46].

Theorem 1.2.4. *Let M be a spacetime. If λ is a causal curve joining the points $p, q \in M$ but not a null pregeodesic, then in any neighbourhood of λ there exists a timelike curve μ connecting the points p and q.*

As an immediate consequence of theorem 1.2.4, we get the following corollary.

Corollary 1.2.5. *If $r \in J^+(q)$ and $q \in I^+(p)$, or also $r \in I^+(q)$ and $q \in J^+(p)$, then we have that $r \in I^+(p)$.*

Proof. In former case, if $q \in I^+(p)$ then there exists a future–directed timelike curve λ_1 joining p and q, and if $r \in J^+(q)$ then there exists a future–directed causal curve λ_2 connecting q with r (if $q = r$ then λ_2 is constant). Then the curve $\lambda = \lambda_1 \cup \lambda_2$ is a future–directed causal curve joining p and r and it is not a null pregeodesic because λ_1 is timelike. By theorem 1.2.4, there exists a timelike curve μ joining p and r that, by construction conserves the same time–orientation of λ. Therefore $r \in I^+(p)$.

The proof for the latter case can be done in an analogous way. □

The previous corollary is also true when we consider the chronological and causal past, and its proof is similar if we interchange the roles of future and past.

Depending on the behaviour of causal curves in M, it is possible to classify the spacetimes according to some conditions about the nature of causal curves. The next classification list is not exhaustive but it is enough for our purpose. It is possible to find a wide explanation about the causality conditions in [53], [48], [8], [25] and [55]. In the next definition we only introduce some conditions, taking into account that if one of them is verified then all the previous conditions are also verified.

Definition 1.2.6. *Let M be a time–oriented spacetime, then*

1. *It is said that M verifies the* chronological condition *or that M is* chronological *if there does not exist closed timelike curves.*

2. *It is said that M verifies the* causal condition *or that M is* causal *if there does not exist closed causal curves.*

3. *We say that M is* strongly causal at $p \in M$ *or verifies the* strong causality condition *if for every neighbourhood U of $p \in M$ there exists a neighbourhood $V \subset U$ of p such that any segment of causal curve with endpoints at V, is wholly contained in U. This means that there is not* almost closed *causal curves at p, that is there exists a neighbourhood V of p such that any causal curve that leaves V does not return to said neighbourhood. We will say that M is strongly causal if it is so for every $p \in M$.*

4. *We say that M is* globally hyperbolic *or verifies the* global hyperbolicity condition *if it causal and $J^+(p) \cap J^-(q)$ is compact for any $p, q \in M$.*

Definition 1.2.7. *We will say that a* naked singularity *occurs at the future (resp. past) of a causal curve λ inextensible to the future (resp. past) if there exists a point $p \in M$ such that $I^-(\lambda) \subset I^-(p)$ (resp. $I^+(\lambda) \subset I^+(p)$).*

In [57], Penrose shows that a strongly causal spacetime M is globally hyperbolic if naked singularities does not exist in M.

Definition 1.2.8. *A future–directed causal curve γ inextensible to the future such that it enters and remains into a compact set K is said to be* totally imprisoned to the future *in K. If γ does not remain in K, but continually re–enters into K, then γ is said to be* partially imprisoned to the future *in K.*

These phenomena of imprisonment can not exist under some causality conditions, as it can be observed in the next proposition found at [25, Prop. 6.4.7].

Proposition 1.2.9. *If there exists a totally or partially imprisoned future–directed causal curve inextensible to the future in some compact set $K \subset M$, then the strong causality condition does not hold on K*

Definition 1.2.10. *A* Cauchy surface *is a topological hypersurface $S \subset M$ such that any inextensible timelike curve intersects S exactly once.*

Proposition 1.2.11. *Let M be a spacetime with a Cauchy surface $S \subset M$ and let $X \in \mathfrak{X}(M)$ be a timelike vector field. If $p \in M$, every maximal integral curve of X passing through p intersects S in a unique point $\sigma(p)$. Then the map $\sigma : M \to S$ is open, continuous and surjective leaving fixed any point of S. Moreover S is connected.*

Proof. We offer the proof of [53, Prop. 14.31]. It is known that the maximal integral curves of X are inextensible. Let $\tilde{\Psi} : D \longrightarrow M$ be the flow of X where D is open in $M \times \mathbb{R}$. Since S is a topological hypersurface of M, then $D_S = (S \times \mathbb{R}) \cap D$ is a topological hypersurface in D and since $\tilde{\Psi}$ is differentiable, then its restriction $\Psi : D_S \longrightarrow M$ is continuous. Moreover S is a Cauchy surface and then $\Psi : D_S \to M$ is bijective. Since the dimensions of D_S and M are the same then Ψ is a homeomorphism. The projection $\pi : S \times \mathbb{R} \to S$ is an open, continuous and surjective map, hence since $\sigma = \pi \circ \Psi^{-1}$, then σ is also open, continuous and surjective and leaves fixed any point of S. Since M is connected then we conclude that $\sigma(M) = S$ is connected. □

An important consequence of proposition 1.2.11 is the topological equivalence of Cauchy surfaces. It is described in the next corollary.

Corollary 1.2.12. *All the Cauchy surfaces in a spacetime M are homeomorphic.*

Proof. We sketch the idea of the proof in [53, Cor. 14.32]. Let S and T be two Cauchy surfaces of M and let X be a timelike vector field. If σ_S and σ_T are the respective retractions built in proposition 1.2.11 for S and T by means of the flow of X, then the restrictions $\sigma_S : T \to S$ and $\sigma_T : S \to T$ are mutually inverses. $\square$

Theorem [53, Th. 14.38] states a relation between Cauchy developments and global hyperbolicity. It claims that given a achronal set A, then the interior of the Cauchy development of A, that is $\mathrm{int}\,(D\,(A))$, if it is not empty, then it is globally hyperbolic. This result can be applied to a Cauchy surface S, and since $D\,(S) = M$ then $\mathrm{int}\,(D\,(A)) = M$, therefore M is globally hyperbolic. So, the existence of a Cauchy surface implies the global hyperbolicity of M.

The next theorem is an important characterization of globally hyperbolic spacetimes.

Theorem 1.2.13. ***(Geroch-Bernal-Sánchez)*** *Any globally hyperbolic spacetime M admits a differentiable spacelike Cauchy surface S, and moreover M is diffeomorphic to $S \times \mathbb{R}$.*

Proof. See [9, Th. 1] for proof. $\square$

Recall that a open neighbourhood U^p of $p \in M$ is called a *normal neighbourhood* if there exists a star–shaped neighbourhood U_0^p of $\mathbf{0}_p \in T_pM$ such that $\exp_p : U_0^p \to U^p$ is a diffeomorphism. The existence of these normal neighbourhoods were shown by J.H.C. Whitehead in [62], and a proof can be seen in [27, p. 133–136]. Moreover, as pointed out in [25, p. 34], a normal neighbourhood can be chosen as a neighbourhood of any of its points. This implies that given two points $r, q \in U^p$ then there is a geodesic segment with endpoints at r and q fully contained in U^p. We will call *convex normal neighbourhood* to such normal neighbourhoods.

According to [48], we have the following definitions and results.

Definition 1.2.14. *Let U, V be open sets in a spacetime M such that $V \subset U$. Then V is said to be* causally convex *in U if any causal curve contained in U with endpoints in V is totally contained in V.*

Theorem 1.2.15. *Let M be a spacetime. For any $p \in M$ and any neighbourhood U of p there exists a neighbourhood U' such that $p \in U' \subset U$ and a sequence of globally hyperbolic nested neighbourhoods $\{V_n\}$ such that $V_{n+1} \subset V_n$ and $\{p\} = \bigcap_n V_n$ all contained in U' and verifying that every V_n is causally convex in U'.*

Proof. See [48, Th. 2.14] for proof. $\square$

It is important to notice that any V_n in the previous theorem 1.2.15 can assumed to be contained in a convex normal neighbourhood. Then, for brevity, we will use the following definition.

Definition 1.2.16. *An open set $V \subset M$ is said to be a* basic open set *or a* basic neighbourhood *of some point, if V is globally hyperbolic, causally convex and contained in a convex normal neighbourhood.*

By theorem 1.2.15, it is possible to give a different, but equivalent, definition of strongly causal spacetimes.

Definition 1.2.17. *A spacetime M is said to be* strongly causal *if for all $p \in M$ and all neighbourhood $U \subset M$ of p there exists a causally convex neighbourhood $V \subset U$ of p. This neighbourhood V, according to theorem 1.2.15 can be considered basic.*

Finally, we finish the present chapter with the next proposition that will be used later.

Proposition 1.2.18. *Let M be a strongly causal spacetime, then for every $p \in M$ there exists a neighbourhood V of p such that if γ is an inextensible causal curve then $\gamma \cap V$ has exactly one connected component.*

Proof. It is a direct consequence of strong causality of M. It is known that for all $p \in M$ there exist a basic neighbourhood V of p. Let γ be a causal curve intersecting V, if $\gamma \cap V$ had more that one connected component, then taking two points $q, r \in \gamma$ contained in different connected components, since γ is connected, there would exist a point $s \in \gamma$ between q and r such that $s \notin V$, contradicting that V is causally convex. $\square$

Chapter 2

The space of light rays

This chapter is intended to be a self–contained text and it presents the construction of the space of light rays and of some of its structures from a basic starting point. In section 2.1 we define the space of light rays $\mathcal{N}$ what will be the space that we will use as framework and in section 2.2 we study its differentiable structure. The characterization of tangent vectors in $T\mathcal{N}$ as Jacobi fields of geodesic variations is done in section 2.3. Sections 2.4 and 2.5 are dedicated to the construction of the canonical contact structure comprised in $\mathcal{N}$.

Section 2.1

Definition of the space $\mathcal{N}$ of light rays

Given a spacetime $(M, \mathbf{g})$, we define the *set of light rays* of $(M, \mathbf{g})$ by

$$\mathcal{N}_{\mathbf{g}} = \{\mathrm{Im}\,(\gamma) \subset M : \gamma \text{ is a maximal null geodesic in } (M, \mathbf{g})\}$$

where $\mathrm{Im}\,(\gamma)$ denotes the image of the curve γ. This definition, a priori, depends on the metric but we will show that it only depends on the conformal class of spacetimes.

First, we need to know how the Levi–Civita connection varies when a conformal factor appears in the metric. The following proposition can be found in [32, Lem. 2.1].

Proposition 2.1.1. *Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two spacetimes with $\overline{\mathbf{g}} \in \mathcal{C}_{\mathbf{g}}$. If ∇ and $\overline{\nabla}$ denote the Levi–Civita connections of $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ respectively, then*

$$\overline{\nabla}_X Y = d\sigma\,(X)\,Y + d\sigma\,(Y)\,X - \mathbf{g}\,(X, Y)\,\mathrm{grad}\,\sigma + \nabla_X Y$$

is verified for all $X, Y \in \mathfrak{X}\,(M)$.

Proof. Since $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ with $\sigma \in \mathfrak{F}\,(M)$, then applying Koszul's formula [53, Th. 3.11] we have that

$$\begin{aligned}
2\overline{\mathbf{g}}\left(\overline{\nabla}_X Y, Z\right) &= X\left(\overline{\mathbf{g}}\left(Y,Z\right)\right) + Y\left(\overline{\mathbf{g}}\left(Z,X\right)\right) - Z\left(\overline{\mathbf{g}}\left(X,Y\right)\right) - \\
&- \overline{\mathbf{g}}\left(X,[Y,Z]\right) + \overline{\mathbf{g}}\left(Y,[Z,X]\right) + \overline{\mathbf{g}}\left(Z,[X,Y]\right) = \\
&= X\left(e^{2\sigma}\mathbf{g}\left(Y,Z\right)\right) + Y\left(e^{2\sigma}\mathbf{g}\left(Z,X\right)\right) - Z\left(e^{2\sigma}\mathbf{g}\left(X,Y\right)\right) - \\
&- e^{2\sigma}\mathbf{g}\left(X,[Y,Z]\right) + e^{2\sigma}\mathbf{g}\left(Y,[Z,X]\right) + e^{2\sigma}\mathbf{g}\left(Z,[X,Y]\right) = \\
&= X\left(e^{2\sigma}\right)\mathbf{g}\left(Y,Z\right) + Y\left(e^{2\sigma}\right)\mathbf{g}\left(Z,X\right) - Z\left(e^{2\sigma}\right)\mathbf{g}\left(X,Y\right) + 2e^{2\sigma}\mathbf{g}\left(\nabla_X Y, Z\right) = \\
&= 2\left[X\left(\sigma\right)\overline{\mathbf{g}}\left(Y,Z\right) + Y\left(\sigma\right)\overline{\mathbf{g}}\left(Z,X\right) - Z\left(\sigma\right)\overline{\mathbf{g}}\left(X,Y\right) + \overline{\mathbf{g}}\left(\nabla_X Y, Z\right)\right] = \\
&= 2\left[d\sigma\left(X\right)\overline{\mathbf{g}}\left(Y,Z\right) + d\sigma\left(Y\right)\overline{\mathbf{g}}\left(Z,X\right) - \mathbf{g}\left(\operatorname{grad}\sigma, Z\right)\overline{\mathbf{g}}\left(X,Y\right) + \overline{\mathbf{g}}\left(\nabla_X Y, Z\right)\right] = \\
&= 2\left[d\sigma\left(X\right)\overline{\mathbf{g}}\left(Y,Z\right) + d\sigma\left(Y\right)\overline{\mathbf{g}}\left(Z,X\right) - \overline{\mathbf{g}}\left(\operatorname{grad}\sigma, Z\right)\mathbf{g}\left(X,Y\right) + \overline{\mathbf{g}}\left(\nabla_X Y, Z\right)\right] = \\
&= 2\overline{\mathbf{g}}\left(d\sigma\left(X\right)Y + d\sigma\left(Y\right)X - \mathbf{g}\left(X,Y\right)\operatorname{grad}\sigma + \nabla_X Y, Z\right)
\end{aligned}$$

obtaining then

$$\overline{\nabla}_X Y = d\sigma\left(X\right)Y + d\sigma\left(Y\right)X - \mathbf{g}\left(X,Y\right)\operatorname{grad}\sigma + \nabla_X Y$$

□

The next lemma is a particular case of the Hilbert's Nullstellensatz (see [24, Th. 1.3A], [4, p. 85]), but we offer an autonomous proof.

Lemma 2.1.2. *The future lightcone $\mathbb{N}_p^+ \subset T_pM$ determines the metric in $p \in M$ except by a constant factor.*

Proof. If $\mathbb{N}_p^+$ is known, since $v \in \mathbb{N}_p^-$ if and only if $-v \in \mathbb{N}_p^+$, then $\mathbb{N}_p$ is also known and hence it is possible to determine if a vector $v \in T_pM$ is timelike, null or spacelike. Then given $u \in T_pM$ timelike and $v \in T_pM$ spacelike, there exists $t \in \mathbb{R}$ such that $u + tv \in T_pM$ is a null vector. Indeed, since $\mathbf{g}\left(u,u\right) < 0$ and $\mathbf{g}\left(v,v\right) > 0$, then the equation

$$\mathbf{g}\left(u+tv, u+tv\right) = \mathbf{g}\left(u,u\right) + 2t\mathbf{g}\left(u,v\right) + t^2\mathbf{g}\left(v,v\right) = 0 \tag{2.1.1}$$

has a positive discriminant, hence it has two solutions $t_1, t_2 \in \mathbb{R}$. These values make the vectors $u + t_1 v$ and $u + t_2 v$ to be null and they only depends, like the values t_1 and t_2, on the lightcone $\mathbb{N}_p$. In this way, t_1 y t_2 are the same for any metric having $\mathbb{N}_p$ as lightcone. So, resolving the equation (2.1.1) we have

$$t_1 = -\tfrac{\mathbf{g}(u,v)}{\mathbf{g}(v,v)} + \sqrt{\tfrac{\mathbf{g}(u,v)^2}{\mathbf{g}(v,v)^2} - \tfrac{\mathbf{g}(u,u)}{\mathbf{g}(v,v)}}$$

$$t_2 = -\tfrac{\mathbf{g}(u,v)}{\mathbf{g}(v,v)} - \sqrt{\tfrac{\mathbf{g}(u,v)^2}{\mathbf{g}(v,v)^2} - \tfrac{\mathbf{g}(u,u)}{\mathbf{g}(v,v)}}$$

then

$$t_1 t_2 = \frac{\mathbf{g}\left(u,u\right)}{\mathbf{g}\left(v,v\right)} \tag{2.1.2}$$

for any metric $\mathbf{g}$ with $\mathbb{N}_p$ as lightcone.

We will use this property to finish the proof. Let us denote by

$$t_v\left(u\right) \equiv t_1 t_2 = \mathbf{g}\left(u,u\right)/\mathbf{g}\left(v,v\right)$$

and let us take any two vectors $w, z \in T_pM$. We have that

$$\mathbf{g}(w,z) = \frac{1}{2}\left(\mathbf{g}(w+z, w+z) - \mathbf{g}(w,w) - \mathbf{g}(z,z)\right)$$

Call $y = w, z, w+z$. If y is timelike, then we have that $\mathbf{g}(y,y)/\mathbf{g}(v,v) = t_v(y)$, but if y is spacelike then

$$\frac{\mathbf{g}(y,y)}{\mathbf{g}(v,v)} = \frac{\mathbf{g}(y,y)}{\mathbf{g}(u,u)}\frac{\mathbf{g}(u,u)}{\mathbf{g}(v,v)} = \frac{t_v(u)}{t_y(u)}$$

Finally, if y is null, then it is clear that $\mathbf{g}(y,y) = 0$. In any of the previous cases, the quotient $\mathbf{g}(w,z)/\mathbf{g}(v,v)$ can be written in terms of $t_v(y)$, $t_v(u)$ and $t_y(u)$ which only depends on the lightcone $\mathbb{N}_p$, therefore it coincides for all metrics with the same lightcone $\mathbb{N}_p$. Therefore, if $\mathbf{g}$ and $\overline{\mathbf{g}}$ are two metrics with the same lightcone, then

$$\frac{\mathbf{g}(w,z)}{\mathbf{g}(v,v)} = \frac{\overline{\mathbf{g}}(w,z)}{\overline{\mathbf{g}}(v,v)}$$

and so we have

$$\mathbf{g}(w,z) = \frac{\mathbf{g}(v,v)}{\overline{\mathbf{g}}(v,v)} \cdot \overline{\mathbf{g}}(w,z)$$

hence $\mathbf{g}(w,z)$ is fully determined except by the factor $\mathbf{g}(v,v)/\overline{\mathbf{g}}(v,v)$. □

Lemma 2.1.2 implies that $\mathbb{N}^+$ only depends on the conformal metric $\mathcal{C}_{\mathbf{g}}$. The previous proof is inspired in [25, p. 60-61] and an alternative proof can be found in [48, Prop. 2.6 and Lem. 2.7]. It is obvious that lemma 2.1.2 is also true for $\mathbb{N}_p^-$.

Proposition 2.1.3. *Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two spacetimes and let $\mathcal{N}_{\mathbf{g}}$ and $\mathcal{N}_{\overline{\mathbf{g}}}$ be their corresponding spaces of light rays. Then $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ are conformally equivalent if and only if $\mathcal{N}_{\mathbf{g}} = \mathcal{N}_{\overline{\mathbf{g}}}$.*

Proof. Assume that $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ are conformally equivalent, that is $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ with $\sigma \in \mathfrak{F}(M)$. By proposition 2.1.1, we have

$$\overline{\nabla}_X Y = d\sigma(X)\, Y + d\sigma(Y)\, X - \mathbf{g}(X,Y)\operatorname{grad}\sigma + \nabla_X Y$$

then, if $X \in \mathfrak{X}(M)$ is a geodesic null vector field related to $\overline{\mathbf{g}}$ then $\overline{\nabla}_X X = 0$ and $\overline{\mathbf{g}}(X,X) = e^{2\sigma}\mathbf{g}(X,X) = 0$. So, we get

$$2d\sigma(X)\, X + \nabla_X X = 0$$

it means that $\nabla_X X = -2d\sigma(X)\, X$ is proportional to X. If we consider the vector field $\widetilde{X} = e^{2\sigma}X$ proportional to X, then the integral curves of $\widetilde{X}$ are reparametrizations of the integral curves of X, and since

$$\begin{aligned}\nabla_{\widetilde{X}}\widetilde{X} &= e^{2\sigma}X\left(e^{2\sigma}\right)X + e^{4\sigma}\nabla_X X = \\ &= e^{2\sigma}X\left(e^{2\sigma}\right)X + e^{4\sigma}\left(-2d\sigma(X)\, X\right) = \\ &= 2e^{2\sigma}e^{2\sigma}X(\sigma)\, X - 2e^{4\sigma}X(\sigma)\, X = 0\end{aligned}$$

then the integral curves of $\widetilde{X}$ are geodesics related to the metric $\mathbf{g}$ and therefore the integral curves of X are pregeodesics in $(M, \mathbf{g})$. Then they describe the light rays of M for both metrics and hence we have that $\mathcal{N}_{\mathbf{g}} = \mathcal{N}_{\overline{\mathbf{g}}}$.

Let us prove the converse. Assume that $\mathcal{N}_{\mathbf{g}} = \mathcal{N}_{\overline{\mathbf{g}}}$. Given a vector $v \in \mathbb{N}_p$ there exists a null geodesic γ related to $\mathbf{g}$ such that $\gamma(0) = p$ and $\gamma'(0) = v$. Since $\mathcal{N}_{\mathbf{g}} = \mathcal{N}_{\overline{\mathbf{g}}}$, then the geodesic $\overline{\gamma}$ related to $\overline{\mathbf{g}}$ such that $\operatorname{Im}\gamma = \operatorname{Im}\overline{\gamma}$ verifies that $\overline{\gamma}'(0) = av$ with $a \neq 0$ is also a null vector related to $\overline{\mathbf{g}}$. Then the lightcones of $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ coincide at any point $p \in M$, hence any vector $v \in TM$ has the same causal character related to both metrics. Consider an open set $B \subset M$ and two vector fields $U, V \in \mathfrak{X}(B)$ such that U is timelike y V spacelike. Given any vector fields $W, Z \in \mathfrak{X}(B)$, by lemma 2.1.2, we have that

$$\frac{\mathbf{g}(W,Z)}{\mathbf{g}(V,V)} = \frac{\overline{\mathbf{g}}(W,Z)}{\overline{\mathbf{g}}(V,V)}$$

where U is necessary to establish the equation (2.1.2) of lemma 2.1.2. So, we get

$$\overline{\mathbf{g}}(W,Z) = \frac{\overline{\mathbf{g}}(V,V)}{\mathbf{g}(V,V)}\mathbf{g}(W,Z)$$

and since the term $\overline{\mathbf{g}}(V,V)/\mathbf{g}(V,V)$ is positive due to V is spacelike for $\mathbf{g}$ as well as for $\overline{\mathbf{g}}$, then denoting

$$\sigma(p) = \frac{1}{2}\log\left(\frac{\overline{\mathbf{g}}(V,V)}{\mathbf{g}(V,V)}\right)$$

we have that $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ as we wanted to show. □

Proposition 2.1.3 permits to state the next definition.

Definition 2.1.4. *Let $(M, \mathcal{C}_{\mathbf{g}})$ be a conformal manifold with* $\dim M = m \geq 3$. *We will name* light ray *to the image $\gamma(I)$ in M of a maximal null geodesic $\gamma : I \to M$ related to any metric* $\mathbf{g} \in \mathcal{C}_{\mathbf{g}}(M)$. *It will be denoted by $[\gamma]$ or γ when there is not possibility of confusion, that is $[\gamma] \in \mathcal{N}$, $\gamma \in \mathcal{N}$ or also $\gamma \subset M$. So, every light ray is equivalent to an unparametrized null geodesic. Then, we will say that the* space of light rays $\mathcal{N}$ of a conformal manifold $(M, \mathcal{C}_{\mathbf{g}})$ *is the set*

$$\mathcal{N} = \{\gamma(I) \subset M / \gamma : I \to M \textit{ is a maximal null geodesic for any metric } \overline{\mathbf{g}} \in \mathcal{C}_{\mathbf{g}}\}$$

Section 2.2

Differentiable structure of $\mathcal{N}$

A more geometric construction of $\mathcal{N}$ is possible, as Low does in [45], from a quotient space of the tangent bundle TM. This construction will allow $\mathcal{N}$ to inherit the topological and differentiable structures of TM.

Let us consider the *geodesic spray* $X_{\mathbf{g}}$ related to the metric $\mathbf{g}$, that is the vector field in TM such that its integral curves define the geodesics in $(M, \mathbf{g})$ and their tangent vectors. So, the canonical projection $\pi_M^{TM} : TM \to M$ maps integral curves of $X_{\mathbf{g}}$ into geodesics of M. Take a coordinate chart $\left(\left(x^k,v^k\right),TU\right)$ in TM such that a vector $v\in TU$ can be written as $v=v^k\frac{\partial}{\partial x^k}$, where x^k with $k=1,\ldots,m$ are coordinates in M. The expression of the geodesic spray $X_{\mathbf{g}}$ in these coordinates is

$$X_{\mathbf{g}}=v^k\frac{\partial}{\partial x^k}-\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k} \tag{2.2.1}$$

where Γ_{ij}^k with $i,j,k=1,\ldots,m$ denotes the Christoffel symbols of Levi–Civita connection ∇ for $\mathbf{g}$.

We claim that $X_{\mathbf{g}}$ is tangent to the bundle $\mathbb{N}$. Indeed, for any geodesic γ, the curve $\gamma'(t)\in T_{\gamma(t)}M$ is an integral curve of $X_{\mathbf{g}}$. Calling $f(v)=\mathbf{g}(v,v)$, then we have $f(\gamma'(t))$ constant, hence $X_{\mathbf{g}}(f)=0$ and therefore $X_{\mathbf{g}}$ is tangent to any level set of f, in particular it is tangent to $\mathbb{N}=f^{-1}(0)$.

Observe that the integral curve of $X_{\mathbf{g}}$ passing through $v\in\mathbb{N}^+$ is projected on the null geodesic $\gamma\subset M$ such that $\gamma(t_0)=\pi_M^{\mathbb{N}}(v)$ and $\gamma'(t_0)=v$, and moreover $\gamma'(t)$ for all t. Then $X_{\mathbf{g}}$ is tangent to $\mathbb{N}^+$.

On the other hand, we define the *Euler field* Δ *in* TM as the vector field in TM dilating the vector fields in M, that is, if $u\in T_pM$ then

$$\Delta(u)=dc\left(\frac{\partial}{\partial t}\right)(0)$$

where $c:\mathbb{R}\rightarrow T_pM$ is defined by $c(t)=e^t u$. In case of $u\in\mathbb{N}_p^+$, since for all $t\in\mathbb{R}$ we have that $e^t u\in\mathbb{N}_p^+$, then c is a curve in $\mathbb{N}_p^+$. Moreover, since

$$c'(t)=dc\left(\frac{\partial}{\partial t}\right)(t)=\Delta(c(t))$$

then c is an integral curve of Δ contained in $\mathbb{N}^+$ if $u\in\mathbb{N}^+$, then the Euler field Δ is tangent to $\mathbb{N}^+$. In the previous coordinates $\left(x^k,v^k\right)$, the field Δ can be expressed by

$$\Delta=v^k\frac{\partial}{\partial v^k} \tag{2.2.2}$$

By expressions (2.2.1) and (2.2.2), it is clear that both $X_{\mathbf{g}}$ and Δ are differentiable vector fields in TM.

Now, we can define the differentiable distribution in $\mathbb{N}^+$ given by $\mathcal{D}=\operatorname{span}\{X_{\mathbf{g}},\Delta\}$. Since

$$\begin{aligned}
[\Delta,X_{\mathbf{g}}]&=\left[v^l\frac{\partial}{\partial v^l},v^k\frac{\partial}{\partial x^k}-\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k}\right]=\\
&=\left[v^l\frac{\partial}{\partial v^l},v^k\frac{\partial}{\partial x^k}\right]-\left[v^l\frac{\partial}{\partial v^l},\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k}\right]=\\
&=v^k\frac{\partial}{\partial x^k}-v^l\left(\frac{\partial\Gamma_{ij}^k}{\partial v^l}v^i v^j+\Gamma_{ij}^k v^j\delta_l^i+\Gamma_{ij}^k v^i\delta_l^j\right)\frac{\partial}{\partial v^k}+\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k}=\\
&=v^k\frac{\partial}{\partial x^k}-\left(v^i\Gamma_{ij}^k v^j+v^j\Gamma_{ij}^k v^i\right)\frac{\partial}{\partial v^k}+\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k}=\\
&=v^k\frac{\partial}{\partial x^k}-\Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k}=X_{\mathbf{g}}\in\mathcal{D}
\end{aligned}$$

then $\mathcal{D}$ is involutive and, by Fröbenius' Theorem [61, Thm. 1.60], it is also integrable. This means that the quotient space $\mathbb{N}^+/\mathcal{D}$ is well defined. Every leaf of $\mathcal{D}$ is the equivalence class consisting of a future–directed null geodesic and all its affine reparametrizations preserving time–orientation, hence the space of light rays $\mathcal{N}$ of M that we want to construct is precisely the quotient space $\mathbb{N}^+/\mathcal{D}$, that is

$$\mathcal{N} = \mathbb{N}^+/\mathcal{D}$$

Remark 2.2.1. *The construction of $\mathcal{N}$ can also be done by the quotient*

$$\mathcal{N} = \mathbb{N}/\mathcal{D}$$

since the distribution $\mathcal{D}$ is still involutive and the null geodesic defined by $v \in \mathbb{N}^+$ has the same image than the one defined by $-v$. Working with $\mathbb{N}^+$ assumes that null geodesics are future–oriented. Along this essay, we will usually work this way.

Lemma 2.2.2. *Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two conformally equivalent spacetimes such that $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$, and let $X_{\mathbf{g}} \in \mathfrak{X}(\mathbb{N}^+)$ and $X_{\overline{\mathbf{g}}} \in \mathfrak{X}(\mathbb{N}^+)$ be their respective geodesics sprays. Then we have that*

$$X_{\overline{\mathbf{g}}} = -2d\sigma \cdot \Delta + X_{\mathbf{g}}$$

Proof. Let us consider the chart $\varphi = \left(x^k, v^k\right)$ defined in $W \subset TM$ as above. Let $\overline{\Gamma}_{ij}^k$ and Γ_{ij}^k be the Christoffel symbols related to the metrics $\overline{\mathbf{g}}$ and $\mathbf{g}$ respectively. So, we have

$$\begin{aligned}
\overline{\Gamma}_{ij}^k &= \frac{1}{2}\overline{g}^{mk}\left(\frac{\partial \overline{g}_{im}}{\partial x^j} + \frac{\partial \overline{g}_{jm}}{\partial x^i} - \frac{\partial \overline{g}_{ij}}{\partial x^m}\right) = \\
&= \frac{1}{2}e^{-2\sigma}g^{mk}\left(\frac{\partial\left(e^{2\sigma}g_{im}\right)}{\partial x^j} + \frac{\partial\left(e^{2\sigma}g_{jm}\right)}{\partial x^i} - \frac{\partial\left(e^{2\sigma}g_{ij}\right)}{\partial x^m}\right) = \\
&= \frac{\partial\sigma}{\partial x^j}g^{mk}g_{im} + \frac{\partial\sigma}{\partial x^i}g^{mk}g_{jm} - \frac{\partial\sigma}{\partial x^m}g^{mk}g_{ij} + \Gamma_{ij}^k = \\
&= \frac{\partial\sigma}{\partial x^j}\delta_i^k + \frac{\partial\sigma}{\partial x^i}\delta_j^k - \frac{\partial\sigma}{\partial x^m}g^{mk}g_{ij} + \Gamma_{ij}^k
\end{aligned}$$

where δ_i^j denotes the Kronecker's delta. So, the geodesic spray $X_{\overline{\mathbf{g}}}$ can be written as

$$\begin{aligned}
X_{\overline{\mathbf{g}}} &= v^k\frac{\partial}{\partial x^k} - \overline{\Gamma}_{ij}^k v^i v^j\frac{\partial}{\partial v^k} = \\
&= v^k\frac{\partial}{\partial x^k} - \frac{\partial\sigma}{\partial x^j}v^k v^j\frac{\partial}{\partial v^k} - \frac{\partial\sigma}{\partial x^i}v^i v^k\frac{\partial}{\partial v^k} + \frac{\partial\sigma}{\partial x^m}g^{mk}g_{ij}v^i v^j\frac{\partial}{\partial v^k} - \Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k} = \\
&= v^k\frac{\partial}{\partial x^k} - \frac{\partial\sigma}{\partial x^j}v^k v^j\frac{\partial}{\partial v^k} - \frac{\partial\sigma}{\partial x^i}v^i v^k\frac{\partial}{\partial v^k} - \Gamma_{ij}^k v^i v^j\frac{\partial}{\partial v^k} = \\
&= -2\frac{\partial\sigma}{\partial x^j}v^k v^j\frac{\partial}{\partial v^k} + X_{\mathbf{g}} = \\
&= -2d\sigma \cdot \Delta + X_{\mathbf{g}}
\end{aligned}$$

as we claimed and where we have used that $g_{ij}v^i v^j = 0$ since $X_{\mathbf{g}}$ is restricted to $\mathbb{N}^+$. □

In order to give differentiable structure to a quotient space, we will need to define what is a regular distribution and to use the proposition 2.2.4 for this purpose.

Definition 2.2.3. *A k–dimensional integrable distribution $\mathcal{D}$ in M is said to be* regular *if for every point in M there exists a coordinate chart (φ, U) adapted to $\mathcal{D}$, that is a chart such that for every leaf $\mathcal{F}$ of the foliation generated by $\mathcal{D}$ there exist $c_{k+1}, \ldots, c_n \in \mathbb{R}$ verifying that $x_j(\mathcal{F} \cap U) = c_j$ for all $j = k+1, \ldots, n$.*

The next proposition and its proof can be found at [11, Prop. 11.4.2].

Proposition 2.2.4. *Let $\mathcal{D}$ be a regular distribution in a differentiable manifold M. Then, a differentiable structure can be provided to the set $\mathcal{F}$ of leaves of $\mathcal{D}$ in such a way the canonical projection $p : M \to \mathcal{F}$ is a submersion.*

Lemma 2.2.2 allows to prove the next proposition.

Proposition 2.2.5. *The differentiable structure of the space of light rays $\mathcal{N}$ of $(M, \mathcal{C}_{\mathbf{g}})$ does not depend on the representative $\overline{\mathbf{g}}$ of the conformal metric $\mathcal{C}_{\mathbf{g}}$.*

Proof. Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two conformally equivalent spacetimes such that $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ and let $X_{\mathbf{g}}, X_{\overline{\mathbf{g}}} \in \mathfrak{X}(\mathbb{N}^+)$ be their corresponding geodesic sprays restricted to $\mathbb{N}^+$. Consider the distributions $\mathcal{D} = \operatorname{span}\{X_{\mathbf{g}}, \Delta\}$ and $\overline{\mathcal{D}} = \operatorname{span}\{X_{\overline{\mathbf{g}}}, \Delta\}$. Then, by lemma 2.2.2 we have

$$\overline{\mathcal{D}} = \operatorname{span}\{X_{\overline{\mathbf{g}}}, \Delta\} = \operatorname{span}\{-2d\sigma \cdot \Delta + X_{\mathbf{g}}, \Delta\} = \operatorname{span}\{X_{\mathbf{g}}, \Delta\} = \mathcal{D}$$

and hence the distribution $\mathcal{D}$ does not depends on the metric $\mathbf{g}$ inside the same conformal metric $\mathcal{C}_{\mathbf{g}}$. Then $\mathcal{N} = \mathbb{N}^+/\mathcal{D}$ only depends on the conformal metric and not on their representatives. □

If we require the space of light rays of M to be a differentiable manifold, it is necessary to ensure that the leaves of the distribution that builds $\mathcal{N}$, are regular submanifolds. This characteristic is not automatically obtained for any spacetime M, as example 2.2.6 shows, so it will be necessary to impose further conditions to ensure it.

Example 2.2.6. *Light rays are not always leaves of a regular distribution. An analogous example can be seen in [44, Ex. 1]. Consider the restriction of the two–dimensional Minkowski spacetime to the rectangle $R = [0, \alpha) \times [0, 1)$ with $\alpha \in \mathbb{R} - \mathbb{Q}$ identifying its borders as $(x, 1) \sim (x, 0)$ for all $x \in [0, \alpha)$ and $(\alpha, t) \sim (0, t)$ for all $t \in [0, 1)$. Then any null geodesic is dense in R and therefore the distribution can not be regular. Figure 2.1 illustrates how the null geodesic γ moves from the point $(0, 0) \in R$ to become dense due to the irrationality of the value α.*

Let us use the proposition 2.2.4 above to show that the space of light rays $\mathcal{N}$ has a differentiable structure. It is possible to find the following result and its proof at [39, Prop. 2.1].

Proposition 2.2.7. *Let M be a strongly causal spacetime, then the distribution above defined by $\mathcal{D} = \operatorname{span}\{X_{\mathbf{g}}, \Delta\}$ is regular and the space of light rays $\mathcal{N}$ inherits from $\mathbb{N}^+$ the structure of differentiable manifold such that $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ defined by $p_{\mathbb{N}^+}(u) = [\gamma_u]$ is a submersion.*

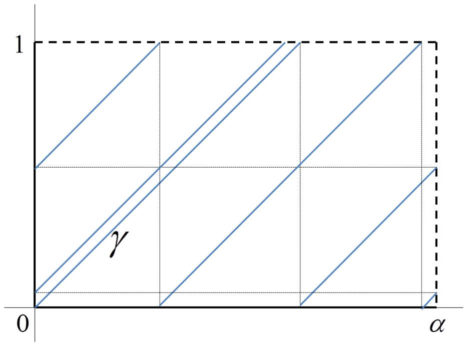

Figure 2.1: $\mathcal{D}$ is not regular.

Proof. We have that $\mathbb{N}^+$ is foliated by the elevation of null geodesics from M. Let $\mathcal{D}$ be the distribution generated by this foliation and consider the canonical projection $\pi_M^{\mathbb{N}^+} : \mathbb{N}^+ \to M$. Given $u \in \mathbb{N}^+$, there exists an adapted coordinate chart (ψ, U) to $\mathcal{D}$ in u. Since $\pi_M^{\mathbb{N}^+}$ is a submersion, then $\pi_M^{\mathbb{N}^+}(U)$ is open in M containing $\pi_M^{\mathbb{N}^+}(u) = p \in M$. By proposition 1.2.18, there exists a neighbourhood V of p such that if γ is a causal curve passing through V, then $\gamma \cap V$ have a unique connected component. So, the elevation of any null geodesic γ to $\mathbb{N}^+$ will intersect $\left(\pi_M^{\mathbb{N}^+}\right)^{-1}(V)$ in exactly one connected component, hence, denoting $W = U \cap \left(\pi_M^{\mathbb{N}^+}\right)^{-1}(V)$, then we have that $\left(\psi|_W, W\right)$ is an adapted chart to $\mathcal{D}$ verifying that each leaf of the generated foliation (that is each null geodesic with its null tangent vector at every point) is regular in W. Now, applying proposition 2.2.4, we conclude that $\mathcal{N}$ inherits from $\mathbb{N}^+$ the differentiable structure and, moreover $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ is a submersion. $\square$

The space $\mathcal{N}$ can also be constructed as a quotient of the bundle of null directions $\mathbb{PN}$ defined below.

In order to construct $\mathcal{N}$ in this way, we need to build $\mathbb{PN}$ as the quotient $\mathbb{N}^+/\mathcal{D}_\Delta$ where $\mathcal{D}_\Delta = \text{span}\{\Delta\}$. First, we will study if $\mathcal{D}_\Delta$ is a regular distribution in $\mathbb{N}^+$. Consider a local chart $\left(V, \varphi = \left(x^1, \ldots x^m\right)\right)$ in M and let $\{E_1, \ldots, E_m\}$ be a orthonormal frame in V such that E_1 is a future timelike vector field. A vector $\xi \in T_pV$ can be written as $\xi = \sum_{j=1}^{m} u^j E_j(p)$ then (ϕ, TV) with

$$\phi\colon TV \to \mathbb{R}^{2m}; \quad \xi \mapsto \left(x^1, \ldots, x^m, u^1, \ldots, u^m\right) \tag{2.2.3}$$

is a coordinate chart in TM. Let us denote by $\mathbb{N}^+(V)$ the restriction of the bundle $\mathbb{N}^+$ to the base V. For $\xi \in \mathbb{N}^+(V)$ we have that $\left(u^1\right)^2 = \sum_{j=2}^{m} \left(u^j\right)^2$ and hence coordinates in $\mathbb{N}^+(V)$ can be given by the map

$$\phi_{\mathbb{N}^+} : \mathbb{N}^+(V) \to \mathbb{R}^{2m-1}; \quad \xi \mapsto \left(x^1, \ldots, x^m, u^2, \ldots, u^m\right) \tag{2.2.4}$$

We have seen above that the Euler field Δ is tangent to $\mathbb{N}^+$ and it determines a differentiable distribution, that being 1–dimensional, is also involutive. Since for all $\xi_0 \in \mathbb{N}^+$ some of the coordinates $u^k(\xi_0)$ with $k = 2, \ldots, m$ does not vanish, then there exists a neighbourhood $W \subset \mathbb{N}^+$ of ξ_0 such that $u^k(\xi) \neq 0$ for all $\xi \in W$. Assuming, without any lack of generality, that $u^2 \neq 0$ in W, a coordinate chart $\overline{\phi}_{\mathbb{N}^+}$ can be defined in W by

$$\overline{\phi}_{\mathbb{N}^+} : \mathbb{N}^+(W) \to \mathbb{R}^{2m-1}; \quad \xi \mapsto \left(x^1, \ldots, x^m, w^2, w^3, \ldots, w^m\right) \in \mathbb{R}^{2m-1} \tag{2.2.5}$$

where $w^2 = u^2$ and $w^k = \frac{u^k}{u^2}$ for $k = 3, \ldots, m$. If $c(t) = e^t \xi$ is the integral curve of Δ passing through $\xi \in \mathbb{N}^+$, and

$$\phi_{\mathbb{N}^+}(\xi) = \left(x_0^1, \ldots, x_0^m, u_0^2, u_0^3, \ldots, u_0^m\right)$$

then

$$\overline{\phi}_{\mathbb{N}^+}(c(t)) = \left(x_0^1, \ldots, x_0^m, e^t u_0^2, \frac{u_0^3}{u_0^2} \ldots, \frac{u_0^m}{u_0^2}\right) \tag{2.2.6}$$

hence $\overline{\phi}_{\mathbb{N}^+}$ is a chart adapted to the integral curves of Δ. Moreover, if $\eta \in \mathbb{N}^+$ verifies

$$\begin{cases} x^k(\eta) = x_0^k & \text{for } k = 1, \ldots, m \\ \\ w^k(\eta) = \frac{u_0^k}{u_0^2} & \text{for } k = 3, \ldots, m \end{cases}$$

then, it is clear that $\eta = e^{t_0}\xi$ for some $t_0 \in \mathbb{R}$. This implies that the distribution $\mathcal{D}_\Delta = \operatorname{span}\{\Delta\}$ is regular. By proposition 2.2.4, the quotient space $\mathbb{N}^+/\mathcal{D}_\Delta$ defined by

$$\mathbb{PN} = \mathbb{N}^+/\mathcal{D}_\Delta = \left\{[\xi] : \eta \in [\xi] \Leftrightarrow \eta = e^t\xi \text{ for some } t \in \mathbb{R} \text{ and } \xi \in \mathbb{N}^+\right\}$$

is a differentiable manifold and, moreover, the canonical projection

$$\begin{array}{rccc} \pi_{\mathbb{PN}}^{\mathbb{N}^+} : & \mathbb{N}^+ & \to & \mathbb{PN} \\ & \xi & \mapsto & [\xi] \end{array}$$

is a submersion.

The next step is to find a regular distribution that allows us to define $\mathcal{N}$ by a quotient. For each vector $u \in \mathbb{N}_p^+$ there exists a null geodesic γ_u such that $\gamma_u(0) = p$ and $\gamma_u'(0) = u$, and given two vectors $u, v \in \mathbb{N}_p^+$ verifying that $v = \lambda u$ with $\lambda > 0$, then the geodesics γ_u and γ_v such that $\gamma_u(0) = \gamma_v(0) = p$ have the property

$$\gamma_v(s) = \gamma_{\lambda u}(s) = \gamma_u(\lambda s)$$

hence they have the same image in M and then $\gamma_v = \gamma_u$ as unparametrized sets in M. This fact implies that the elevations to $\mathbb{PN}$ of the null geodesics of M define a foliation $\mathcal{D}_G$. Two directions $[u], [v] \in \mathbb{PN}$ belong to the same leaf of the foliation $\mathcal{D}_G$ if for the vectors $v \in \mathbb{N}_p^+$ and $u \in \mathbb{N}_q^+$ there exist null geodesics γ_1 and γ_2 and values $t_1, t_2 \in \mathbb{R}$ verifying

$$\begin{cases} \gamma_1(t_1) = p \in M \\ \gamma_1'(t_1) = v \in \mathbb{N}_p^+ \end{cases} \quad \text{and} \quad \begin{cases} \gamma_2(t_2) = q \in M \\ \gamma_2'(t_2) = u \in \mathbb{N}_q^+ \end{cases}$$

such that there is a reparametrization h verifying $\gamma_1 = \gamma_2 \circ h$.

Hence, the space of leaves of $\mathcal{D}_G$ in $\mathbb{PN}$ coincides with $\mathcal{N}$, that is,

$$\mathcal{N} = \mathbb{PN}/\mathcal{D}_G$$

The map

$$\begin{array}{rccc} p_{\mathbb{PN}}: & \mathbb{PN} & \longrightarrow & \mathcal{N} \\ & [u] & \longmapsto & [\gamma_u] \end{array}$$

is well defined, since $\gamma_{\lambda u}(s) = \gamma_u(\lambda s)$ as seen above, and it verifies the identity

$$p_{\mathbb{PN}}\left([\gamma_u'(s)]\right) = [\gamma_u] \in \mathcal{N}$$

for all s.

Remark 2.2.8. *Proposition 2.2.7 can be formulated for the bundle $\mathbb{PN}$ instead of $\mathbb{N}^+$, because the proof is,* mutatis mutandis, *the same, where in this case $\mathcal{D}_G$ is a regular distribution and a differentiable structure is also inherited from $\mathbb{PN}$ such that $p_{\mathbb{PN}}: \mathbb{PN} \to \mathcal{N}$ is a submersion. In fact, there exist an unique differential structure in $\mathcal{N}$ such that $p_{\mathbb{PN}}: \mathbb{PN} \to \mathcal{N}$, as well as $p_{\mathbb{N}^+}: \mathbb{N}^+ \to \mathcal{N}$ are submersions. In [45, Thm. 1], this result is shown for the subbundle $\mathbb{N}^{+*}$ of the cotangent bundle T^*M.*

Now, we will describe a generic way to construct coordinate charts in $\mathcal{N}$. First, for any subset $W \subset M$, we define

$$\mathbb{N}^+(W) = \left\{\xi \in \mathbb{N}^+ : \pi_M^{\mathbb{N}^+}(\xi) \in W \subset M\right\}$$

$$\mathbb{PN}(W) = \left\{[\xi] \in \mathbb{PN} : \pi_M^{\mathbb{PN}}([\xi]) \in W \subset M\right\}.$$

By theorem 1.2.15, we can take $V \subset M$ as a basic open set. Let $\mathcal{U}$ be the image of the projection $p_{\mathbb{N}}: \mathbb{N}^+(V) \mapsto \mathcal{N}$. Since $\mathbb{N}^+(V)$ is open in $\mathbb{N}^+$ and $p_{\mathbb{N}}$ is a submersion, then $\mathcal{U} \subset \mathcal{N}$ is open. Moreover, since V is globally hyperbolic, then we can fix a smooth spacelike Cauchy surface $C \subset V$. So, each null geodesic passing through V intersects C in a unique point and since $p_{\mathbb{N}^+} = p_{\mathbb{PN}} \circ \pi_{\mathbb{PN}}^{\mathbb{N}^+}$, this ensures that

$$\mathcal{U} = p_{\mathbb{N}}\left(\mathbb{N}^+(V)\right) = p_{\mathbb{N}}\left(\mathbb{N}^+(C)\right) = p_{\mathbb{N}} \circ \pi_{\mathbb{PN}}^{\mathbb{N}^+}\left(\mathbb{N}^+(C)\right) = p_{\mathbb{PN}}\left(\mathbb{PN}(C)\right) = p_{\mathbb{PN}}\left(\mathbb{PN}(V)\right).$$

Since C is a regular differentiable submanifold of V, then the bundles $\mathbb{N}^+(C)$ and $\mathbb{PN}(C)$ are also regular differentiable submanifolds of $\mathbb{N}^+(V)$ and $\mathbb{PN}(V)$ respectively, and moreover the map $\sigma = p_{\mathbb{PN}}|_{\mathbb{PN}(C)}: \mathbb{PN}(C) \mapsto \mathcal{U}$ is a differentiable bijection. The map $p_{\mathbb{PN}}$ is a submersion verifying that for any $[\xi] \in \mathbb{PN}(V)$, the kernel of $(dp_{\mathbb{PN}})_{[\xi]}$ is the 1–dimensional subspace generated by the tangent vectors to curves defining light rays, that is, curves $\lambda(s) = [\gamma'(s)] \in \mathbb{PN}_{\gamma(s)}$ where γ is a null geodesic and

$$[\gamma'(s)] = \{\lambda\gamma'(s) : \lambda \in \mathbb{R}\}.$$

Being C a spacelike hypersurface, the kernel of $\left(dp_{\mathbb{PN}}|_{\mathbb{PN}(C)}\right)_{[\xi]} = d\sigma_{[\xi]}$ is trivial, hence $d\sigma_{[\xi]}$ is a surjective linear map between vector spaces of the same dimension, then it is also bijective and therefore σ is a diffeomorphism. So, we have the following diagram

$$\begin{tikzcd} \mathbb{PN}(V) \arrow[r, "p_{\mathbb{PN}}"] & \mathcal{U} \\ \mathbb{PN}(C) \arrow[u, hook, "\text{inc}"] \arrow[ur, "\sigma"'] & \end{tikzcd} \tag{2.2.7}$$

If ϕ is any coordinate chart for $\mathbb{PN}(C)$ then $\phi \circ \sigma^{-1}$ is a coordinate chart for $\mathcal{U} \subset \mathcal{N}$.

Observe that if M is time–orientable, there exists a non–vanishing future timelike vector field $T \in \mathfrak{X}(M)$. Then we can define the submanifold $\Omega^T(C) \subset \mathbb{N}^+(C)$ by

$$\Omega^T(C) = \left\{ \xi \in \mathbb{N}^+(C) : \mathbf{g}(\xi, T) = -1 \right\}.$$

We have that $\pi_{\mathbb{PN}}^{\mathbb{N}^+} : \mathbb{N}^+ \to \mathbb{PN}$ is a submersion such that the kernel of the differential $d\pi_{\mathbb{PN}}^{\mathbb{N}^+}$ at any point $\xi \in \mathbb{N}^+$ is generated by $\Delta(\xi)$. If we consider the restriction $\left.\pi_{\mathbb{PN}}^{\mathbb{N}^+}\right|_{\Omega^T(C)} : \Omega^T(C) \to \mathbb{PN}(C)$, it is clear that it is a bijection. Moreover, since

$$\ker\left(\left(\left. d\pi_{\mathbb{PN}}^{\mathbb{N}^+}\right|_{\Omega^T(C)}\right)_{\xi}\right) = \{\mathbf{0}\}$$

at any point ξ, and due to $\dim\left(\Omega^T(C)\right) = \dim\left(\mathbb{PN}(C)\right) = 2m-3$, then $\left.\pi_{\mathbb{PN}}^{\mathbb{N}^+}\right|_{\Omega^T(C)}$ is a diffeomorphism. So, we have the following diagram

$$\mathcal{N} \supset \mathcal{U} \leftrightarrow \mathbb{PN}(C) \leftrightarrow \Omega^T(C) \hookrightarrow \mathbb{N}^+(C) \hookrightarrow \mathbb{N}^+ \hookrightarrow TM \tag{2.2.8}$$

where $\leftrightarrow$ and $\hookrightarrow$ represent diffeomorphisms and inclusions respectively.

Then, the composition of the diffeomorphism $\mathcal{U} \to \Omega^T(C)$ with the restriction of a coordinate chart in TM to the vectors in $\Omega^T(C)$, can be used to construct a coordinate chart in $\mathcal{N}$.

Remark 2.2.9. *By construction of the diffeomorphism $\sigma : \mathbb{PN}(C) \to \mathcal{U} \subset \mathcal{N}$, if M is globally hyperbolic then it is possible to choose $V = M$ and C a global Cauchy surface. In this case we have that $\sigma : \mathbb{PN}(C) \to \mathcal{N}$ is a global diffeomorphism.*

If there exists a non-vanishing $X \in \mathfrak{X}(C)$, then $\mathbb{PN}(C)$ is a trivial fibre bundle because it is possible to construct a global section taking X and a non-vanishing timelike vector field $T \in \mathfrak{X}(M)$. Since X is spacelike then for any $p \in C$ there exist $\alpha_p > 0$ such that $T_p + \alpha_p X_p \in T_pM$ is a null vector. Then $s : C \to \mathbb{PN}(C)$ defined by

$$s(p) = [T_p + \alpha_p X_p] \in \mathbb{PN}_p \subset \mathbb{PN}(C)$$

is a global section and therefore

$$\mathcal{N} \simeq C \times \mathbb{S}^{m-2}.$$

If we require the space of light rays of M to be a differentiable manifold, it remains to ensure that $\mathcal{N}$ is a Hausdorff topological space. Again, it is not verified for any strongly causal spacetime M as we can check in example 2.2.10, so we need to state conditions to ensure it.

Example 2.2.10. *$\mathcal{N}$ is not Hausdorff. Consider the 2–dimensional Minkowski spacetime and remove the point $(1,1)$. Clearly, M is strongly causal. Let $\{\tau_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n \mapsto \infty} \tau_n = 0$. Then the sequence of null geodesic given by $\lambda_n(s) = (s, \tau_n + s)$ with $s \in (-\infty, \infty)$ converges to two different null geodesics, $\mu_1(s) = (s,s)$ with $s \in (-\infty, 1)$ and $\mu_2(s) = (s,s)$ with $s \in (1, \infty)$. Figure 2.2 illustrates this example.*

A sufficient condition to ensure that $\mathcal{N}$ is Hasudorff is the absence of naked singularities, as next proposition shows. But we will see in example 2.2.12 that it is not a necessary condition.

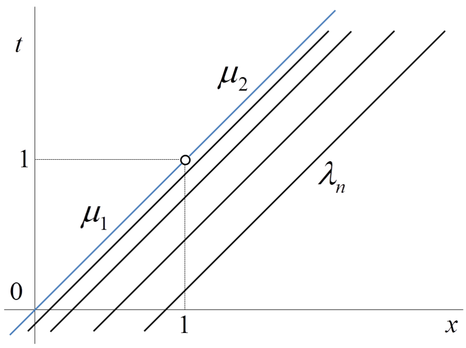

Figure 2.2: $\mathcal{N}$ is not Hausdorff.

Proposition 2.2.11. *Let M be a strongly causal spacetime and $\mathcal{N}$ its corresponding space of light rays. If $\mathcal{N}$ is not Hausdorff then M possesses a naked singularity.*

Proof. We will follow the proof of [39, Prop. 2.2]. If $\mathcal{N}$ is not Hausdorff, then there exists two light rays $\gamma_1, \gamma_2 \in \mathcal{N}$ such that any pair of neighbourhoods $U_1, U_2 \subset \mathcal{N}$ of γ_1 and γ_2 respectively verifies that $U_1 \cap U_2 \neq \varnothing$. Hence, it is possible to build a sequence $\{\mu_n\} \subset \mathcal{N}$ such that γ_1 and γ_2 are their limits. If we consider the same sequence as curves in M, we can take points $p_1 \in \gamma_1 \subset M$ and $p_2 \in \gamma_2 \subset M$ and corresponding neighbourhoods V_1 and V_2 such that $V_1 \cap V_2 = \varnothing$. This is possible since M is actually Hausdorff. We can assume without any lack of generality that $\mu_n \cap V_i \neq \varnothing$ for all n with $i = 1, 2$. Let us take points $q_n^i \in \mu_n \cap V_i$ with $i = 1, 2$ such that p_i is a limit point of the sequence$\left\{q_n^i\right\}$. Since each light ray μ_n is a causal curve, we can consider that $q_n^2 \in J^+\left(q_n^1\right)$ for all n. If $r \in I^+(p_2)$ then $I^-(r)$ is a neighbourhood of p_2 and, hence there exists n_0 such that $q_n^2 \in I^-(r)$ for all $n > n_0$. Moreover, since $q_n^2 \in J^+\left(q_n^1\right)$ then $q_n^1 \in I^-(r)$, therefore $p_1 \in \overline{I^-(r)}$. Now if we take $w \in I^-(p_1)$ then $I^+(w)$ is a neighbourhood of p_1 and it must intersect $I^-(r)$, hence $w \in I^-(r)$ but, since it does not depends on the chosen point $p_1 \in \gamma_1$, then any point of $z \in I^-(\gamma_1)$ verifies that $z \in I^-(r)$. Consequently $I^-(\gamma_1) \subset I^-(r)$ and since γ_1 is an inextensible causal curve then there exists a naked singularity in M. $\square$

Example 2.2.12. *Let $\mathbb{M}$ be the 3–dimensional Minkowski spacetime described by coordinates (t, x, y) and equipped with the metric $\mathbf{g} = -dt \otimes dt + dx \otimes dx + dy \otimes dy$. The hypersurface $C \equiv \{t = 0\}$ is a spacelike Cauchy surface. The corresponding space of light rays $\mathcal{N}_{\mathbb{M}}$ is diffeomorphic to the bundle of circumferences on C, that is, $\mathcal{N}_{\mathbb{M}} \simeq C \times \mathbb{S}^1$.*

Now, consider the restriction $\mathbb{B} = \left\{(t, x, y) \in \mathbb{M} : t^2 + x^2 + y^2 < 1\right\}$. It is clear that $\mathbb{B}$ is strongly causal.

First, we will see that $\mathbb{B}$ is not globally hyperbolic. Consider the inextensible null geodesics in $\mathbb{B}$ given by

$$\gamma_1(s) = \left(s, \frac{7}{5} - s, 0\right) \qquad s \in \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\gamma_2(\tau) = \left(\tau, \frac{7}{5} + \tau, 0\right) \qquad \tau \in \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

It is easy to see that any point of γ_1 *is in the chronological future of any point of* γ_2. *Indeed, the curve* $\mu(u) = \gamma_2(\tau) + u \cdot (\gamma_1(s) - \gamma_2(\tau))$ *is a future–directed timelike geodesic connecting* $\gamma_2(\tau)$ *to* $\gamma_1(s)$ *since*

$$\mu'(u) = (s - \tau, s + \tau, 0)$$

and

$$\mathbf{g}(\mu', \mu') = 4s\tau < 0$$

for all $s \in \left(\frac{3}{5}, \frac{4}{5}\right)$ *and* $\tau \in \left(-\frac{4}{5}, -\frac{3}{5}\right)$. *If a spacelike Cauchy surface* $\Omega \subset \mathbb{B}$ *exists, then* $\Omega \cap \gamma_i \neq \varnothing$ *for* $i = 1, 2$, *and then* Ω *would have timelike related points, but this is not possible in a Cauchy surface. Therefore* $\mathbb{B}$ *is not globally hyperbolic.*

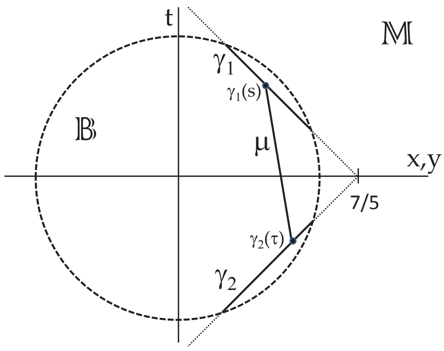

Figure 2.3: $M = \mathbb{B}$ is naked singular and Hausdorff.

We have already shown that $\mathbb{B}$ *is nakedly singular, because for any* $s \in \left(\frac{3}{5}, \frac{4}{5}\right)$ *we have that*

$$I^-(\gamma_2) \subset I^-(\gamma_1(s))$$

Finally, we will show that the space of light rays $\mathcal{N}_{\mathbb{B}}$ *of* $\mathbb{B}$ *is Hausdorff. It is clear that* $\mathcal{N}_{\mathbb{B}} \subset \mathcal{N}_{\mathbb{M}}$. *Consider* $\gamma \in \mathcal{N}_{\mathbb{B}}$. *As a curve in* $\mathcal{N}_{\mathbb{M}} = C \times \mathbb{S}^1$ *we denote*

$$\gamma = (x_0, y_0, \theta_0)$$

We can parametrize γ *as* $\gamma(s) = (s, x_0 + s\cos\theta_0, y_0 + s\sin\theta_0)$ *and since* $\gamma \in \mathcal{N}_{\mathbb{B}}$, *then there exists* $s_0 \in \mathbb{R}$ *such that*

$$s_0^2 + (x_0 + s_0\cos\theta_0)^2 + (y_0 + s_0\sin\theta_0)^2 < 1 \tag{2.2.9}$$

Since inequality (2.2.9) is an open condition, then there exist $\alpha, \beta, \delta, \epsilon \in \mathbb{R}$ *verifying*

$$s^2 + (x + s\cos\theta)^2 + (y + s\sin\theta)^2 < 1$$

for any (t, x, y, θ) *with*

$$\begin{aligned} s &\in (s_0 - \alpha, s_0 + \alpha) \\ x &\in (x_0 - \beta, x_0 + \beta) \\ y &\in (y_0 - \delta, y_0 + \delta) \\ \theta &\in (\theta_0 - \epsilon, \theta_0 + \epsilon) \end{aligned}$$

Then $\mathcal{N}_{\mathbb{B}}$ *is open in* $\mathcal{N}_{\mathbb{M}}$. *Since* M *is globally hyperbolic, then* $\mathcal{N}_{\mathbb{M}}$ *is Hausdorff and therefore* $\mathcal{N}_{\mathbb{B}}$ *is also Hausdorff.*

Example 2.2.12 shows that the absence of naked singularities is a condition too strong for a strongly causal spacetime M. Moreover in this case, M becomes globally hyperbolic as Penrose proved in [57].

A suitable condition to avoid the behavior of light rays in the paradigmatic example 2.2.10 but to permit naked singularities similar to the ones in example 2.2.12 is the condition of *null pseudo–convexity.*

Definition 2.2.13. *A spacetime M is said to be* null pseudo–convex *if for any compact $K \subset M$ there exists a compact $K' \subset M$ such that any null geodesic segment γ with endpoints in K is totally contained in K'.*

In [40], Low states the equivalence of null pseudo–convexity of M and the Hausdorffness of $\mathcal{N}$ for a strongly causal spacetime M.

We offer a different and straightforward proof of the directed result. For the converse, see [40, Prop. 3.2] and the paragraph below its proof.

Proposition 2.2.14. *If M is strongly causal and null pseudo–convex then $\mathcal{N}$ is Hausdorff.*

Proof. Let us suppose that $\mathcal{N}$ is not Hausdorff, then there exist $\gamma_1 \neq \gamma_2 \in \mathcal{N}$ such that any open neighbourhoods $\mathcal{U}^{\gamma_i} \subset \mathcal{N}$ of γ_i for $i = 1, 2$ verify $\mathcal{U}^{\gamma_1} \cap \mathcal{U}^{\gamma_2} \neq \varnothing$. Consider any $p \in \gamma_1$ and $q \in \gamma_2$ and take neighbourhoods $\mathcal{U}^{\gamma_1}$ and $\mathcal{U}^{\gamma_2}$ defined by diffeomorphisms $\Omega^T(C_i) \to \mathcal{U}^{\gamma_i}$ where

$$\Omega^T(C_i) = \left\{v \in \mathbb{N}^+(C_i) : \mathbf{g}(v, T) = -1\right\}$$

for a non-vanishing timelike vector field $T \in \mathfrak{X}(M)$ and where C_i are Cauchy surfaces of relatively compact basic neighbourhoods $V^i \subset M$ for $i = 1, 2$ of p and q respectively. Now, we take nested sequences $\left\{\mathcal{U}_n^i\right\}$ of relatively compact neighbourhoods of γ_i such that $\mathcal{U}_n^i \subset \mathcal{U}^{\gamma_i}$ and $\mathcal{U}_n^i \mapsto \{\gamma_i\}$ for $i = 1, 2$. Then, for any n, there exists $\lambda_n \in \mathcal{U}_n^1 \cap \mathcal{U}_n^2$ and hence a sequence $\{\lambda_n\}$ such that $\lambda_n \mapsto \gamma_1$ and $\lambda_n \mapsto \gamma_2$. This means that there exist sequences $\{u_n\} \subset \Omega^T(C_1)$ and $\{v_n\} \subset \Omega^T(C_2)$ such that $\left(\lambda_n^1\right)'(0) = u_n$ and $\left(\lambda_n^2\right)'(0) = v_n$ with $u_n \mapsto \gamma_1'(0)$ and $v_n \mapsto \gamma_2'(0)$ and where λ_n^1 and λ_n^2 are the parametrizations of λ_n corresponding to $\Omega^T(C_1)$ and $\Omega^T(C_2)$ respectively. Since λ_n^1 and λ_n^2 are parametrizations of the same λ_n, then there exists $\alpha_n, \beta_n \in \mathbb{R}$ such that $\lambda_n^2(s) = \lambda_n^1(\alpha_n s + \beta_n)$.

We can consider $\left(\lambda_n^1\right)'$ and $\left(\lambda_n^1\right)'$ as integral curves of the flow Φ of the geodesic spray $X_{\mathbf{g}}$ in $\mathbb{N}$, so

$$\Phi(t, u_n) = \left(\lambda_n^1\right)'(t)$$
$$\Phi(s, v_n) = \left(\lambda_n^2\right)'(s)$$

and therefore

$$\alpha_n \Phi(\beta_n, u_n) = \Phi(0, v_n)$$

Since M is assumed to be null pseudo–convex, then for the compact $K = \overline{V^1} \cup \overline{V^2}$ there is a compact $K' \subset M$ such that any null geodesic segment with endpoints in K is totally contained in K'. Due to M is strongly causal, there exists $\tau \in \mathbb{R}$ such that $\pi_M^{TM}(\Phi_t(\gamma_1'(0))) \notin K'$ for all $t \geq \tau$. Observe that for a fixed $t \in \mathbb{R}$ such that $\Phi_t(\gamma_1'(0))$ is defined, there is a subsequence of $\{u_n\}$ such that $\Phi_t(u_k) \mapsto \Phi_t(\gamma_1'(0))$. In particular, also for $t = \tau$, then there is a subsequence such that $\pi_M^{TM}(\Phi_\tau(u_k)) \notin K'$. Since M is null pseudo–convex and

$$\pi_M^{TM}(\Phi_{\beta_k}(u_k)) = \pi_M^{TM}(\Phi_0(v_k)) \in K'$$

then we have that there exists a subsequence $\{\beta_m\}$ such that $\beta_m < \tau$ and therefore there exist a convergent subsequence such that $\beta_m \mapsto \beta \in [0, \tau]$. But we have that

$$\begin{array}{l} \Phi(\beta_m, u_m) \mapsto \Phi(\beta, \gamma_1'(0)) = \gamma_1'(\beta) \neq \mathbf{0} \\ \Phi(0, v_m) \mapsto \gamma_2'(0) \neq \mathbf{0} \end{array}$$

and since $\alpha_m \Phi(\beta_m, u_m) = \Phi(0, v_m)$ then, due to the convergence of $\{\Phi(\beta_m, u_m)\}$ and $\{\Phi(0, v_m)\}$ to non–zero vectors, it implies that there exists a convergent subsequence of $\{\alpha_m\}$ such that $\alpha_m \mapsto \alpha \in \mathbb{R}$. Then $\alpha \gamma_1'(\beta) = \gamma_2'(0)$, whence $\gamma_1 = \gamma_2 \in \mathcal{N}$ obtaining a contradiction. Therefore $\mathcal{N}$ is Hausdorff. $\square$

From now on, we will assume that M is a strongly causal and null pseudo–convex spacetime unless others conditions are pointed out.

Section 2.3

Tangent bundle of $\mathcal{N}$

To take advantage of the geometry and topology of $\mathcal{N}$ it is needed to have a suitable characterization of the tangent spaces $T_\gamma \mathcal{N}$ for any $\gamma \in \mathcal{N}$. We will proceed as follows: first, fix an auxiliary representative metric $\mathbf{g} \in \mathcal{C}$ where $\mathcal{C}$ is the conformal metric in M. We will define *geodesic variations* (in particular, *variations by light rays*), *initial* and *Jacobi fields*, explaining the relation between both concepts (in lemmas 2.3.3, 2.3.4, 2.3.5, 2.3.8 and proposition 2.3.7). Then, in proposition 2.3.9, we will characterize tangent vectors of TM by Jacobi fields. Second, we will keep an eye on how the initial fields changes when we change the corresponding variation by light rays (see lemma 2.3.10 to lemma 2.3.14). Finally, in proposition 2.3.15, we will get the main aim of this section identifying tangent vectors of $\mathcal{N}$ with some equivalence classes of Jacobi fields.

Definition 2.3.1. *A differentiable map $\mathbf{x} : (a, b) \times (\alpha, \beta) \to M$ is said to be a* variation of a segment of curve $c : (\alpha, \beta) \to M$ *if $c(t) = \mathbf{x}(s_0, t)$ for some $s_0 \in (a, b)$. We will say that $V^{\mathbf{x}}_{s_0}$ is the* initial field of $\mathbf{x}$ *in $s = s_0$ if*

$$V^{\mathbf{x}}_{s_0}(t) = d\mathbf{x}_{(s_0,t)} \left(\frac{\partial}{\partial s} \right)_{(s_0,t)} = \left. \frac{\partial \mathbf{x}(s, t)}{\partial s} \right|_{(s_0,t)} \in T_{c(t)} M$$

defining a vector field along c.

We will say that $\mathbf{x}$ is a geodesic variation *if any longitudinal curve of $\mathbf{x}$, that is $c^{\mathbf{x}}_s = \mathbf{x}(s, \cdot)$ for $s \in (a, b)$, is a geodesic.*

If the longitudinal curves $c^{\mathbf{x}}_s : (\alpha, \beta) \to M$ are regular curves covering segments of light rays, then $\mathbf{x} : (a, b) \times (\alpha, \beta) \to M$ is said to be a variation by light rays.

Moreover, a variation by light rays $\mathbf{x}$ is said to be a variation by light rays of $\gamma \in \mathcal{N}$ *if γ is a longitudinal curve of $\mathbf{x}$.*

Notation 2.3.2. *It is possible to identify a given segment of null geodesic $\gamma : (-\delta, \delta) \to M$, with a slight abuse in the notation, to the light ray in $\mathcal{N}$ defined by it. So, if $\mathbf{x} = \mathbf{x}(s, t)$ is a variation by light rays, we can denote by $\gamma^{\mathbf{x}}_s \subset M$ the null pregeodesics of the variation and also by $\gamma^{\mathbf{x}}_s \in \mathcal{N}$ the light rays they define.*

Consider a geodesic curve $\mu(t)$ in a spacetime $(M, \mathbf{g})$. Given $J \in \mathfrak{X}_\mu$, we will abbreviate the notation $J' = \frac{DJ}{dt}$ and $J'' = \frac{D}{dt}\frac{DJ}{dt} = \frac{D^2 J}{dt^2}$. We can define the *Jacobi equation* by

$$J'' + R(J, \mu')\mu' = 0 \tag{2.3.1}$$

where R is the Riemann tensor. We will name the solutions of the equation (2.3.1) by *Jacobi field* along μ. So, the set of Jacobi fields along μ is then defined by

$$\mathcal{J}(\mu) = \{J \in \mathfrak{X}_\mu : J'' + R(J, \mu')\mu' = 0\} \tag{2.3.2}$$

The linearity of $\frac{D}{dt}$ and R provides a vector space structure to $\mathcal{J}(\mu)$. Indeed, for $\alpha, \beta \in \mathbb{R}$ and $J, K \in \mathcal{J}(\gamma)$ we have

$$\begin{gathered}\frac{D}{dt}\frac{D}{dt}(\alpha J + \beta K) + R((\alpha J + \beta K), \mu')\mu' = \\ = \frac{D}{dt}(\alpha J' + \beta K') + \alpha R(J, \mu')\mu' + \beta R(K, \mu')\mu' = \\ = \alpha J'' + \beta K'' + \alpha R(J, \mu')\mu' + \beta R(K, \mu')\mu' = \\ = \alpha(J'' + R(J, \mu')\mu') + \beta(K'' + R(K, \mu')\mu') = \\ = \alpha \cdot 0 + \beta \cdot 0 = 0\end{gathered}$$

then $\alpha J + \beta K$ is a Jacobi field and hence $\mathcal{J}(\mu)$ is a vector subspace of $\mathfrak{X}_\mu$.

The relation between geodesic variations and Jacobi fields is expounded in next lemma.

Lemma 2.3.3. *If $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ is a geodesic variation of a geodesic γ, then the initial field $V^{\mathbf{x}}$ is a Jacobi field along γ.*

Proof. See [53, Lem. 8.3]. □

A Jacobi field along a geodesic γ is fully defined by its initial values at any point of γ as lemma 2.3.4 claims, and moreover it also implies that the vector space $\mathcal{J}(\mu)$ is isomorphic to $T_pM \times T_pM$ therefore $\dim(\mathcal{J}(\gamma)) = 2\dim(M) = 2m$.

Lemma 2.3.4. *Let γ be a geodesic in M such that $\gamma(0) = p$ and $u, v \in T_pM$. Then there exists a only Jacobi field J along γ such that $J(0) = u$ and $\frac{DJ}{dt}(0) = v$.*

Proof. See [53, Lem. 8.5]. □

Next lemma characterizes the Jacobi fields of a particular type of variation. This type will be the general case for the variations by light rays studied below.

Lemma 2.3.5. *Let M be a spacetime, $\gamma : (-\delta, \delta) \to M$ a geodesic segment, $\lambda : (-\epsilon, \epsilon) \to M$ a curve verifying $\lambda(0) = \gamma(0)$, and $W(s)$ a vector field along λ such that $W(0) = \gamma'(0)$. Then the Jacobi field J along γ defined by the geodesic variation*

$$\mathbf{x}(s, t) = \exp_{\lambda(s)}(tW(s))$$

verifies that

$$\begin{cases} J(0) = \lambda'(0) \\ J'(0) = \frac{DW}{ds}(0) \end{cases}$$

Proof. First, the vector $\frac{\partial \mathbf{x}}{\partial s}(0,0)$ is the tangent vector of the curve $\mathbf{x}(s,0)$ at $s=0$, and since $\mathbf{x}(s,0)=\exp_{\lambda(s)}(0\cdot W(s))=\exp_{\lambda(s)}(0)=\lambda(s)$, then we have

$$J(0)=\frac{\partial \mathbf{x}}{\partial s}(0,0)=\frac{d\lambda}{ds}(0)=\lambda'(0).$$

On the other hand, $\frac{D}{ds}\frac{\partial \mathbf{x}}{\partial t}(0,0)$ is the covariant derivative of the vector field $\frac{\partial \mathbf{x}}{\partial t}(s,0)=W(s)$ for $s=0$ along the curve $\mathbf{x}(s,0)=\lambda(s)$. Then

$$J'(0)=\frac{DJ}{dt}(0)=\frac{D}{dt}\frac{\partial \mathbf{x}}{\partial s}(0,0)=\frac{D}{ds}\frac{\partial \mathbf{x}}{\partial t}(0,0)=\frac{DW}{ds}(0).$$

as required. □

Remark 2.3.6. *It can be observed that given a geodesic variation $\mathbf{x}=\mathbf{x}(s,t)$ such that J is the corresponding Jacobi field at $s=0$, if we change the geodesic parameters such that $\overline{\mathbf{x}}(s,\tau)=\mathbf{x}(s,a\tau+b)$ for $a>0$ and $b\in\mathbb{R}$, then the initial values of the Jacobi field $\overline{J}$ of $\overline{\mathbf{x}}$ at $s=\frac{-b}{a}$ verify*

$$\overline{J}(-b/a)=\frac{\partial \overline{\mathbf{x}}}{\partial s}(0,-b/a)=\frac{\partial \mathbf{x}}{\partial s}(0,0)=J(0)$$

and also

$$\begin{aligned}\overline{J}'(-b/a)&=\left.\frac{D}{d\tau}\right|_{(0,-b/a)}\frac{\partial \overline{\mathbf{x}}}{\partial s}(s,\tau)=\left.\frac{D}{ds}\right|_{(0,-b/a)}\frac{\partial \overline{\mathbf{x}}}{\partial \tau}(s,\tau)=\\&=\left.\frac{D}{ds}\right|_{(0,-b/a)}\frac{\partial \mathbf{x}}{\partial \tau}(s,a\tau+b)=\left.\frac{D}{ds}\right|_{(0,0)}a\frac{\partial \mathbf{x}}{\partial t}(s,a\tau+b)=\\&=a\left.\frac{D}{ds}\right|_{(0,0)}\frac{\partial \mathbf{x}}{\partial t}(s,a\tau+b)=a\left.\frac{D}{dt}\right|_{(0,0)}\frac{\partial \mathbf{x}}{\partial s}(s,a\tau+b)=\\&=aJ'(0)\end{aligned}$$

If we denote by $Y(\tau)=J(a\tau+b)$, then it is trivial to see that $Y(-b/a)=J(0)$ and $Y'(-b/a)=aJ'(0)$, therefore $Y=\overline{J}$ and this implies that changing the geodesic parameter does not modify the Jacobi field as a geometric object.

Although the following proposition is proven in [8, Lem. 10.9] for timelike geodesic, the same proof is valid for any geodesic.

Proposition 2.3.7. *Given a geodesic γ in $(M,\mathbf{g})$ and a Jacobi field $J\in\mathcal{J}(\gamma)$ along γ, then $\mathbf{g}(J(t),\gamma'(t))=a+bt$ is verified.*

Proof. Deriving $\mathbf{g}(J(t),\gamma'(t))$, we obtain

$$\left.\frac{d}{dt}\right|_t\mathbf{g}(J,\gamma')=\mathbf{g}\left(\left.\frac{D}{dt}\right|_t J,\gamma'\right)+\mathbf{g}\left(J,\left.\frac{D}{dt}\right|_t\gamma'\right)=\mathbf{g}\left(\left.\frac{D}{dt}\right|_t J,\gamma'\right)$$

and so

$$\left.\frac{d^2}{dt^2}\right|_t \mathbf{g}(J,\gamma') = \mathbf{g}\left(\left.\frac{D^2}{dt^2}\right|_t J,\gamma'\right) + \mathbf{g}\left(\left.\frac{D}{dt}\right|_t J, \left.\frac{D}{dt}\right|_t \gamma'\right) = $$
$$= \mathbf{g}\left(\left.\frac{D^2}{dt^2}\right|_t J,\gamma'\right) = \mathbf{g}\left(-R(J,\gamma')\gamma',\gamma'\right) = 0.$$

where the anti–symmetric property, [53, Prop. 3.36 (3)], of the curvature tensor R has been used. Then, $\left.\frac{d}{dt}\right|_t \mathbf{g}(J,\gamma') = b$ constant and therefore $\mathbf{g}(J(t), \gamma'(t)) = a + bt$. □

We will need the following technical lemma. It shows that the information contained in the tangent vector of a curve $v \subset TM$ coincides with the one in the covariant derivative of v as vector field along its base curve in M.

Lemma 2.3.8. *If $u_0 \in T_pM$, then the map*

$$\begin{array}{rccc} A: & T_{u_0}TM & \to & T_pM \times T_pM \\ & \xi & \mapsto & \left(\left(\pi_M^{TM}\circ u\right)'(0), \frac{Du}{ds}(0)\right) \end{array}$$

is a linear isomorphism, where $u \subset TM$ is a differentiable curve verifying $u'(0) = \xi$.

Proof. Let us consider coordinates $\left(x^1,\ldots,x^m\right)$ in a neighbourhood of $p \in M$ to build the coordinates $\left(x^1,\ldots,x^m,v^1,\ldots,v^m\right)$ in a neighbourhood $W \subset TM$ containing u_0 in such way that $w \in W$ can be written as $w = \sum_{k=1}^m v^k \left(\frac{\partial}{\partial x^k}\right)_q$. Consider a differentiable curve $u : (-\delta,\delta) \to W \subset TM$ such that $u'(0) = \xi$ and so $u(0) = u_0$.

We denote $\alpha = \pi_M^{TM} \circ u$ and $a^k = x^k \circ \alpha$ for $k = 1,\ldots,m$. So, u can be expressed by $u(s) = \sum_{k=1}^m u^k(s) \left(\frac{\partial}{\partial x^k}\right)_{\alpha(s)}$. Then, $\xi \in T_{u_0}TM$ can be written as

$$\xi = u'(0) = \sum_{k=1}^m \frac{da^k}{ds}(0)\left(\frac{\partial}{\partial x^k}\right)_{u_0} + \sum_{k=1}^m \frac{du^k}{ds}(0)\left(\frac{\partial}{\partial v^k}\right)_{u_0}$$

If u_1 and u_2 are two differentiable curves verifying $\xi = u_1'(0) = u_2'(0)$ then, it is trivial to see that $u_1(0) = u_2(0)$, $\frac{da_1^k}{ds}(0) = \frac{da_2^k}{ds}(0)$ and $\frac{du_1^k}{ds}(0) = \frac{du_2^k}{ds}(0)$. Thus, denoting by Γ_{ij}^k the Christoffel symbols, we get

$$\frac{du_1^k}{ds}(0) + \Gamma_{ij}^k(p)\,u_1^i(0)\,\frac{da_1^k}{ds}(0) = \frac{du_2^k}{ds}(0) + \Gamma_{ij}^k(p)\,u_2^i(0)\,\frac{da_2^k}{ds}(0)$$

and hence

$$\left(\left(\pi_M^{TM}\circ u_1\right)'(0), \frac{Du_1}{ds}(0)\right) = \left(\left(\pi_M^{TM}\circ u_2\right)'(0), \frac{Du_2}{ds}(0)\right)$$

Therefore the map A is well–defined.

Since A can be written in coordinates by

$$\left(\frac{da^k}{ds}(0), \frac{du^k}{ds}(0)\right) \mapsto \left(\frac{da^k}{ds}(0), \frac{du^k}{ds}(0) + \Gamma_{ij}^k(p)\,u^i(0)\,\frac{da^k}{ds}(0)\right)$$

then A is clearly linear and its matrix relative to the previous coordinates is

$$\mathbf{A} = \begin{pmatrix} I_m & 0 \\ G & I_m \end{pmatrix}$$

where $G = (G_{ik}) = \left(\Gamma_{ij}^k (p) u^j (0)\right) \in \mathbb{R}^{m\times m}$ and $I_m \in \mathbb{R}^{m\times m}$ is the m–dimensional identity matrix. Trivially, because $\mathbf{A}$ is not singular, then A is an isomorphism. □

It is possible to identify any tangent vector $\xi \in TTM$ with a Jacobi field along the geodesic γ defined by the exponential of the vector $u = \pi_{TM}^{TTM}(\xi) \in TM$. As an immediate consequence of the previous lemmas, we have the following result.

Proposition 2.3.9. *Given a vector $u_0 \in T_pM$ and consider the geodesic γ_{u_0} defined by $\gamma_{u_0}(t) = \exp_p(tu_0)$. Let $u : (-\delta, \delta) \to TM$ be a differentiable curve such that $u(0) = u_0$ and $u'(0) = \xi$. If $J \in \mathcal{J}(\gamma_{u_0})$ is the Jacobi field of the geodesic variation given by $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tu(s))$ where $\alpha = \pi_M^{TM} \circ u$, then the map*

$$\begin{array}{rccc} \zeta : & T_{u_0}TM & \to & \mathcal{J}(\gamma_{u_0}) \\ & \xi & \mapsto & J \end{array}$$

is a well–defined linear isomorphism.

Proof. By lemmas 2.3.8, 2.3.5 and 2.3.4, we have that ζ can be obtained by composition of isomorphisms given by

$$\begin{array}{ccccc} T_{u_0}TM & \to & T_pM \times T_pM & \to & \mathcal{J}(\gamma_{u_0}) \\ \xi & \mapsto & \left(\left(\pi_M^{TM} \circ u\right)'(0), \frac{Du}{ds}(0)\right) & \mapsto & J \end{array}$$

□

Now, we will focus on the variations by light rays and the initial fields they define.

Next lemma claims that there exist a change of parameter such that any variation by light rays can be transformed in a geodesic variation by light rays. So, lemma 2.3.3 can be used.

Lemma 2.3.10. *Let $\mathbf{x} = \mathbf{x}(s,t)$ be a variation by light rays in $(M, \mathcal{C})$ such that $\gamma_s(t) = \mathbf{x}(s,t)$ defines its light rays. Fixed any metric $\mathbf{g} \in \mathcal{C}$ then there exists a differentiable function $h = h(s,\tau)$ such that the light rays parametrized as $\overline{\gamma}_s = \gamma_s(h(s,\tau))$ are null geodesics related to $\mathbf{g}$.*

Proof. Since each γ_s is a segment of light ray then $\gamma_s = \gamma_s(t)$ is a pregeodesic related to $\mathbf{g}$. Hence

$$\frac{D\gamma_s'(t)}{dt} = \frac{D}{dt}\frac{\partial \mathbf{x}}{\partial t}(s,t) = f(s,t)\,\gamma_s'(t)$$

where f is differentiable and $\frac{D}{dt}$ denotes the covariant derivative related to $\mathbf{g}$ along $\gamma_s(t)$. We look for the function $h = h(s,\tau)$ such that $\overline{\gamma}_s = \gamma_s \circ h$ is geodesic. For any s, for convenience, we will call $h_s(\tau) = h(s,\tau)$, $h_s'(\tau) = \frac{\partial h(s,\tau)}{\partial \tau}$ and $h_s''(\tau) = \frac{\partial^2 h(s,\tau)}{\partial \tau^2}$. Since h is a change of parameter for every s, we can assume that $\frac{\partial h(s,t)}{\partial \tau} \neq 0$ for every (s,t). So,

$$0 = \frac{D\overline{\gamma}_s'(\tau)}{d\tau} = \frac{Dh_s'(\tau)\,\gamma'(h_s(\tau))}{d\tau} = h_s''(\tau)\,\gamma_s'(h_s(\tau)) + h_s'(\tau)\,\frac{D\gamma_s'(h_s(\tau))}{d\tau} =$$

$$= h_s''(\tau)\,\gamma_s'(h_s(\tau)) + (h_s'(\tau))^2 \frac{D\gamma_s'(h_s(\tau))}{dt} =$$

$$= h_s''(\tau)\,\gamma_s'(h_s(\tau)) + (h_s'(\tau))^2 f(s, h_s(\tau))\,\gamma_s'(h_s(\tau))$$

hence

$$h_s''(\tau) + (h_s'(\tau))^2 f(s, h_s(\tau)) = 0$$

and therefore

$$\frac{h_s''(\tau)}{h_s'(\tau)} = -h_s'(\tau)\, f(s, h(s,\tau))$$

With no lack of generality, we assume that $h_s(0) = 0$ and $h_s'(0) = 1$ for any s, and then integrating

$$\log h_s'(\tau) = -\int_0^{h_s(\tau)} f(s,y)\, dy$$

$$h_s'(\tau) = e^{-\int_0^{h_s(\tau)} f(s,y)dy}$$

and calling $t = h_s(\tau)$ then

$$h_s'\left(h_s^{-1}(t)\right) = e^{-\int_0^t f(s,y)dy}$$

It is known that $\left(h_s^{-1}\right)'(t) = \frac{1}{h_s'\left(h_s^{-1}(t)\right)}$, then we have

$$\left(h_s^{-1}\right)'(t) = e^{\int_0^t f(s,y)dy}$$

and we conclude that

$$h_s^{-1}(t) = \int_0^t e^{\int_0^x f(s,y)dy} dx \tag{2.3.3}$$

is the inverse of the change of parameter h_s for each γ_s. Define $k(s,t) = h_s^{-1}(t)$ and the map $T(s,t) = (s, k(s,t))$. By the expression (2.3.3), T is clearly differentiable, and since the jacobian matrix of T verifies

$$|JT| = \begin{vmatrix} 1 & 0 \\ \frac{\partial k}{\partial s} & \frac{\partial k}{\partial t} \end{vmatrix} = \frac{\partial k}{\partial t} = e^{\int_0^t f(s,y)dy} > 0$$

then T is invertible with T^{-1} differentiable. A trivial computation shows that

$$T^{-1}(s,\tau) = (s, h(s,t))$$

therefore, since T^{-1} is differentiable, then h is also so. □

Lemma 2.3.11 shows that any differentiable curve $\Gamma \subset \mathcal{N}$ defines a variation by light rays $\mathbf{x}$ such that the longitudinal curves of $\mathbf{x}$ corresponds to points in Γ. This variation is not unique by construction.

Lemma 2.3.11. *Given a differentiable curve $\Gamma:(-\epsilon,\epsilon)\rightarrow\mathcal{N}$ such that $\Gamma(s)=\gamma_s\subset M$, then there exists a variation by light rays $\mathbf{x}:(-\epsilon,\epsilon)\times(-\delta,\delta)\rightarrow M$ verifying*

$$\mathbf{x}(s,t)=\gamma_s(t)$$

for all $(s,t)\in(-\epsilon,\epsilon)\times(-\delta,\delta)$. Moreover, the variation $\mathbf{x}$ can be written as

$$\mathbf{x}(s,t)=\exp_{\pi_M^{\mathbb{N}^+}(v(s))}(tv(s))$$

where $v:(-\epsilon,\epsilon)\rightarrow\mathbb{N}^+(C)$ is a differentiable curve.

Proof. Consider the restriction $\pi=\left.\pi_{\mathbb{PN}}^{\mathbb{N}^+}\right|_{\mathbb{N}^+(C)}:\mathbb{N}^+(C)\rightarrow\mathbb{PN}(C)$ and the diffeomorphism $\sigma:\mathbb{PN}(C)\rightarrow\mathcal{U}$ in the diagram (2.2.7), where $\mathcal{U}\subset\mathcal{N}$ is an open neighbourhood of $\gamma_0\in\mathcal{N}$ and $V\subset M$ is a basic open with Cauchy surface $C\subset V$, in such a way the following diagram arise

$$\begin{tikzcd}
\mathbb{PN}(C) \arrow[r, "\sigma"] & \mathcal{U} \\
\mathbb{N}^+(C) \arrow[u, "\pi"] \arrow[ur, "\sigma\circ\pi"'] &
\end{tikzcd} \tag{2.3.4}$$

Also consider the canonical projection $\pi_M^{\mathbb{N}^+}:\mathbb{N}^+\rightarrow M$ as well as the exponential map $\exp:(-\delta,\delta)\times\mathbb{N}^+\rightarrow M$ defined by $\exp(t,v)=\exp_{\pi_M^{\mathbb{N}^+}(v)}(tv)$. Fix $\epsilon>0$ such that $\Gamma(s)\in\mathcal{U}$ for all $s\in(-\epsilon,\epsilon)$ and let $z:\mathbb{PN}(C)\rightarrow\mathbb{N}^+(C)$ be a section of π that, without restriction of generality, can be considered a global section due to the locality of π. Naming $v(s)=z\circ\sigma^{-1}\circ\Gamma(s)$ for $s\in(-\epsilon,\epsilon)$, then we can define a variation $\mathbf{x}:(-\epsilon,\epsilon)\times(-\delta,\delta)\rightarrow M$ by $\mathbf{x}(s,t)=\exp(t,v(s))=\exp_{\pi_M^{\mathbb{N}^+}(v(s))}(tv(s))$. By construction as a composition of differentiable maps, $\mathbf{x}$ is differentiable. Moreover, since $v(s)$ is the initial vector of the geodesic $\gamma_s^{\mathbf{x}}$ defined by $\mathbf{x}(s,t)=\gamma_s^{\mathbf{x}}(t)$, then

$$\gamma_s^{\mathbf{x}}=\sigma\circ\pi(v(s))=\sigma\circ\pi\circ z\circ\sigma^{-1}\circ\Gamma(s)=\sigma\circ\sigma^{-1}\circ\Gamma(s)=\Gamma(s) \tag{2.3.5}$$

for all $s\in(-\epsilon,\epsilon)$, and the lemma follows. $\square$

Lemma 2.3.12. *Given a variation $\mathbf{x}:(-\epsilon,\epsilon)\times(-\delta,\delta)\rightarrow M$ by light rays such that $\mathbf{x}(s,t)=\gamma_s^{\mathbf{x}}(t)$, then the curve $\Gamma^{\mathbf{x}}:I\rightarrow\mathcal{N}$ verifying $\Gamma^{\mathbf{x}}(s)=\gamma_s^{\mathbf{x}}$ is differentiable.*

Proof. Let $\mathbf{x}:(-\epsilon,\epsilon)\times(-\delta,\delta)\rightarrow M$ be a variation by light rays such that $\gamma_s^{\mathbf{x}}(t)=\mathbf{x}(s,t)$. Then the curve

$$\lambda(s)=d\mathbf{x}_{(s,0)}\left(\frac{\partial}{\partial t}\right)_{(s,0)}\in\mathbb{N}^+$$

is clearly differentiable. If $p_{\mathbb{N}^+}:\mathbb{N}^+\rightarrow\mathcal{N}$ is the submersion of proposition 2.2.7, then $p_{\mathbb{N}^+}\circ\lambda:I\rightarrow\mathcal{N}$ is differentiable in $\mathcal{N}$ by composition of differentiable maps. Since

$$p_{\mathbb{N}^+}\circ\lambda(s)=p_{\mathbb{N}^+}\left((\gamma_s^{\mathbf{x}})'(0)\right)=\gamma_s^{\mathbf{x}}=\Gamma^{\mathbf{x}}(s).$$

then $\Gamma^{\mathbf{x}}$ is also differentiable. $\square$

Let us adopt the notation used in lemma 2.3.12 and call $\Gamma^{\mathbf{x}}$ the curve in $\mathcal{N}$ defined by the variation $\mathbf{x}$ by light rays such that if $\mathbf{x}(s,t) = \gamma_s^{\mathbf{x}}(t)$ then $\Gamma^{\mathbf{x}}(s) = \gamma_s^{\mathbf{x}} \in \mathcal{N}$.

Although the variations defined in lemma 2.3.11 are not unique, lemma 2.3.13 shows that all they define the same initial field except by a term in the direction of γ'.

Lemma 2.3.13. *Let $\overline{\mathbf{x}} : I \times \overline{H} \to M$ and $\mathbf{x} : I \times H \to M$ be variations by light rays such that $\Gamma^{\overline{\mathbf{x}}}(s) = \gamma_s^{\overline{\mathbf{x}}}$ and $\Gamma^{\mathbf{x}}(s) = \gamma_s^{\mathbf{x}}$ with $\gamma_0^{\overline{\mathbf{x}}} = \gamma_0^{\mathbf{x}} = \gamma \in \mathcal{N}$ and providing the same parameter for γ. Let us denote by $\overline{J}$ and J the initial fields over γ of $\overline{\mathbf{x}}$ and $\mathbf{x}$ respectively. If $\Gamma^{\overline{\mathbf{x}}} = \Gamma^{\mathbf{x}}$ then $\overline{J} = J\,(\mathrm{mod}\gamma')$.*

Proof. We have that $\overline{\mathbf{x}}(s,t) = \gamma_s^{\overline{\mathbf{x}}}(t)$ and $\mathbf{x}(s,\tau) = \gamma_s^{\mathbf{x}}(\tau)$. By lemma 2.3.10, we can assume without any lack of generality, that $\gamma_s^{\overline{\mathbf{x}}}$ are null geodesics for the metric $\mathbf{g} \in \mathcal{C}$ giving new parameters if necessary. If $\Gamma^{\overline{\mathbf{x}}} = \Gamma^{\mathbf{x}}$ then $\gamma_s^{\overline{\mathbf{x}}} = \gamma_s^{\mathbf{x}}$ for all $s \in I$. Then there exist a differentiable function $h_s(t) = h(s,t)$ such that $\overline{\mathbf{x}}(s,t) = \mathbf{x}(s, h(s,t))$. Hence we have that

$$\frac{\partial \overline{\mathbf{x}}(s,t)}{\partial s} = \frac{\partial \mathbf{x}(s,h(s,t))}{\partial s} + \frac{\partial h(s,t)}{\partial s} \cdot \frac{\partial \mathbf{x}(s,h(s,t))}{\partial \tau}$$

then if $s = 0$

$$\overline{J}(t) = J(h(0,t)) + \frac{\partial h}{\partial s}(0,t) \cdot \gamma'(t)$$

Since $\gamma_0^{\overline{\mathbf{x}}} = \gamma_0^{\mathbf{x}}$ are parametrized as the same geodesic, then $h(0,t) = t$ and therefore $\overline{J} = J\,(\mathrm{mod}\gamma')$. $\square$

For a fixed auxiliary metric $\mathbf{g} \in \mathcal{C}$, lemma 2.3.10 permit us to work with geodesic variations of null geodesics. The difference between using variations of light rays or geodesic variations is an extra term in their initial fields in the direction of γ' as lemma 2.3.13 shows.

We will need the following lemma at the proof of proposition 2.3.15, that is the main result of the current section.

Lemma 2.3.14. *Given two null geodesic variations $\mathbf{x} : I \times H \to M$ and $\overline{\mathbf{x}} : \overline{I} \times \overline{H} \to M$ such that $\Gamma^{\mathbf{x}}(0) = \Gamma^{\overline{\mathbf{x}}}(0) = \gamma$. Let us denote by J and $\overline{J}$ their corresponding Jacobi fields at $0 \in I$ and $0 \in \overline{I}$ of $\mathbf{x}$ and $\overline{\mathbf{x}}$ respectively. If $(\Gamma^{\mathbf{x}})'(0) = \left(\Gamma^{\overline{\mathbf{x}}}\right)'(0)$ then $J = \overline{J}\,(\mathrm{mod}\gamma')$.*

Proof. Due to we want to compare the Jacobi fields J and $\overline{J}$ on γ, we can assume without any lack of generality that $\mathbf{x}$ and $\overline{\mathbf{x}}$ provide the same geodesic parameter for γ, then by lemmas 2.3.11 and 2.3.13, we can consider that $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tu(s))$ and $\overline{\mathbf{x}}(r,t) = \exp_{\overline{\alpha}(r)}(t\overline{u}(r))$ where $u = u(0) = \overline{u}(0)$ and also $p = \alpha(0) = \overline{\alpha}(0)$.

Moreover, we can assume the diagram (2.3.4) holds.

$$\begin{tikzcd}
\mathbb{PN}(C) \arrow[r, "\sigma"] & \mathcal{U} \\
\mathbb{N}^+(C) \arrow[u, "\pi"] \arrow[ur, "\sigma \circ \pi"'] &
\end{tikzcd}$$

Since $(\Gamma^{\mathbf{x}})'(0) = \left(\Gamma^{\overline{\mathbf{x}}}\right)'(0)$ then, by expression (2.3.5) in the proof of lemma 2.3.11, we have

$$d\sigma_{[u(0)]} \circ d\pi_{u(0)}(u'(0)) = d\sigma_{[\overline{u}(0)]} \circ d\pi_{\overline{u}(0)}(\overline{u}'(0)) \Leftrightarrow d\pi_{u(0)}(u'(0)) = d\pi_{\overline{u}(0)}(\overline{u}'(0))$$

Observe that $[u(0)] = [\overline{u}(0)]$ and thus, $d\pi_{u(0)} = d\pi_{\overline{u}(0)}$, and its kernel is the subspace generated by the tangent vector at $s = 0$ of the curve $c(s) = e^s u(0)$, hence

$$u'(0) = \overline{u}'(0) + \mu c'(0) \tag{2.3.6}$$

with $\mu \in \mathbb{R}$. By lemma 2.3.8, we have that

$$\begin{cases} \alpha'(0) = \overline{\alpha}'(0) \\ \frac{Du}{ds}(0) = \frac{D\overline{u}}{dr}(0) + \mu \frac{Dc}{ds}(0) \end{cases} \Rightarrow \begin{cases} \alpha'(0) = \overline{\alpha}'(0) \\ \frac{Du}{ds}(0) = \frac{D\overline{u}}{dr}(0) + \mu\gamma'(0) \end{cases}$$

therefore we conclude that $J = \overline{J}\,(\mathrm{mod}\gamma')$. $\square$

Let us fix an auxiliary metric $\mathbf{g} \in \mathcal{C}$ and a light ray $\gamma \in \mathcal{N}$ parametrized as null geodesic related to $\mathbf{g}$. Again, by lemma 2.3.10, we can assume that $\mathbf{x}(s,t)$ is geodesic variation of $\gamma = \gamma_0^{\mathbf{x}} \in \mathcal{N}$ in such a way that $J(t) = V_0^{\mathbf{x}}(t)$ is the Jacobi field over γ corresponding to the initial field of $\mathbf{x}$ and $\frac{\partial \mathbf{x}}{dt}(s,t) = (\gamma_s^{\mathbf{x}})'(t)$. So, it provides that $\mathbf{g}\left((\gamma_s^{\mathbf{x}})'(t), (\gamma_s^{\mathbf{x}})'(t)\right) = 0$ for all (s,t) in the domain of $\mathbf{x}$, hence

$$0 = \left.\frac{\partial}{\partial s}\right|_{(0,t)} \mathbf{g}\left((\gamma_s^{\mathbf{x}})'(t), (\gamma_s^{\mathbf{x}})'(t)\right) = 2\mathbf{g}\left(\left.\frac{D}{ds}\right|_{(0,t)} \frac{\partial \mathbf{x}}{dt}(s,t), \frac{\partial \mathbf{x}}{dt}(0,t)\right) =$$

$$= 2\mathbf{g}\left(\left.\frac{D}{dt}\right|_{(0,t)} \frac{\partial \mathbf{x}}{ds}(s,t), \frac{\partial \mathbf{x}}{dt}(0,t)\right) = \left.\frac{\partial}{\partial t}\right|_{(0,t)} \mathbf{g}\left(V_s^{\mathbf{x}}(t), (\gamma_s^{\mathbf{x}})'(t)\right) =$$

$$= \left.\frac{d}{dt}\right|_t \mathbf{g}\left(V_0^{\mathbf{x}}(t), \gamma'(t)\right) = \left.\frac{d}{dt}\right|_t \mathbf{g}\left(J(t), \gamma'(t)\right)$$

then the geodesic variations by light rays of γ verify that their Jacobi fields J fulfil

$$\mathbf{g}\left(J(t), \gamma'(t)\right) = c \tag{2.3.7}$$

with $c \in \mathbb{R}$ constant. By lemma 2.3.13, the expression (2.3.7) is also true for any variation by light rays of γ, not necessarily geodesic.

Then, we define the set of Jacobi fields of variations by light rays by

$$\mathcal{J}_L(\gamma) = \{J \in \mathcal{J}(\gamma) : \mathbf{g}(J, \gamma') = c \text{ constant}\}$$

Since $\mathbf{g}(\alpha J + \beta K, \gamma') = \alpha \mathbf{g}(J, \gamma') + \beta \mathbf{g}(K, \gamma')$ for all $\alpha, \beta \in \mathbb{R}$ and every $J, K \in \mathcal{J}_L(\gamma)$ then $\mathcal{J}_L(\gamma)$ is a vector subspace of $\mathcal{J}(\gamma)$, and by proposition 2.3.7, it verifies that $\dim(\mathcal{J}_L(\gamma)) = 2\dim(M) - 1 = 2m - 1$.

Observe that since

$$\left.\frac{d}{dt}\right|_t \mathbf{g}\left(J(t), \gamma'(t)\right) = \mathbf{g}\left(\frac{DJ}{dt}(t), \gamma'(t)\right) + \mathbf{g}\left(J(t), \frac{D\gamma'}{dt}(t)\right) = \mathbf{g}\left(J'(t), \gamma'(t)\right)$$

then we have

$$\mathbf{g}\left(J'(t), \gamma'(t)\right) = 0 \tag{2.3.8}$$

for all t and $J \in \mathcal{J}_L(\gamma)$.

Now, we define subsets of $\mathcal{J}(\gamma)$ given by

$$\widehat{\mathcal{J}_0}(\gamma) = \{J(t) = bt\gamma'(t) : b \in \mathbb{R}\}$$
$$\widehat{\mathcal{J}_0'}(\gamma) = \{J(t) = a\gamma'(t) : a \in \mathbb{R}\}$$

It is trivial to see that $\widehat{\mathcal{J}_0}(\gamma) \subset \mathcal{J}_L(\gamma)$ and $\widehat{\mathcal{J}_0'}(\gamma) \subset \mathcal{J}_L(\gamma)$.

Moreover, observe that for any $\beta_1, \beta_2 \in \mathbb{R}$ and any $J_1, J_2 \in \widehat{\mathcal{J}_0}(\gamma)$, if $J_1(t) = b_1 t\gamma'(t)$ and $J_2(t) = b_2 t\gamma'(t)$ then

$$\beta_1 J_1(t) + \beta_2 J_2(t) = (\beta_1 b_1 + \beta_2 b_2)\, t\gamma'(t) \in \widehat{\mathcal{J}_0}(\gamma)$$

hence $\widehat{\mathcal{J}_0}(\gamma)$ is a vector subspace of $\mathcal{J}_L(\gamma)$ such that $\dim\left(\widehat{\mathcal{J}_0}(\gamma)\right) = 1$. Analogously, for any $\beta_1, \beta_2 \in \mathbb{R}$ and any $J_1, J_2 \in \widehat{\mathcal{J}_0'}(\gamma)$, verifying $J_1(t) = a_1\gamma'(t)$ and $J_2(t) = a_2\gamma'(t)$ then

$$\beta_1 J_1(t) + \beta_2 J_2(t) = (\beta_1 a_1 + \beta_2 a_2)\, \gamma'(t) \in \widehat{\mathcal{J}_0'}(\gamma)$$

hence $\widehat{\mathcal{J}_0'}(\gamma)$ is also a 1–dimensional vector subspace of $\mathcal{J}_L(\gamma)$.

If $J \in \widehat{\mathcal{J}_0}(\gamma) \cap \widehat{\mathcal{J}_0'}(\gamma)$, then its initial values must verify

$$\begin{cases} J(0) = 0 \\ J'(0) = b\gamma'(0) \end{cases} \quad \text{and} \quad \begin{cases} J(0) = a\gamma'(0) \\ J'(0) = 0 \end{cases}$$

then $a = b = 0$ and therefore $\widehat{\mathcal{J}_0}(\gamma) \cap \widehat{\mathcal{J}_0'}(\gamma) = \{\mathbf{0}\}$. So, we can define the direct sum

$$\mathcal{J}_0(\gamma) = \widehat{\mathcal{J}_0}(\gamma) \oplus \widehat{\mathcal{J}_0'}(\gamma) = \{J(t) = (a + bt)\,\gamma'(t) : a, b \in \mathbb{R}\}$$

being the vector subspace of Jacobi fields proportional to γ' verifying $\dim(\mathcal{J}_0(\gamma)) = 2$.

Now, we can define the quotient vector space

$$\mathcal{L}(\gamma) = \mathcal{J}_L(\gamma)/\mathcal{J}_0(\gamma) = \{[J] : K \in [J] \Leftrightarrow K = J + J_0 \text{ such that } J_0 \in \mathcal{J}_0(\gamma)\}$$

whose dimension is $\dim(\mathcal{L}(\gamma)) = \dim(\mathcal{J}_L(\gamma)) - \dim(\mathcal{J}_0(\gamma)) = 2\dim(M) - 3$. The elements of $\mathcal{L}(\gamma)$ will be denoted by $[J] \equiv J\,(\mathrm{mod}\gamma')$ and we will say that $K = J\,(\mathrm{mod}\gamma')$ when $[K] = [J]$.

The differentiable structure of $\mathcal{N}$ has been built in section 2.2 from the one in $\mathbb{PN}(C)$ where C is a local spacelike Cauchy surface. So, we will identify the tangent space $T_\gamma\mathcal{N}$ with some quotient space of $\mathcal{J}_L(\gamma)$ via a tangent space of $\mathbb{PN}(C)$.

Proposition 2.3.15. *Given $\xi \in T_{\gamma_{u_0}}\mathcal{N}$ such that $\Gamma'(0) = \xi$ for some curve $\Gamma \subset \mathcal{N}$. Let $\mathbf{x} = \mathbf{x}(s,t)$ be a variation by light rays of γ_{u_0} verifying that $\Gamma^{\mathbf{x}} = \Gamma$ such that $J \in \mathcal{L}(\gamma_{u_0})$ is the Jacobi field over γ_{u_0} of $\mathbf{x}$. If $\zeta : T_{\gamma_{u_0}}\mathcal{N} \to \mathcal{L}(\gamma_{u_0})$ is the map defined by*

$$\overline{\zeta}(\xi) = J\left(\mathrm{mod}\gamma'_{u_0}\right)$$

then $\overline{\zeta}$ is well–defined and a linear isomorphism.

Proof. By lemma 2.3.14, $\overline{\zeta}$ is well–defined.

We have seen in section 2.2 that for a basic open set $V \subset M$ such that $C \subset V$ is a smooth local spacelike Cauchy surface, the diagram (2.2.8) given by

$$\mathcal{N} \supset \mathcal{U} \simeq \mathbb{PN}(C) \simeq \Omega^X(C) \hookrightarrow \mathbb{N}^+(C) \hookrightarrow \mathbb{N}^+ \hookrightarrow TM$$

holds. Proposition 2.3.9 shows that $\zeta : T_u TM \to \mathcal{J}(\gamma_u)$ is a linear isomorphism for any $u \in TM$. In order to complete the proof, we will restrict ζ from $T_u TM$ up to $T_{[u]}\mathbb{PN}(C)$ step by step, identifying the corresponding subspace of $\mathcal{J}(\gamma_u)$ image of the map. By definition of $\mathcal{J}_L(\gamma)$, it is clear that $\zeta|_{\mathbb{N}^+} : T_u\mathbb{N}^+ \to \mathcal{J}_L(\gamma_u)$ is a linear isomorphism. Since $\mathbb{N}^+(C)$ is a local submanifold of $\mathbb{N}^+$ of codimension 1 such that for any future–directed null geodesic γ, the curve $c(s) = \gamma'(s) \in \mathbb{N}^+$ intersects transversally to $\mathbb{N}^+(C)$, then the image of the restriction of the isomorphism ζ of proposition 2.3.9 to $T_{u_0}\mathbb{N}^+(C)$ is a vector subspace $S \subset \mathcal{J}_L(\gamma_{u_0})$ of the same codimension and transverse (that is, linearly independent) to the vector subspace $\widehat{\mathcal{J}}'_0(\gamma_{u_0})$, which is generated by the Jacobi field J of the variation

$$\mathbf{x}(s,t) = \exp_{\gamma_{u_0}(s)}\left(t\gamma'_{u_0}(s)\right)$$

By lemma 2.3.5, we have that $J(0) = \gamma'_{u_0}(0)$ and $J'(0) = 0$, hence $J(t) = \gamma'_{u_0}(t)$. Observe that it is clear that the linear map

$$\begin{array}{ccc} S & \to & \mathcal{J}_L(\gamma_{u_0}) / \widehat{\mathcal{J}}'_0(\gamma_{u_0}) \\ J & \mapsto & [J] \end{array}$$

is an isomorphism.

On the other hand, let $v : (-\epsilon, \epsilon) \to \mathbb{N}^+(C)$ be a differentiable curve such that $v(0) = u_0$ and let us denote by $\alpha = \pi_M^{\mathbb{N}^+} \circ v$ its projection on $C \subset M$. Consider the variation by light rays defined by $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tv(s))$ where J is the Jacobi field of $\mathbf{x}$ along γ_{u_0}. By lemma 2.3.5, we have that $J(0) = \alpha'(0)$ and $J'(0) = \frac{Dv}{ds}(0)$. If $\lambda : (-\epsilon, \epsilon) \to \mathbb{R}$ is a non–vanishing differentiable function where $\lambda(0) = 1$, again by lemma 2.3.5, the Jacobi field $\overline{J}$ corresponding to the variation

$$\overline{\mathbf{x}}(s,t) = \exp_{\alpha(s)}(t\lambda(s)v(s)).$$

verifies

$$\begin{cases} \overline{J}(0) = \alpha'(0) \\ \overline{J}'(0) = \left.\frac{D\lambda(s)v(s)}{ds}\right|_{s=0} = \lambda(0)\frac{Dv(0)}{ds} + \lambda'(0)v(0) \end{cases}$$

then we have

$$\begin{cases} \overline{J}(0) = J(0) \\ \overline{J}'(0) = J'(0) + \lambda'(0)\gamma'(0) \end{cases}$$

This shows that for all the curves $c \subset \mathbb{N}^+(C)$ such that $c(s)$ is proportional to $v(s) \in \mathbb{N}^+(C)$, their tangent vectors are in correspondence with the same equivalence class in $S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right)$, but this implies that

$$\begin{array}{ccc} T_{[u_0]}\mathbb{PN}(C) & \to & S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right) \\ [v(0)]' & \mapsto & [J] \end{array}$$

is an isomorphism, where we have denoted $[v(0)]' = \left.\frac{d}{ds}\right|_{s=0}[v(s)]$ and $[v(s)] \in \mathbb{PN}(C)$. Since there is a diffeomorphism $\sigma : \mathbb{PN}(C) \to \mathcal{U} \subset \mathcal{N}$, then $T_{[u_0]}\mathbb{PN}(C)$ is isomorphic to $T_{\gamma_{u_0}}\mathcal{N}$ therefore, since $\mathbf{x}(s,t) = \gamma_{v(s)}(t) = \exp_{\alpha(s)}(tv(s))$ with $\gamma_{v(0)} = \gamma_{u_0}$, and moreover $(\Gamma^{\mathbf{x}})'(0) = (\Gamma^{\mathbf{x}})'(0) = \xi$ then the map

$$\begin{array}{ccc} T_{\gamma_{u_0}}\mathcal{N} & \to & S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right) \\ \xi & \mapsto & [J] \end{array}$$

is a linear isomorphism.

Recall that we have denoted $\mathcal{J}_0(\gamma_{u_0}) = \widehat{\mathcal{J}}_0(\gamma_{u_0}) \oplus \widehat{\mathcal{J}}'_0(\gamma_{u_0})$. Observe that the linear map $q : S \to \mathcal{J}_L(\gamma_{u_0}) / \mathcal{J}_0(\gamma_{u_0})$ defined by $q(J) = [J]$ verifies that

$$q(J) = [0] \Leftrightarrow J(t) = (a + bt)\,\gamma'_{u_0}(t) \Leftrightarrow J \in S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})$$

then $S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right)$ is isomorphic to $\mathcal{L}(\gamma_{u_0}) = \mathcal{J}_L(\gamma_{u_0}) / \mathcal{J}_0(\gamma_{u_0})$. This shows that

$$\begin{array}{rccl} \overline{\zeta}: & T_{\gamma_{u_0}}\mathcal{N} & \to & \mathcal{L}(\gamma_{u_0}) = \mathcal{J}_L(\gamma_{u_0}) / \mathcal{J}_0(\gamma_{u_0}) \\ & \xi & \mapsto & [J] \end{array}$$

is a linear isomorphism. The proof is complete. $\square$

Proposition 2.3.15 allows to see the vectors of the tangent space $T_\gamma\mathcal{N}$ as Jacobi fields of variations by light rays. We will use, from now on, this characterization when working with tangent vectors of $\mathcal{N}$.

Observe that the construction of $\mathcal{L}(\gamma)$ depends on the parametrization of γ as well as the used metric $\mathbf{g}$ but, by proposition 2.3.15, it is clear that all of the characterizations of $T_\gamma\mathcal{N}$ as some $\mathcal{L}(\gamma)$ are isomorphic in the class of the conformal metric $\mathcal{C}$, in fact, they are realizations of $T_\gamma\mathcal{N}$.

Section 2.4

The canonical contact structure in $\mathcal{N}$

In this section, we will show the existence of a canonical contact structure in $\mathcal{N}$ inherited from the kernel of the *canonical 1–form* of T^*M.

There exists a canonical distribution of hyperplanes in $T\mathcal{N}$. Indeed, let us consider the diffeomorphism $\sigma : \mathbb{PN}(C) \to \mathcal{U} \subset \mathcal{N}$ of diagram (2.2.7) given by $\sigma([u]) = \gamma_{[u]}$. Given $x \in C \subset M$, the image by σ of the fibre $\mathbb{PN}_x$ is written by

$$X = \sigma(\mathbb{PN}_x) \subset \mathcal{U}$$

and it is clearly diffeomorphic to $\mathbb{S}^{m-2}$. This image will be studied in deep in chapter 3 under the name of *sky of x*.

Consider any $\gamma \in X$, then $J \in T_\gamma X$ can be defined by a tangent vector of a curve $\Gamma \subset X$. Then, if $\Gamma : (-\epsilon, \epsilon) \to X \subset \mathcal{N}$ is a differentiable curve such that $\Gamma(0) = \gamma$, then by lemma 2.3.11 it is possible to construct a geodesic variation of γ given by

$$\mathbf{f}(s,t) = \exp_x(tv(s))$$

where $v(s) \in \mathbb{N}_x^+$ for all $s \in (-\epsilon, \epsilon)$. By lemma 2.3.5, $J = \Gamma'(0) \in T_\gamma X$ verifies $J(0) = 0$ and $J'(0) = \frac{Dv}{ds}(0)$. Then,

$$T_\gamma X = \{J \in T_\gamma\mathcal{N} : J(s_0) = 0\,(\mathrm{mod}\gamma'(s_0)) \text{ with } \gamma(s_0) = x\} \tag{2.4.1}$$

If we define

$$\mathcal{H}_\gamma = \{J \in \mathcal{L}(\gamma) : \mathbf{g}(J, \gamma') = 0\} \tag{2.4.2}$$

then it is trivial to see that

$$T_\gamma X \subset \mathcal{H}_\gamma$$

for any $x \in \gamma$ where, with a slight abuse on the notation identifying $T_\gamma X \simeq \zeta\left(T_\gamma X\right)$, we have used the characterization of $T_\gamma \mathcal{N}$ at proposition 2.3.15.

Choosing another point $y \in \gamma$ close enough to avoid being conjugate to x, then if $Y = \sigma\left(\mathbb{PN}_y\right)$ we have

$$T_\gamma X \cap T_\gamma Y = \{\mathbf{0}_\gamma\}$$

Then

$$T_\gamma X \oplus T_\gamma Y \subset \mathcal{H}_\gamma$$

for any pair of non–conjugate points $x, y \in \gamma$ and since $\dim\left(T_\gamma X \oplus T_\gamma Y\right) = \dim H_\gamma = 2m-4$, therefore

$$\mathcal{H}_\gamma = T_\gamma X \oplus T_\gamma Y \tag{2.4.3}$$

for any pair of non–conjugate points $x, y \in \gamma$. Hence $\mathcal{H}_\gamma$ is a hyperplane that, trivially, does not depend on the representative $\mathbf{g}$ of the conformal metric $\mathcal{C}$ because $T_\gamma X \oplus T_\gamma Y$ neither do. Then, the distribution of hyperplanes

$$\mathcal{H} = \bigcup_{\gamma \in \mathcal{N}} \mathcal{H}_\gamma$$

is conformal.

In the following sections, we will show that $\mathcal{H} \subset T\mathcal{N}$ is a contact structure. We will do it in two different ways. First, in section 2.4.2, passing the distribution of hyperplanes to TM before pushing it down to $\mathcal{N}$ through the chain of inclusions (2.2.8). This way is pointed out by Low but done from T^*M in [44].

In section 2.5, we will build the contact structure using symplectic reduction in two different ways.

In order to carry out this task in a self–contained way, we will introduce some basic elements of symplectic and contact geometry (see references [1], [3] and [37]) and observe how the construction of $\mathcal{N}$ can be done from T^*M.

2.4.1 Elements of symplectic geometry in T^*M

Definition 2.4.1. *Let E be a k–dimensional vector space over $\mathbb{R}$ and $\omega : E \times E \to \mathbb{R}$ a skew–symmetric and non–degenerated bilinear map, then the pair (E, ω) is called a* symplectic vector space*.*

Given any vector subspace $W \subset E$, we define the *symplectic orthogonal of* W by

$$W^\perp = \{v \in E : \omega(v, u) = 0 \text{ for all } u \in W\}$$

W is said to be *symplectic* if $\omega|_{W \times W}$ is non–degenerated, or equivalently $W \cap W^\perp = \{\mathbf{0}\}$.

Whenever $\omega|_{W \times W} \equiv 0$ or equivalently $W \subset W^\perp$, we will say that W is *isotropic*.

We will say that W is *coisotropic* if $W^\perp \subset W$.

Any isotropic and coisotropic subspace $W \subset E$ is called *lagrangian*.

This previous definitions pass directly to the scope of manifolds.

Definition 2.4.2. *A pair* (P, ω) *is called a* symplectic manifold *whenever* P *is a differenciable manifold equipped with a non–degenerated and closed 2–form* $\omega \in \Lambda^2(P)$*. We will say that* ω *is the* symplectic 2–form *of* P*.*

We also can talk about *symplectic*, *isotropic*, *coisotropic* and *lagrangian submanifolds*, $S \subset P$ when $W = T_pS \subset T_pP = E$ can be classified in the corresponding vector subspace for all $p \in S$.

Remark 2.4.3. *It is known, see for example [1, Prop. 3.1.3 and 3.1.5], that* ω *is non–degenerated if and only if* $\dim(P) = 2k$ *and* $\omega^k = \omega \wedge \cdots \wedge \omega \in \Lambda^{2k}(P)$ *is a* volume form, *that is* ω^k *does not vanish at any point* $q \in P$*. Then* ω *is degenerated when restricted to an odd–dimensional submanifold (or vector subspace).*

Consider a differentiable manifold M and take a coordinate chart (U, ϕ) in M such that if $q \in U \subset M$ then $\phi(q) = \left(x^1, \ldots, x^m\right)$, hence for $\alpha \in T^*M$ we can write

$$\alpha_q = \sum_{k=1}^{m} p_k dx^k$$

and therefore $\left(x^k, p_k\right)$ are coordinates in $T^*U \subset T^*M$.

If $\pi = \pi_M^{T^*M} : T^*M \to M$ denotes the canonical projection, we can define the 1–form $\theta \in \mathfrak{X}^*(T^*M)$ pointwise at every $\alpha \in T^*M$ by

$$\theta_\alpha = (d\pi_\alpha)^* \alpha$$

Consequently we have

$$\theta_\alpha(\xi) = \left((d\pi_\alpha)^* \alpha\right)(\xi) = \alpha\left((d\pi_\alpha)\xi\right) \tag{2.4.4}$$

for $\xi \in T_\alpha(T^*M)$. In the previous coordinates, we can write

$$\theta = \sum_{k=1}^{m} p_k dx^k \tag{2.4.5}$$

This 1–form θ is called the *canonical* or *tautological 1–form*.

Now, the 2–form ω given by

$$\omega = -d\theta$$

defines a symplectic 2–form in T^*M, that can be expressed by

$$\omega = \sum_{k=1}^{m} dx^k \wedge dp_k$$

Definition 2.4.4. *A vector field* $X \in \mathfrak{X}(P)$ *of a symplectic manifold* (P, ω) *is said to be a* Liouville vector field *if it verifies*

$$\mathcal{L}_X \omega = \omega$$

Definition 2.4.5. *Given a symplectic manifold* (P,ω) *and a smooth function* $H : P \to \mathbb{R}$*, then the only vector field* $X_H \in \mathfrak{X}(P)$ *verifying*

$$i_{X_H}(\omega) = dH$$

is called the hamiltonian vector field *associated to* H*. This function* H *will be called the* hamiltonian function.

In case of $P = T^*M$, for a hamiltonian function $H : T^*M \to \mathbb{R}$, using the equality $i_{X_H}(\omega) = dH$, it is possible to express the corresponding hamiltonian vector field $X_H \in \mathfrak{X}(T^*M)$ as

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i}$$

Now, we want to construct $\mathcal{N}$ again, but this time from T^*M. First, consider the diffeomorphism

$$\begin{array}{rccc} \widehat{\mathbf{g}}: & TM & \to & T^*M \\ & \xi & \mapsto & \mathbf{g}(\xi,\cdot) \end{array} \tag{2.4.6}$$

and denote by $\mathbb{N}^{+*}$ the image of the restriction of $\widehat{\mathbf{g}}$ to $\mathbb{N}^{+}$, that is

$$\mathbb{N}^{+*} = \widehat{\mathbf{g}}\left(\mathbb{N}^{+}\right) = \left\{\alpha = \widehat{\mathbf{g}}(\xi) \in T^*M : \xi \in \mathbb{N}^{+}\right\}$$

In an analogous manner as done in section 2.2 to define the Euler field Δ in TM, we can define the *Euler field* $\mathcal{E} \in \mathfrak{X}(T^*M)$ by

$$\mathcal{E}(\alpha) = dc\left(\frac{\partial}{\partial t}\right)(0)$$

where $\alpha \in T_p^*M$ and $c : \mathbb{R} \to T_p^*M$ verifies that $c(t) = e^t\alpha$. The curve c is an integral curve of $\mathcal{E}$ because

$$c'(t) = dc\left(\frac{\partial}{\partial t}\right)(t) = \mathcal{E}(c(t))$$

In the previous coordinates, $\mathcal{E}$ can be written as

$$\mathcal{E} = p_k\frac{\partial}{\partial p_k}$$

So, for every $\alpha \in \mathbb{N}^{+*}$ the integral curve $c(t) = e^t\alpha$ is contained in $\mathbb{N}^{+*}$, therefore $\mathcal{E}$ is tangent to $\mathbb{N}^{+*}$.

Moreover, if ω is the symplectic 2–form of T^*M it is trivial to see that

$$-i_{\mathcal{E}}\omega = \theta \tag{2.4.7}$$

where θ is the tautological 1–form in T^*M and hence

$$\mathcal{L}_{\mathcal{E}}\omega = i_{\mathcal{E}}d\omega + d(i_{\mathcal{E}}\omega) = d(-\theta) = -d\theta = \omega \tag{2.4.8}$$

therefore $\mathcal{E}$ is a Liouville vector field. In fact, $\mathcal{E}$ sometimes is called the *Liouville* or *Euler–Liouville vector field*.

Consider now the hamiltonian function defined by

$$\begin{array}{rccl} H: & T^*M & \to & \mathbb{R} \\ & \alpha & \mapsto & \frac{1}{2}\mathbf{g}\left(\widehat{\mathbf{g}}^{-1}(\alpha), \widehat{\mathbf{g}}^{-1}(\alpha)\right) \end{array} \tag{2.4.9}$$

defining the hamiltonian vector field given by

$$X_H = g^{ki}p_i\frac{\partial}{\partial x^k} - \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}p_ip_j\frac{\partial}{\partial p_i}$$

Lemma 2.4.6. *Let $X_{\mathbf{g}}, \Delta \in \mathfrak{X}(TM)$ be the the geodesic spray and Euler field of TM and $X_H, \mathcal{E} \in \mathfrak{X}(T^*M)$ the hamiltonian vector field and Euler field of T^*M. Then we have that $\widehat{\mathbf{g}}_*(\Delta) = \mathcal{E}$ and $\widehat{\mathbf{g}}_*(X_{\mathbf{g}}) = X_H$.*

Proof. If we take any $\xi \in T^*M$ and $\alpha = \widehat{\mathbf{g}}(\xi)$, then the integral curve $c(t) = e^t\xi$ of Euler field Δ in TM is transformed by $\widehat{\mathbf{g}}$ as

$$\widehat{\mathbf{g}}(c(t)) = \mathbf{g}(c(t), \cdot) = \mathbf{g}\left(e^t\xi, \cdot\right) = e^t\mathbf{g}(\xi, \cdot) = e^t\widehat{\mathbf{g}}(\xi) = e^t\alpha \in T^*M$$

being an integral curve of Euler field $\mathcal{E}$ in T^*M. Then, for any $\xi \in T^*M$ we have that

$$\widehat{\mathbf{g}}_*(\Delta(\xi)) = \mathcal{E}(\widehat{\mathbf{g}}(\xi))$$

is verified, therefore this implies $\widehat{\mathbf{g}}_*(\Delta) = \mathcal{E}$.

On the other hand, the equations of the integral curves of X_H are

$$\begin{cases} \dfrac{dx^k}{ds} = g^{ki}p_i \\ \\ \dfrac{dp_k}{ds} = -\dfrac{1}{2}\dfrac{\partial g^{ij}}{\partial x^k}p_ip_j \end{cases} \tag{2.4.10}$$

From the first equation of (2.4.10) we have that $p_i = g_{ik}\frac{dx^k}{ds}$. Since $\delta^j_m = g_{mi}g^{ij}$ where δ^j_m is the Kronecker's delta, then deriving we obtain

$$0 = \frac{\partial\left(g_{mi}g^{ij}\right)}{\partial x^k} = \frac{\partial g_{mi}}{\partial x^k}g^{ij} + g_{mi}\frac{\partial g^{ij}}{\partial x^k}$$

then

$$\frac{\partial g^{ij}}{\partial x^k} = -g^{im}\frac{\partial g_{ml}}{\partial x^k}g^{lj}$$

and substituting it in the second equation of (2.4.10) we get

$$\begin{cases} p_k = g_{kj}\dfrac{dx^j}{ds} \\ \\ \dfrac{dp_k}{ds} = \dfrac{1}{2}g^{im}\dfrac{\partial g_{ml}}{\partial x^k}g^{lj}p_ip_j = \dfrac{1}{2}\dfrac{\partial g_{ml}}{\partial x^k}g^{im}p_ig^{lj}p_j \end{cases} \tag{2.4.11}$$

Now, deriving the first equation of (2.4.11) with respect to s and equalling the result to the second equation we have that

$$\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} + g_{kj}\frac{d^2x^j}{ds^2} = \frac{1}{2}\frac{\partial g_{ml}}{\partial x^k}g^{im}p_ig^{lj}p_j$$

then

$$\begin{aligned} g_{kj}\frac{d^2x^j}{ds^2} &= \frac{1}{2}\frac{\partial g_{ml}}{\partial x^k} g^{im} p_i g^{lj} p_j - \frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} = \\ &= \frac{1}{2}\frac{\partial g_{ml}}{\partial x^k}\frac{dx^m}{ds}\frac{dx^l}{ds} - \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\frac{dx^i}{ds}\frac{dx^j}{ds} - \frac{1}{2}\frac{\partial g_{ik}}{\partial x^j}\frac{dx^i}{ds}\frac{dx^j}{ds} = \\ &= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^j}\right)\frac{dx^i}{ds}\frac{dx^j}{ds} = \\ &= -g_{ka}\Gamma^a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \end{aligned}$$

hence we can conclude that

$$\frac{d^2x^k}{ds^2} = -\Gamma^k_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}$$

for $k = 1, \ldots, m$ being the geodesic's equations. Therefore, system (2.4.10) can be written as

$$\begin{cases} \dfrac{dx^k}{ds} = g^{ki} p_i \\ \\ \dfrac{d^2x^k}{ds^2} = -\Gamma^k_{ij}\dfrac{dx^i}{ds}\dfrac{dx^j}{ds} \end{cases} \tag{2.4.12}$$

Since the integral curves of the hamiltonian vector field X_H coincide with the ones of the geodesic spray $X_{\mathbf{g}}$, then it is immediate to deduce that

$$\widehat{\mathbf{g}}_* \left(X_{\mathbf{g}}\right) = X_H$$

□

It is possible to show that $\widehat{\mathbf{g}}_* \left(X_{\mathbf{g}}\right) = X_H$ introducing the *fiber derivative* as done in [1, Th. 3.6.2]. For brevity, we have used coordinates.

The next corollary is an immediate consequence of lemma 2.4.6 and the construction of $\mathcal{N}$ done in section 2.2.

Corollary 2.4.7. *The space of light rays $\mathcal{N}$ of M can be built by the quotient*

$$\mathcal{N} = \mathbb{N}^{+*}/\mathcal{D}^*$$

where $\mathcal{D}^*$ is the distribution generated by the vector fields $\mathcal{E}$ and X_H, that is $\mathcal{D}^* = \operatorname{span}\{\mathcal{E}, X_H\}$.

This corollary 2.4.7 is also true for $\mathcal{N} = \mathbb{N}^*/\mathcal{D}^*$ since $\alpha \in \mathbb{N}^{+*}$ and $-\alpha \in \mathbb{N}^{-*}$ define the same light ray, in the same way $v \in \mathbb{N}^+$ and $-v \in \mathbb{N}^-$ also do it.

The expression (2.4.12) in proof of lemma 2.4.6 also shows that the null geodesic defined by $\alpha \in \mathbb{N}^*$ coincides to the null geodesic defined by $v \in \mathbb{N}$ if and only if $\widehat{\mathbf{g}}\left(v\right) = \alpha$, because the first equation has to be verified. Then we have the following commutative diagram.

$$\begin{tikzcd} \mathbb{N}^* \arrow[r, "p_{\mathbb{N}^*}"] & \mathcal{N} \\ \mathbb{N} \arrow[u, "\widehat{\mathbf{g}}"] \arrow[ur, "p_{\mathbb{N}}"'] & \end{tikzcd} \tag{2.4.13}$$

Next, we will introduce some basic definitions and results in contact geometry that we will need later. [3, Appx. 4] and [37, Ch. 5] can be consulted for more details.

Definition 2.4.8. *Given a n–dimensional differentiable manifold P, a* contact element *in P is a $(n-1)$–dimensional subspace $\mathcal{H}_q \subset T_qP$. The point $q \in P$ is called the* contact point *of $\mathcal{H}_q$.*

We will say that a distribution of hyperplanes *$\mathcal{H}$ in a differentiable manifold M is a map $\mathcal{H}$ defined in M such that for every $q \in M$ we have that $\mathcal{H}(q) = \mathcal{H}_q$ is a contact element at q.*

Lemma 2.4.9. *Every differentiable distribution of hyperplanes $\mathcal{H}$ can be written locally as the kernel of 1–form.*

Proof. We will follow the proof in [20, Lem. 1.1.1]. Consider the quotient bundles $\pi : TP \to TP/\mathcal{H}$ and $\overline{\pi} : T^*P \to (TP/\mathcal{H})^*$ and observe that $\pi^*(\overline{\pi}(\beta)) = \beta$ for any $\beta \in T^*P$. Recall that every bundle is locally trivial, this means there exists local sections. Take a non–zero local section $\alpha : U \subset (TP/\mathcal{H})^* \to T^*P$ of $\overline{\pi}$. For any $\eta \in (TP/\mathcal{H})^*$ we have that $\alpha(\eta)$ is a 1–form in TP such that $\overline{\pi} \circ \alpha(\eta) = \eta$. Thus, for $X \in T(TP)$ we have

$$\pi^*\eta(X) = \eta(\pi_* X) = \overline{\pi} \circ \alpha(\eta)(\pi_* X) = \pi^*(\overline{\pi} \circ \alpha(\eta))(X) = \alpha(\eta)(X)$$

Then,

$$X \in \mathcal{H} \Leftrightarrow \eta(\pi_* X) = 0 \Leftrightarrow \alpha(\eta)(X) = 0$$

for all $\eta \in (TP/\mathcal{H})^*$, therefore $\ker(\alpha|_U) = \mathcal{H}$. $\square$

It is clear that if a differentiable distribution of hyperplanes $\mathcal{H}$ is defined locally by the 1–form $\alpha \in \mathfrak{X}^*(P)$ then, for every non–vanishing function $f \in \mathfrak{F}(P)$ the 1–form $f\alpha$ also defines $\mathcal{H}$ since α and $f\alpha$ have the same kernel.

Definition 2.4.10. *A* contact structure *$\mathcal{H}$ in a $(2n+1)$–dimensional differentiable manifold P is a maximally non–integrable smooth field of contact elements. If $\mathcal{H} = \ker(\eta) \subset TP$ with $\eta \in \mathfrak{X}^*(P)$, the* condition of maximal non–integrability *can be written as*

$$\eta \wedge (d\eta)^n \neq 0$$

Such 1–form η which locally defines $\mathcal{H}$ is named a contact form *and we will say that (P, η) is a* contact manifold*.*

If $\mathcal{H}$ is defined by a global contact form, we will say that $\mathcal{H}$ is a cooriented contact structure*.*

An equivalent way to determine if a distribution of hyperplanes $\mathcal{H} = \ker(\eta)$ is a contact structure is the following result. See [3] and [12] for more details.

Lemma 2.4.11. *If $\mathcal{H}$ is a distribution of hyperplanes in P such that it is locally defined by $\mathcal{H} = \ker(\eta)$, then $d\eta|_{\mathcal{H}}$ is non–degenerated if and only if $\eta \wedge (d\eta)^n \neq 0$.*

Proof. Since $\dim(\mathcal{H}_q) = 2n$, then we can take $v \in T_qP$ such that $T_qP = \operatorname{span}\{v\} \oplus \mathcal{H}_q$. Take a basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2n}\}$ in T_qP such that $\mathbf{e}_0 \in \operatorname{span}\{v\}$ and $\mathbf{e}_j \in \mathcal{H}_q$ for $j = 1, \ldots, 2n$. Due to $\eta(\mathbf{e}_j) = 0$ for $j = 1, \ldots, 2n$, then we have

$$\eta \wedge (d\eta)^n (\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2n}) = \eta(\mathbf{e}_0)(d\eta)^n(\mathbf{e}_1, \ldots, \mathbf{e}_{2n})$$

and since $\eta(\mathbf{e}_0) \neq 0$, then

$$\eta \wedge (d\eta)^n \neq 0 \Leftrightarrow (d\eta)^n|_{\mathcal{H}} \neq 0$$

being equivalent to $d\eta|_{\mathcal{H}}$ is non–degenerated. □

Lemma 2.4.12. *If α is a contact form in P, then $f\alpha$ is also a contact form for every non–vanishing differentiable function $f \in \mathfrak{F}(P)$.*

Proof. Observe that α and $f\alpha$ have the same kernel. In order to show that $f\alpha$ is maximally non–integrable, we will proceed by induction. First, observe that

$$f\alpha \wedge d(f\alpha) = f\alpha \wedge (df \wedge \alpha + f d\alpha) = f\alpha \wedge df \wedge \alpha + f\alpha \wedge f d\alpha =$$

$$= -f\alpha \wedge \alpha \wedge df + f^2\alpha \wedge d\alpha = f^2\alpha \wedge d\alpha$$

Assume that

$$f\alpha \wedge (d(f\alpha))^{k-1} = f^k\alpha \wedge (d\alpha)^{k-1}$$

Then we have

$$\begin{aligned} f\alpha \wedge (d(f\alpha))^k &= f\alpha \wedge (d(f\alpha))^{k-1} \wedge d(f\alpha) = \\ &= f^k\alpha \wedge (d\alpha)^{k-1} \wedge d(f\alpha) = \\ &= f^k\alpha \wedge (d\alpha)^{k-1} \wedge (df \wedge \alpha + f d\alpha) = \\ &= f^k\alpha \wedge (d\alpha)^{k-1} \wedge df \wedge \alpha + f^k\alpha \wedge (d\alpha)^{k-1} \wedge f d\alpha = \\ &= f^k (d\alpha)^{k-1} \wedge \alpha \wedge \alpha \wedge df + f^{k+1}\alpha \wedge (d\alpha)^k = \\ &= f^{k+1}\alpha \wedge (d\alpha)^k \end{aligned}$$

whence we have proven for all n

$$f\alpha \wedge (d(f\alpha))^n = f^{n+1}\alpha \wedge (d\alpha)^n$$

Therefore, for non–vanishing f, if $\alpha \wedge (d\alpha)^n \neq 0$ then $f\alpha \wedge (d(f\alpha))^n \neq 0$. □

2.4.2

Constructing the contact structure of $\mathcal{N}$

Consider the tautological 1–form $\theta \in \mathfrak{X}^*(T^*M)$. The diffeomorphism $\widehat{\mathbf{g}} : TM \to T^*M$ allows to carry away θ to TM by pull–back. Let $\pi_M^{TM} : TM \to M$ and $\pi_M^{T^*M} : T^*M \to M$ be the canonical projections, since $\pi_M^{TM} = \pi_M^{T^*M} \circ \widehat{\mathbf{g}}$, then it is verified

$$\left(d\pi_M^{TM}\right)_v(\xi) = \left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}(v)} \left(\widehat{\mathbf{g}}_*(\xi)\right)$$

for all $\xi \in T_vTM$. If we define $\theta_{\mathbf{g}} = \widehat{\mathbf{g}}^*\theta \in \mathfrak{X}^*(TM)$ then, using the expression (2.4.4), if $\xi \in T_vTM$ we have

$$\begin{aligned}(\theta_{\mathbf{g}})_v(\xi) &= (\widehat{\mathbf{g}}^*\theta)_v(\xi) = \theta_{\widehat{\mathbf{g}}(v)}(\widehat{\mathbf{g}}_*(\xi)) = \\ &= \widehat{\mathbf{g}}(v)\left(\left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}(v)}(\widehat{\mathbf{g}}_*(\xi))\right) = \widehat{\mathbf{g}}(v)\left(\left(d\pi_M^{TM}\right)_v(\xi)\right) = \\ &= \mathbf{g}\left(v, \left(d\pi_M^{TM}\right)_v(\xi)\right)\end{aligned} \tag{2.4.14}$$

For a given basic open set $V \subset M$ equipped with coordinates $\left(x^1, \ldots, x^m\right)$ such that $v \in TV$ is written as $v = v^i \frac{\partial}{\partial x^i}$, then $\left(x^i, v^i\right)$ are coordinates in TV. By expression (2.4.5), we can write

$$\theta_{\mathbf{g}} = g_{ij} v^i dx^j$$

Let us denote by $\mathcal{H}^{TV} = \ker(\theta_{\mathbf{g}})$, that is a distribution of hyperplanes in $TV \subset TM$. This implies that $\dim\left(\mathcal{H}_v^{TV}\right) = 2m-1$ for every $v \in TV$.

As we seen in section 2.2, we have the chain of inclusions (2.2.8):

$$\Omega \hookrightarrow \mathbb{N}^+(C) \hookrightarrow \mathbb{N}^+(V) \hookrightarrow TV \tag{2.4.15}$$

where $\Omega = \Omega^X(C) = \{v \in \mathbb{N}^+ \mid g(v, X) = -1\}$ for a non-vanishing timelike vector field $X \in \mathfrak{X}(M)$. Observe that if $v \in \Omega$ is the representative of the class of equivalence $[v] \in \mathbb{PN}(C)$, then clearly the following maps

$$\begin{array}{ccccc} \Omega & \longrightarrow & \mathbb{PN}(C) & \longrightarrow & \mathcal{U} \subset \mathcal{N} \\ v & \mapsto & [v] & \mapsto & \gamma_v \end{array} \tag{2.4.16}$$

are diffeomorphisms.

Then, we will see that the pullback of $\theta_{\mathbf{g}}$ by the inclusion $\Omega \hookrightarrow TV$ defines a 1–form $\theta_{\mathbf{g}}|_{\Omega}$, and therefore a distribution of hyperplanes, in Ω. This 1–form and its kernel can be passed on $\mathcal{U} \subset \mathcal{N}$ obtaining the 1–form θ_0 looked for.

To obtain a suitable formula of θ_0 we will proceed projecting the distribution of hyperplanes in TM up to Ω step by step.

First, observe that the restriction of $\mathcal{H}^{TV}$ to $T\mathbb{N}^+(V)$, denoted by $\mathcal{H}^{\mathbb{N}^+(V)}$, is again a distribution of hyperplanes. Indeed, if $c : (-\epsilon, \epsilon) \to \mathbb{N}^+(V)$ is a differentiable curve such that

$$\begin{cases} \alpha(s) = \pi_M^{\mathbb{N}^+}(c(s)) \text{ is a timelike curve} \\ v = c(0) \in \mathbb{N}^+(V) \\ \xi = c'(0) \in T_v\mathbb{N}^+(V) \end{cases}$$

then, by using expression (2.4.14)

$$\theta_{\mathbf{g}}(\xi) = \mathbf{g}(v, \alpha'(0)) \neq 0$$

since v is null and $\alpha'(0)$ timelike. This implies that $\xi \notin \mathcal{H}_v^{TV}$. So, we have that $T_vTV = span\{\xi\} \oplus \mathcal{H}_v^{TV}$ and since $span\{\xi\} \subset T_v\mathbb{N}^+(V)$ and $\mathcal{H}_v^{\mathbb{N}^+(V)} = \mathcal{H}_v^{TV} \cap T_v\mathbb{N}^+(V)$ then we have that

$$\dim\left(\mathcal{H}_v^{\mathbb{N}^+(V)}\right) = 2m-2$$

therefore $\mathcal{H}^{\mathbb{N}^+(V)}$ is a distribution of hyperplanes in $\mathbb{N}^+(V)$.

The next step is to restrict $\mathcal{H}^{\mathbb{N}^+(V)}$ to $T\mathbb{N}^+(C)$, where C is a Cauchy surface of V. Again, as done above, if $\gamma : I \to M$ is a null geodesic verifying $\gamma(0) \in C$ and $\gamma'(0) = v \in \mathbb{N}^+(C)$, since the vector subspace

$$\{v\}^\perp = \left\{u \in T_{\gamma(0)}M : \mathbf{g}(v,u) = 0\right\}$$

is $(m-1)$–dimensional and $v = \gamma'(0) \in \{v\}^\perp$, then $\dim\left(\{v\}^\perp \cap T_{\gamma(0)}C\right) = m-2$. Hence, we can pick up a vector $\eta \in T_{\gamma(0)}C$ such that $T_{\gamma(0)}C = \mathit{span}\{\eta\} \oplus \left(\{v\}^\perp \cap T_{\gamma(0)}C\right)$. Now, we can choose a differentiable curve $c : (-\epsilon, \epsilon) \to \mathbb{N}^+(C)$ verifying

$$\begin{cases} c(0) = v \in \mathbb{N}^+(C) \\ c'(0) = \kappa \in T_v\mathbb{N}^+(C) \\ \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa) = \lambda\eta \text{ for } \lambda \neq 0 \end{cases}$$

then

$$\theta_{\mathbf{g}}(\kappa) = \mathbf{g}\left(v, \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa)\right) = \mathbf{g}(v, \lambda\eta) \neq 0$$

because $\eta \notin \{v\}^\perp$, and this shows that $\kappa \notin \mathcal{H}_v^{\mathbb{N}^+(V)}$. Then $T_v\mathbb{N}^+(V) = \mathit{span}\{\kappa\} \oplus \mathcal{H}_v^{\mathbb{N}^+(V)}$ and since $\mathit{span}\{\kappa\} \subset T_v\mathbb{N}^+(C)$ and $\mathcal{H}_v^{\mathbb{N}^+(C)} = \mathcal{H}_v^{\mathbb{N}^+(V)} \cap T_v\mathbb{N}^+(C)$, then it follows

$$\dim\left(\mathcal{H}_v^{\mathbb{N}^+(C)}\right) = \dim\left(T_v\mathbb{N}^+(C)\right) - 1 = 2m-3$$

thus $\mathcal{H}^{\mathbb{N}^+(C)}$ is a distribution of hyperplanes in $\mathbb{N}^+(C)$.

It is possible to repeat the previous argument to show that the restriction of $\mathcal{H}^{\mathbb{N}^+(C)}$ to $T\Omega$ defines a distribution of hyperplanes. In fact, consider some $\eta \in T_{\gamma(0)}C$ in the same condition as before and take a differentiable curve $c : (-\epsilon, \epsilon) \to \Omega$ verifying

$$\begin{cases} c(0) = v \in \Omega \\ c'(0) = \kappa \in T_v\Omega \\ \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa) = \lambda\eta \text{ for } \lambda \neq 0 \end{cases}$$

then again

$$\theta_{\mathbf{g}}(\kappa) = \mathbf{g}(v, \lambda\eta) \neq 0$$

showing that $\kappa \notin \mathcal{H}_v^{\mathbb{N}^+(C)}$. Then $T_v\mathbb{N}^+(C) = \mathit{span}\{\kappa\} \oplus \mathcal{H}_v^{\mathbb{N}^+(C)}$ and since $\mathit{span}\{\kappa\} \subset T_v\Omega$ then we have that

$$\dim\left(\mathcal{H}_v^\Omega\right) = \dim\left(T_v\Omega\right) - 1 = 2m-4$$

thus $\mathcal{H}^\Omega$ is a distribution of hyperplanes in $\Omega \subset \mathbb{N}^+(C)$.

By this process of restriction from TV to Ω we have passed $\mathcal{H}^{TV} \subset TTV$ as a distribution of hyperplanes $\mathcal{H}^\Omega \subset T\Omega \subset TTV$. Moreover since $\mathcal{H}^{TV} = \ker(\theta_{\mathbf{g}})$ and $\mathcal{H}^\Omega = T\Omega \cap \mathcal{H}^{TV}$ then

$$\mathcal{H}^\Omega = \ker\left(\theta_{\mathbf{g}}|_\Omega\right)$$

where $\theta_{\mathbf{g}}|_\Omega$ denotes the restriction of $\theta_{\mathbf{g}}$ to Ω. This fact is important in order to show that $\mathcal{H}^\Omega$ is a contact structure.

Then, using the diffeomorphisms in (2.4.16), $\mathcal{H}^\Omega$ passes to $\mathcal{U} \subset \mathcal{N}$ as a distribution of hyperplanes of dimension $2m-4$. Let us denote by $\mathcal{H} \subset T\mathcal{N}$ such distribution.

Proposition 2.4.13. *Assuming the previous notation, if $X \in \mathfrak{X}(M)$ is a given global non–vanishing timelike vector field and $\mathcal{U} \subset \mathcal{N}$ is open as above, then the distribution of hyperplanes*

$$\mathcal{H}(\mathcal{U}) = \{[J] \in T_\gamma \mathcal{U} : \mathbf{g}(\gamma'(0), J(0)) = 0 \text{ with } \mathbf{g}(\gamma'(0), X) = -1\} \tag{2.4.17}$$

is a contact structure.

Proof. Since $\omega = -d\theta$, then taking the exterior derivative on $\theta_{\mathbf{g}}$ we obtain

$$\omega_{\mathbf{g}} = -d\theta_{\mathbf{g}}$$

therefore we have

$$\omega_{\mathbf{g}} = -d\left(g_{ij} v^i dx^j\right) = -g_{ij} dv^i \wedge dx^j - \frac{\partial g_{ij}}{\partial x^k} v^i dx^k \wedge dx^j$$

then it can be written by

$$\omega_{\mathbf{g}} = g_{ij} dx^j \wedge dv^i + \frac{\partial g_{ij}}{\partial x^k} v^i dx^j \wedge dx^k \tag{2.4.18}$$

It can be shown, see [1, Th. 3.2.13], that $\omega_{\mathbf{g}}$ is a symplectic 2–form in TM.

Consider two curves $u_n(s) = u_n^i(s) \left(\frac{\partial}{\partial x^i}\right)_{\alpha_n(s)} \in TM$ where $n = 1, 2$ such that

$$\begin{aligned} \alpha_n'(s) &= a_n^i(s) \left(\tfrac{\partial}{\partial x^i}\right)_{\alpha_n(s)} \\ u_n'(s) &= a_n^i(s) \left(\tfrac{\partial}{\partial x^i}\right)_{u_n(s)} + \tfrac{du_n^i}{ds}(s) \left(\tfrac{\partial}{\partial v^i}\right)_{u_n(s)} \end{aligned}$$

and recall that

$$\frac{Du_n}{ds} = \left(\frac{du_n^k}{ds} + \Gamma_{ij}^k a_n^i u_n^j\right) \left(\frac{\partial}{\partial x^k}\right)_{\alpha_n}$$

calling $\frac{D^k u_n}{ds} = \frac{du_n^k}{ds} + \Gamma_{ij}^k a_n^i u_n^j$ to the k–th component of $\frac{Du_n}{ds}$. If $u = u_1(0) = u_2(0)$ and $\xi_n = u_n'(0)$ for $n = 1, 2$, then we have that

$$\begin{aligned} \omega_{\mathbf{g}}(\xi_1, \xi_2) &= g_{ij} a_1^i \tfrac{du_2^j}{ds} - g_{ij} a_2^j \tfrac{du_1^i}{ds} + \tfrac{\partial g_{ij}}{\partial x^k} u^i a_1^j a_2^k - \tfrac{\partial g_{ij}}{\partial x^k} u^i a_1^k a_2^j = \\ &= g_{ij} a_1^i \left(\tfrac{D^j u_2}{ds} - \Gamma_{lr}^j a_2^l u^r\right) - g_{ij} a_2^j \left(\tfrac{D^i u_1}{ds} - \Gamma_{lr}^i a_1^l u^r\right) + \left(\tfrac{\partial g_{ij}}{\partial x^k} - \tfrac{\partial g_{ik}}{\partial x^j}\right) u^i a_1^j a_2^k = \\ &= g_{ij} a_1^i \tfrac{D^j u_2}{ds} - g_{ij} a_2^j \tfrac{D^i u_1}{ds} + \left(g_{kl} \Gamma_{ji}^l - g_{jl} \Gamma_{ki}^l + \tfrac{\partial g_{ij}}{\partial x^k} - \tfrac{\partial g_{ik}}{\partial x^j}\right) u^i a_1^j a_2^k = \\ &= g_{ij} a_1^i \tfrac{D^j u_2}{ds} - g_{ij} a_2^j \tfrac{D^i u_1}{ds} = \\ &= \mathbf{g}\left(\alpha_1'(0), \tfrac{Du_2}{ds}(0)\right) - \mathbf{g}\left(\alpha_2'(0), \tfrac{Du_1}{ds}(0)\right) \end{aligned} \tag{2.4.19}$$

where we have used that $g_{kl} \Gamma_{ji}^l = \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k}\right)$.

Since the exterior derivative commutes with the restriction to submanifolds, then

$$\omega_{\mathbf{g}}|_{\Omega} = -\left(d\theta_{\mathbf{g}}\right)|_{\Omega} = -d\left(\theta_{\mathbf{g}}|_{\Omega}\right)$$

Proposition 2.3.9 permits to transmit $\theta_{\mathbf{g}}|_{\Omega}, \omega_{\mathbf{g}}|_{\Omega}$ to $\mathcal{L}(\gamma_u)$ pointwise. Calling θ_0 and ω_0 the resultant forms, then for $[J], [J_1], [J_2] \in \mathcal{L}(\gamma_u)$ we have

$$\theta_0([J]) = \mathbf{g}(\gamma_u'(0), J(0)) \tag{2.4.20}$$

where γ_u is parametrized such that $\gamma_u'(0) \in \Omega$, and

$$\omega_0([J_1],[J_2]) = \mathbf{g}(J_1(0), J_2'(0)) - \mathbf{g}(J_2(0), J_1'(0)) \tag{2.4.21}$$

In order to prove that $\mathcal{H}$ is a contact structure, we will show that $\omega_0|_{\mathcal{H}\times\mathcal{H}}$ is non–degenerated. Consider $[J_1],[J_2] \in \mathcal{H}$, then the initial values of J_1 and J_2 in expression (2.4.21) verify

$$\begin{cases} \mathbf{g}(J_i(0), \gamma_u'(0)) = 0 \\ \mathbf{g}(J_i'(0), \gamma_u'(0)) = 0 \end{cases} \tag{2.4.22}$$

for $i = 1,2$, that is $J_i(0), J_i'(0) \in \{\gamma_u'(0)\}^{\perp} = \left\{v \in T_{\gamma_u(0)}M : \mathbf{g}(v, \gamma_u'(0)) = 0\right\}$.

Assume that $\omega_0([J_1],[J_2]) = 0$ for a given $[J_1] \in \mathcal{H}$ and all $[J_2] \in \mathcal{H}$, then in particular, it is also so for $[J_2]$ verifying $J_2'(0) = 0$, then

$$\omega_0([J_1],[J_2]) = 0 \Rightarrow \mathbf{g}(J_2(0), J_1'(0)) = 0$$

Since $J_1'(0) \in \{\gamma_u'(0)\}^{\perp}$, the only vector $J_1'(0)$ such that $\mathbf{g}(J_2(0), J_1'(0)) = 0$ for all $J_2(0) \in \{\gamma_u'(0)\}^{\perp}$ is, by definition of $\{\gamma_u'(0)\}^{\perp}$, the vector $J_1'(0) = 0\,(\mathrm{mod}\gamma_u'(0))$.

On the other hand, for $[J_2]$ verifying $J_2(0) = 0$ we have

$$\omega_0([J_1],[J_2]) = 0 \Rightarrow \mathbf{g}(J_1(0), J_2'(0)) = 0$$

and again, since $J_1(0) \in \{\gamma_u'(0)\}^{\perp}$ then the only vector $J_1(0)$ such that $\mathbf{g}(J_1(0), J_2'(0)) = 0$ for all $J_2'(0) \in \{\gamma_u'(0)\}^{\perp}$ is $J_1(0) = 0\,(\mathrm{mod}\gamma_u'(0))$.

Thus, the only $[J_1] \in \mathcal{H}$ such that $\omega_0([J_1],[J_2]) = 0$ for all $[J_2] \in \mathcal{H}$ is $J_1 = 0\,(\mathrm{mod}\gamma_u')$, therefore $\omega_0|_{\mathcal{H}\times\mathcal{H}}$ is non–degenerated. This shows that $\mathcal{H}$ is a contact structure in $\mathcal{N}$. □

Let us take $\gamma \in \mathcal{U}\cap\mathcal{V}$, since in general $\frac{d}{dt}\mathbf{g}(\gamma'(t), X(\gamma(t))) \neq 0$, then there are different parameter for γ in order to write $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\mathcal{V})$ as in expression (2.4.17). If we consider that $\gamma = \gamma(t)$ and $\overline{\gamma} = \overline{\gamma}(\tau)$ are the parametrizations of $\gamma \in \mathcal{U}\cap\mathcal{V}$ such that $\overline{\gamma}(\tau) = \gamma(a\tau + b)$ verifying

$$\begin{cases} \mathbf{g}(\gamma'(0), X) = -1 \\ \mathbf{g}(\overline{\gamma}'(0), X) = -1 \end{cases}$$

By definition of $\mathcal{J}_L(\overline{\gamma})$, we have that $\mathbf{g}\left(\overline{J}(\tau), \overline{\gamma}'(\tau)\right)$ is constant, therefore

$$\mathbf{g}\left(\overline{J}(0), \overline{\gamma}'(0)\right) = \mathbf{g}\left(\overline{J}(-b/a), \overline{\gamma}'(-b/a)\right) = \mathbf{g}(J(0), \gamma'(0))$$

as seen in remark 2.3.6, whence since $\overline{\gamma}(-b/a) = \gamma(0)$ we have

$$\mathbf{g}\left(\overline{J}(0), \overline{\gamma}'(0)\right) = 0 \Leftrightarrow \mathbf{g}\left(\overline{J}(-b/a), \overline{\gamma}'(-b/a)\right) = 0 \Leftrightarrow \mathbf{g}(J(0), \gamma'(0)) = 0$$

The same argument above is valid to prove that $\mathcal{H}_\gamma$ does not depends on the timelike vector field used to define Ω, because it only affects to the parametrization of γ. This shows that $\mathcal{H}_\gamma$ is well defined and does not depends on the neighbourhood used in its construction.

At this point, we have a covering $\{\mathcal{U}_\delta\}_{\delta\in I} \subset \mathcal{N}$ and, for any $\delta \in I$, also a local 1–form θ_0^δ defining the contact structure $\mathcal{H}$. If we take a partition of unity $\{\chi_\delta\}_{\delta\in I}$ subordinated to the covering $\{\mathcal{U}_\delta\}_{\delta\in I}$ then we can define a global 1–form by

$$\theta_0([J]) = \sum_{\delta\in I} \chi_\delta([J]) \cdot \theta_0^\delta([J]) \tag{2.4.23}$$

then the contact structure $\mathcal{H}$ is cooriented since θ_0 is global and, by lemma 2.4.11, remains maximally non-integrable.

Moreover, although the expression of $\mathcal{H}_\gamma$ in proposition 2.4.13 depends on the representative $\mathbf{g}$ of the conformal metric, it coincides with the definition of hyperplane distribution of (2.4.2) that is conformal, then $\mathcal{H}$ does not depends on the specific metric itself, but only on the conformal manifold $(M, \mathcal{C})$.

Section 2.5

The contact structure of $\mathcal{N}$ and symplectic reduction

The celebrated Theorem of Marsden–Weinstein [47] claims that a $2m$–dimensional symplectic manifold P, in which a Lie group G acts preserving the symplectic form ω and possessing an equivariant momentum map, can be reduced into another $(2m-2r)$–dimensional symplectic manifold P_μ, called the *Marsden–Weinstein reduction of P with respect to μ*, where μ is an element of the dual of the Lie algebra of G and r is the dimension of the coadjoint orbit passing through μ. Moreover, the Hamiltonian H in P is also reduced to a hamiltonian H_μ of P_μ such that the integral curves of the hamiltonian vector field $X_{H_\mu} \in \mathfrak{X}(P_\mu)$ carry the relevant information to describe the integral curves of $X_H \in \mathfrak{X}(P)$.

Although it is possible to derive the contact structure of $\mathcal{N}$ using Marsden–Weinstein reduction, as indicated by Low in [44] and [45] as well as Keshin and Tabachnikov in [30], we can also choose a different path to achieve it. This new way is simpler because we do not need the full extent of Marsden–Weinstein reduction theorem but just a simplified version of it, and also it is more general because it is not necessary to assume of the existence of a group action. In fact, it is an equivalent but more elegant manner to obtain $\mathcal{H}$ to the way used in section 2.4.2. Actually, the setting we will use is a particular instance of the scheme called *generalized symplectic reduction* (see [13] and references therein). We will carry it out in next section 2.5.1.

Finally, as illustration of the construction by Marsden–Weinstein reduction of the contact structure of $\mathcal{N}$ done in the literature (see for example [30], [44] and [45]), we offer the missing details in section 2.5.3.

2.5.1

Coisotropic reduction of $\mathbb{N}^+$

The main result of the present section, theorem 2.5.7, that is, the construction of the contact structure $\mathcal{H}$ by reduction of $\mathbb{N}^+$, is based on some elementary algebraic facts that we will develop below.

Lemma 2.5.1. *Let $\pi : M \to N$ be a submersion, then $\pi^* : \Lambda^p(N) \to \Lambda^p(M)$ is injective.*

Proof. It is known that π^* is linear (see [49, Prop. 2.10]). Consider $\theta \in \Lambda^p(N)$ such that $\theta_y = f_{i_1,\ldots,i_p}\mathbf{e}^{i_1} \wedge \cdots \wedge \mathbf{e}^{i_p}$ for $y = \pi(x) \in N$, and $\left\{\mathbf{e}^1, \ldots, \mathbf{e}^n\right\} \subset T_y^*N$ the dual basis of

$\{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset T_yN$. Since π is a submersion, then $d\pi_x : T_xM \to T_yN$ is surjective, then for every choice of $\{i_1, \ldots, i_p\}$ we can choose $\alpha_k \in T_xM$ such that

$$d\pi_x (\alpha_k) = \mathbf{e}_{i_k} \in T_yN$$

with $k = 1, \ldots, p$.

So, if $\pi^*\theta = \mathbf{0}$ then

$$0 = (\pi^*\theta)_x (\alpha_1, \ldots, \alpha_p) = \theta_y (d\pi_x (\alpha_1), \ldots, d\pi_x (\alpha_p)) = \theta_y \left(\mathbf{e}^{i_1}, \ldots, \mathbf{e}^{i_p}\right) = f_{i_1,\ldots,i_p}$$

hence $f_{i_1,\ldots,i_p} = 0$ for all $y \in N$ and every choice of $\{i_1, \ldots, i_p\}$. Then we have that $\theta = \mathbf{0}$ and thus $\ker(\pi^*) = \{\mathbf{0}\}$ and therefore π^* is injective. □

Lemma 2.5.2. *Let (E, ω) be a symplectic vector space and $W \subset E$ a subspace. Then $\dim E = \dim W^\perp + \dim W$.*

Proof. Consider the linear map $l : E \to W^*$ given by $l(v) = \omega(v, \cdot)|_W$ where W^* denotes the dual of W. Then observe

$$\ker(l) = \{v \in E : \omega(v, \cdot)|_W = \mathbf{0}\} = \{v \in E : \omega(v, u) = 0 \text{ for all } u \in W\} = W^\perp$$

and since ω is non–degenerated also

$$\mathrm{Im}(l) = \{\omega(v, \cdot)|_W : v \in E\} = W^*$$

Hence

$$\dim E = \dim \ker(l) + \dim \mathrm{Im}(l) = \dim W^\perp + \dim W^* = \dim W^\perp + \dim W.$$

□

Corollary 2.5.3. *Let E be a symplectic vector space and $W \subset E$ any hyperplane. Then W is coisotropic.*

Proof. Since the codimension of W is 1, then W is odd–dimensional, hence $\omega|_{W\times W}$ is degenerated (see remark 2.4.3). Then we have that $W \cap W^\perp \neq \{\mathbf{0}\}$, and by lemma 2.5.2, $\dim W^\perp = 1$ therefore $W^\perp \subset W$. □

Corollary 2.5.4. *Let E be a symplectic vector space and $W \subset E$ any hyperplane. Then the quotient space $W/W^\perp$ inherits a canonical symplectic form $\overline{\omega}$ defined by the expression:*

$$\overline{\omega}(u_1 + W^\perp, u_2 + W^\perp) = \omega(u_1, u_2), \qquad \textit{for all } u_1, u_2 \in W. \tag{2.5.1}$$

Proof. By corollary 2.5.3, W is coisotropic and $\dim W^\perp = 1$, then $W/W^\perp$ is a even–dimensional quotient vector space. Moreover, by definition of $\overline{\omega}$, it is trivial to see that $\overline{\omega}$ is non–degenerated. □

The following theorem 2.5.5 states that any hypersurface on a symplectic manifold is coisotropic and that, provided that the quotient space is a manifold, the space of leaves of its *characteristic foliation*, inherits a symplectic structure. Such space of leaves is thus the reduced symplectic manifold we are seeking for and it will be called the *coisotropic reduction* of the hypersurface S.

Theorem 2.5.5. *Let (P, ω) be a symplectic manifold and $i\colon S \to P$ be a hypersurface, i.e., an immersed manifold with codimension 1. Then:*

1. *The symplectic form ω induces a 1–dimensional distribution K on S, called the* characteristic distribution *of ω, defined by $K_x = \ker i^*\omega_x = (T_xS)^\perp \subset T_xS$.*

2. *If we denote by $\mathcal{K}$ the 1–dimensional foliation defined by the distribution K and $\overline{S} = S/\mathcal{K}$ has the structure of a quotient manifold, i.e., the canonical projection map $\rho\colon S \to S/\mathcal{K}$ is a submersion, then there exists a unique symplectic 2–form $\overline{\omega}$ on $\overline{S}$ such that $\rho^*\overline{\omega} = i^*\omega$.*

3. *If $\omega = -d\theta$ and there exists $\overline{\theta}$ a 1–form on $\overline{S}$ such that $\rho^*\overline{\theta} = i^*\theta$, then $\overline{\omega} = -d\overline{\theta}$.*

Proof. The proof of (1) is just the restriction of the algebraic statements above to $W = T_xS \subset E = T_xP$.

The statement (2) is consequence of corollary 2.5.4 taking again $W = T_xS \subset E = T_xP$. If $\overline{\omega}_{\rho(x)}$ is the symplectic 2–form in $T_{\rho(x)}\overline{S}$ defined by expression (2.5.1), then for any $U_1, U_2 \in T_xS$ we have

$$(\rho^*\overline{\omega})_x (U_1, U_2) = \overline{\omega}_{\rho(x)} (\rho_* (U_1), \rho_* (U_2)) = \overline{\omega}_{\rho(x)} (U_1 + K, U_2 + K) = \omega_x (U_1, U_2)$$

for every $x \in S$, then we have

$$\rho^*\overline{\omega} = \omega|_{S\times S} = i^*\omega$$

By lemma 2.5.1, ρ^* is injective, therefore $\overline{\omega}$ is unique. Since ω is closed and $d(i^*\omega) = i^*(d\omega)$ then $i^*\omega$ is also closed. Then

$$\mathbf{0} = d(i^*\omega) = d(\rho^*\overline{\omega}) = \rho^*(d\overline{\omega})$$

and since ρ^* is injective because of lemma 2.5.1, then $d\overline{\omega} = \mathbf{0}$ and therefore $\overline{\omega}$ is closed.

The proof of (3) is trivial because $\rho^*\overline{\omega} = i^*\omega = i^*(-d\theta) = -di^*\theta = -d\rho^*\overline{\theta} = \rho^*(-d\overline{\theta})$ and since ρ is a submersion, then lemma 2.5.1 implies that $\overline{\omega} = -d\overline{\theta}$. □

In addition to the previous reduction mechanism, we will use the following one to use it for hyperplane distributions in order to define the sought contact structure.

Theorem 2.5.6. *Let $(P, \omega = -d\theta)$ be an exact symplectic manifold and $\pi : P \to N$ be a submersion on a manifold N such that* $\dim N = \dim P - 1$ *and verifying that it projects the hyperplane distribution $H = \ker\theta$, that is there exists a hyperplane distribution H^N in N such that for any $x \in P$, $d\pi_x(H_x) = H^N_{\pi(x)}$. Then H^N defines a contact structure on N.*

Proof. Notice that due to theorem 2.5.5, we have that $\ker d\pi_x = H_x^{\perp}$ and ω induces a symplectic form $\overline{\omega}_x$ in $H_x/H_x^{\perp}$. Moreover since $H_x/H_x^{\perp} \simeq H^N_{\pi(x)}$, then it inherits the symplectic form $\overline{\omega}_x$. Consider a local section σ of the submersion π and the 1–form $\sigma^*\theta$ in N. For any $v \in H_y^N$ with $y \in N$ we have that $\sigma_* v \in H_{\sigma(y)}$, then

$$\sigma^*\theta\,(v) = \theta\,(\sigma_* v) = 0$$

thus $\ker \sigma^*\theta = H^N$. Moreover, given $u, v \in H_y^N$ for $y \in N$, then

$$-d\,(\sigma^*\theta)_y\,(u,v) = (\sigma^*\,(-d\theta))_y\,(u,v) = (\sigma^*\omega)_y\,(u,v) =$$

$$= \omega_{\sigma(y)}\,(d\sigma_y\,(u)\,, d\sigma_y\,(v)) = \overline{\omega}_y\,(u,v)$$

because of corollary 2.5.4 and where $d\sigma_y\,(u)\,, d\sigma_y\,(v) \in H_{\sigma(y)}$. Therefore $-d(\sigma^*\theta)$ coincides with the symplectic 2–form $\overline{\omega}$ when restricted to H^N, and since $\overline{\omega}$ is non–degenerated, then applying lemma 2.4.11 we conclude that H^N is a contact structure. □

The two previous results, theorems 2.5.5 and 2.5.6, hold the key to understand how the quotient space $\mathcal{N}$ inherits a canonical contact structure. Consider again a spacetime $(M, \mathbf{g})$ and the canonical identification provided by the metric $\widehat{\mathbf{g}}\colon \widehat{T}M \to \widehat{T}^*M$ defined in equation (2.4.6), which is just the Legendre transform corresponding to the Lagrangian function

$$\begin{array}{cccc} L: & TM & \to & \mathbb{R} \\ & v & \mapsto & L\,(v) = \frac{1}{2}\mathbf{g}\,(v,v). \end{array} \tag{2.5.2}$$

Observe that if H is the hamiltonian function defined in (2.4.9) then

$$H \circ \widehat{\mathbf{g}}\,(v) = \frac{1}{2}\mathbf{g}\left(\widehat{\mathbf{g}}^{-1}\,(\widehat{\mathbf{g}}\,(v))\,, \widehat{\mathbf{g}}^{-1}\,(\widehat{\mathbf{g}}\,(v))\right) = \frac{1}{2}\mathbf{g}\,(v,v) = L\,(v)$$

for any $v \in \widehat{T}M$.

As we discussed at section 2.4.2 we can propagate to TM the canonical 1–form θ as well as the symplectic 2–form ω defined on T^*M by pull–back through the diffeomorphism $\widehat{\mathbf{g}}$, then we obtain

$$\left\{ \begin{array}{l} \theta_{\mathbf{g}} = \widehat{\mathbf{g}}^*\theta \\ \omega_{\mathbf{g}} = \widehat{\mathbf{g}}^*\omega = -d\theta_{\mathbf{g}} \end{array} \right.$$

in such a way that $\left(\widehat{T}M, \omega_{\mathbf{g}}\right)$ becomes a symplectic manifold. Moreover $\mathbb{N}^+ \subset \widehat{T}M$ defines a hypersurface, hence by theorem 2.5.5 we can construct its coisotropic reduction.

We will denote by $\mathcal{N}_s^+$ the space of equivalence classes of future–oriented null geodesics that differ by a translation of the parameter. Thus two parametrized null geodesics $\gamma_1\,(t)$, $\gamma_2\,(\tau)$ are equivalent if there exists a real number s such that $\gamma_2\,(\tau) = \gamma_1\,(t+s)$. The equivalence class of future–directed null geodesics containing the parametrized geodesic $\gamma(t)$ such that $\gamma'(0) = v$ will be denoted by γ_v. The space $\mathcal{N}_s^+$ is sometimes called the space of *future–directed scaled null geodesic* and describes equivalence classes of null geodesics distinguishing different scale parametrizations. Clearly there is a natural projection $\pi : \mathcal{N}_s^+ \to \mathcal{N}$ defined by $\pi(\gamma_v) = [\gamma_v] = \gamma_{[v]}$.

Theorem 2.5.7. *Let $(M, \mathbf{g})$ be a spacetime, then:*

1. *The characteristic distribution $K = \ker \omega_{\mathbf{g}} \mid_{\mathbb{N}^+}$ is generated by the restriction of the geodesic spray $X_{\mathbf{g}}$ to $\mathbb{N}^+$ and $\mathbb{N}^+/K$ can be identified naturally with the space of scaled null geodesics $\mathcal{N}_s^+$.*
2. *If M is strongly causal, $\mathcal{N}_s^+$ is a quotient manifold of $\mathbb{N}^+$, and it becomes a symplectic manifold with the canonical reduced symplectic structure obtained by coisotropic reduction of $\omega_{\mathbf{g}}$.*

Proof. In order to prove (1), we just check that $\omega_{\mathbf{g}}(X_{\mathbf{g}}, Y) = 0$. Indeed, we have that

$$\begin{aligned}\omega_{\mathbf{g}}\left(X_{\mathbf{g}}, Y\right) = \widehat{\mathbf{g}}^* \omega\left(X_{\mathbf{g}}, Y\right) &= \omega\left(\widehat{\mathbf{g}}_*\left(X_{\mathbf{g}}\right), \widehat{\mathbf{g}}_*\left(Y\right)\right) = \\ &= \omega\left(X_H, \widehat{\mathbf{g}}_*\left(Y\right)\right) = i_{X_H}\omega\left(\widehat{\mathbf{g}}_*\left(Y\right)\right) = \\ &= dH\left(\widehat{\mathbf{g}}_*\left(Y\right)\right) = Y\left(H \circ \widehat{\mathbf{g}}\right) = \\ &= Y\left(L\right) = 0\end{aligned}$$

for all $Y \in T\mathbb{N}^+$ because $\mathbb{N}^+ \subset \mathbb{N} = L^{-1}(\mathbf{0})$ (see equation (1.2.1)).

Notice that the flow Φ_t of the geodesic spray $X_{\mathbf{g}}$ is such that $\Phi_s(\gamma(t)) = \gamma(t+s)$ where $\gamma(t)$ is a parametrized geodesic. Then the quotient $\mathbb{N}^+/K$ corresponds exactly to the notion of scaled null geodesic before. We will denote, as before, by $\rho : \mathbb{N}^+ \to \mathcal{N}_s^+$ the canonical projection and, with the notations above, we get simply that $\rho(v) = \gamma_v$.

Since M is strongly causal, the proof of (2) mimics the proof of proposition 2.2.7 (see also remark 2.2.8). Hence due to (2) in theorem 2.5.5, we conclude that the quotient manifold inherits a canonical symplectic structure by coisotropic reduction of $\omega_{\mathbf{g}}$. □

2.5.2 *Symplectic reduction*

We will introduce the Theorem of Marsden-Weinstein in order to apply it to construct $\mathcal{N}$ by symplectic reduction from T^*Mas it is mentioned in [30], [44] and [45].

Definition 2.5.8. *Let (P, ω) be a connected symplectic manifold and G a Lie group with Lie algebra $\mathfrak{g}$. An action $\Phi : G \times P \to P$ is said to be a* symplectic action *if for each $g \in G$ the map $\Phi_g : P \to P$ defined by $\Phi_g(p) = \Phi(g, p)$ verifies $\Phi_g^*\omega = \omega$.*

A map $J : P \to \mathfrak{g}^*$ will be called a momentum map *for the action Φ if for every $\eta \in \mathfrak{g}$*

$$d\hat{J}\left(\eta\right) = i_{\eta_P}\left(\omega\right)$$

where $\hat{J}(\eta) : P \to \mathbb{R}$ is defined by $\hat{J}(\eta)(p) = J(p)(\eta)$ and η_P is the infinitesimal generator of the action corresponding to η.

Definition 2.5.9. *A momentum map J of an action $\Phi : G \times P \to P$ is said to be Ad^*–*equivariant *if Φ is compatible with J, that is*

$$J\left(\Phi_g\left(p\right)\right) = Ad^*_{g^{-1}} J\left(p\right)$$

for all $p \in P$ and $g \in G$, where $Ad^*_{g^{-1}}$ denotes de co–adjoint action associated to G.

Example 2.5.10. *Consider a hamiltonian function $H : T^*M \to \mathbb{R}$ and its corresponding hamiltonian vector field X_H. The flow $\Phi : \mathbb{R} \times T^*M \to T^*M$ of X_H is an action of the Lie group $\mathbb{R}$ (with the usual addition) since it trivially verifies for all $t, s \in \mathbb{R}$ and $\alpha \in T^*M$*

$$\Phi(0, \alpha) = \alpha$$

$$\Phi(t + s, \alpha) = \Phi(t, \Phi(s, \alpha))$$

for being a flow. Moreover, we have that

$$\left.\frac{d}{dt}\right|_{t=0} \Phi_t^* \omega = \mathcal{L}_{X_H} \omega = i_{X_H}(d\omega) + d(i_{X_H}(\omega)) = 0$$

since both ω and $i_{X_H}\omega$ are exact, hence closed. This implies that $\Phi_t^*\omega$ is constant, then $\Phi_t^*\omega = \Phi_0^*\omega = \omega$ and therefore Φ is a symplectic action.

Since $G = \mathbb{R}$ is the Lie group of the action Φ, then we have that $\mathfrak{g} \simeq \mathbb{R} \simeq \mathfrak{g}^*$.

Next, we will see that a hamiltonian function H is a moment map. Indeed, we define the maps $\hat{J}(k) : T^*M \to \mathbb{R}$ by $\hat{J}(k) = kH$ and then we have

$$J(\alpha)(k) = \hat{J}(k)(\alpha) = (kH)(\alpha)$$

Now, consider $k \in \mathbb{R} \simeq \mathfrak{g}$, then its infinitesimal generator at $\alpha \in T^*M$ is

$$k_{T^*M}(\alpha) = \left.\frac{d}{dt}\right|_{t=0} \Phi(\exp(t \cdot k), \alpha) = \left.\frac{d}{dt}\right|_{t=0} \Phi(tk, \alpha) = kX_H(\alpha)$$

Then

$$i_{kX_H}(\omega) = k \cdot i_{X_H}(\omega) = k \cdot dH = d(kH) = d\hat{J}(k)$$

hence J is a momentum map. For any $H(\alpha) \in \mathbb{R} \simeq \mathfrak{g}^*$ we have that $H(\alpha)(k) = kH(\alpha) = J(\alpha)(k)$ for all $k \in \mathbb{R} \simeq \mathfrak{g}$ then

$$H(\alpha) = J(\alpha)$$

for all $\alpha \in T^*M$, then $J = H$ and therefore H is a momentum map.

Recall that, in this case, the exponential in G, the adjoint and the co–adjoint actions, that is exp, *Ad_t and Ad_t^* respectively, are the corresponding identity maps. Since*

$$Ad_{-t}^*(k)(X_n) = k(Ad_{-t}(X_n)) = k(X_n)$$

for $k \in \mathbb{R} \simeq \mathfrak{g}^*$, then the co–adjoint action $\overline{\Phi}_t = Ad_{-t}^*$ verifies

$$\overline{\Phi}_t(k) = k \in \mathfrak{g}^*$$

for every $t \in \mathbb{R}$. This implies that the stabilizer of k is

$$G_k = \left\{ t \in \mathbb{R} : \overline{\Phi}_t(k) = k \right\} = \mathbb{R}$$

It is trivial to see that H is Ad^*–equivariant because

$$H(\Phi_t(\alpha)) = H(\alpha) = Ad_{-t}^* H(\alpha)$$

is verified automatically since H is invariant by the flow Φ_t of X_H.

Theorem 2.5.11. ***(Marsden-Weinstein-Meyer Reduction Theorem)*** *Let (P,ω) be a symplectic manifold on which the Lie group G acts symplectically and let $J : P \to \mathfrak{g}^*$ be an Ad^*–equivariant momentum map for this action. Assume $\mu \in \mathfrak{g}^*$ is a regular value of J and that the isotropy group G_μ under the Ad^* action on $\mathfrak{g}^*$ acts freely and properly on $J^{-1}(\mu)$. Then $P_\mu = J^{-1}(\mu)/G_\mu$ has a unique symplectic form ω_μ with the property*

$$\pi_\mu^* \omega_\mu = i_\mu^* \omega$$

where $\pi_\mu : J^{-1}(\mu) \to P_\mu$ is the canonical projection and $i_\mu : J^{-1}(\mu) \to P$ the inclusion.

Proof. See [1, Th. 4.3.1]. □

Consider the restriction of the hamiltonian function $H : T^*M \to \mathbb{R}$ defined in (2.4.9) by $H(\alpha) = \frac{1}{2}\mathbf{g}\left(\widehat{\mathbf{g}}^{-1}(\alpha), \widehat{\mathbf{g}}^{-1}(\alpha)\right)$ to the subbundle $\widehat{T}^*M = \{\alpha \in T^*M : \alpha \neq 0\}$. Let $\Phi : \mathbb{R} \times \widehat{T}^*M \to \widehat{T}^*M$ be the action defined by the flow of X_H. In example 2.5.10 we have seen that Φ is a symplectic action with Ad^*–equivariant momentum map H.

Recall that

$$H \circ \widehat{\mathbf{g}}(v) = \frac{1}{2}\mathbf{g}(v,v) = L(v)$$

for any $v \in \widehat{T}M$. We saw by means of the expression (1.2.2) that $0 \in \mathbb{R}$ is a regular value of $L(v) = \frac{1}{2}\mathbf{g}(v,v)$ therefore, since $\widehat{\mathbf{g}}^{-1}$ is a diffeomorphism, $0 \in \mathbb{R}$ is also a regular value of H. Moreover, if $\alpha \in H^{-1}(0)$ with $\widehat{\mathbf{g}}(v) = \alpha \neq 0$ then

$$\alpha \in H^{-1}(0) \Leftrightarrow H(\alpha) = 0 \Leftrightarrow H \circ \widehat{\mathbf{g}}(v) = 0 \Leftrightarrow \mathbf{g}(v,v) = 0 \Leftrightarrow v \in \mathbb{N}$$

therefore we have that

$$H^{-1}(0) = \mathbb{N}^*$$

Given $\xi \in \widehat{T}M$, let us denote by $\alpha_\xi \in \widehat{T}^*M$ the element such that $\alpha_\xi = \widehat{\mathbf{g}}(\xi)$. So, denote by c_ξ the integral curve of $X_{\mathbf{g}}$ passing through ξ and c_{α_ξ} the corresponding integral curves of X_H. We have seen in lemma 2.4.6 above that $\widehat{\mathbf{g}}(c_\xi(t)) = c_{\alpha_\xi}(t)$ and moreover $\pi_M^{T^*M}\left(c_{\alpha_\xi}(t)\right) = \pi_M^{TM}(c_\xi(t)) = \mu(t)$. If $\xi \in \mathbb{N}$ then $\alpha_\xi \in \mathbb{N}^* \subset \widehat{T}^*M$. Since μ is a light ray and M is strongly causal, then μ can not have any loop and hence it is injective, therefore c_{α_ξ} is also injective. So, if $s \neq t$ then

$$c_{\alpha_\xi}(s) \neq c_{\alpha_\xi}(t) \Rightarrow \Phi(s, \alpha_\xi) \neq \Phi(t, \alpha_\xi) \Rightarrow \Phi_s(\alpha_\xi) \neq \Phi_t(\alpha_\xi)$$

and hence the restriction of the action Φ is free for every $\alpha \in \mathbb{N}^*$.

Now, we will show that Φ is proper. Consider the sequences $\{t_n\} \subset \mathbb{R}$ and $\{\alpha_n\} \subset \mathbb{N}^*$ such that

$$\alpha_n \mapsto \alpha \in \mathbb{N}^* \qquad \text{and} \qquad \Phi_{t_n}(\alpha_n) \mapsto \beta \in \mathbb{N}^*$$

and take two relatively compact neighbourhoods $U^\alpha, U^\beta \subset \mathbb{N}^*$ of α and β respectively. Since M is assumed to be null pseudo–convex, then for the compact $K = \pi_M^{T^*M}\left(\overline{U^\alpha} \cup \overline{U^\beta}\right)$ there is a compact $K' \subset M$ such that any null geodesic segment with endpoints in K is totally contained in K'. Due to M is strongly causal, there exists $\tau \in \mathbb{R}$ such that $\pi_M^{T^*M}(\Phi_t(\alpha)) \notin K'$ for all $t \geq \tau$. Observe that for a fixed $t \in \mathbb{R}$ such that $\Phi_t(\alpha)$ is defined, there is a subsequence of $\{\alpha_n\}$ such that $\Phi_t(\alpha_k) \mapsto \Phi_t(\alpha)$. In particular, also for $t = \tau$, then there is a subsequence such that $\pi_M^{T^*M}(\Phi_\tau(\alpha_m)) \notin K'$. Since M is null pseudo–convex and $\pi_M^{T^*M}(\Phi_{t_m}(\alpha_m)) \in K'$ then we have that $t_m < \tau$ and therefore there exist a convergent subsequence of $\{t_m\}$. Hence Φ is proper.

Now, we can apply theorem 2.5.11 on the action defined by the flow of the hamiltonian vector field X_H with momentum map H. In this way, we can ensure that

$$\mathcal{N}_s = \mathbb{N}^*/\overline{\mathcal{D}}$$

is a symplectic manifold equipped with the 2–form $\overline{\omega}$ verifying $\pi^*\overline{\omega} = j^*\omega$ where $\pi : \mathbb{N}^* \to \mathcal{N}_s$ is the canonical projection and $j : \mathbb{N}^* \to T^*M$ the inclusion and where $\overline{\mathcal{D}}$ is the distribution generated by the integral curves of X_H restricted to $\mathbb{N}^*$.

$\mathcal{N}_s$ is sometimes named the space of *scaled null geodesic* and describes null geodesics distinguishing different parametrizations. Since $\mathbb{N}^{+*} = \widehat{\mathbf{g}}(\mathbb{N}^+)$ and $\mathbb{N}^{-*} = \widehat{\mathbf{g}}(\mathbb{N}^-)$ are disjoint, then we can define

$$\mathcal{N}_s^+ = \mathbb{N}^{+*}/\overline{\mathcal{D}} \qquad \text{and} \qquad \mathcal{N}_s^- = \mathbb{N}^{-*}/\overline{\mathcal{D}}$$

where $\mathcal{N}_s^+$ is called the *space of future–directed scaled null geodesic*. The definitions of $\mathcal{N}_s^+$ done in the present section and in section 2.5.1 are equivalent in virtue of $\widehat{\mathbf{g}}(\mathbb{N}^+) = \mathbb{N}^{+*}$ and $\widehat{\mathbf{g}}_*(X_{\mathbf{g}}) = X_H$, so we will abuse of the notation and denote both constructions by $\mathcal{N}_s^+$. Moreover, we will keep the notation and we will denote by $\overline{\omega}$, H,π, ... the restrictions to $\mathcal{N}_s^+$ and $\mathbb{N}^{+*}$ of the same objects in $\mathcal{N}_s$ and $\mathbb{N}^*$.

Consider $\widehat{\pi} : \mathbb{N}^+ \to \mathcal{N}_s^+$ defined by $\widehat{\pi} = \pi \circ \widehat{\mathbf{g}}$ and given $v \in \mathbb{N}^+$, denote by $\widehat{\pi}(v) = \gamma_v \in \mathcal{N}_s^+$ the null geodesic defined by $\begin{cases} \gamma_v(0) = p \in M \\ \gamma_v'(0) = v \in \mathbb{N}_p^+ \end{cases}$. It is clear that $\widehat{\pi}(\gamma_v'(s)) = \widehat{\pi}(\gamma_v'(0))$ for all s in the domain of γ_v.

So, given $v, w \in \mathbb{N}^+$ such that $\widehat{\pi}(v) = \widehat{\pi}(w)$ then

$$\gamma_v(s) = \gamma_w(s + a)$$

for some $a \in \mathbb{R}$. If Δ is the Euler vector field in $\mathbb{N}^+$ and $c_v(t) = e^t v$ and $c_w(t) = e^t w$ the corresponding integral curves of Δ passing by v and w respectively, then

$$(d\widehat{\pi})_w(\Delta(w)) = (d\widehat{\pi})_w(c_w'(0)) = \left.\frac{d}{dt}\right|_{t=0} \left(\widehat{\pi}\left(e^t w\right)\right) =$$

$$= \left.\frac{d}{dt}\right|_{t=0} \left(\widehat{\pi}\left(e^t \gamma_w'(0)\right)\right) = \left.\frac{d}{dt}\right|_{t=0} \left(\widehat{\pi}\left(e^t \gamma_v'(a)\right)\right) = \left.\frac{d}{dt}\right|_{t=0} \left(\widehat{\pi}\left(e^t \gamma_v'(0)\right)\right) =$$

$$= \left.\frac{d}{dt}\right|_{t=0} \left(\widehat{\pi}\left(e^t v\right)\right) = (d\widehat{\pi})_v(c_v'(0)) = (d\widehat{\pi})_v(\Delta(v))$$

therefore $(d\widehat{\pi})_w(\Delta(w)) = (d\widehat{\pi})_v(\Delta(v))$ for all $v, w \in \widehat{\pi}^{-1}(\gamma_v) = \widehat{\pi}^{-1}(\gamma_w)$. This implies that the push–forward $\overline{\Delta} = \widehat{\pi}_*(\Delta) \in \mathfrak{X}(\mathcal{N}_s^+)$ is well defined, thus since $\widehat{\mathbf{g}}_*(\Delta) = \mathcal{E}$, we have that

$$\overline{\mathcal{E}} = \pi_*(\mathcal{E}) = \pi_* \circ \widehat{\mathbf{g}}_*(\Delta) = \widehat{\pi}_*(\Delta) = \overline{\Delta} \in \mathfrak{X}\left(\mathcal{N}_s^+\right)$$

is also well defined.

In next two lemmas, we identify the tautological 1–form and the Liouville vector field of $\mathcal{N}_s^+$. We check that both result from the ones in T^*M.

Lemma 2.5.12. *Let* $\pi : \mathbb{N}^{+*} \to \mathcal{N}_s^+$ *be the canonical projection,* $j : \mathbb{N}^{+*} \to \widehat{T}^*M$ *the inclusion and* $\mathcal{E} \in \mathfrak{X}\left(\widehat{T}^*M\right)$ *the Euler field. Given* $\overline{\mathcal{E}} = \pi_*(\mathcal{E})$*, if we define* $\overline{\theta} = -i_{\overline{\mathcal{E}}}\overline{\omega}$ *then we have that* $\pi^*\overline{\theta} = j^*\theta$*.*

Proof. Given $X \in \mathfrak{X}(\mathbb{N}^{+*})$ and $\alpha \in \mathbb{N}^{+*}$, then

$$\begin{aligned}\pi^*\overline{\theta}(X_\alpha) = \overline{\theta}\left((d\pi)_\alpha(X_\alpha)\right) = -i_{\overline{\mathcal{E}}}\overline{\omega}\left((d\pi)_\alpha(X_\alpha)\right) = -\overline{\omega}\left((d\pi)_\alpha(\mathcal{E}_\alpha), (d\pi)_\alpha(X_\alpha)\right) = \\ = -\left(\pi^*\overline{\omega}\right)_\alpha(\mathcal{E}_\alpha, X_\alpha) = -\left(j^*\omega\right)_\alpha(\mathcal{E}_\alpha, X_\alpha) = -\left.\omega\right|_{\mathbb{N}^*}(\mathcal{E}_\alpha, X_\alpha) = \left.\theta\right|_{\mathbb{N}^{+*}}(X_\alpha) = \\ = (j^*\theta)(X_\alpha)\end{aligned}$$

therefore $\pi^*\overline{\theta} = j^*\theta$ concluding the proof. □

Lemma 2.5.13. *With the same notation of lemma 2.5.12 it is verified that* $\mathcal{L}_{\overline{\mathcal{E}}}\overline{\omega} = \overline{\omega}$*,* $d\overline{\theta} = -\overline{\omega}$ *and* $\mathcal{L}_{\overline{\mathcal{E}}}\overline{\theta} = \overline{\theta}$*.*

Proof. First, by the Cartan's formula we have that

$$\mathcal{L}_{\overline{\mathcal{E}}}\overline{\omega} = i_{\overline{\mathcal{E}}}d\overline{\omega} + d\left(i_{\overline{\mathcal{E}}}\overline{\omega}\right) = d\left(i_{\overline{\mathcal{E}}}\overline{\omega}\right) = -d\overline{\theta}$$

Let $\alpha \in \mathbb{N}^{+*}$ such that $\gamma = \pi(\alpha)$. Given $\overline{X}, \overline{Y} \in \mathfrak{X}(\mathcal{N}_s^+)$ and $X_\alpha, Y_\alpha \in T_\alpha\mathbb{N}^{+*}$ such that $\overline{X}_\gamma = (d\pi)_\alpha(X_\alpha)$ and $\overline{Y}_\gamma = (d\pi)_\alpha(Y_\alpha)$, then

$$\begin{aligned}\mathcal{L}_{\overline{\mathcal{E}}}\overline{\omega}\left(\overline{X}_\gamma, \overline{Y}_\gamma\right) &= -d\overline{\theta}\left(\overline{X}_\gamma, \overline{Y}_\gamma\right) = -d\overline{\theta}\left((d\pi)_\alpha(X_\alpha), (d\pi)_\alpha(Y_\alpha)\right) = -\pi^*\left(d\overline{\theta}\right)(X_\alpha, Y_\alpha) = \\ &= -d\left(\pi^*\overline{\theta}\right)(X_\alpha, Y_\alpha) = -d\left(j^*\theta\right)(X_\alpha, Y_\alpha) = j^*\left(-d\theta\right)(X_\alpha, Y_\alpha) = \\ &= j^*\omega(X_\alpha, Y_\alpha) = \pi^*\overline{\omega}(X_\alpha, Y_\alpha) = \overline{\omega}\left(\overline{X}_\gamma, \overline{Y}_\gamma\right)\end{aligned}$$

hence $\mathcal{L}_{\overline{\mathcal{E}}}\overline{\omega} = -d\overline{\theta} = \overline{\omega}$.

Finally, observe that

$$\overline{\theta}\left(\overline{\mathcal{E}}\right) = \overline{\theta}\left(\pi_*(\mathcal{E})\right) = \left(\pi^*\overline{\theta}\right)(\mathcal{E}) = (j^*\theta)(\mathcal{E}) = \left.\theta\right|_{\mathbb{N}^{+*}}(\mathcal{E}) = 0.$$

Now, we have

$$\mathcal{L}_{\overline{\mathcal{E}}}\overline{\theta}\left(\overline{X}_\gamma\right) = i_{\overline{\mathcal{E}}}d\overline{\theta}\left(\overline{X}_\gamma\right) + d\left(i_{\overline{\mathcal{E}}}\overline{\theta}\right)\left(\overline{X}_\gamma\right) = -i_{\overline{\mathcal{E}}}\overline{\omega}\left(\overline{X}_\gamma\right) + d\left(\overline{\theta}\left(\overline{\mathcal{E}}\right)\right)\left(\overline{X}_\gamma\right) = \overline{\theta}\left(\overline{X}_\gamma\right)$$

therefore $\mathcal{L}_{\overline{\mathcal{E}}}\overline{\theta} = \overline{\theta}$. □

2.5.3 Contact structure of $\mathcal{N}$ by symplectic reduction

It is possible to state $\mathcal{N}$ as the base manifold of a principal bundle of the space $\mathcal{N}_s^+$. Next lemma give us the result.

Lemma 2.5.14. $\pi : \mathcal{N}_s^+ \to \mathcal{N}$ *is a principal bundle with structural group the multiplicative group* $\mathbb{R}_+ = ((0, \infty), \cdot)$*.*

Proof. We will show that the flow $\Phi : \mathbb{R}_+ \times \mathcal{N}_s \to \mathcal{N}_s$ defined by the vector field $\overline{\mathcal{E}} \in \mathfrak{X}(\mathcal{N})$ defined in lemma 2.5.12 is a right action acting freely and properly.

It is clear that Φ is a right action and it can be written by

$$\Phi(t, \gamma_v) = \gamma_{tv}$$

where γ_u denotes the future–directed null geodesic defined by the null vector $u \in \mathbb{N}^+$. It is well known that for any $\lambda \in \mathbb{R}_+$ it is verified that $\gamma_{\lambda u}(s) = \gamma_u(\lambda s)$. Then, the equality $\Phi(t, \gamma_v) = \gamma_v$, that is $\gamma_{tv} = \gamma_v$, implies that $tv = v$ whence $t = 1$ is the only solution. Therefore the action is free.

Now, consider two sequences $\{\gamma_{v_n}\}$ and $\{\Phi_{t_n}(\gamma_{v_n})\}$ converging to γ_v and γ_u respectively. Since $\Phi_{t_n}(\gamma_{v_n}) = \gamma_{t_n v_n}$ and again $\gamma_{t_n v_n}(s) = \gamma_{v_n}(t_n s)$ then γ_u and γ_v have the same image as parametrized curve in M, then for every s we have $\gamma_u(s) = \gamma_v(\overline{t}s)$ for some $\overline{t} \in \mathbb{R}_+$. So we have that $u = \overline{t}v$ and hence $t_n v_n \mapsto \overline{t}v$. Since $v \neq 0$ and $\begin{cases} v_n \mapsto v \\ t_n v_n \mapsto \overline{t}v \end{cases}$ then we have that $t_n \mapsto \overline{t}$. This shows that the action Φ is proper. Then we have shown that $\pi : \mathcal{N}_s^+ \to \mathcal{N}$ is a principal bundle with structural group $\mathbb{R}_+$. □

Next proposition 2.5.15 shows that the distribution of hyperplanes in $\mathcal{N}_s^+$ defined by the kernel of $\overline{\theta}$ descends to a distribution of hyperplanes in $\mathcal{N}$ defined by the kernel of a 1–form θ_0. In the literature, for a fibre bundle $\pi : P \to M$ and a 1–form β in P, the existence of a 1–form β_0 in M such that $\pi^*\beta_0 = \lambda$ is said that β is *projectable* to β_0 by π. So, we will show that $\overline{\theta}$ is projectable to θ_0.

Proposition 2.5.15. *Let $\pi : \mathcal{N}_s^+ \to \mathcal{N}$ be the principal bundle of lemma 2.5.14, then there exists a 1–form θ_0 in $\mathcal{N}$ such that $\pi^*\theta_0 = \overline{\theta}$.*

Proof. First, observe that since $\mathcal{L}_{\overline{\mathcal{E}}}\overline{\omega} = \overline{\omega}$ then $\overline{\mathcal{E}}$ is a Liouville vector field. Denote by Φ_t the flow of $\overline{\mathcal{E}}$ and recall that $\mathcal{N}$ is the space of orbits of the flow Φ. Consider the 1–form $\overline{\theta}$ in $\mathcal{N}_s^+$. Since $\overline{\theta} = -i_{\overline{\mathcal{E}}}\overline{\omega}$ and $\overline{\omega}$ is the symplectic form of $\mathcal{N}_s^+$, then $\overline{\theta}$ is not zero. Let us call $\overline{\mathcal{H}} = \ker(\overline{\theta})$, then $\overline{\mathcal{H}}$ is a distribution of hyperplanes, that means $\dim(\overline{\mathcal{H}}) = 2m - 3$ at every point since $\dim(\mathcal{N}_s^+) = 2m - 2$. Observe that the fibres of $\pi : \mathcal{N}_s^+ \to \mathcal{N}$ are the integral curves of $\overline{\mathcal{E}}$ and moreover the differential $d\pi_\gamma$ has rank $2m - 3$ for any $\gamma \in \mathcal{N}_s^+$. Since

$$\overline{\theta}(\overline{\mathcal{E}}) = -i_{\overline{\mathcal{E}}}\overline{\omega}(\overline{\mathcal{E}}) = -\overline{\omega}(\overline{\mathcal{E}}, \overline{\mathcal{E}}) = 0$$

then $\overline{\mathcal{E}} \in \overline{\mathcal{H}}$, and thus $\dim\left(d\pi_\gamma\left(\ker(\overline{\theta})\right)\right) = 2m - 2$. This implies that $\mathcal{H} = \pi_*(\overline{\mathcal{H}})$ is a distribution of hyperplanes at every $\gamma \in \mathcal{N}$ and, by remark 2.4.3, there exists a 1–form $\theta_0 \in \mathfrak{X}^*(\mathcal{N})$ such that $\ker(\theta_0) = \mathcal{H}$. Since

$$\pi^*\theta_0(v) = 0 \Leftrightarrow \theta_0(\pi_*(v)) = 0 \Leftrightarrow \pi_*(v) \in \mathcal{H} \Leftrightarrow v \in \overline{\mathcal{H}} \Leftrightarrow \overline{\theta}(v) = 0$$

then we conclude that $\overline{\theta} = \pi^*\theta_0$. □

The distribution of hyperplanes in $\mathcal{N}$ defined in proposition 2.5.15 is, in fact, a contact structure. This result can be found, for example in [44], [45] and [30].

Theorem 2.5.16. *$\mathcal{N}$ is equipped with a canonical contact structure.*

Proof. By proposition 2.5.15, we have that there exists $\theta_0 \in \mathfrak{X}^*(\mathcal{N})$ such that $\mathcal{H} = \ker(\theta_0)$ is a distribution of hyperplanes. Now, we have that

$$\begin{aligned}\pi^*\left(\theta_0 \wedge (d\theta_0)^{m-2}\right) &= \pi^*(\theta_0) \wedge \pi^*\left((d\theta_0)^{m-2}\right) = \overline{\theta} \wedge (d\pi^*(\theta_0))^{m-2} = \\ &= \overline{\theta} \wedge \left(d\overline{\theta}\right)^{m-2} = -i_{\overline{\mathcal{E}}}\overline{\omega} \wedge (-1)^{m-2}\,\overline{\omega}^{m-2} = \\ &= (-1)^{m-1}\, i_{\overline{\mathcal{E}}}\overline{\omega} \wedge \overline{\omega}^{m-2} = \frac{(-1)^{m-1}}{m-1} i_{\overline{\mathcal{E}}}\overline{\omega}^{m-1} \neq 0\end{aligned}$$

since, by remark 2.4.3, $\overline{\omega}^{m-1}$ is a volume form in $\mathcal{N}_s$. This implies that $\mathcal{H}$ is a contact structure in $\mathcal{N}$. □

Next, we will look for an expression for the local contact forms defining the contact structure $\mathcal{H} \subset T\mathcal{N}$. Recall that a coordinate chart $\psi : \mathcal{U} \subset \mathcal{N} \to \mathbb{R}^{2m-3}$ can be defined via the diffeomorphism $\mathcal{U} \to \Omega^T(C)$ of diagram (2.2.8), where $\Omega^T(C)$ is an embedded submanifold of $TV \subset TM$ with $V \subset M$ a basic open set with Cauchy surface C. Then we have the following diagram

$$\begin{tikzcd} \mathcal{N} \supset \mathcal{U} \arrow[r, "\overline{z}"] \arrow[d, "\psi"'] & TV \arrow[d, "\phi"] \\ \mathbb{R}^{2m-3} \supset B_0 \arrow[r, "z"'] & B \subset \mathbb{R}^{2m} \end{tikzcd} \tag{2.5.3}$$

where $\overline{z} = \phi^{-1} \circ z \circ \psi$. The image of the embedding $\overline{z}$ is contained in $\mathbb{N}^+(C)$, and moreover if $p_{\mathbb{N}} : \mathbb{N}^+ \to \mathcal{N}$ is the canonical projection, then $p_{\mathbb{N}} \circ \overline{z}([\gamma]) = [\gamma]$ for all $[\gamma] \in \mathcal{U} \subset \mathcal{N}$. Then $\overline{z}$ is a local section of $p_{\mathbb{N}}$.

By lemma 2.5.12, proposition 2.5.15 and theorem 2.5.16, we have that for $\xi \in T\mathbb{N}^+$

$$\theta_\alpha(\xi) = (\theta_0)_{p_{\mathbb{N}^*}(\alpha)}\left((dp_{\mathbb{N}^*})_\alpha(\xi)\right)$$

then, by diagram in (2.4.13), $p_{\mathbb{N}} = p_{\mathbb{N}^*} \circ \widehat{\mathbf{g}}$, and hence we can write for $J \in T_{[\gamma]}\mathcal{N} \subset T\mathcal{U}$

$$\theta_{\widehat{\mathbf{g}}\circ\overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}} \circ \overline{z}\right)_{[\gamma]}(J)\right) = (\theta_0)_{[\gamma]}(J)$$

On the other hand, by definition of the tautological 1–form θ and since $\pi_M^{TM} = \pi_M^{T^*M} \circ \widehat{\mathbf{g}}$, we have

$$\begin{aligned}\theta_{\widehat{\mathbf{g}}\circ\overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}} \circ \overline{z}\right)_{[\gamma]}(J)\right) &= \widehat{\mathbf{g}} \circ \overline{z}([\gamma])\left(d\left(\widehat{\mathbf{g}} \circ \overline{z}\right)_{[\gamma]}(J)\right) = \\ &= \mathbf{g}\left(\left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}\circ\overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}} \circ \overline{z}\right)_{[\gamma]}(J)\right), \overline{z}([\gamma])\right) = \\ &= \mathbf{g}\left(\left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}(J)\right), \overline{z}([\gamma])\right) = \\ &= \mathbf{g}\left(J(0), \gamma'(0)\right)\end{aligned}$$

where $\overline{z}([\gamma]) \in \mathbb{N}^+(C)$ is a vector defining the light ray $[\gamma] \in \mathcal{U}$, so we have considered the null geodesic γ such that $\gamma'(0) = \overline{z}([\gamma])$. On the other hand, observe that if $J = \Gamma'(0) \in T_{[\gamma]}\mathcal{N}$ where Γ is a smooth curve in $\mathcal{N}$ with $\Gamma(0) = [\gamma]$, then

$$\left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}(J)\right) = \left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}(\Gamma'(0))\right) = \left(\pi_M^{TM} \circ \overline{z} \circ \Gamma\right)'(0)$$

where $\pi_M^{TM} \circ \overline{z} \circ \Gamma$ is the curve in M where $\overline{z} \circ \Gamma \subset \mathbb{N}^+(C)$ rest. By lemma 2.3.5, we have that $\left(\pi_M^{TM} \circ \overline{z} \circ \Gamma\right)'(0) = J(0)$ and therefore we claim that

$$(\theta_0)_{[\gamma]}(J) = \mathbf{g}(J(0), \gamma'(0))$$

and then

$$J \in \mathcal{H} \Longleftrightarrow \mathbf{g}(J(0), \gamma'(0)) = 0.$$

It is clear that this characterization does not depends neither on the representative of the conformal metric $\mathcal{C}$ nor on the parametrization of γ in virtue of lemma 2.3.13.

Again, since the expression of the local 1–form θ_0 defining the contact structure $\mathcal{H}$ coincides with the one constructed in section 2.4.2, then the same used arguments to show that $\mathcal{H}$ is cooriented and conformal remain valid.

Chapter 3

The space of skies

In this chapter we will deal with a structure associated to the space of light rays $\mathcal{N}$ of M consisting of a family of compact submanifolds of $\mathcal{N}$ in correspondence with points of M.

Given a point $x \in M$, the set of light rays passing through x will be called the *sky* of x and will be denoted by $S(x)$ or X, i.e.

$$X = S(x) = \{\gamma \in \mathcal{N} : x \in \gamma \subset M\}. \tag{3.0.1}$$

Fixed $x \in M$, notice that the light rays $\gamma \in S(x)$ are in one–to–one correspondence with the elements in the fibre $\mathbb{PN}_x = \left(\pi_M^{\mathbb{PN}}\right)^{-1}(x) \subset \mathbb{PN}$, hence the sky $S(x)$ of any point $x \in M$ is diffeomorphic to the standard sphere $\mathbb{S}^{m-2} \simeq \mathbb{PN}_x$. Now, it is possible to define the *space of skies* by

$$\Sigma = \{S(x) \subset \mathcal{N} : x \in M\} \tag{3.0.2}$$

and the *sky map* by

$$\begin{array}{rccc} S: & M & \to & \Sigma \\ & x & \mapsto & S(x) \end{array}$$

The map S is, by definition of Σ, surjective. When S is a bijection, its inverse map is called the *parachute map* and it will be denoted by $P = S^{-1} : \Sigma \to M$. An important part of this chapter will be devoted to the study of the natural topological and differentiable structures induced in the space of skies Σ considered as a collection of subsets of $\mathcal{N}$. In order to understand better the structures inherited by Σ we need to analyse the structure of $T\mathcal{N}$.

Consider $X = S(x) \in \Sigma$, then for any $\gamma \in X$, the tangent space $T_\gamma X$ can be characterized by

$$T_\gamma X = \left\{J \in T_\gamma \mathcal{N} : J(s_0) = 0\,(\mathrm{mod}\gamma'(s_0)) \text{ with } \gamma(s_0) = x = S^{-1}(X)\right\} \tag{3.0.3}$$

as done in (2.4.1).

Recall that in a $(2n+1)$–dimensional contact manifold P with contact structure $\mathcal{H}$, a n–dimensional submanifold $N \subset P$ is said to be *legendrian* if $TN \subset \mathcal{H}$. Then observe that for every $J \in T_\gamma \mathcal{N}$ we have $\mathbf{g}(J, \gamma')$ is constant, and if moreover $J \in T_\gamma X$, since $J(s_0) = 0\,(\mathrm{mod}\gamma'(s_0))$ for some s_0, then $\mathbf{g}(J, \gamma') = 0$. This implies that $T_\gamma X \subset \mathcal{H}_\gamma$, that is, the Jacobi fields tangent to some sky $X \in \Sigma$ are in the contact structure $\mathcal{H}$ of $\mathcal{N}$ and therefore any sky is a legendrian submanifold of $\mathcal{N}$. Moreover, again as in (2.4.3), the contact hyperplanes are generated by tangent vectors of two non–conjugated points $x, y \in \gamma$ along the same $\gamma \in \mathcal{N}$, that is

$$H_\gamma = T_\gamma X \oplus T_\gamma Y. \tag{3.0.4}$$

Definition 3.0.17. *A spacetime M is said to* separate skies *or to be* sky separating *if the sky map is injective, that is if $S(x) = S(y)$ then $x = y$.*

Given a spacetime M, theorem 1.2.15 implies the existence of a basic neighbourhood V at any point $p \in M$, so by the causal convexity and normality of V there is not two different null geodesics connecting two different points $q_1, q_2 \in V$. Then, skies of points of V can not coincide and therefore M locally separates skies. So, this property is clearly natural but it is easy to find an example of a spacetime which is not globally sky separating.

Example 3.0.18. *Consider the 3–dimensional* Einstein's cylinder *given by $M = \mathbb{R} \times \mathbb{S}^2$ equipped with the metric defined by*

$$ds^2 = -dt^2 + \sin^2\phi \;\; d\theta^2 + d\phi^2$$

where $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi]$ represent the longitude and colatitude respectively of the sphere $\mathbb{S}^2$. If we rename the coordinates as $x^0 = t$, $x^1 = \theta$ and $x^2 = \phi$, the non–vanishing Christoffel symbols in M can be written by

$$\begin{aligned} \Gamma^2_{11} &= -\cos\phi\sin\phi \\ \Gamma^1_{21} &= \Gamma^1_{12} = \frac{\cos\phi}{\sin\phi} \end{aligned}$$

therefore the null geodesics in M are given by $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ where γ_1 is a geodesic in $\mathbb{R}$ and γ_2 a geodesic in $\mathbb{S}^2$ such that their parameters match to make a null geodesic of γ.

Since γ_2 describes a maximal circumference in $\mathbb{S}^2$, then it is periodic and all geodesics passing through a point $p = (t_0, \theta_0, \phi_0)$ also pass through the point $q = (t_0 + L, \theta_0, \phi_0)$ where L is the period of γ_2. Then $S(p) = S(q)$ but $p \neq q$ and therefore M does not separate skies.

If $x \neq y \in M$ are points such that $S(x) = S(y)$, then every outgoing (or incoming because we do not distinguish future and past in this context) light ray from y refocuses to the point x. In [31], Kinlaw names the lack of skies separation *strong refocusing* property. Related to this, Low introduces in [38], [44] and [45], the concept of weak refocusing that Kinlaw studies widely in [31].

Definition 3.0.19. *Let M be a strongly causal spacetime. We will say that* refocusing *or* weak refocusing *at $x \in M$ occurs if there exists an open neighborhood V of x such that for all open U with $x \in U \subset V$, there exists $y \notin V$ such that all light rays through y enter U. In case there is not refocusing at any $x \in M$, then we shall say that M is* non–refocusing.

In order to establish the topological equivalence between M and Σ in [31], the required hypotheses are M is strongly causal, null pseudo–convex and non–refocusing, but we will show in section 3.5 that we can replace the property of non–refocusing by the skies separating one. Trivially, the former hypothesis implies the latter one. Figure 3.1 shows the difference between both concepts.

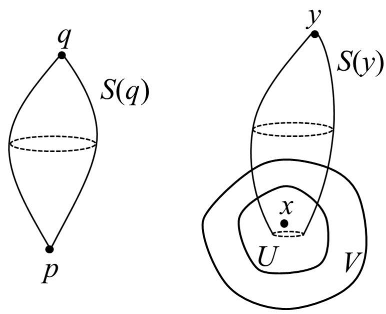

Figure 3.1: Skies of p and q are not separated (strong refocusing). Weak refocusing at x.

Unless otherwise stated, throughout this chapter 3 we will assume that M is a strongly causal, null pseudo–convex and sky separating spacetime.

Section 3.1

Coordinates in $T\mathcal{N}$

In this section we will construct a smooth atlas suitable to describe the vectors in $T\mathcal{N}$ by the initial values of the Jacobi field they represent. We will show that this atlas is compatible with the one defined by the canonical coordinates in $T\mathcal{N}$, and it will be a very helpful tool used in the study of the differentiable structure of Σ.

First, consider an atlas for M with local charts $\left(V, \varphi = \left(x^1, \ldots x^m\right)\right)$ such that V is basic open set and, without lack of generality, the local hypersurface $C \subset V$ defined by $x^1 = 0$ is a smooth spacelike Cauchy surface in V, then each null geodesic passing through V intersects C at exactly one point. Motivated by the diagram (2.2.8) we have the following one

$$\begin{tikzcd} \mathbb{PN}\left(C\right) \arrow[r, "\sigma"] & \mathcal{U} \subset \mathcal{N} \\ \Omega^T\left(C\right) \arrow[u, "\rho"] \arrow[ur, "\sigma \circ \rho"'] & \end{tikzcd} \tag{3.1.1}$$

where, recall that we denote $\Omega^T\left(W\right) = \left\{v \in \mathbb{N}^+\left(W\right) : \mathbf{g}\left(v, T\right) = -1\right\}$ for $W \subset M$ and $T \in \mathfrak{X}\left(M\right)$ a non–vanishing future timelike vector field, and where the maps σ and ρ are diffeomorphisms.

Then, in order to construct said coordinate chart in $\mathcal{N}$, we will build a chart in $\Omega^T(C)$ by restriction of a chart in TM and we will use it as a chart in $\mathcal{N}$ via the diffeomorphism $(\sigma \circ \rho)^{-1}$.

Let $\{E_1, \ldots, E_m\} \subset \mathfrak{X}(V)$ be an orthonormal frame such that $E_1 = T$ is a non–vanishing future timelike vector field in V. If $\xi \in T_pV$ is written by $\xi = \sum_{j=1}^{m} u^j E_j(p)$ then (TV, ϕ) with:

$$\begin{array}{rccc} \phi\colon & TV & \to & \mathbb{R}^{2m} \\ & \xi & \mapsto & \left(x^1, \ldots, x^m, u^1, \ldots, u^m\right) \end{array}$$

is a local coordinate chart in TM.

For $\xi \in \mathbb{N}^+(V)$ we have $\left(u^1\right)^2 = \sum_{j=2}^{m} \left(u^j\right)^2$ so, a coordinate chart in $\mathbb{N}^+(V)$ is given by the map

$$\xi \mapsto \left(x^1, \ldots, x^m, u^2, \ldots, u^m\right) \in \mathbb{R}^{2m-1}$$

If we consider $\xi \in \Omega^T(V)$ then

$$\mathbf{g}(\xi, T) = -1 \Rightarrow \mathbf{g}\left(u^j E_j, E_1\right) = -1 \Rightarrow u^1 = 1$$

then $\left(u^2, \ldots, u^m\right)$ lies in $\mathbb{S}^{m-2}$ and describes a null direction. So, for example, we can take $u^2 = \sqrt{1 - (u^3)^2 - \cdots - (u^m)^2}$ to obtain the coordinate chart

$$\begin{array}{ccc} \Omega^T(V) & \to & \mathbb{R}^{2m-2} \\ \xi & \mapsto & \left(x^1, \ldots, x^m, u^3, \ldots, u^m\right) \end{array} \tag{3.1.2}$$

and the restriction to $\Omega^T(C)$ verifies $x^1 = 0$ and therefore give us the following chart

$$\begin{array}{rccc} \overline{\phi}\colon & \Omega^T(C) & \to & \mathbb{R}^{2m-3} \\ & \xi & \mapsto & \left(x^2, \ldots, x^m, u^3, \ldots, u^m\right) \end{array} \tag{3.1.3}$$

thus a coordinate chart $(\psi, \mathcal{U})$ in $\mathcal{N}$ can be defined by

$$\begin{array}{rccc} \psi = (\sigma \circ \rho)^{-1} \circ \overline{\phi}\colon & \mathcal{U} & \to & \mathbb{R}^{2m-3} \\ & \gamma & \mapsto & \left(x^2, \ldots, x^m, u^3, \ldots, u^m\right) = (\mathbf{x}, \mathbf{u}) \end{array} \tag{3.1.4}$$

where $\gamma \in \mathcal{U}$ is represented by the null geodesic verifying $\gamma(0) = p \in C \subset V$ and $\gamma'(0) = E_1(p) + u^2 E_2(p) + \cdots + u^m E_m(p) \in \Omega^T(C)$.

Now, we will define an atlas on $T\mathcal{N}$ by using the open sets $T\mathcal{U}$. Thus, in order to complete a chart in $T\mathcal{U}$, we will add the coordinates for the tangent vectors at every light ray $\gamma \in \mathcal{N}$ with coordinates $\mathbf{x}, \mathbf{u}$. This can be done by using the initial values at $t = 0$ for Jacobi's equation (2.3.1) whose solutions are the Jacobi fields along γ. Thus if $J \in T_\gamma\mathcal{N}$ then

$$\begin{cases} J(0) = \sum_{j=1}^{m} w^j E_j(p) \\ J'(0) = \sum_{j=1}^{m} v^j E_j(p) \end{cases}$$

define J, so a chart in $T\mathcal{U}$ is given by the map $\overline{\psi}\colon T\mathcal{U} \to \mathbb{R}^{4m-6}$:

$$\overline{\psi}(J) = \left(\mathbf{x}, \mathbf{u}; \left\langle w^1, \ldots, w^m \right\rangle, \left\langle v^1, \ldots, v^m \right\rangle\right) = (\mathbf{x}, \mathbf{u}; \mathbf{w}, \mathbf{v}) \in \mathbb{R}^{4m-6}\,, \tag{3.1.5}$$

with $\mathbf{w} = \langle w^1, \ldots, w^m \rangle$ and $\mathbf{v} = \langle v^1, \ldots, v^m \rangle$ denoting respectively,

$$\begin{cases} \mathbf{w} = \left(w^1, \ldots, w^m\right) (\mathrm{mod}\gamma') \\ \mathbf{v} = \left(v^1, \ldots, v^m\right) (\mathrm{mod}\gamma') \end{cases}$$

where $\left(a^1, \ldots, a^m\right) (\mathrm{mod}\gamma') = \sum\limits_{j=1}^{m} a^j E_j(p) (\mathrm{mod}\gamma'(0))$.

We may define $m-2$ independent coordinates from $(v^1, \ldots, v^m)$ and $m-1$ from $(w^1, \ldots, w^m)$ as follows. Notice that because of equation (2.3.8), $J'(0)$ is orthogonal to $\gamma'(0)$, so $v^1 = v^2u^2 + \cdots + v^m u^m$. Then, we may consider the Jacobi field $\overline{J}$ representative of $J \in T\mathcal{N}$ as the one verifying

$$\overline{J}(0) = J(0) - w^1\gamma'(0) = \left(w^2 - w^1u^2\right) E_2 + \cdots + \left(w^m - w^1u^m\right) E_m \tag{3.1.6}$$

$$\overline{J}'(0) = J'(0) - v^1\gamma'(0) = \left(v^2 - v^1u^2\right) E_2 + \cdots + \left(v^m - v^1u^m\right) E_m \tag{3.1.7}$$

therefore the coordinates $\mathbf{w}$ and $\mathbf{v}$ can be written as

$$\begin{cases} \mathbf{w} = \left(\overline{w}^2, \ldots, \overline{w}^m\right) \\ \mathbf{v} = \left(\overline{v}^3, \ldots, \overline{v}^m\right) \end{cases} \tag{3.1.8}$$

where $\overline{w}^k = w^k - w^1u^k$ and $\overline{v}^k = v^k - v^1u^k$ for $k = 1, \ldots, m$. Finally notice that since $\left(u^2, \ldots, u^m\right) \neq (0, \ldots, 0)$ then there exist $j = 2, \ldots, m$ such that $u^j \neq 0$. If, for instance $u^2 \neq 0$, then we have $\overline{v}^2 = -\frac{1}{u^2}\sum_{j=3}^m \overline{v}^j u^j$ since $v^1 = v^2u^2 + \cdots + v^m u^m$. So, we will denote, with an slight abuse of notation, by $(\mathbf{x}, \mathbf{u}; \mathbf{w}, \mathbf{v})$ the $4m-6$ independent coordinates thus constructed on $T\mathcal{U}$.

It is possible to show the compatibility between the local charts $(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ and the canonical atlas defined on the tangent bundle $T\mathcal{N}$ over the open sets $T\mathcal{U}$ with canonical coordinates $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$. This would imply that the local charts $(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ and $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$ are in the same maximal atlas.

In order to show it, let us describe how the coordinates $(\mathbf{w}, \mathbf{v})$ depends on the canonical ones. Let us consider the coordinate chart $(\psi, \mathcal{U})$ in $\mathcal{N}$ given by (3.1.4) where $\gamma(0) \in C$ for each $\gamma \in \mathcal{U}$. So, let $\Gamma_1(s) \in \mathcal{U} \subset \mathcal{N}$, $s \in (-\epsilon, \epsilon)$, be a curve such that its coordinates are

$$\psi(\Gamma_1(s)) = \left(x_0^2, \ldots, x_0^m, \alpha^3(s), \ldots, \alpha^m(s)\right)$$

This curve corresponds to a geodesic variation $\mathbf{f}(s,t)$ such that

$$\lambda(s) = \mathbf{f}(s, 0) = p \in M$$

for every s because the coordinates $\mathbf{x} = \left(x_0^2, \ldots, x_0^m\right)$ remain constant, and moreover the curve $\beta(s) = \partial\mathbf{f}(s,t)/\partial t \in T_pM$ is given by

$$\beta(s) = E_1(p) + \alpha^2(s) E_2(p) + \alpha^3(s) E_3(p) + \ldots + \alpha^m(s) E_m(p).$$

Hence $\mathbf{f}$ can be written by the expression similar to the one in lemma 2.3.5

$$\mathbf{f}(s,t) = \exp_p(t\beta(s)) .$$

Calling J the Jacobi field of $\mathbf{f}$, then by lemma 2.3.5 we have that

$$\begin{cases} J(0) = 0 \\ J'(0) = \beta'(0) \end{cases} \tag{3.1.9}$$

Now, if we consider a curve $\Gamma_2 \subset \mathcal{N}$ such that its coordinates are

$$\psi\left(\Gamma_2\left(s\right)\right) = \left(x^2\left(s\right), \ldots, x^m\left(s\right), u_0^3, \ldots, u_0^m\right)$$

This curve corresponds to a geodesic variation $\mathbf{f}\left(s,t\right)$ verifying

$$\lambda\left(s\right) = \mathbf{f}\left(s,0\right) \in C \subset M$$

The fact of the coordinates $\mathbf{u}^k = u_0^k$ remain constant implies that

$$W\left(s\right) = \frac{\partial \mathbf{f}}{\partial t}\left(s,0\right) = E_1\left(\lambda\left(s\right)\right) + u_0^2 E_2\left(\lambda\left(s\right)\right) + \ldots + u_0^m E_m\left(\lambda\left(s\right)\right) \tag{3.1.10}$$

and $W(s)$ belongs to $T_{\lambda(s)}M$. So, the geodesic variation $\mathbf{f}$ corresponding to Γ_2 can be written by

$$\mathbf{f}\left(s,t\right) = \exp_{\lambda(s)}\left(tW\left(s\right)\right)$$

Again, if J is the Jacobi field of $\mathbf{f}$, then by lemma 2.3.5

$$\left\{\begin{array}{l} J\left(0\right) = \lambda'\left(0\right) \\ J'\left(0\right) = \frac{DW}{ds}\left(0\right)\,. \end{array}\right. \tag{3.1.11}$$

Choosing curves Γ_1 and Γ_2 such that $\Gamma_1'\left(0\right) = \left(\frac{\partial}{\partial u^i}\right)_{\Gamma_1(0)}$ and $\Gamma_2'\left(0\right) = \left(\frac{\partial}{\partial x^j}\right)_{\Gamma_2(0)}$ respectively with $i = 3, \ldots, m$ and $j = 2, \ldots, m$, then we have that the change from canonical coordinates $(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}})$ to the coordinates $(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ verifies

$$\left(\begin{array}{c} \mathbf{w} \\ \mathbf{v} \end{array}\right) = \left(\begin{array}{c} \overline{w}^i \\ \overline{v}^j \end{array}\right) = \left(\begin{array}{cc} A & 0 \\ B & I_{m-2} \end{array}\right)\left(\begin{array}{c} \dot{\mathbf{x}} \\ \dot{\mathbf{u}} \end{array}\right) \tag{3.1.12}$$

with $i = 3, \ldots, m$ and $j = 2, \ldots, m$. The matrix $I_{m-2} \in \mathbb{R}^{(m-2)\times(m-2)}$ is the identity matrix and $B \in \mathbb{R}^{(m-2)\times(m-1)}$ is the matrix whose $(k-1)$–th column is the vector containing the $\mathbf{v}$–coordinates of $\frac{DW_k}{ds}\left(0\right)$ with $k = 2, \ldots, m$ with

$$W_k\left(s\right) = E_1\left(\lambda_k\left(s\right)\right) + u_0^2 E_2\left(\lambda_k\left(s\right)\right) + \ldots + u_0^m E_m\left(\lambda_k\left(s\right)\right) \tag{3.1.13}$$

and $\lambda_k\left(s\right)$ a curve such that $x^j\left(\lambda_k\left(s\right)\right) = x_0^j$ are constant for $j \neq k$ and $x^k\left(\lambda_k\left(s\right)\right) = x_0^k + s$.

Since $J\left(0\right) = \lambda_k'\left(0\right) = \left(\partial/\partial x^k\right)_{\lambda_k(0)} = \sum_{j=1}^m w_k^j E_j$ then we have that $\overline{w}^j = w_k^j - w_k^1 u^j$ for $j = 2, \ldots, m$. This implies that the matrix A is given by

$$A = \left(w_k^j - w_k^1 u^j\right); \qquad j,k = 2, \ldots, m\,. \tag{3.1.14}$$

Calling $\mathbb{V} = \operatorname{span}\left\{E_j\left(\lambda_k\left(0\right)\right)\right\}_{j=2,\ldots,m}$, the projection $\pi_{\mathbf{u}} : T_{\lambda_k(0)}M \to \mathbb{V}$ is given by

$$\pi_{\mathbf{u}}\left(\eta\right) = \eta - \mathbf{g}\left(\eta, E_1\right)\gamma'\left(0\right)\,,$$

where we have taken $\gamma'\left(0\right) = E_1 + u^2 E_2 + \cdots + u^m E_m$. The matrix $\widetilde{A}$ of $\pi_{\mathbf{u}}$ relative to the basis $\left\{\left(\partial/\partial x_k\right)_{\lambda_k(0)}\right\}_{k=1,\ldots,m}$ in $T_{\lambda_k(0)}M$ and $\left\{E_j\left(\lambda_k\left(0\right)\right)\right\}_{j=2,\ldots,m}$ in $\mathbb{V}$ is

$$\widetilde{A} = \left(w_k^j - w_k^1 u^j\right); \qquad j = 2, \ldots, m, \quad k = 1, \ldots, m\,.$$

We have that $\mathbb{V}$ and $\mathbb{V}_c = \operatorname{span}\left\{\left(\frac{\partial}{\partial x_k}\right)_{\lambda_k(0)}\right\}_{k=2,\ldots,m}$ are spacelike by construction, $\ker \pi_{\mathbf{u}} = \operatorname{span}\{\gamma'(0)\}$ and the matrix of the restriction $\pi_{\mathbf{u}}|_{\mathbb{V}_c}$ is A, then $\pi_{\mathbf{u}}|_{\mathbb{V}_c}$ is an isomorphism and therefore A is regular. Hence, the matrix in (3.1.12) describing the change of coordinates along the fibers of the tangent bundle $T\mathcal{N}$ is regular and differentiable, then the change of coordinates

$$(\mathbf{x}, \mathbf{u}, \dot{\mathbf{x}}, \dot{\mathbf{u}}) \longleftrightarrow (\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$$

is also differentiable. This also shows that $(\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ is a coordinate chart of the canonical differentiable structure of $T\mathcal{N}$.

Section 3.2

Topology on Σ

We will start this section defining a natural topology on the space of skies Σ induced by the topology of $\mathcal{N}$.

Notation 3.2.1. *Let $\mathcal{U} \subset \mathcal{N}$ be an open set, then we denote by $\Sigma(\mathcal{U}) \subset \Sigma$, the set of all skies $X \in \Sigma$ such that $X \subset \mathcal{U}$.*

Lemma 3.2.2. *The collection of sets*

$$\mathfrak{B}(X) = \{\Sigma(\mathcal{U}) \subset \Sigma : \mathcal{U} \subset \mathcal{N} \text{ is open with } X \subset \mathcal{U}\}$$

is a topological basis of Σ at X.

Proof. First, given $X \in \Sigma$, there exists $x \in M$ such that $S(x) = X$. If $\sigma : \mathbb{P}(C) \to \mathcal{U} \subset \mathcal{N}$ is the diffeomorphism of diagram (3.1.1) used to define a coordinate chart, where C is a local Cauchy surface of a basic open set $V \subset M$ such that $x \in C$, then trivially we have that $X \in \mathcal{U}$.

On the other hand, if $\Sigma(\mathcal{U}), \Sigma(\mathcal{V}) \in \mathfrak{B}(X)$ since clearly we have $\Sigma(\mathcal{U}) \cap \Sigma(\mathcal{V}) = \Sigma(\mathcal{U} \cap \mathcal{V})$ and due to $\mathcal{U} \cap \mathcal{V}$ is open in $\mathcal{N}$, then $\Sigma(\mathcal{U} \cap \mathcal{V}) \in \mathfrak{B}(X)$. This implies that given $X \in \Sigma(\mathcal{U}) \cap \Sigma(\mathcal{V})$ then taking $\mathcal{W} = \mathcal{U} \cap \mathcal{V}$, there exists $\Sigma(\mathcal{W}) \in \mathcal{B}(X)$ such that $\Sigma(\mathcal{W}) \subset \Sigma(\mathcal{U}) \cap \Sigma(\mathcal{V})$. Therefore $\mathfrak{B}(X)$ is a topological basis for Σ at X. ☐

The previous lemma justifies the following definition.

Definition 3.2.3. *The topology $\mathfrak{T}$ in Σ generated by the bases*

$$\mathfrak{B}(X) = \{\Sigma(\mathcal{U}) \subset \Sigma : \mathcal{U} \subset \mathcal{N} \text{ is open with } X \subset \mathcal{U}\}$$

will be called the reconstructive *or* Low's topology *of Σ.*

The reconstructive topology provides good properties to the sky map S.

Proposition 3.2.4. *Given a spacetime M with space of skies Σ equipped with the reconstructive topology, then the sky map $S : M \to \Sigma$ is continuous.*

Proof. We will see that if $\mathcal{U} \subset \mathcal{N}$ is an open subset, then $V = S^{-1}(\Sigma(\mathcal{U})) \subset M$ is also open. Thus if $S(x) \subset \mathcal{U}$ we must show that there exists an open subset $V^x \subset M$ such that $S(y) \subset \mathcal{U}$ for all $y \in V^x$. Let us suppose that this is not the case. Then, in virtue of theorem 1.2.15, we can choose a family of compact basic neighbourhoods $\{V_n^x\}$ such that $V_{n+1}^x \subset V_n^x$ with local Cauchy surfaces C_n, $C_{n+1} \subset C_n$, such that $\{x\} = \bigcap_n C_n$, and points $y_n \in V_n^x$ with $S(y_n) \not\subset \mathcal{U}$. Hence, there exist $\gamma_n \in \mathcal{N}$ with $y_n \in \gamma_n$, but $\gamma_n \notin \mathcal{U}$. If $\gamma_n \cap C_n = \{x_n\}$ and $x_n = \gamma_n(0)$, then $\lim x_n = x$ and, since the space of directions $\mathbb{PN}$ over a compact set is compact, then there exists a convergent subsequence $\{[\gamma_k'(0)]\} \subset \mathbb{PN}$ to some $[u] \in \mathbb{PN}_x$. Denoting $\sigma : \mathbb{PN}(C_1) \to \mathcal{U}_1 \subset \mathcal{U}$ the restriction of the diffeomorphism of diagram (3.1.1) and $\gamma = \sigma([u]) \in \mathcal{N}$, we have shown that $\lim \gamma_k = \gamma = \sigma([u])$, but then $\gamma \in S(x) \subset \mathcal{U}$, and because $\mathcal{U}$ is open, there exists k such that $\gamma_k \in \mathcal{U}$ obtaining a contradiction. $\square$

Now, in order to show that S is open, we will choose a slightly more restrictive hypothesis: being non–refocusing instead of being sky separating.

Proposition 3.2.5. *If M is a non–refocusing spacetime with space of skies Σ equipped with the reconstructive topology, then the sky map $S : M \to \Sigma$ is open.*

Proof. In case of S open, for any open $V \subset M$ and all $x \in V$, there exists an open set $\mathcal{U} \subset \mathcal{N}$ such that $S(x) \subset \mathcal{U}$, and $\Sigma(\mathcal{U}) \subset S(V)$. Let us suppose this does not occur. Taking a family of globally hyperbolic open sets $\{V_n^x\}$ such that $V_{n+1}^x \subset V_n^x \subset V$ with local Cauchy surfaces C_n, $C_{n+1} \subset C_n$, such that $\{x\} = \bigcap_n C_n$ for all n. The sets

$$\mathcal{U}_n = \{\gamma \in \mathcal{N} : \gamma \cap V_n^x \neq \emptyset\}$$

are such that $\Sigma(\mathcal{U}_n) \not\subset S(V)$, hence there exists $x_n \in M$ with $S(x_n) \subset \mathcal{U}_n$ and $x_n \notin V$. Then for all V_n^x there exists $x_n \notin V_n^x$ such that for all $\gamma \in \mathcal{N}$ with $x_n \in \gamma$, then $\gamma \cap V_n^x \neq \emptyset$ and M is refocusing at x. This contradicts the hypotheses, therefore S is open. $\square$

Since the property of being non–refocusing implies being sky separating, due to propositions 3.2.4 and 3.2.5, we have the topological equivalence between M and Σ by the following corollary.

Corollary 3.2.6. *If M is a non–refocusing spacetime and its space of skies Σ is equipped with the reconstructive topology, then the sky map $S : M \to \Sigma$ is a homeomorphism.*

For any basic neighbourhood $V \subset M$ (see theorem 1.2.15) and any $x, y \in V$, then there exists a unique geodesic segment joining x and y. Let us consider the open $U = S(V) = \{S(x) : x \in V\}$, then for every $S(x) = X \neq Y = S(y) \in U$ and $\gamma \in X \cap Y$ verifying $T_\gamma X \cap T_\gamma Y \neq \{0\}$, by expression (3.0.3), there exist a Jacobi field J such that $J(s_0) = J(s_1) = 0$ where $x = \gamma(s_0)$ and $y = \gamma(s_1)$, but that is not possible since V is convex normal (see [53, Prop. 10.10]). So, in this case we have that $X = Y$ and the next definition is justified.

Definition 3.2.7. *A set $U \subset \Sigma$ is called* null conjugated *if there exist $X \neq Y \in U$ and $\gamma \in X \cap Y$ such that $T_\gamma X \cap T_\gamma Y \neq \{0\}$. We will say that $U \subset \Sigma$ is* null non–conjugated *in other case.*

Since S is bijective, we can extend the null conjugation property to M, then we will say that $V \subset M$ is null conjugated *if and only if $U = S(V) \subset \Sigma$ also is so.*

If M is a non–refocusing spacetime, observe that basic neighbourhood of $x \in M$ set up a basis for the topology of M at x, then by corollary 3.2.6, null non–conjugated neighbourhoods of Σ also constitute a basis for the topology of Σ.

It gives us a further condition for null non–conjugated sets to constitute a basis for the reconstructive topology. Recall that if N is manifold, we denote by $\widehat{T}N$ its reduced tangent bundle, this is, $\widehat{T}N = \cup_{x\in N}\widehat{T}_xN$ where $\widehat{T}_xN = T_xN - \{\mathbf{0}_x\}$. It will also help in the construction of a differentiable structure in Σ.

Theorem 3.2.8. *Let $V \subset M$ be a relatively compact basic open set. Then $U = S(V) \subset \Sigma$ is null non–conjugated and $\widehat{U} = \bigcup\limits_{X\in U} \widehat{T}X \subset T\mathcal{N}$ is a regular submanifold of $\widehat{T}\mathcal{N}$, where $\mathcal{U} = \bigcup\limits_{X\in U} X$.*

Proof. Let $V \subset M$ be a relatively compact basic open set such that $U = S(V) \subset \Sigma$. Since V is basic, then it is contained in a convex normal neighbourhood, therefore it is clear that U is null non–conjugated.

We will use the local coordinate chart $\psi\colon \mathcal{U} \to \mathbb{R}^{2m-3}$ described by equation (3.1.4) on $\mathcal{U}$, with $\mathcal{U} = \bigcup_{X\in U} X = \bigcup_{x\in V} S(x)$. Without any lack of generality, because of the properties of V, we can also consider the coordinates $\varphi = \left(x^1,\ldots,x^m\right)$ and the orthonormal frame $\{E_1,\ldots,E_m\}$ in V used to construct the coordinates $\overline{\psi} = (\mathbf{x},\mathbf{u},\mathbf{w},\mathbf{v})$ of $T\mathcal{N}$ in equation (3.1.5) in order to construct a coordinate chart $\overline{\varphi} : \widehat{U} \to \mathbb{R}^{3m-4}$ such that

$$\overline{\varphi}(J) = (x,u,v) = \left(x^1,x^2,\ldots,x^m,u^3,\ldots,u^m,\left\langle v^1\ldots,v^m\right\rangle\right) \in \mathbb{R}^{3m-4} \tag{3.2.1}$$

is analogous to the chart $\overline{\psi}$ in (3.1.5), and where $\widehat{U} = \bigcup\limits_{X\in U} \widehat{T}X$, with $J_0' = \sum\limits_{j=1}^{m} v^j E_j(x)$ and again $v = \left\langle v^1\ldots,v^m\right\rangle = \left(v^1,\ldots,v^m\right)(\mathrm{mod}\gamma')$. Notice that because of equation (3.0.3) if J is tangent to a sky $S(q)$, $\gamma(0) = q$, then $J(0) = 0\,(\mathrm{mod}\gamma'(0))$, hence the local chart $\overline{\varphi}$ is analogous to the chart $\overline{\psi}$ setting $\mathbf{w} = 0$ but the coordinate x describes the point $q \in V$ where J vanishes. Observe that if $J(0) = 0\,(\mathrm{mod}\gamma'(0))$, trivially we can choose a representative $\overline{J}$ such that $\overline{J}(0) = 0$.

Now, we will show that the map $\overline{\varphi}$ gives a differentiable structure to $\widehat{U}$ which does not depend on the chart φ nor the orthonormal frame chosen in V.

1. First, we will prove that the inclusion $i : \widehat{U} \hookrightarrow T\mathcal{U} \subset T\mathcal{N}$ is differentiable. Recall the diagram (2.2.7)

$$\begin{tikzcd} \mathbb{PN}(V) \arrow[r, "p_{\mathbb{PN}}"] & \mathcal{U} \subset \mathcal{N} \\ \mathbb{PN}(C) \arrow[u, hook, "\mathrm{inc}"] \arrow[ur, "\sigma"'] & \end{tikzcd}$$

   where $p_{\mathbb{PN}}$ is a submersion and σ a diffeomorphism. Observe the construction of the coordinates (x,u) of $\widehat{U}$ and $(\mathbf{x},\mathbf{u})$ of $T\mathcal{N}$ from the coordinates of $\mathbb{PN}^+(V) \simeq \Omega^T(V)$ and $\mathbb{PN}^+(C) \simeq \Omega^T(C)$ in equations (3.1.2) and (3.1.3) respectively. Since

$$\sigma_{VC} = \sigma^{-1} \circ p_{\mathbb{PN}} : \mathbb{PN}^+(V) \mapsto \mathbb{PN}^+(C) \tag{3.2.2}$$

then σ_{VC} is a submersion. The expression in coordinates of σ_{VC} is given by $(\mathbf{x}(x,u), \mathbf{u}(x,u))$, hence $\mathbf{x}(x,u)$ and $\mathbf{u}(x,u)$ are differentiable functions. If $\mathbf{x} = (x^2, \ldots, x^m)$, we will denote $(0, \mathbf{x}) = (0, x^2, \ldots, x^m)$. Consider then

$$p(x,u) = \varphi^{-1}(0, \mathbf{x}(x,u)) \in C \subset V$$

and

$$W(x,u) = E_1(p(x,u)) + u^2(x,u)\, E_2(p(x,u)) + \cdots + u^m(x,u)\, E_m(p(x,u))$$

where $\mathbf{u}(x,u) = (u^3(x,u), \ldots, u^m(x,u))$ and $u^2 = \sqrt{1-(u^3)^2 - \cdots - (u^m)^2}$. For any (x,u) we define the following map

$$h(t,x,u) = \exp_{p(x,u)}(tW(x,u))$$

It is clear that h is differentiable by composition of differentiable maps, and for fixed (x_0, u_0) the curve $\gamma_{(x_0,u_0)}(t) = h(t, x_0, u_0)$ is a null geodesic such that $\gamma_{(x_0,u_0)}(0) \in C$. For any of these geodesics, we have the initial value problem of Jacobi fields given by the Jacobi equation (2.3.1) with initial data

$$J(\tau) = 0, \qquad J'(\tau) = \xi\,, \tag{3.2.3}$$

with τ in the domain of $\gamma_{(x,u)}$ and $\xi \in T_{\gamma_{(x,u)}(\tau)}M$. See sketched scheme in figure 3.2.

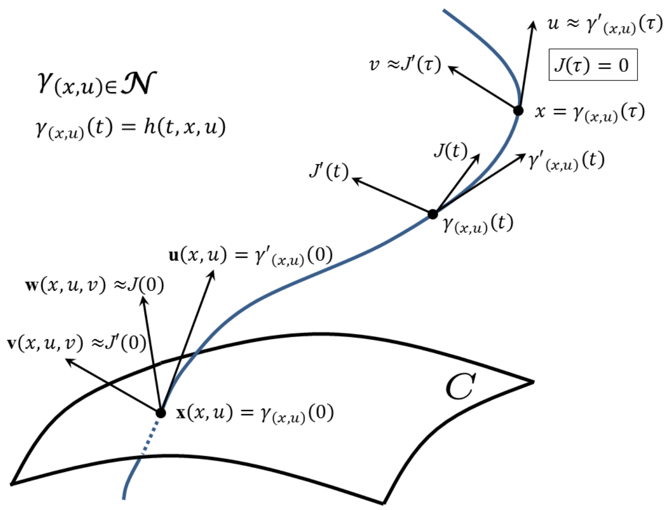

Figure 3.2: Coordinates in $T\mathcal{U}$ and $\widehat{U}$.

If we express the Jacobi field J as $J = \alpha^k \partial/\partial x^k$, then Jacobi equation (2.3.1) can be written by a system of differential equations

$$\begin{aligned} \frac{d^2\alpha^k}{dt^2} \; &+ \; \frac{d\alpha^i}{dt}\left(\Gamma^k_{ij}\frac{\partial h^j}{\partial t}\right) + \alpha^i \frac{d}{dt}\left(\Gamma^k_{ij}\frac{\partial h^j}{\partial t}\right) + \\ &+ \; \Gamma^k_{ln}\left(\frac{d\alpha^l}{dt} + \Gamma^l_{ij}\alpha^i \frac{\partial h^j}{\partial t}\right)\frac{\partial h^n}{\partial t} - \alpha^n \frac{\partial h^i}{\partial t}\frac{\partial h^j}{\partial t} R^k_{jni} = 0 \end{aligned}$$

for $k = 1, \dots, m$ where, for brevity, we write $h^j = x^j \circ h$, $\Gamma^k_{ij} = \Gamma^k_{ij}(h(t,x,u))$ and $R^k_{jni} = R^k_{jni}(h(t,x,u))$.

If we transform this second order system into a first order one by using the standard transformation $y^k = \alpha^k$ and $y^{m+k} = d\alpha^k/dt$ for $k = 1, \dots, m$ then, the initial value problem (2.3.1)-(3.2.3) has the form:

$$\frac{dy}{dt} = f(t,y,x,u), \qquad y(\tau) = \overline{\xi}\,. \tag{3.2.4}$$

Let us denote by $y\left(t,x,u,\tau,\overline{\xi}\right)$ the solution of (3.2.4), corresponding to a Jacobi field $J_{\tau,\overline{\xi}} \in \widehat{U}$ along the null geodesic $\gamma_{(x,u)}$ with $J_{\tau,\overline{\xi}}(\tau) = 0$ and $J'_{\tau,\overline{\xi}}(\tau) = \xi$. By construction, for each (x,u) there exists a unique τ such that

$$\varphi(h(\tau,x,u)) = x.$$

We will write this function as $\tau(x,u)$ and it is possible to show easily that this τ is differentiable applying the implicit function theorem to the map $F(t,x,u) = \varphi(h(t,x,u)) - x$. The solution $y\left(0,x,u,\tau(x,u),\overline{\xi}\right)$ gives us the values of $J_{\tau,\overline{\xi}}(0)$ and $J'_{\tau,\overline{\xi}}(0)$, and therefore it provides the coordinates $\mathbf{w}(x,u,v)$ and $\mathbf{v}(x,u,v)$. The theorem on the regular dependence of solutions of initial value problems with parameter (see for instance [23, ch. 5]), claims that $y\left(0,x,u,\tau(x,u),\overline{\xi}\right)$ is a differentiable function depending smoothly on the data $\left(x,u,\overline{\xi}\right)$, therefore $\mathbf{w}(x,u,v)$ and $\mathbf{v}(x,u,v)$ are differentiable functions of (x,u,v). This proves that $i : \widehat{U} \hookrightarrow T\mathcal{U}$ is differentiable.

2. The second step in this proof is to show that $i : \widehat{U} \hookrightarrow T\mathcal{U}$ is an immersion. For this purpose we will show that any regular curve in $\widehat{U}$ is transformed by i into a regular curve in $T\mathcal{U}$. Let us consider a regular curve $c(s) \in \widehat{U}$ with $s \in (-\varepsilon, \varepsilon)$. This means that $c(s) = J_s$ is a Jacobi field along a light ray γ_s (parametrized as a null geodesic) verifying $J_s(t_s) = 0$, and $J'_s(t_s) = \xi(s)$ is not proportional to $\gamma'_s(t_s)$. We will prove that $i_*(c'(0)) \neq 0$ if $c'(0) \neq 0$, that is

$$c'(0) \neq 0 \Rightarrow (i \circ c)'(0) \neq 0$$

This curve c can be written in coordinates as $\overline{\varphi}(c(s)) = (x(s), u(s), v(s))$ with $\overline{\varphi}(c(0)) = (x_0, u_0, v_0)$ and it has a differentiable image in $T\mathcal{U}$. The inclusion i transforms the coordinates of c as

$$\overline{\psi} \circ i \circ (\overline{\varphi})^{-1}(x(s), u(s), v(s)) =$$

$$= (\mathbf{x}(x(s), u(s)), \mathbf{u}(x(s), u(s)), \mathbf{w}(x(s), u(s), v(s)), \mathbf{v}(x(s), u(s), v(s)))$$

The map $(\mathbf{x}(x,u), \mathbf{u}(x,u))$ coincides with the expression in coordinates of map $\sigma_{VC} = \sigma^{-1} \circ p_{\mathbb{PN}} : \mathbb{PN}^+(V) \mapsto \mathbb{PN}^+(C)$ of equation (3.2.2), which is a submersion, then its differential has maximal rank $2m-3$ and codimension 1. If the curve with coordinates $(x(s), u(s))$ is transversal to the fibre of σ_{VC} at $s = 0$, then obviously $(i \circ c)'(0) \neq 0$. In other case, we can take c (defining $c'(0)$) as a regular curve verifying that $c(s) = J_s$ lies on a fixed null geodesic γ, then

$$\overline{\psi} \circ i \circ (\overline{\varphi})^{-1}(x(s), u(s), v(s)) =$$

$$= (\mathbf{x}(x_0, u_0), \mathbf{u}(x_0, u_0), \mathbf{w}(x_0, u_0, v(s)), \mathbf{v}(x_0, u_0, v(s)))$$

where $(\mathbf{x}, \mathbf{u})$ remains constant for every s. Then the differential

$$\left(d\mathbf{x}_{c(0)}(c'(0)), d\mathbf{u}_{c(0)}(c'(0))\right) = (0, 0).$$

This regular curve c is a curve of Jacobi fields $J_s \in \widehat{U}$ along the null geodesic γ such that $J_s(t_0 + s) = 0$ and $J_s'(t_0 + s) = \xi(s)$ for $s \in (-\epsilon, \epsilon)$ and hence $\xi(s)$ is a vector field along γ non-proportional to γ' at $s = 0$. We can assume, without any lack of generality that $t_0 = 0$ and the local Cauchy surface C associated with the chart $\overline{\psi}$ contains $\gamma(0)$. We have that $J_0(0) = 0$. So,

$$\left.\frac{d}{ds}\right|_{s=0} J_s(0) = \lim_{s\mapsto 0} \frac{J_s(0) - J_0(0)}{s} = \lim_{s\mapsto 0} \frac{J_s(0)}{s}$$

By [8, Prop. 10.16], we have that $J_s(t) = \left(\exp_{\gamma(s)}\right)_* \left((t-s)\,\tau_{(t-s)\gamma'(s)} J_s'(s)\right)$ where for $v \in T_{\gamma(s)}M$, the map $\tau_v : T_{\gamma(s)}M \to T_v T_{\gamma(s)}M$ is the canonical isomorphism. Then

$$\left.\frac{d}{ds}\right|_{s=0} J_s(0) = \lim_{s\mapsto 0} \frac{1}{s} \left(\exp_{\gamma(s)}\right)_* \left((-s)\,\tau_{(-s)\gamma'(s)}\xi(s)\right) =$$

$$= \lim_{s\mapsto 0} \left(\exp_{\gamma(s)}\right)_* \left(\left(\frac{-s}{s}\right) \tau_{(-s)\gamma'(s)}\xi(s)\right) = \lim_{s\mapsto 0} \left(\exp_{\gamma(s)}\right)_* \left(-\tau_{(-s)\gamma'(s)}\xi(s)\right) =$$

$$= \left(\exp_{\gamma(0)}\right)_* \left(-\tau_0 \xi(0)\right) = -\xi(0)$$

Hence, we state that

$$\left.\frac{d}{ds}\right|_{s=0} J_s(0) = -\xi(0)$$

Since $\xi(0)$ is not proportional to $\gamma'(0)$, then $d\mathbf{w}_{c(0)}(c'(0)) \neq 0$, and this implies that $i \circ c$ is a regular curve for $s = 0$. Therefore i is an immersion.

3. In the last step of this proof, we will show that $\widehat{U} \subset T\mathcal{U}$ is a regular submanifold. Let us consider the system of ordinary differential equations (3.2.4) for Jacobi fields in $\widehat{U}$. We will denote its solution by $y\left(t, x, u, \tau, \overline{\xi}\right)$. If the origin of the parameter t of equation (3.2.4) is lying in the local Cauchy surface C, we can write the Jacobi field J such that $J(\tau) = 0$ and $J'(\tau) = \xi$ as the solution $y\left(t, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi}\right)$, where $\mathbf{x} = \left(0, x^2, \ldots, x^m\right)$ which can be identified with the adapted coordinates $\mathbf{x}$ to C in equation (3.1.3). Then, the pair $(\mathbf{x}, \mathbf{u})$ are the coordinates of a point in $\mathbb{PN}^+(C)$ and therefore, they determine the null geodesic $\gamma_{(\mathbf{x},\mathbf{u})}$. In fact, $y\left(\tau, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi}\right)$ corresponds to the values $J(\tau) = 0$ and $J'(\tau) = \xi$. Moreover, $y\left(0, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi}\right)$ represents the values $J(0)$ and $J'(0)$ which are lying in C, therefore $y\left(0, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi}\right)$ is equivalent to give the coordinates $\overline{\psi}(J) = (\mathbf{x}, \mathbf{u}, \mathbf{w}, \mathbf{v})$ of J in $T\mathcal{N}$. Since V is relatively compact and due to the existence of flow boxes of non-vanishing differentiable vector fields, we can assume, without any lack of generality, that there exist a compact interval I neighbourhood of 0 such that the parameter of any null geodesic defined by $\eta = E_1(p) + u^2 E_2(p) + \cdots + u^m E_m(p) \in \mathbb{N}_p^+(C)$ with $p \in C \subset V$ running through V is defined for $t \in I$. Now, let us consider an arbitrary sequence $\{J_n\} \subset \widehat{U} \subset T\mathcal{N}$

converging to $J_\infty \in \widehat{U} \subset T\mathcal{N}$ in $T\mathcal{N}$. Proving that $\{J_n\}$ converges to J_∞ in $\widehat{U}$ is sufficient in order to show that $\widehat{U} \subset T\mathcal{U}$ is a regular submanifold.

The Jacobi fields J_n and J_∞ are fields along the null geodesics $\gamma_{(\mathbf{x}_n,\mathbf{u}_n)}$ and $\gamma_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}$ respectively and moreover there exist $t_n, t_\infty \in I$ such that $J_n(t_n)$ and $J_\infty(t_\infty)$ are proportional to $\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)$ and $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)$ respectively for every positive integer n. If their coordinates in $T\mathcal{N}$ are $\overline{\psi}(J_n) = (\mathbf{x}_n, \mathbf{u}_n, \mathbf{w}_n, \mathbf{v}_n)$ and $\overline{\psi}(J_\infty) = (\mathbf{x}_\infty, \mathbf{u}_\infty, \mathbf{w}_\infty, \mathbf{v}_\infty)$ respectively, then we have that

$$\lim_{n\mapsto\infty} \overline{\psi}(J_n) = \overline{\psi}(J_\infty)$$

or equivalently

$$\lim_{n\mapsto\infty} y\left(0, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n\right) = y\left(0, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty\right)$$

Again because of the theorem on the regular dependence of solutions of initial value problems with parameters, the solution $y\left(t, \mathbf{x}, \mathbf{u}, \tau, \overline{\xi}\right)$ differentiably depends on the variables $\left(t, x, u, \tau, \overline{\xi}\right)$, therefore

$$\lim_{n\mapsto\infty} y\left(t, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n\right) = y\left(t, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty\right)$$

This implies that

$$\lim_{n\mapsto\infty} J_n(t) = J_\infty(t)$$

Since I is compact, the sequence $\{t_n\} \subset I$ has a convergent subsequence, so we can assume that $\{t_n\}$ itself verifies that $\lim_{n\mapsto\infty} t_n = \overline{t} \in I$. Then we have that

$$\lim_{n\mapsto\infty} y\left(t_n, \mathbf{x}_n, \mathbf{u}_n, t_n, \overline{\xi}_n\right) = y\left(\overline{t}, \mathbf{x}_\infty, \mathbf{u}_\infty, t_\infty, \overline{\xi}_\infty\right)$$

hence

$$\begin{array}{l} \lim_{n\mapsto\infty} J_n(t_n) = J_\infty\left(\overline{t}\right) \\ \lim_{n\mapsto\infty} J'_n(t_n) = J'_\infty\left(\overline{t}\right) \end{array}$$

Since $J_n(t_n)$ is proportional to $\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)$ for every positive integer n, then $J_\infty\left(\overline{t}\right)$ is also proportional to $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)$, but $\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}$ is a null geodesic without conjugate points, therefore $\overline{t} = t_\infty$. This gives us

$$\lim_{n\mapsto\infty} J'_n(t_n) = J'_\infty(t_\infty)$$

Recall that the coordinates of $\widehat{U}$ are given by $\overline{\varphi} = (x, u, v)$ where $\varphi = \left(x^1, \ldots, x^m\right)$ is the chart in V. Then

$$\lim_{n\mapsto\infty} \overline{\varphi}(J_n) = \lim_{n\mapsto\infty} \left(\varphi\left(\gamma_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)\right), \left[\gamma'_{(\mathbf{x}_n,\mathbf{u}_n)}(t_n)\right], \langle J'_n(t_n)\rangle\right) =$$

$$= \left(\varphi\left(\gamma_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)\right), \left[\gamma'_{(\mathbf{x}_\infty,\mathbf{u}_\infty)}(t_\infty)\right], \langle J'_\infty(t_\infty)\rangle\right) = \overline{\varphi}(J_\infty)$$

So, the sequence $\{J_n\}$ converges to J_∞ in $\widehat{U}$.

This completes the proof. $\square$

Denote $\mathcal{U} = \bigcup_{X \in U} X$ for an open set $U \subset \Sigma$, then we can define the family

$$\mathfrak{R} = \{U \subset \Sigma \text{ null non–conjugated} : \widehat{U} = \bigcup_{X \in U} \widehat{T}X \text{ is a regular submanifold of } \widehat{T}\mathcal{N}\} \tag{3.2.5}$$

Corollary 3.2.9. *If M is a non–refocusing spacetime, then the family of open sets in $\mathfrak{R}$ constitutes a basis for the reconstructive topology of Σ.*

Proof. Given $X \in \Sigma$ such that $x = S^{-1}(X) \in M$ there exist $V \subset M$ relatively compact basic neighbourhood of x. Since M is non–refocusing, by corollary 3.2.6, $U = S(V) \subset \Sigma$ is open and since V is basic, then U is null non–conjugate. By theorem 3.2.8, $U \in \mathfrak{R}$. Recall that sets like V form a basis for the topology of M, then corollary 3.2.6, since S is a homeomorphism, then open sets in $\mathfrak{R}$ constitute a basis for the reconstructive topology. □

Observe that in the definition of $\mathfrak{R}$ there are not implicit or explicit references to M. So, it is appropriate to recover the strongly causal conformal manifold M from $\mathcal{N}$ and Σ, because we will have to use structures in $\mathcal{N}$ and Σ defined independently from M. Anyway, in section 3.5, we will refine this basis without using the non–refocusing hypothesis.

Section 3.3

Some types of special curves

This section is devoted to introduce a class of curves in $\mathcal{N}$ and its counterpart class in M. Both will be useful tools in next sections. After the description of such curves in section 3.3.1, we will give some results in section 3.3.2 in relation to causality properties in M in which those special curves arise.

3.3.1

Celestial and dust curves

We will introduce a class of curves that are going to play a fundamental role in order to establish some results such as weakening the hypotheses of corollary 3.2.6 or characterizing the causality of M in terms of $\mathcal{N}$ among others.

Definition 3.3.1. *A tangent vector $J \neq \mathbf{0}_\gamma$ at $T_\gamma\mathcal{N}$ will be called a* celestial vector *if there exists a sky $X \in \Sigma$ such that $J \in T_\gamma X \subset T\mathcal{N}$. We will denote the set of all* celestial vectors *by $\widehat{\Sigma} \subset T\mathcal{N}$. With the notation introduced in theorem 3.2.8, we write $\widehat{\Sigma} = \bigcup_{X \in \Sigma} \widehat{T}X \subset \widehat{T}\mathcal{N}$.*

A differentiable curve $\Gamma : I \to \mathcal{N}$ is called a celestial curve *if $\Gamma'(s) \in \widehat{\Sigma}$ for every $s \in I$. We will denote the set of celestial curves by $\mathfrak{C}(\mathcal{N})$.*

Next proposition allows us to understand that celestial curves can be described by a particular class of geodesic variations.

Proposition 3.3.2. *If the curve* $\Gamma : [0,1] \to \mathcal{N}$ *with* $\Gamma(s) = \gamma_s \in \mathcal{N}$ *is celestial then there exists a null curve* $\mu : [0,1] \to M$ *such that* $\gamma_s(\tau) = \exp_{\mu(s)}(\tau\sigma(s))$ *where* $\sigma(s) \in \mathbb{N}^+_{\mu(s)}$ *is a differentiable curve proportional to* $\mu'(s)$ *wherever* μ *is regular.*

Proof. Let $\Gamma : [0,1] \to \mathcal{N}$ be a celestial curve with $\Gamma(s) = \gamma_s$. Let $s_0 \in [0,1]$ and $t_0 \in \mathbb{R}$ such that $\Gamma'(s_0) \in T_{\gamma_{s_0}} S(\gamma_{s_0}(t_0))$ and a local chart $(\widehat{U}, \overline{\varphi})$, with $\overline{\varphi} = (x, u, v)$ as in (3.2.1) with $\Gamma'(s_0) \in \widehat{U}$ such that (V, φ) is the local chart containing $\gamma_{s_0}(t_0) \in M$ used to define $\overline{\varphi}$. We will denote again by $\{E_1, \ldots, E_m\}$ the orthonormal frame in V used to define the coordinates u and v in $\overline{\varphi}$.

Consider the neighbourhood $I \subset \mathbb{R}$ of s_0 such that $\Gamma'(s) \in \widehat{U}$ for all $s \in I$, thus we have that

$$\overline{\varphi}(\Gamma'(s)) = (x(\Gamma'(s)), u(\Gamma'(s)), v(\Gamma'(s))) \in \mathbb{R}^m \times \mathbb{R}^{m-2} \times \mathbb{R}^{m-2}$$

is a smooth curve. The coordinates x and u describe the light rays supporting the Jacobi fields, thus we can reconstruct the curve Γ from them. Notice that the curve $\mu(s) = \varphi^{-1} \circ x(\Gamma'(s)) \in M$ is smooth. Then consider the curve in $\mathbb{N}^+$ given by:

$$\sigma(s) = E_1(\mu(s)) + u^2(\Gamma'(s))E_2(\mu(s)) + \cdots + u^m(\Gamma'(s))E_m(\mu(s)) \in \mathbb{N}^+_{\mu(s)}$$

Clearly, $\sigma(s)$ is smooth, then the geodesic variation:

$$\mathbf{f}(s,\tau) = \exp_{\mu(s)}(\tau\sigma(s)) = \overline{\gamma}_s(\tau)$$

reconstructs the curve $\Gamma(s)$.

Because $\mathbf{f}(s,0) = \mu(s)$, by lemma 2.3.5, the Jacobi field $\overline{J}_s$ along $\overline{\gamma}_s$ defined by $\mathbf{f}(s,\tau)$ satisfies that $\overline{J}_s(0) = \mu'(s)$ (we choose now $t_0 = 0$). Moreover, since Γ is a celestial curve, hence tangent to $S(\mu(s))$ at $\Gamma(s)$, then $\overline{J}_s(0) = 0 \, (\mathrm{mod} \overline{\gamma}'_s(0))$ and therefore $\overline{J}_s(0) = \lambda_s \overline{\gamma}'_s(0)$ for some $\lambda_s \in \mathbb{R}$. Then we conclude that μ' is proportional to $\overline{\gamma}'_s(0)$, hence also to $\sigma(s)$.

Finally, due to the compactness of Γ, the curves μ and σ can be extended to the full interval $[0,1]$. □

The previous proposition describes a celestial curve Γ as a pair $(\mu, \sigma) \subset M \times \mathbb{N}^+$ where μ is a null curve that cannot be geodesic because in this case Γ would not be regular. Moreover the regularity of μ is not guaranteed at all, in fact, it is possible to exhibit examples of celestial curves such that μ stops for $s \in [\alpha, \beta] \subset \mathbb{R}$, where $\alpha = \beta$ is not excluded. So, while μ remains at $\mu(s) = p \in M$, the curve $\sigma(s)$ moves smoothly in $\mathbb{N}^+_p$. The time–orientation of μ is not guaranteed neither, as the next example shows.

Example 3.3.3. *Let* $\mathbb{M}^3$ *be the 3–dimensional Minkowski spacetime with coordinates given by* $(t,x,y) \in \mathbb{R}^3$ *and metric* $\mathbf{g} = -dt \otimes dt + dx \otimes dx + dy \otimes dy$. *Let us denote its space of light rays by* $\mathcal{N}$. *Consider the geodesic variation*

$$\mathbf{f}(s,\tau) = \gamma_s(\tau) = \left(\tau + \frac{1}{2}s^2, s\sin s + (1+\tau)\cos s, -s\cos s + (1+\tau)\sin s\right)$$

where $s \in [-\epsilon, \epsilon]$, $\tau \in (-\delta, \delta)$, *thus* $\Gamma(s) = \gamma_s$ *is a curve in* $\mathcal{N}$. *Since*

$$J_s(\tau) = \frac{\partial \mathbf{f}}{\partial s}(s,\tau) = s(1, \cos s, \sin s) + (0, -\tau \sin s, \tau \cos s)$$

for any s *we have that* $\tau = 0$ *implies that* $J_s(0) = s\gamma_s'(0)$, *then* $J_s \in TS(\gamma_s(0))$ *and therefore* Γ *is a celestial curve. For this curve,* μ *is defined as*

$$\mu(s) = \mathbf{f}(s, \tau(s)) = \mathbf{f}(s,0) = \left(\frac{1}{2}s^2, s\sin s + \cos s, -s\cos s + \sin s\right)$$

hence,

$$\mu'(s) = (s, s\cos s, s\sin s) = s(1, \cos s, \sin s)$$

and μ *is a null curve since*

$$\mathbf{g}(\mu'(s), \mu'(s)) = 0$$

It is trivial to observe that μ *is not a regular curve when* $s = 0$ *and the* s *factor in* μ' *changes the time–orientation of* μ*: if* $s < 0$ *then* μ *is past–oriented and if* $s > 0$ *then* μ *is future–oriented.*

By construction, the curve μ in proposition 3.3.2 verifies that $\Gamma'(s) \in \widehat{T}S(\mu(s))$ for all $s \in [0,1]$, then it runs the points in M such that the celestial curve Γ is tangent to their skies, or in other words, μ is the trail in M left by the celestial curve Γ.

As a consequence of proposition 3.3.2, we have the following corollary.

Corollary 3.3.4. *Given a celestial curve* $\Gamma : [0,1] \to \mathcal{N}$ *such that* $\Gamma'(s_0) \in \widehat{T}S(p_0)$, $0 \leq s_0 \leq 1$, *then the curve* μ: $[0,1] \to M$ *of the previous proposition 3.3.2, is unique verifying* $\mu(s_0) = p_0 \in M$.

Proof. Consider that there exists $\mu_1, \mu_2 : [0,1] \to M$ associated to Γ in the sense of proposition 3.3.2 and verifying $\mu_1(s_0) = \mu_2(s_0) = p_0$ for $s_0 \in [0,1]$. Let us define the set $A = \{s \in [0,1] : \mu_1(s) = \mu_2(s)\}$. Clearly, A is not empty and closed in $[0,1]$. Consider a basic neighbourhood $U \subset M$ of p_0. Since U is open, then there exist $\delta > 0$ such that $\mu_i((s_0 - \delta, s_0 + \delta)) \subset U$ for $i = 1, 2$ (eventually if $s_0 = 0$ then we consider $\mu_i([0,\delta)) \subset U$ and analogously for $s_0 = 1$). Let us suppose that for $s \in (s_0 - \delta, s_0 + \delta)$ we have that $\mu_1(s) \neq \mu_2(s)$ and since U is causally convex, then the segment of the light ray $\Gamma(s) = \gamma_s \in \mathcal{N}$ connecting $\mu_1(s)$ and $\mu_2(s)$ is totally contained in U and, moreover since $\Gamma'(s) \in \widehat{T}S(\mu_1(s)) \cap \widehat{T}S(\mu_2(s))$, then the points $\mu_1(s)$ and $\mu_2(s)$ are mutually conjugated along γ_s but, in virtue of [53, Prop. 10.10], this is not possible in a normal neighbourhood contradicting U is normal. Then we have that $\mu_1(s) = \mu_2(s)$ and hence the set A is also open in $[0,1]$. Since A is open, closed and not empty in $[0,1]$ then $A = [0,1]$ and we conclude that $\mu_1 = \mu_2$. □

Proposition 3.3.2 and corollary 3.3.4 allow to state the following definition.

Definition 3.3.5. *Given a celestial curve* $\Gamma \in \mathfrak{C}(\mathcal{N})$ *such that* $\Gamma'(s_0) \in \widehat{T}X$ *where* $X = S(x)$, *then the only curve* μ *verifying* $\mu(s_0) = x$ *of corollary 3.3.4 is called the* dust *or* trail *of* Γ *through* x.

As mentioned above, the dust $\mu : [a,b] \to M$ associated to the celestial curve $\Gamma : [a,b] \to \mathcal{N}$ could stop for any $s \in I$ for some closed interval $I \subset [a,b]$ in which Γ runs a fixed sky, so μ does not provide information about how Γ moves among skies while μ is stopped. Then, cutting away the intervals of the domain where μ stops, we keep the essential information about Γ. The price we will pay is that this new curve will not be differentiable but just piecewise differentiable. The resultant curve μ_X^Γ will be called the *essential dust* or *essential trail* and both μ and μ_X^Γ have the same image in M. We will deal with the construction of the essential dust μ_X^Γ in lemma 3.3.7.

In order to characterize in some way the essential dusts, we will introduce the notion of twisted null curve as follows.

Definition 3.3.6. *A continuous curve $\mu : [a,b] \to M$ will be called a* piecewise twisted null curve *if there exists a partition $a = s_0 < s_1 < \ldots < s_k = b$ such that for every $i = 1, \ldots, k$:*

(i.) $\mu|_{(s_{i-1},s_i)}$ *is differentiable.*

(ii.) $\mathbf{g}(\mu'(s), \mu'(s)) = 0$ *for all* $s \in (s_{i-1}, s_i)$.

(iii.) $\mu'(s)$ *and* $\frac{D\mu'}{ds}(s)$ *are linearly independent for all* $s \in (s_{i-1}, s_i)$.

We say that μ is future–directed (past–directed) if $\mu|_{(s_{i-1},s_i)}$ is future–directed (past-directed) for all $i = 1, \ldots, k$. If $k = 1$ then μ will be simply called twisted null curve.

Now, in next lemma, we will show that a essential dust can be identified with a piecewise twisted null curve.

Lemma 3.3.7 (μ-Lemma)**.** *Let $\Gamma : [0,1] \to \mathcal{N}$ be a celestial curve such that $\Gamma'(0) \in \widehat{T}X_0$ with $X_0 \in \Sigma$. Then there exists a unique curve $\chi_{X_0}^\Gamma : [0,1] \to \Sigma$ such that it is continuous in Low's topology and verifies $\chi_{X_0}^\Gamma(0) = X_0$ and $\Gamma'(s) \in \widehat{T}\chi_{X_0}^\Gamma(s)$. Moreover, the essential dust curve $\mu_{X_0}^\Gamma$ is a piecewise twisted null curve in M running along the image of $S^{-1} \circ \chi_{X_0}^\Gamma$.*

Conversely, given a regular twisted null curve μ: $[0,1] \to M$ such that $\mu(0) = x_0 = S^{-1}(X_0)$, $\mu'(0) \neq 0 \neq \mu'(1)$, then the curve $\Gamma^\mu : [0,1] \to \mathcal{N}$ defined by the variation of null geodesics $\mathbf{f} : [0,1] \times I \to M$ such that

$$\mathbf{f}(s,t) = \exp_{\mu(s)}(t\mu'(s)) = \Gamma^\mu(s)|_t$$

is celestial with $(\Gamma^\mu)'(0) \in \widehat{T}X_0$ and $\chi_{X_0}^\Gamma(s) = S(\mu(s))$.

Proof. Let $\Gamma : [0,1] \to \mathcal{N}$ be a celestial curve such that $\Gamma(s) = \gamma_s \in \mathcal{N}$ and $\Gamma'(0) \in \widehat{T}X_0$ with $X_0 = S(x_0) \in \Sigma$. Let $\mu : [0,1] \to M$ be the dust of Γ through p_0 constructed in corollary 3.3.4. There exists a partition

$$\{0 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_{n-1} \leq b_{n-1} < a_n \leq b_n = 1\} \subset [0,1]$$

such that

$$\gamma_s(t) = \exp_{\mu(s)}(t\sigma(s)) \tag{3.3.1}$$

where $\sigma : [0,1] \to \mathbb{N}$ is a differentiable curve verifying $\sigma(s) = \lambda_k(s)\mu'(s)$ for $s \in (b_k, a_{k+1})$ and λ_k differentiable with $k = 1, \ldots, n-1$. This curve μ also verifies $\mu(s) = p_k \in M$ for all $s \in [a_k, b_k]$.

Now, we can define the curve $\chi^{\Gamma}_{X_0} = S \circ \mu : [0,1] \to \Sigma$. Recall that for an open set $\mathcal{U} \subset \mathcal{N}$ containing a sky $X \in \Sigma$, the set of all skies contained in $\mathcal{U}$ is denoted as $\Sigma(\mathcal{U})$. By the definition of the Low's topology, the set $\Sigma(\mathcal{U})$ is open in Σ and these collection of open sets forms a basis at X.

In order to show that $\chi^{\Gamma}_{X_0}$ is continuous, we will show that, given any $\mathcal{U} \subset \mathcal{N}$ containing a sky $S(\mu(s)) \in \Sigma$ then $\left(\chi^{\Gamma}_{X_0}\right)^{-1}(\Sigma(\mathcal{U}))$ is open in $[0,1]$ is verified. So, take any $s \in [0,1]$ and consider an open set $\mathcal{U} \subset \mathcal{N}$ such that $\chi^{\Gamma}_{X_0}(s) \subset \mathcal{U}$ and then $\chi^{\Gamma}_{X_0}(s) \in \Sigma(\mathcal{U})$. Choose a collection of nested intervals $I^s_n \subset \mathbb{R}$ such that $\{s\} = \bigcap_n I^s_n$. Let us suppose that there exists $s_n \in I^s_n$ such that $\chi^{\Gamma}_{X_0}(s_n) \notin \Sigma(\mathcal{U})$. Then there is a light ray $\gamma_n \in \chi^{\Gamma}_{X_0}(s_n) \in \Sigma$ such that $\gamma_n \notin \mathcal{U}$. Recall that a light ray is fully determined by a point $p \in M$ and a direction $[v] \in \mathbb{PN}_p$, so γ_n can be defined by $\mu(s_n) \in \gamma_n \subset M$ and a null direction $[v_n] \in \mathbb{PN}_{\mu(s_n)}$. Since $\lim \mu(s_n) = \mu(s)$ and due to the compactness of the fibres $\mathbb{PN}_{\mu(s_n)}$, then with no lack of generality taking a subsequence of $[v_n]$ if necessary, there exists a direction $[v] \in \mathbb{PN}_{\mu(s)}$ defining, together with $\mu(s)$, the light ray γ such that $\lim \gamma_n = \gamma \in \chi^{\Gamma}_{X_0}(s) \subset \mathcal{U}$.

But since $\mathcal{U}$ is open, there exists an integer n_0 such that for every $n > n_0$ we have that $\gamma_n \in \mathcal{U}$ contradicting that $\chi^{\Gamma}_{X_0}(s_n) \notin \Sigma(\mathcal{U})$. Therefore there exist I^s_n such that $\chi^{\Gamma}_{X_0}(s_n) \in \Sigma(\mathcal{U})$ and hence $\left(\chi^{\Gamma}_{X_0}\right)^{-1}(\Sigma(\mathcal{U}))$ is open in $[0,1]$.

To obtain the essential dust $\mu^{\Gamma}_{X_0}$ from the dust μ, we will cut off the segments $\mu|_{(a_k,b_k)}$ from μ and glue together the segments $\mu|_{[b_k,a_{k+1}]}$. We call $c_1 = 0$ and for every $k = 1,\ldots,n-1$, let us define $c_{k+1} = a_{k+1} - \sum_{i=1}^{k}(b_i - a_i) \in [0,1]$ and consider the change of parameter $h_k : [c_k, c_{k+1}] \to [b_k, a_{k+1}]$ defined by $h_k(\tau) = \tau + a_{k+1} - c_{k+1}$. Since μ is differentiable and h_k is a diffeomorphism for every $k = 1,\ldots,n-1$ then $\overline{\mu}_k(\tau) = \mu \circ h_k(\tau)$ is differentiable for $\tau \in (c_k, c_{k+1})$. Moreover, since $\overline{\mu}'_k(\tau) = \mu'(h_k(\tau))$ then

$$\mathbf{g}\left(\overline{\mu}'_k(\tau), \overline{\mu}'_k(\tau)\right) = \mathbf{g}\left(\mu'_k(h_k(\tau)), \mu'_k(h_k(\tau))\right) = 0$$

for $\tau \in (c_k, c_{k+1})$. Also, the covariant derivatives verify

$$\frac{D\overline{\mu}'_k(\tau)}{d\tau} = h''_k(\tau)\,\mu'(h_k(\tau)) + \left(h'_k(\tau)\right)^2 \frac{D\mu'(h_k(\tau))}{ds} = \frac{D\mu'(h_k(\tau))}{ds}$$

then denoting J_s as the Jacobi field along γ_s defined by the variation (3.3.1), we have $J_s(0) = \mu'(s)$ and

$$J'_s(0) = \frac{D\sigma(s)}{ds} = \frac{D(\lambda_k(s)\,\mu'(s))}{ds} = \lambda'_k(s)\,\mu'(s) + \lambda_k(s)\frac{D\mu'(s)}{ds}$$

for $s \in (b_k, a_{k+1})$. Since Γ is celestial, then $J'_s \neq 0 \,(\mathrm{mod}\gamma'_s)$ and so, $\frac{D\mu'(s)}{ds}$ is not proportional to $\mu'(s)$ for $s \in (b_k, a_{k+1})$, therefore $\frac{D\overline{\mu}'_k(\tau)}{d\tau}$ and $\overline{\mu}'_k(\tau)$ are linearly independent for $\tau \in (c_k, c_{k+1})$. We have shown that for any $k = 1,\ldots,n-1$ the curves $\overline{\mu}_k$ are twisted null curves. Since $h_k^{-1}(a_{k+1}) = h_{k+1}^{-1}(b_{k+1})$ then all the segments $\overline{\mu}_k$ glue together continuously. Therefore we can define, with no ambiguity, the curve $\mu^{\Gamma}_{X_0} : [0,a] \to M$ such that $\mu^{\Gamma}_{X_0}(\tau) = \overline{\mu}_k(\tau)$ if $\tau \in [c_k, c_{k+1}]$ for $k = 1,\ldots,n-1$ and $[0,a] = \cup_{k=1}^{n-1}[c_k, c_{k+1}]$. This essential dust curve $\mu^{\Gamma}_{X_0}$ is then a piecewise twisted null curve associated to the partition $\{0 = c_1 < c_2 < \cdots < c_n = a\} \subset [0,a]$ and it is unique except by reparametrization.

Conversely, let us consider a twisted null curve $\mu : [0,1] \to M$ such that $\mu(0) = x_0 = S^{-1}(X_0)$. Then, we can define the variation of null geodesics $\mathbf{f} : [0,1] \times I \to M$ such that

$$\mathbf{f}(s,t) = \exp_{\mu(s)}(t\mu'(s)) = \gamma_s(t)$$

which verifies $\gamma_s'(0) = \mu'(s)$. Now, define the curve $\Gamma^\mu(s) = \gamma_s \in \mathcal{N}$ for every $s \in [0,1]$. The Jacobi field J_s of the variation $\mathbf{f}$ along γ_s verifies $J_s(0) = \mu'(s) = \gamma_s'(0)$ and $J_s'(0) = \frac{D\mu'}{ds}(s)$ and, since μ is twisted null then $\frac{D\mu'}{ds}$ is not proportional to γ_s'. Therefore $(\Gamma^\mu)'(s) = J_s\,(\mathrm{mod}\gamma_s') \neq 0\,(\mathrm{mod}\gamma_s')$ and hence

$$(\Gamma^\mu)'(s) \in \widehat{T}S(\gamma_s(0)) = \widehat{T}S(\mu(s))$$

then Γ^μ is celestial. □

3.3.2 *Twisted null curves and causality in M*

It is widely known that the endpoints of any future–directed timelike curve $\lambda : [a,b] \to M$ can be joined by a future–directed piecewise null geodesic. This fact follows from [25, Prop. 6.7.1] and the fact of that, by compactness and theorem 1.2.15, $\lambda \subset M$ can be covered by the finite union of globally hyperbolic neighbourhoods. Moreover, by theorem 1.2.4, if there is a future–directed causal curve λ connecting $p \in M$ to $q \in M$ such that λ is not a null geodesic, then there is a future–directed timelike curve β connecting $p \in M$ to $q \in M$. These results permit to characterize the causality in M: $q \in I^+(p)$ if and only if there exists a future–directed piecewise null geodesic joining p to q.

Now, in theorem 3.3.11 of this section, we will show the existence of an analogue characterization of the causality in M in terms of piecewise twisted null curves. This result, in addition to be interesting by itself, will be useful in order to weaken the hypotheses of proposition 3.2.5 in a forthcoming section.

To prove theorem 3.3.11 we will proceed in several steps. First, in lemma 3.3.8, it will be shown that points in a 3–dimensional spacetime, locally connected by timelike geodesic can also be connected by a twisted null curve. Next, in lemma 3.3.9, we will extend the same statement for any m–dimensional spacetime with $m \geq 3$. In proposition 3.3.10 we will show that the local connection by twisted null curves can be done by piecewise twisted null curves at the large. Finally, Twisted null curve theorem 3.3.11 permits the required characterization of the causality of M in terms of piecewise twisted null curves.

Lemma 3.3.8. *Let M be a 3–dimensional spacetime and $\gamma\colon I \to M$ be a future–directed timelike geodesic. Then there exists $\delta > 0$ such that for any $t \in (t_0, t_0 + \delta]$, there exists a future–directed twisted null curve μ joining $\gamma(t_0)$ to $\gamma(t)$.*

Proof. Given the future–directed timelike geodesic $\gamma\colon I \to M$ and $t_0 \in I$, it is known, e. g. by [34, sec. 97] and [55, def. 7.13], that there exists a synchronous coordinate system $(U, \phi = (t,x,y))$ with $\gamma(t_0) \subset U$ in which the metric $\mathbf{g}$ of M can be written as

$$(g_{ij}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & g_{11} & g_{12} \\ 0 & g_{12} & g_{22} \end{pmatrix}$$

where $g_{ij} \equiv g_{ij}(t,x,y)$ for $i,j = 1,2$, U is contained in a convex normal neighbourhood and the expression of the geodesic γ in these coordinates is $\phi(\gamma(s)) = (s,0,0) \in \mathbb{R}^3$. For a point $\gamma\left(\overline{t}\right) \in U$, it is possible to find $R > 0$ such that the compact set

$$U_0 = \left\{(t, x, y) : x^2 + y^2 \leq R^2, t_0 \leq t \leq \overline{t}\right\}$$

is contained in U.

As candidates for the required twisted null curve, we will study curves μ_r such that

$$\phi\left(\mu_r\left(s\right)\right) = \left(f_r\left(s\right), r\left(1 - \cos s\right), r \sin s\right)$$

where $0 \leq r \leq R/2$ and $f_r = f_r\left(s\right)$ is a function. If μ_r is a null curve, then $\mathbf{g}\left(\mu_r', \mu_r'\right) = 0$ and therefore

$$-\left(f_r'\left(s\right)\right)^2 + r^2 g_{11} \sin^2 s + 2r^2 g_{12} \sin s \cos s + r^2 g_{22} \cos^2 s = 0$$

where $g_{ij} = g_{ij}\left(\phi\left(\mu_r\left(s\right)\right)\right)$. Thus, we have a first order ordinary differential equation which describes a null curve passing through $\gamma\left(t_0\right)$

$$\begin{cases} f_r'\left(s\right) = r\sqrt{g_{11} \sin^2 s + 2g_{12} \sin s \cos s + g_{22} \cos^2 s} \\ f_r\left(0\right) = t_0 \end{cases} \tag{3.3.2}$$

Since the metric in the hypersurfaces $\{t = c\}$ with $t_0 \leq c \leq \overline{t}$ is positive definite, then the term under the square root in (3.3.2) is always positive. Moreover, since $f_r' > 0$ then μ_r is future.

Let us show that we can find $r > 0$ such that μ_r is twisted. A simple calculation gives

$$(d\phi)_{\mu_r(s)}\left(\frac{D\mu_r'}{ds}\left(s\right)\right) = \left(f_r'' + r^2\varphi_0\left(r, s\right), r\cos s + r^2\varphi_1\left(r, s\right), -r\sin s + r^2\varphi_2\left(r, s\right)\right),$$

where $\varphi_i = \varphi_i\left(r, s\right)$, $i = 0, 1, 2$, are continuous functions in U depending on the Christoffel symbols and the components of μ_r'. In order to show that $\frac{D\mu_r'}{ds}$ and μ_r' are linearly independent, it is enough to see that the determinant of their components x, y does not cancel out, so

$$\begin{vmatrix} r\cos s + r^2\varphi_1\left(r, s\right) & r \sin s \\ -r\sin s + r^2\varphi_2\left(r, s\right) & r\cos s \end{vmatrix} = r^2\left(1 + r\left(\varphi_1\left(r, s\right)\cos s + \varphi_2\left(r, s\right)\sin s\right)\right)$$

hence, since φ_1 and φ_2 are continuous in U, they are also bounded in the compact set U_0 and there exists $r_0 \leq R/2$ such that

$$1 + r\left(\varphi_1\left(r, s\right)\cos s + \varphi_2\left(r, s\right)\sin s\right) \neq 0$$

for all $r \in (0, r_0]$, and in this case, $\frac{D\mu_r'}{ds}$ and μ_r' are linearly independent.

At this moment, we have seen that μ_r is a twisted null curve passing through $\gamma\left(t_0\right)$ for $0 < r \leq r_0$, and it remains to show that there exists $\delta > 0$ such that for every $t \in (t_0, t_0 + \delta]$ there is $r \in (0, r_0]$ verifying μ_r also passes through $\gamma\left(t\right)$.

Now, we want to prove that for every $r \in (0, r_0]$ there exists $s_r > 0$ such that $f_r\left(s_r\right) = \overline{t}$. Given $r \in (0, r_0]$, we define $\omega_r = \sup\left\{s : f_r\left(s\right) \text{ exists}\right\}$. Let us assume that $\lim\limits_{s \mapsto \omega_r} f_r\left(s\right) = c \leq \overline{t}$. In case of $\omega_r < +\infty$, the solution $\overline{f}_r$ of equation (3.3.2) verifying the initial condition $\overline{f}_r\left(\omega_r\right) = c$ would coincide with $f_r = f_r\left(s\right)$ for $s < \omega_r$ contradicting the maximality of f_r up to ω_r because in that case f_r could be extended beyond $s = \omega_r$. On the other hand, if $\omega_r = +\infty$, the derivability of f_r would imply that $\lim_{s\mapsto+\infty} f_r'(s) = 0$ and hence the curve solution μ_r would approximate to the curve β_r verifying

$$\beta_r(s) = (c, r(1-\cos s), r\sin s) \in U_0$$

in TM, i.e. for every $s_0 \in \mathbb{R}$ the sequence $\{s_n = s_0 + 2\pi n\}_{n\in\mathbb{N}}$ would verify

$$\lim_{n\mapsto+\infty} \mu_r(s_n) = \beta_r(s_0) \qquad \text{and} \qquad \lim_{n\mapsto+\infty} \mu_r'(s_n) = \beta_r'(s_0)$$

By the continuity of the metric $\mathbf{g}$ then we have

$$\lim_{n\mapsto+\infty} \mathbf{g}(\mu_r'(s_n), \mu_r'(s_n)) = \mathbf{g}(\beta_r'(s_0), \beta_r'(s_0)) \neq 0$$

since β_r is contained in the spacelike hypersurface $\{t = c\}$, but this contradicts that $\mathbf{g}(\mu_r', \mu_r') = 0$. Therefore, independently from ω_r, for every $r \in (0, r_0]$ we have that $\lim_{s\mapsto\omega_r} f_r(s) > \overline{t}$ and hence, for all $r \in (0, r_0]$ there exists $s_r \in (0, \omega_r)$ such that $f_r(s_r) = \overline{t}$.

Since the functions g_{ij} are continuous in U for $i,j = 1,2$, then their restrictions to the compact set U_0 reach their maximum, therefore there exists $M_{ij} > 0$ such that $|g_{ij}(t,x,y)| \leq M_{ij}$ for $(t,x,y) \in U_0$. Then,

$$\begin{aligned} 0 < f_r'(s) &= r\sqrt{g_{11}\sin^2 s + 2g_{12}\sin s\cos s + g_{22}\cos^2 s} \leq \\ &\leq r\sqrt{|g_{11}\sin^2 s| + 2|g_{12}\sin s\cos s| + |g_{22}\cos^2 s|} \leq \\ &\leq r\sqrt{M_{11} + 2M_{12} + M_{22}} = rM \end{aligned}$$

where $M = \sqrt{M_{11} + 2M_{12} + M_{22}} \in \mathbb{R}$ is independent from r and s. So integrating, we have that $t_0 \leq f_r(s) \leq rMs + t_0$ and therefore

$$\overline{t} = f_r(s_r) \leq rMs_r + t_0 \tag{3.3.3}$$

that implies

$$\frac{\overline{t} - t_0}{rM} \leq s_r \tag{3.3.4}$$

then there exists $\rho \in (0, r_0]$ small enough such that $s_r \geq 2\pi$ for all $r \in (0, \rho]$ and hence the parameter s of f_r can be extended beyond $s = 2\pi$ for all $r \in (0, \rho]$. Since $f_\rho'(s) > 0$ then $f_\rho(s) > t_0$ for all $s > 0$, therefore there exists $\delta > 0$ such that $f_\rho(2\pi) = t_0 + \delta$. So, by the inequality (3.3.3) we have

$$t_0 \leq f_r(2\pi) \leq 2\pi rM + t_0$$

hence $\lim_{r\mapsto 0} f_r(2\pi) = t_0$ and for every $t \in (t_0, t_0 + \delta]$ there exists $r \in (0, \rho]$ such that

$$\mu_r(0) = (t_0, 0, 0) = \phi(\gamma(t_0))$$

$$\mu_r(2\pi) = (f_r(2\pi), 0, 0) = (t, 0, 0) = \phi(\gamma(t))$$

therefore we have shown that there exists $\delta > 0$ such that for every $t \in (t_0, t_0 + \delta]$ the points $\gamma(t_0)$ and $\gamma(t)$ can be connected by some future–directed twisted null curve μ_r. Analogously, this construction can be done to obtain a future–directed twisted null curve joining $\gamma(t)$ to $\gamma(t_0)$ for all $t \in [t_0 - \delta, t_0)$. □

Lemma 3.3.9. *The statement of Lemma 3.3.8 is true in a m–dimensional spacetime M.*

Proof. We can find a synchronous coordinate system (U, ϕ) with $\phi = (t, x_1, \ldots, x_{m-1})$ (as done previously) such that the expression of the geodesic γ in these coordinates is $\phi(\gamma(s)) = (s, 0, \ldots, 0) \in \mathbb{R}^m$, so this chart is adapted to γ. Consider the restriction

$$V = \{(t, x_1, \ldots, x_{m-1}) : x_i = 0, i = 3, \ldots, m-1\} \subset \phi(U)$$

then $N = \phi^{-1}(V) \subset M$ is a 3–dimensional manifold embedded in M. Moreover, by [53, Lem. 4.3] we have that Levi-Civita connection in N coincides with the orthogonal projection over N of the Levi-Civita connection in M, hence we have $\frac{D^N}{ds} = \tan\left(\frac{D}{ds}\right)$ where $\frac{D^N}{ds}$ and $\frac{D}{ds}$ denote the covariant derivatives in N and M respectively. So the geodesics in M contained in N are also geodesics in N and the restriction $(N, \phi|_N = (t, x_1, x_2))$ of the synchronous coordinate system is still a synchronous coordinate system for N. Then, since γ is a geodesic contained in N, by lemma 3.3.8, there exists $\delta > 0$ and a future–directed twisted null curve $\mu \subset N$ such that μ joins $\gamma(t_0)$ to $\gamma(t_0 + \delta)$. Since the metric in N is the restriction of the metric in M, then μ as curve in M is also null. Finally, since μ' and $\frac{D^N \mu'}{ds} = \tan\left(\frac{D\mu'}{ds}\right)$ are lineally independent in $T_{\mu(s)}N$ then, it is immediate that μ' and $\frac{D\mu'}{ds}$ are lineally independent in $T_{\mu(s)}M$. Therefore, we have shown that there exists $\delta > 0$ and μ a future–directed twisted null curve in M joining $\gamma(t_0)$ to $\gamma(t_0 + \delta)$. $\square$

A direct consequence of lemmas 3.3.8 and 3.3.9 is the following proposition.

Proposition 3.3.10. *Let $\gamma : I \to M$ be a future–directed timelike geodesic. Then, for any $t_0, t_1 \in I$, there exists a future–directed piecewise twisted null curve μ joining $\gamma(t_0)$ to $\gamma(t_1)$.*

Proof. By lemma 3.3.9, for all $t \in [t_0, t_1]$ there exists an open interval $I_t = [t - \delta_t, t + \delta_t] \subset [t_0, t_1]$ relative to $[t_0, t_1]$ such that $\gamma(t)$ can be joined to $\gamma(u)$ with $u \in I_t$ by means of a piecewise twisted null curve. By the compactness of $[t_0, t_1]$, we can extract a finite covering $\{I_n\}_{n=1,\ldots,N}$ such that, with no lack of generality, verifies $I_i \cap I_k \neq \varnothing \Leftrightarrow k = i \pm 1$. We can choose a partition

$$\{t_0 = a_1 < b_1 < \cdots < a_{N-1} < b_{N-1} < a_N = t_1\}$$

such that $a_i \in I_i$ and $b_i \in I_i \cap I_{i+1}$ and therefore there exists future–directed twisted null curves joining $\gamma(a_i)$ to $\gamma(b_i)$ and $\gamma(b_i)$ to $\gamma(a_{i+1})$ for $i = 1, \ldots, N-1$. The union of these curves forms a future–directed piecewise twisted null curve connecting $\gamma(t_0)$ to $\gamma(t_1)$. $\square$

Finally, we can proceed with theorem 3.3.11.

Theorem 3.3.11 (Twisted null curve theorem)**.** *Let $p, q \in M$ such that $q \in I^+(p)$, then there exists a future–directed piecewise twisted null curve μ joining p to q.*

Proof. Consider $p, q \in M$ such that $q \in I^+(p)$, then there exists a continuous future–directed timelike curve λ connecting p and q. By compactness of λ between p and q, there exists a finite covering $\{W_k\}_{k=1,\ldots,K}$ of globally hyperbolic open sets contained in convex normal neighbourhoods, then it is possible to built a continuous curve γ joining p and q formed by segments $\gamma_k \subset W_k$ of future–directed timelike geodesics with endpoints at λ. So γ becomes a future–directed piecewise timelike geodesic.

By proposition 3.3.10, the endpoints of the timelike geodesic segments γ_k of γ can be connected by a future–directed piecewise twisted null curve μ_k. Since γ is continuous, we can paste all μ_k to obtain another piecewise twisted null curve μ joining p and q. □

As an immediate corollary of theorem 3.3.11 and the causality theorem 1.2.4 we have the following result.

Corollary 3.3.12. *$q \in I^+(p)$ if and only if there exists a future–directed piecewise twisted null curve μ joining p to q.*

Observe that all the results of this section are valid for any spacetime M without any further hypotheses.

Section 3.4

Causality in Σ and legendrian isotopies

According to the previous corollary 3.3.12 and the μ–lemma 3.3.7, we can translate the causal character of curves in M into curves of skies (as legendrian submanifolds of $\mathcal{N}$) signed by the 1–form of equation (2.4.23) defining the cooriented contact structure $\mathcal{H}$ of $\mathcal{N}$.

To achieve this purpose, we need to introduce some background about contact geometry that will be related to causality properties of spacetimes.

Let $(Y, \mathcal{H})$ be a co-oriented $(2n-1)$–dimensional contact manifold with contact distribution $\mathcal{H} = \ker \alpha$ where $\alpha \in T^*Y$ is a contact 1–form defining the co-orientation. A differentiable family $\{\Lambda_s\}_{s\in[0,1]}$ of legendrian submanifolds is called a *legendrian isotopy*. It is possible to describe a legendrian isotopy by a parametrization $F : \Lambda_0 \times [0,1] \to Y$ verifying $F(\Lambda_0 \times \{s\}) = \Lambda_s \subset Y$ where $s \in [0,1]$. Notice that we are assuming that the map $F_s \colon \Lambda_0 \to \Lambda_s$, given by $F_s(\lambda) = F(\lambda, s)$ is a diffeomorphism for all $s \in [0,1]$.

Definition 3.4.1. *A parametrization F of a legendrian isotopy is said to be* non–negative *if $(F^*\alpha)\left(\frac{\partial}{\partial s}\right) \geq 0$ and* non–positive *if $(F^*\alpha)\left(\frac{\partial}{\partial s}\right) \leq 0$.*

Definition 3.4.2. *We will say that two legendrian isotopies are* equivalent *if their corresponding parametrizations $F, \widetilde{F} : \Lambda_0 \times [0,1] \to Y$ verify $F(\Lambda_0 \times \{s\}) = \widetilde{F}(\Lambda_0 \times \{s\})$ for every $s \in [0,1]$.*

Next lemma shows that the sign of signed legendrian isotopies is independent of the parametrization.

Lemma 3.4.3. *Let $F, \widetilde{F} : \Lambda_0 \times [0,1] \to Y$ be two parametrizations of a legendrian isotopy $\{\Lambda_s\}_{s\in[0,1]}$. If F is non-negative (respectively non-positive) then so is $\widetilde{F}$.*

Proof. Let us consider a legendrian isotopy $\{\Lambda_s\}_{s\in[0,1]}$ given by two parametrizations $F, \widetilde{F} : \Lambda_0 \times [0,1] \to Y$. Let us define the maps $F_s, \widetilde{F}_s : \Lambda_0 \to \Lambda_s \subset Y$ for $s \in [0,1]$ by $F_s(\lambda) = F(\lambda, s)$ as before. Then we have that

$$F(\lambda, s) = \widetilde{F}(\varphi(\lambda, s), s)$$

where $\varphi(\lambda, s) = \widetilde{F}_s^{-1} \circ F(\lambda, s)$. To check that φ is differentiable, consider the differentiable map $\Upsilon : \Lambda_0 \times [0,1] \to \mathcal{N} \times [0,1]$ defined by $\Upsilon(z, s) = \left(\widetilde{F}(z, s), s\right)$ whose differential at any (z, s) is given by:

$$d\Upsilon_{(z,s)} = \begin{pmatrix} d\widetilde{F}_{(z,s)} \\ \mathrm{Id}_s \end{pmatrix} = \begin{pmatrix} \left(d\widetilde{F}_s\right)_z & * \\ 0 & \mathrm{Id}_s \end{pmatrix}$$

and since $\widetilde{F}_s$ is a diffeomorphism, then $(d\Upsilon)_{(z,s)}$ is an isomorphism, therefore by the inverse function theorem, Υ is a local difeomorphism onto its image in (z, s) and φ can be written locally as:

$$\varphi(z, s) = \pi \circ \Upsilon^{-1}(F(z, s), s)$$

where $\pi : \Lambda_0 \times [0,1] \to \Lambda_0$ is the canonical projection.

Defining $\phi : \Lambda_0 \times [0,1] \to \Lambda_0 \times [0,1]$ as $\phi(\lambda, s) = (\varphi(\lambda, s), s)$, we have

$$\begin{aligned} dF_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} &= d\left(\widetilde{F} \circ \phi\right)_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} \\ &= d\widetilde{F}_{(\varphi(\lambda,s),s)}\left(d\phi_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right) \\ &= d\widetilde{F}_{(\varphi(\lambda,s),s)}\left(\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)} + d\varphi_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right). \end{aligned} \tag{3.4.1}$$

Notice that, since $d\varphi_{(\lambda,s)}(\partial/\partial s) \in T_{\varphi(\lambda,s)}\Lambda_0$, then

$$d\widetilde{F}_{(\varphi(\lambda,s),s)} d\varphi_{(\lambda,s)}(\partial/\partial s) \in T_{(\varphi(\lambda,s),s)}\Lambda_s \subset \mathcal{H}$$

and therefore

$$\alpha\left(d\widetilde{F}_{(\varphi(\lambda,s),s)} d\varphi_{(\lambda,s)}(\partial/\partial s)\right) = 0$$

Now, applying α to both sides of equation (3.4.1) we get:

$$\alpha\left(dF_{(\lambda,s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right) = \alpha\left(d\widetilde{F}_{(\varphi(\lambda,s),s)}\left(\frac{\partial}{\partial s}\right)_{(\lambda,s)}\right)$$

hence

$$(F^*\alpha)\left(\frac{\partial}{\partial s}\right) = \alpha\left(F_*\left(\frac{\partial}{\partial s}\right)\right) = \alpha\left(\widetilde{F}_*\left(\frac{\partial}{\partial s}\right)\right) = \left(\widetilde{F}^*\alpha\right)\left(\frac{\partial}{\partial s}\right)$$

therefore the sign of the parametrizations F and $\widetilde{F}$ coincides. $\square$

Recall that, in case of $Y = \mathcal{N}$ the space of light rays of a conformal manifold $(M, \mathcal{C})$, the co–orientation is defined by using the criterion that the sign of $J\,(\mathrm{mod}\gamma') \in T_\gamma \mathcal{N}$ is the sign of $\mathbf{g}\,(J, \gamma')$, which is unambiguously determined for vectors J in the class $[J] = J + \mathcal{J}_0\,(\gamma)$, where $\mathbf{g} \in \mathcal{C}$ and $\gamma \in \mathcal{N}$ is suitably parametrized according to equation (2.4.17).

As said at the beginning of chapter 3, for any $x_0 \in M$ the sky $X_0 = S(x_0) \in \Sigma$ is a legendrian submanifold of $\mathcal{N}$ diffeomorphic to $S_0 = \{[u] : u \in \mathbb{N}^+_{x_0}\} = \mathbb{PN}^+_{x_0} \simeq \mathbb{S}^{m-2}$, then given a legendrian isotopy $\{X_s\}_{s\in[0,1]}$ where X_s is the sky of $x_s \in M$ for $s \in [0,1]$, a parametrization F for it can be found of the form $F : S_0 \times [0,1] \to \mathcal{N}$ as we show in the next lemma.

Lemma 3.4.4. *Any differentiable curve $\mu : [0,1] \to M$ defines a legendrian isotopy parametrized by the function $F^\mu : S_0 \times [0,1] \to \mathcal{N}$ given by:*

$$F^\mu\,([u]\,, t) = \gamma_{[u_s]}$$

with $S_0 = \{[u] : u \in \mathbb{N}^+_{\mu(0)}\}$ and $u_s \in \mathbb{N}^+_{\mu(s)}$ the parallel transport of $u \in \mathbb{N}^+_{\mu(0)}$ along γ. Moreover F^μ is a legendrian isotopy of skies and $F^\mu_s(S_0) = S(\mu(s))$.

Proof. Let $\mathbf{g} \in \mathcal{C}$ be a metric in M and let $\mathcal{P} : T_{\mu(0)}M \times [0,1] \to TM$ be the parallel transport with respect to the Levi–Civita connection defined by $\mathbf{g}$ along μ given by $\mathcal{P}\,(u, s) = u_s \in T_{\mu(s)}M$. It is widely known that $\mathcal{P}$ is differentiable and the map $\mathcal{P}_s : T_{\mu(0)}M \to T_{\mu(s)}M$ defined by $\mathcal{P}_s\,(u) = \mathcal{P}\,(u, s)$ is a linear isometry. Let us also consider the submersion $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ given by $p_{\mathbb{N}^+}\,(u) = \gamma_{[u]}$. By composition of differentiable maps, $p_{\mathbb{N}^+} \circ \mathcal{P}$ is differentiable and due to the linearity of $\mathcal{P}$ it induces a map F^μ on the quotient space $\mathbb{PN}^+$.

Moreover, since $\mathcal{P}_s$ is a linear isometry, then

$$\mathbf{g}\,(u_s, u_s) = \mathbf{g}\,(u, u) = 0$$

for every $u \in \mathbb{N}^+$ and any metric $\mathbf{g} \in \mathcal{C}$, therefore $u_s \in \mathbb{N}^+_{\mu(s)}$ and $\mathcal{P}_s\left(\mathbb{N}^+_{\mu(0)}\right) = \mathbb{N}^+_{\mu(s)}$. For $s \in [0,1]$ we have

$$\begin{aligned} F^\mu\,(S_0 \times \{s\}) &= \{F^\mu\,([u]\,, s) \in \mathcal{N} : u \in \mathbb{N}^+_{\mu(0)}\} = \{\gamma_{[u_s]} \in \mathcal{N} : u \in \mathbb{N}^+_{\mu(0)}\} = \\ &= \{\gamma_{[v]} \in \mathcal{N} : v \in \mathbb{N}^+_{\mu(s)}\} = S\,(\mu\,(s)) \end{aligned}$$

Hence, F^μ is a legendrian isotopy. □

A converse result of lemma 3.4.4 can be the following one.

Lemma 3.4.5. *Let $F : S_0 \times [0,1] \to \mathcal{N}$ be a legendrian isotopy such that $F\,(S_0 \times \{s\}) = S\,(\mu\,(s)) \in \Sigma$. Then the curve $\mu : [0,1] \to M$ is differentiable and F is equivalent to F^μ.*

Proof. Let us define the map $F_s : S_0 \to S\,(\mu\,(s)) \subset \mathcal{N}$ given by $F_s\,(z) = F\,(z, s)$ for $s \in [0,1]$. It is clear that F_s is differentiable for any $s \in [0,1]$. Now, take any $z_0 \in S_0$ and $\xi \in T_{z_0}S_0$. Since F and F_s are differentiable maps, then the curve

$$j\,(s) = (dF_s)_{z_0}\,(\xi) \in T_{F(z_0,s)}S\,(\mu\,(s))$$

is also differentiable in $\widehat{T\mathcal{N}}$ and $j(s)$ is a Jacobi field along the null geodesic $F(z_0, s) \in \mathcal{N}$ for each $s \in [0,1]$. Let $s_0 \in [0,1]$ and $V \subset M$ a basic neighbourhood of $\mu(s_0)$ such that $U = S(V)$. Consider coordinate charts $\left(\widehat{U}, \overline{\varphi} = (x,u,v)\right)$ and $(V, \varphi = x)$ as in theorem 3.2.8. Then, since j is differentiable and, by theorem 3.2.8, $\widehat{U}$ is regular submanifold of $\widehat{T\mathcal{N}}$ with $j(s_0) \in \widehat{U}$, then we conclude that $j(s) \in \widehat{U}$ for s close to s_0, and since μ can be written locally at $\mu(s_0)$ as composition of differentiable maps

$$\mu(s) = \varphi^{-1} \circ x(j(s)) \in V$$

therefore μ is differentiable. □

Now, we need a simple result on the geometry of causal vectors on Lorentz manifolds that we state as the following technical lemma.

Lemma 3.4.6. *Let M be a Lorentz manifold and $p \in M$. If $v \neq 0$ is a vector in T_pM verifying $\mathbf{g}(u,v) \geq 0$ for any $u \in \mathbb{N}_p^+$ future, then v is causal past.*

Proof. First, we will see that if $v \in T_pM$ is spacelike, then there exists $u \in \mathbb{N}_p^+$ verifying $\mathbf{g}(u,v) < 0$. So, let $v \in T_pM$ be spacelike and take some $z \in T_pM$ timelike future, then since $\mathbf{g}(z,z) < 0$ and $\mathbf{g}(v,v) > 0$, the equation

$$\mathbf{g}(z+\lambda v, z+\lambda v) = \mathbf{g}(z,z) + 2\lambda \mathbf{g}(z,v) + \lambda^2 \mathbf{g}(v,v) = 0$$

has two solutions λ_1, λ_2 due to $(2\mathbf{g}(z,v))^2 - 4\mathbf{g}(z,z)\,\mathbf{g}(v,v) > 0$. These solutions can be written as

$$\lambda_1 = -\frac{\mathbf{g}(z,v)}{\mathbf{g}(v,v)} + \sqrt{\frac{\mathbf{g}(z,v)^2}{\mathbf{g}(v,v)^2} - \frac{\mathbf{g}(z,z)}{\mathbf{g}(v,v)}}$$
$$\lambda_2 = -\frac{\mathbf{g}(z,v)}{\mathbf{g}(v,v)} - \sqrt{\frac{\mathbf{g}(z,v)^2}{\mathbf{g}(v,v)^2} - \frac{\mathbf{g}(z,z)}{\mathbf{g}(v,v)}}$$

For $i = 1,2$, let $u_i = z + \lambda_i v$ be the corresponding null vectors. We have that

$$\mathbf{g}(u_i, v) = \mathbf{g}(z,v) + \lambda_i \mathbf{g}(v,v) = (-1)^{i+1}\,\mathbf{g}(v,v)\sqrt{\frac{\mathbf{g}(z,v)^2}{\mathbf{g}(v,v)^2} - \frac{\mathbf{g}(z,z)}{\mathbf{g}(v,v)}}$$

hence $\mathbf{g}(u_2, v) < 0$.

Let us see now that u_2 is null future. Since

$$\mathbf{g}(u_1, u_2) = 2\left[\mathbf{g}(z,z) - \frac{\mathbf{g}(v,z)^2}{\mathbf{g}(v,v)}\right] < 0$$

therefore u_1 and u_2 are in the same lightcone. Moreover

$$\mathbf{g}(u_i, z) = \mathbf{g}(v,v)\left[\frac{\mathbf{g}(z,z)}{\mathbf{g}(v,v)} - \frac{\mathbf{g}(z,v)^2}{\mathbf{g}(v,v)^2}\right] \pm \sqrt{\frac{\mathbf{g}(z,v)^2}{\mathbf{g}(v,v)^2} - \frac{\mathbf{g}(z,z)}{\mathbf{g}(v,v)}}\,\mathbf{g}(z,v)$$

with the positive sign corresponding to $i = 1$ and the negative to $i = 2$. It can be observed that if $\mathbf{g}(z,v) > 0$ then $\mathbf{g}(u_2, z) < 0$ therefore u_2 is in the same lightcone of z, hence u_2

is null future. In case of $\mathbf{g}(z,v)<0$ we have that $\mathbf{g}(u_1,z)<0$, then u_1 (and also u_2) is in the same lightcone of z, therefore u_1 and u_2 are null future.

At this point, we have proven the equivalent result: If for any $u\in T_pM$ null future $\mathbf{g}(u,v)\geq 0$ is verified, then $v\in T_pM$ is causal. But if v is causal future, then $\mathbf{g}(u,v)\leq 0$, hence $v=0$ contradicting the hypothesis, therefore v must be causal past. □

The time–orientation of any causal curve is related to the sign of the legendrian isotopy it defines as we show in the following proposition.

Proposition 3.4.7. *The curve μ is causal past–directed (respectively causal future–directed) if and only if F^{μ} is a non–negative (respectively non–positive) legendrian isotopy.*

Proof. Let us suppose that μ is causal past–directed. Since $F^{\mu}([u],s)=\gamma_{[u_s]}$ then giving a geodesic parameters to the light ray $\gamma_{[u_s]}$ we can write

$$F^{\mu}([u],s)(t)=\gamma_{[u_s]}(t)=\exp_{\mu(s)}(tu_s)$$

which is a null geodesic variation of the light ray $\gamma_{\left[u_{s_0}\right]}$ for every $s_0\in[0,1]$. By lemma 2.3.5, we have that the Jacobi field $J_{s_0}(t)$ defined by this geodesic variation verifies that $J_{s_0}(0)=\mu'(s_0)$ and $J'_{s_0}(0)=\frac{D}{ds}\big|_{s=s_0}u_s$, and since u_s is the parallel transport of u along μ, then $J'_{s_0}(0)=0$. Hence, since

$$F^{\mu}_{*}\left(\frac{\partial}{\partial s}\right)_{([u],s_0)}=\left.\frac{\partial}{\partial s}\right|_{([u],s_0)}F^{\mu}([u],s)=\left.\frac{\partial}{\partial s}\right|_{(s_0,t)}\left(\exp_{\mu(s)}(tu_s)\right)=J_{s_0}(t)$$

we have that

$$\alpha\left(F^{\mu}_{*}\left(\frac{\partial}{\partial s}\right)\right)_{([u],s_0)}=\alpha\left(J_{s_0}(t)\right)=\mathbf{g}\left(J_{s_0}(t),\gamma'_{\left[u_{s_0}\right]}(t)\right)=$$

$$=\mathbf{g}\left(J_{s_0}(0),\gamma'_{\left[u_{s_0}\right]}(0)\right)=\mathbf{g}\left(\mu'(s_0),u_{s_0}\right)\geq 0$$

since $\mu'(s_0)$ is causal past where it does not vanish and u_{s_0} is null future. This shows that F^{μ} is a non-negative legendrian isotopy.

Now, let us suppose that F^{μ} is non-negative. So, we have as before

$$F^{\mu}([u],s)(t)=\gamma_{[u_s]}(t)=\exp_{\mu(s)}(tu_s)$$

then if $\alpha\left(F^{\mu}_{*}\left(\frac{\partial}{\partial s}\right)\right)_{([u],s_0)}\geq 0$ for any $([u],s_0)$, we have that

$$0\leq\alpha\left(F^{\mu}_{*}\left(\frac{\partial}{\partial s}\right)\right)_{([u],s_0)}=\mathbf{g}\left(\mu'(s_0),u_{s_0}\right).$$

Then because of lemma 3.4.6 we obtain that $\mu'(s_0)$ is causal past provided that $\mu'(s_0)\neq 0$ with $s_0\in[0,1]$. □

Now, we get the following relation between causal curve and legendrian isotopies.

Corollary 3.4.8. *A legendrian isotopy of skies $\{S(\mu(s))\}_{s\in[0,1]}$ is non-negative if and only if the curve $\mu : [0,1] \to M$ is causal past–directed.*

Proof. By lemma 3.4.5, a legendrian isotopy of skies $F : S_0 \times [0,1] \to \mathcal{N}$ defines a differentiable curve $\mu : [0,1] \to M$ such that F is equivalent to F^{μ}. By lemma 3.4.3, F^{μ} is non–negative, then proposition 3.4.7 shows that every regular segment of μ is causal past–directed, therefore μ is causal past–directed because is the union of causal past–directed segments. □

The previous result permits to transmit the causality of M to Σ. Any causal curve $\mu : [0,1] \to M$ defines a legendrian isotopy of skies $F^{\mu} : S_0 \times [0,1] \to \mathcal{N}$. Since $F^{\mu}_s(S_0) = S(\mu(s))$, then we can define a curve of skies $\chi : [0,1] \to \Sigma$ given by $\chi(s) = F^{\mu}_s(S_0) = S(\mu(s))$. Using corollary 3.4.8, it is possible to define a partial order $\leq_{\Sigma}$ in Σ induced by the causal relation $\leq$ in M in such a way

$$x \leq y \Longleftrightarrow X \leq_{\Sigma} Y$$

where $X = S(x)$ and $Y = S(y)$, that is, $X \leq_{\Sigma} Y$ if and only if there is a non–positive legendrian isotopy of skies $F : S_0 \times [0,1] \to \mathcal{N}$ such that $F_0(S_0) = X$ and $F_1(S_0) = Y$.

Section 3.5

Regular sets and differentiable structure in the space of skies

We will need suitable neighbourhoods in Σ to define a smooth atlas, because null non–conjugated neighbourhoods are not good enough to construct coordinated charts.

We have been obtained bases for the topology on the space of skies Σ in section 3.2 by selecting the family $\mathfrak{R}$ introduced in equation (3.2.5). Now, we will propose a refinement of the properties of $\mathfrak{R}$ to obtain the same topology in Σ than $\mathfrak{R}$ does. We will call such neighbourhoods *regular neighbourhoods*. Working with regular neighbourhoods will permit us to define a differentiable structure in Σ and to weaken the hypothesis of being non–refocusing to being just sky–separating in order to show the statement of previous proposition 3.2.5.

First, let us introduce the properties needed to define regular neighbourhoods.

Let $W \subset \Sigma$ be a non-empty set satisfying the conditions:

1. W is null non–conjugated and

$$\widehat{W} = \bigcup_{X\in W} \widehat{T}X \subset \widehat{T}\mathcal{N}$$

is a regular $(3m-4)$–dimensional submanifold of $\widehat{T}\mathcal{N}$.

2. Let $\widehat{\mathcal{D}}$ be the distribution in $\widehat{W}$ whose leaves are $\widetilde{X} = \widehat{T}X$. Then the space of leaves $\widetilde{W} = \left\{\widetilde{X} : X \in W\right\} = \widehat{W}/\widehat{\mathcal{D}}$ is a differentiable quotient manifold.

It is clear that in this case, $\widetilde{W}$ can be identified with W via the bijective map

$$\begin{array}{cccc} \Theta: & W & \rightarrow & \widetilde{W} \\ & X & \mapsto & \widetilde{X} \end{array} \tag{3.5.1}$$

and hence W inherits the quotient topology such that

$$U \subset W \text{ is open} \Leftrightarrow \widehat{U} = \bigcup_{X\in U} \widehat{T}X \subset \widehat{W} \text{ is open},$$

and also a differentiable structure from $\widetilde{W}$. So, we will denote W equipped with the previous structure as $W^{(\sim)} \simeq \widetilde{W}$.

3. For every $X_0 \in W$ and every celestial curve $\Gamma : I_\epsilon \to \mathcal{N}$ such that $\Gamma'(0) \in \widehat{T}X_0$,

   (a) there exists $0 < \delta \in I_\epsilon$ such that $\Gamma'((-\delta, \delta)) \subset \widehat{W}$.

   (b) the curve $\chi^{\Gamma}_{X_0} : I_\delta \to W^{(\sim)}$ defined in lemma 3.3.7 is differentiable.

4. Given $\widetilde{X}, \widetilde{Y} \in \widetilde{W}$, for any causal curve $\chi\colon [a,b] \to \Sigma$, joining X and Y, then $\chi(s) \in W$ for all $s \in [a,b]$.

Now we are ready to state the next definition:

Definition 3.5.1. *A not–empty subset $W \subset \Sigma$ is said to be a* regular set*, and denoted by $W \subset_{\mathrm{reg}} \Sigma$, if it verifies conditions (1) to (4) above.*

It is important to observe that both, the definition of regular subset and the differentiable structure of $W^{(\sim)} \simeq \widetilde{W}$, depend only on $\mathcal{N}$ and Σ.

Next, let us show that the class of regular subsets in Σ is not empty.

Proposition 3.5.2. *Let $V \subset M$ be a relatively compact basic open set, then $U = S(V) \subset_{\mathrm{reg}} \Sigma$ is regular. Moreover, $S\colon V \to U^{(\sim)}$ is a diffeomorphism.*

Proof. Let $V \subset M$ be a relatively compact basic open set, since V is contained in a convex normal neighbourhood, then trivially $\widehat{T}X \cap \widehat{T}Y = \varnothing$ for all $X \neq Y \in U$. Moreover, by theorem 3.2.8 then $\widehat{U}$ is a regular manifold of $\widehat{T}\mathcal{N}$. Hence, condition (1) is verified.

In order to prove condition (2), observe first that any $X \in U$ is a regular submanifold of $\mathcal{N}$, therefore $\widehat{T}X$ is a regular submanifold of $\widehat{T}\mathcal{N}$. Denote $\widetilde{U} = \{\widetilde{X} = \widehat{T}X : X \in U\}$ and define the map $\widetilde{S} : V \to \widetilde{U}$ given by $\widetilde{S}(x) = \widetilde{S(x)}$. Since $\widehat{U}$ is a regular submanifold of $\widehat{T}\mathcal{U}$ which is an open set of $\widehat{T}\mathcal{N}$ and since $\widehat{T}X \cap \widehat{T}Y \neq \varnothing$ for all $X \neq Y \in U$, then we have that $\widehat{U}$ is foliated by $\{\widehat{T}X : X \in U\}$, i.e. by $\widetilde{U}$. Denoting the distribution induced by that foliation as $\widehat{\mathcal{D}}$, we have that $\widetilde{U} = \widehat{U}/\widehat{\mathcal{D}}$ inherits a smooth structure because the chart $\overline{\varphi}$ defined by eq. (3.2.1) along the proof of theorem 3.2.8 is adapted to $\widehat{\mathcal{D}}$. Hence $\widetilde{S} : V \to \widetilde{U}$ is a diffeomorphism. Moreover, since U is null non–conjugated, then the map $U \to \widetilde{U}$ defined by $X \mapsto \widetilde{X}$ is a bijection, and it allows to identify U with $\widetilde{U}$. Therefore U inherits from $\widetilde{U}$ its structure of differentiable manifold and this implies that $S : V \to U^{(\sim)}$ is a diffeomorphism.

Lemma 3.3.7 trivially implies (3a) and permits to construct the curve $\chi^{\Gamma}_{X_0}$ as the following composition of differentiable maps

$$\begin{array}{ccccccc} & \Gamma & & \pi & & \Theta^{-1} & \\ I_\delta & \longrightarrow & \widehat{U} & \longrightarrow & \widetilde{U} & \longrightarrow & U^{(\sim)} \\ s & \mapsto & \Gamma'(s) & \mapsto & \widehat{T}\chi^{\Gamma}_{X_0}(s) & \mapsto & \chi^{\Gamma}_{X_0}(s) \end{array}$$

then (3b) is verified.

Finally, in order to verify (4), we know that $\Gamma'(a) \in \widehat{T}X$, $\Gamma'(b) \in \widehat{T}Y$ and $X, Y \in U$, by lemma 3.3.7, there exists a piecewise twisted null curve $\mu : [a,b] \to M$ such that $\mu(a) = x \in V$ and $\mu(b) = y \in V$. Since V is causally convex, then μ is fully contained in V and therefore $\chi = S \circ \mu$ is fully contained in $U = S(V)$. So, we conclude that $U \subset_{\text{reg}} \Sigma$. □

We may call *elementary regular sets* in Σ to the regular sets $U = S(V)$ with V relatively compact basic open.

Now, we will need to prove a technical lemma.

Lemma 3.5.3. *Given $W \subset_{\text{reg}} \Sigma$ a regular set and $X_0 = S(x_0) \in W$, then for any twisted null curve $\mu : I_\epsilon \to M$ such that $\mu(0) = x_0$ there exists $\delta > 0$ verifying that $\mu((-\delta,\delta)) \subset S^{-1}(W)$.*

Proof. Consider $X_0 = S(x_0) \in W \subset_{\text{reg}} \Sigma$, then by lemma 3.3.7, there exists a celestial curve $\Gamma : I_\epsilon \to \mathcal{N}$ and a continuous curve $\chi^{\Gamma}_{X_0} : I_\epsilon \to \Sigma$ such that $\chi^{\Gamma}_{X_0} = S \circ \mu$. Since W is regular, then there exists $\delta > 0$ such that $\chi^{\Gamma}_{X_0} : (-\delta,\delta) \subset I_\epsilon \to W^{(\sim)}$ is differentiable. Then we have

$$\mu((-\delta,\delta)) = S^{-1} \circ \chi^{\Gamma}_{X_0}((-\delta,\delta)) \subset S^{-1}\left(W^{(\sim)}\right) = S^{-1}(W).$$

□

Then, it is easy to prove the following result.

Theorem 3.5.4. *Let $W \subset_{\text{reg}} \Sigma$ be a regular set, then $S^{-1}(W)$ is open in M.*

Proof. Given $W \subset_{\text{reg}} \Sigma$ and consider $X_0 \in W$ such that $x_0 = S^{-1}(X_0) \in M$. Take a future–directed twisted null curve $\mu : I_\epsilon \to M$ with $\mu(0) = x_0$, then by lemma 3.5.3, there exists $\delta > 0$ verifying that $\mu((-\delta,\delta)) \subset S^{-1}(W)$. Without any lack of generality, we can assume that δ is small enough for $V = I^+(\mu(-\delta)) \cap I^-(\mu(\delta))$ being globally hyperbolic and causally convex. Observe that $x_0 \in V$ and for any $p \in V$, we have that $p \in I^+(\mu(-\delta))$, then by theorem 3.3.11, for any $p \in V$ there exists a future–directed piecewise twisted null curve μ_p connecting $\mu(-\delta)$ and $\mu(\delta)$ passing through p (see figure 3.3). Now, since W is regular, then by property (4), the curve $\chi_p = S \circ \mu_p$ is fully contained in W, therefore $p \in S^{-1}(W)$ and hence $V \subset S^{-1}(W)$ and $S^{-1}(W)$ is open in M. □

It is interesting to point out that whenever M is globally hyperbolic, then any non-empty $V = I^+(\mu(-\delta)) \cap I^-(\mu(\delta))$ is automatically globally hyperbolic and the conclusion of the theorem is reached easily without referring to the previous results.

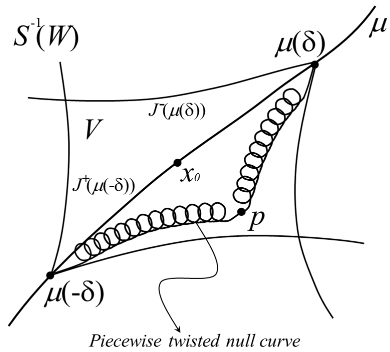

Figure 3.3: Scheme of proof of theorem 3.5.4.

If we provide the topology induced by regular sets to Σ, then proposition 3.5.2 clearly implies that the sky map S is open. The continuity of S trivially follows from theorem 3.5.4, then both results make obvious the following corollary analogue to corollary 3.2.6.

Corollary 3.5.5. *If Σ is equipped with the topology generated by regular sets, then the sky map $S : M \to \Sigma$ is an homeomorphism.*

The importance of the corollary 3.5.5 is that it has been proven without the assumption on M of being non–refocusing because the definition of regular sets only depends on M solely to ensure that M is a Hausdorff manifold and S is injective, that is M is required to be just strongly causal, null pseudo–convex and sky separating.

Moreover, in addition to corollary 3.2.6, they imply that the reconstructive topology in Σ coincides with the topology generated by regular sets, therefore any basis for the topology generated by regular sets is also a basis for the reconstructive topology.

Since for every $x \in M$ there is a basis for the topology of M consisting of basic neighbourhoods V of x, then by proposition 3.5.2, theorem 3.5.4 and corollary 3.5.5, the images of such bases are also bases for the reconstructive topology of Σ, and moreover the sky map is a local diffeomorphism at any point x. Since S is a bijection, then it is a global diffeomorphism. We summarize it all in the following corollary.

Corollary 3.5.6. *The family of regular sets $\{W : W \subset_{\text{reg}} \Sigma\}$ is a basis for the reconstructive topology of Σ. Moreover, there exists a unique differentiable structure in Σ compatible with the manifolds $W^{(\sim)} \subset \Sigma$ that makes of $S : M \to \Sigma$ a diffeomorphism.*

3.5.1 *Non–refocusing hypothesis is superfluous*

We can take advantage of corollary 3.5.5 to show that any strongly causal, null pseudo–convex and sky separating spacetime M is also non–refocusing. In order to do it, we need to observe that the sky map S is open assuming those hypotheses.

According to definition 3.2.3 of Low's topology generated by the bases $\mathfrak{B}(X)$, such topology can be realized by particular bases on M, as following lemma shows. It corroborates the relation between neighbourhood basis of M and its space of skies Σ.

Lemma 3.5.7. *Let $\mathcal{B}(x)$ be a neighbourhood basis consisting of basic open sets. For any $U \in \mathcal{B}(x)$, denote by $\mathcal{U} = \{\gamma \in \mathcal{N} : \gamma \cap U \neq \varnothing\}$. Then $\{\Sigma(\mathcal{U}) : U \in \mathcal{B}(x)\}$ is a neighbourhood basis of $S(x) \in \Sigma$.*

Proof. Because the bundle $\mathbb{PN}(M) \to M$ is locally trivial, let us take a neighbourhood $V \subset M$ of $x \in M$ such that there is a diffeomorphism $\varphi : V \times \mathbb{S}^{m-2} \to \mathbb{PN}(V)$ with $\varphi\left(\{y\} \times \mathbb{S}^{m-2}\right) = \mathbb{PN}_y$ for all $y \in V$.

Consider the map $\sigma : \mathbb{PN}(V) \to \mathcal{V} \subset \mathcal{N}$ defined by $\sigma([v]) = \gamma_{[v]}$. It is clear that σ is continuous and hence $\overline{\sigma} = \sigma \circ \varphi : V \times \mathbb{S}^{m-2} \to \mathcal{V}$ is also so. Observe that

$$S(x) = \overline{\sigma}\left(\{x\} \times \mathbb{S}^{m-2}\right),$$

and $\overline{\sigma}(V \times \mathbb{S}^{m-2}) = \mathcal{V}$.

Now, take any open $\mathcal{W} \subset \mathcal{V}$ containing the sky $S(x)$, then

$$\{x\} \times \mathbb{S}^{m-2} \subset \overline{\sigma}^{-1}(S(x)) \subset \overline{\sigma}^{-1}(\mathcal{W})$$

Since $\overline{\sigma}$ is continuous then $\overline{\sigma}^{-1}(\mathcal{W})$ is open in $V \times \mathbb{S}^{m-2}$.

For any $(y,q) \in V \times \mathbb{S}^{m-2}$ there exists a neighbourhood basis whose elements are $U^{(y,q)} = K^y \times H^q$ where $K^y \subset V$ and $H^q \subset \mathbb{S}^{m-2}$ are open neighbourhoods of $y \in V$ and $q \in \mathbb{S}^{m-2}$ respectively. Then for any $(x,q) \in \{x\} \times \mathbb{S}^{m-2}$, there exist $U^{(y,q)}$ with $(x,q) \in U^{(y,q)} \subset \overline{\sigma}^{-1}(W)$. Since $\{x\} \times \mathbb{S}^{m-2}$ is compact, then there exists a finite subcovering $\{U_j = K_j \times H_j\}_{j=1,\ldots,n} \subset \overline{\sigma}^{-1}(\mathcal{W})$. Then

$$\{x\} \times \mathbb{S}^{m-2} \subset \bigcup_{j=1}^{n} U_j \subset \overline{\sigma}^{-1}(\mathcal{W})$$

Observe that $K_0 = \bigcap_{j=1}^{n} K_j$ is an open neighbourhood of x and $\bigcup_{j=1}^{n} H_j = \mathbb{S}^{m-2}$.

Since $\mathcal{B}(x)$ is a neighbourhood basis of $x \in M$, there exists $U \in \mathcal{B}(x)$ such that $U \subset K_0$.

For any $(y,q) \in U \times \mathbb{S}^{m-2}$, we have that

$$(y,q) \in U \times \bigcup_{j=1}^{n} H_j$$

therefore there exists j such that $q \in H_j$ and since $y \in K_0 \subset K_j$, then $(y,q) \in U_j \subset \overline{\sigma}^{-1}(W)$. This implies that

$$\{x\} \times \mathbb{S}^{m-2} \subset U \times \mathbb{S}^{m-2} \subset \overline{\sigma}^{-1}(\mathcal{W}).$$

and hence

$$S(x) \subset \overline{\sigma}\left(U \times \mathbb{S}^{m-2}\right) \subset \mathcal{W}$$

and since $\mathcal{U} = \overline{\sigma}\left(U \times \mathbb{S}^{m-2}\right)$ then

$$S(x) \in \Sigma(\mathcal{U}) \subset \Sigma(\mathcal{W})$$

is verified. Then $\{\Sigma(\mathcal{U}) : U \in \mathcal{B}(x)\}$ is a neighbourhood basis of $S(x) \in \Sigma$ as we claimed. □

A direct consequence of the previous lemma is the following:

Theorem 3.5.8. *Let M be a strongly causal, null pseudo–convex, space-time separating skies such that it is refocusing at x, then the sky map $S\colon M \to \Sigma$ is not open.*

Proof. We will show that there exists a sequence $\{x_n\}$ in M that does not converge to x but $S(x_n)$ converges to $S(x)$ in Σ. This contradicts the statement that S is open.

Because M is refocusing at x there exists an open neighbourhood $W \subset M$ of x such that for every open neighbourhood $V \subset W$ of x there is $y \notin W$ such that every light ray passing through y enters V. Let us choose a sequence of globally hyperbolic neighbourhoods $V_n^x \subset W$ of x such that $\cap_n V_n^x = \{x\}$. More specifically, let $\sigma(t)$ be a timelike curve contained in a basic neighbourhood $U \subset W$ of x and let a_n (respect. b_n) be a sequence of points on σ, in the past (future) of x, such that $a_n \to x$ (respect. $b_n \to x$). Now we choose the sequence of open neighbourhoods as $V_n^x = I^+(a_n) \cap I^-(b_n)$.

Then for any V_n^x in the previous sequence there exists $x_n \notin W$ such that $\gamma \cap V_n^x \neq \emptyset$ and $x_n \in \gamma \in \mathcal{N}$. Hence, since $x_n \notin W$ for all n, then x_n cannot converge to x.

On the other hand, considering the open subsets $\mathcal{U}_n = \{\gamma \in \mathcal{N} : \gamma \cap V_n^x \neq \emptyset\}$, and because of lemma 3.5.7, it is clear that $\Sigma(\mathcal{U}_n)$ define a neighbourhood basis at $S(x)$ in Σ, and because $S(x_n) \in \Sigma(\mathcal{U}_n)$ then we conclude that $S(x_n) \to S(x)$. ☐

Now, it is easy to conclude. For any strongly causal, null pseudo–convex and sky separating spacetime M, corollary 3.5.5 claims that S is open, and by theorem 3.5.8, we have that M is non–refocusing. Then, the following result is proven.

Corollary 3.5.9. *If M is a strongly causal, null pseudo-convex, spacetime such that the skies of M separate events, then M is non-refocusing.*

Section 3.6

The reconstruction theorem

In this section we will discuss the conditions under a conformal manifold can be reconstructed from its spaces of light rays and skies. A space that could be reconstructed from these data should have the property that "isomorphic" data must provide the same reconstruction. This observation leads to the following definition.

Definition 3.6.1. *Let $(M, \mathcal{C})$, $\left(\overline{M}, \overline{\mathcal{C}}\right)$ be two strongly causal manifolds and $(\mathcal{N}, \Sigma)$, $\left(\overline{\mathcal{N}}, \overline{\Sigma}\right)$ the corresponding pairs of spaces of light rays and skies. We say that a map $\phi : \mathcal{N} \to \overline{\mathcal{N}}$* preserves skies *if $\phi(X) \in \overline{\Sigma}$ for any $X \in \Sigma$. Moreover, $(M, \mathcal{C})$ will be said to be* recoverable *if for any $\left(\overline{\mathcal{N}}, \overline{\Sigma}\right)$, the spaces of light rays and skies corresponding to another strongly causal manifold $\left(\overline{M}, \overline{\mathcal{C}}\right)$, and $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ a diffeomorphism preserving skies, then the map*

$$\varphi = \overline{P} \circ \phi \circ S : M \to \overline{M}$$

is a conformal diffeomorphism on its image, where $\overline{P} : \overline{\Sigma} \to \overline{M}$ is the parachute map to $\overline{M}$.

Lemma 3.6.2. *Let $(M,\mathcal{C})$ and $\left(\overline{M},\overline{\mathcal{C}}\right)$ be two strongly causal manifolds and let $(\mathcal{N},\Sigma)$ and $\left(\overline{\mathcal{N}},\overline{\Sigma}\right)$ be the corresponding pairs of spaces of light rays and skies. If $\phi:\mathcal{N}\to\overline{\mathcal{N}}$ is a diffeomorphism preserving skies then the induced map $\Phi:\Sigma\to\overline{\Sigma}$ defined by $\Phi(X)=\phi(X)$ is injective, open, continuous and a diffeomorphism onto its range.*

Proof. Obviously, Φ is well defined and injective. To show that Φ is continuous, consider and open set $\overline{U}\subset\overline{\Sigma}$ and denote $U=\Phi^{-1}(\overline{U})$. Since $\overline{U}$ is open, there exists an open set $\overline{\mathcal{W}}\subset\overline{\mathcal{N}}$ such that any sky $\overline{X}\subset\overline{\mathcal{W}}$ is in $\overline{U}$. Since ϕ is a diffeomorphism, then $\mathcal{W}=\phi^{-1}(\overline{\mathcal{W}})$ is an open set in $\mathcal{N}$ and every sky $X\subset\mathcal{W}$ verifies that $\phi(X)\subset\overline{\mathcal{W}}$ and, therefore $\Phi(X)\in\overline{U}$. This implies that $U=\Sigma(\mathcal{W})$ and U is open in Σ.

Now we show Φ is an open map. Consider $X\in\Sigma$ and $\overline{X}=\phi(X)\in\overline{\Sigma}$. Because of corollary 3.5.6 and the continuity of Φ there exist regular neighbourhoods $U\subset\Sigma$ of X and $\overline{U}\subset\overline{\Sigma}$ of $\overline{X}$ such that $\Phi(U)\subset\overline{U}$. Then $\phi(\mathcal{U})\subset\overline{\mathcal{U}}$ with $U=\Sigma(\mathcal{U})$ and $\overline{U}=\Sigma(\overline{\mathcal{U}})$. Hence, because $\phi\colon\mathcal{N}\to\overline{\mathcal{N}}$ is a diffeomorphism, then $\phi_*\colon T\mathcal{N}\to T\overline{\mathcal{N}}$ is also a diffeomorphism and the restriction $\phi_*\colon\widehat{T\mathcal{U}}\to\widehat{T\overline{\mathcal{U}}}$ is a diffeomorphism onto its image. It can be restricted again to $\phi_*:\widehat{U}\to\widehat{\overline{U}}$ since

$$\phi_*\left(\widehat{U}\right)=\phi_*\left(\bigcup_{X\in U}\widehat{T}X\right)=\bigcup_{X\in U}\phi_*\left(\widehat{T}X\right)=\bigcup_{X\in U}\widehat{T}\phi(X)\subset\widehat{\overline{U}},$$

and the fact that $\widehat{U}$ and $\widehat{\overline{U}}$ are regular submanifolds of $\widehat{T\mathcal{U}}$ and $\widehat{T\overline{\mathcal{U}}}$ respectively.

Denoting by $\widehat{\mathcal{D}}=\{\widehat{T}X:X\in U\}$, and $\widehat{\overline{\mathcal{D}}}=\{\widehat{T\overline{X}}:\overline{X}\in\overline{U}\}$ the distributions in $\widehat{U}$ and $\widehat{\overline{U}}$, we see that $\phi_*\left(\widehat{\mathcal{D}}\right)=\widehat{\overline{\mathcal{D}}}$. Therefore $\phi_*:\widehat{U}\to\widehat{\overline{U}}$ induces a smooth map

$$\overline{\phi_*}\colon\widehat{U}/\widehat{\mathcal{D}}\to\widehat{\overline{U}}/\widehat{\overline{\mathcal{D}}}$$

and we have the following commutative diagram:

$$\begin{array}{ccc}
\widehat{U} & \xrightarrow{\phi_*} & \widehat{\overline{U}}\\
\downarrow & & \downarrow\\
\widehat{U}/\widehat{\mathcal{D}} & \xrightarrow{\overline{\phi_*}} & \widehat{\overline{U}}/\widehat{\overline{\mathcal{D}}}\\
\downarrow & & \downarrow\\
U & \xrightarrow{\Phi} & \overline{U}
\end{array} \tag{3.6.1}$$

Notice that the lower vertical arrows are diffeomorphisms because of proposition 3.5.2, therefore we conclude that $\Phi:U\to\overline{U}$, is injective, smooth with nonsingular differential, hence it is open and a diffeomorphism onto its image. □

Restricting the map Φ of lemma 3.6.2 to its image, $\Phi:\Sigma\to\Phi(\Sigma)$ then it is clear that Φ is bijective, open and continuous, hence is a homeomorphism. This homeomorphism induces, in virtue of corollary 3.5.6, the homeomorphism $\varphi=\overline{P}\circ\Phi\circ S$ onto an open set of $\overline{M}$. So, we can assume, with no lack of generality that $\overline{\Sigma}=\Phi(\Sigma)$ and $\overline{M}=\overline{P}\circ\Phi(\Sigma)$.

Theorem 3.6.3. *Let $(M,\mathcal{C})$ be a strongly causal, null pseudo–convex and sky separating spacetime then M is recoverable.*

Proof. Let $\left(\overline{M},\overline{\mathcal{C}}\right)$ be another strongly causal manifold with $\left(\overline{\mathcal{N}},\overline{\Sigma}\right)$ its corresponding spaces of light rays and skies, and $\phi : \mathcal{N} \to \overline{\mathcal{N}}$ a diffeomorphism such that $\phi(\Sigma) = \overline{\Sigma}$. Then because of lemma 3.6.2 we conclude that $\Phi : \Sigma \to \overline{\Sigma}$ is a diffeomorphism. So, in virtue of corollary 3.5.6, the map $\varphi = \overline{P} \circ \Phi \circ S : M \to \overline{M}$ is a diffeomorphism too.

Now, we need to show that φ maps light rays of M into light rays of $\overline{M}$. We can consider all the light rays in the skies of a given light ray γ, denoted as

$$S(\gamma) = \{\beta \in \mathcal{N} : \exists\, X \in \Sigma \text{ such that } \gamma, \beta \in X\}.$$

Then $\Phi(S(\gamma)) = \phi(S(\gamma)) = \{\phi(\beta) \in \overline{\mathcal{N}} : \exists\, X \in \Sigma \text{ such that } \gamma, \beta \in X\}$, and since ϕ is a diffeomorphism preserving skies:

$$\Phi(S(\gamma)) = \{\phi(\beta) \in \overline{\mathcal{N}} : \exists\, \Phi(X) \in \overline{\Sigma} \text{ such that } \phi(\gamma), \phi(\beta) \in \Phi(X)\}.$$

Therefore $\Phi(S(\gamma)) = \overline{S}(\phi(\gamma))$. So, it implies

$$\varphi(\gamma) = \overline{P} \circ \Phi \circ S(\gamma) = \overline{P} \circ \overline{S} \circ \phi(\gamma) = \phi(\gamma) \in \overline{\mathcal{N}}$$

is a light ray, that is, φ maps light rays into light rays. By proposition 2.1.3, φ is a conformal diffeomorphism. □

Section 3.7

Celestial curves and reconstruction theorem

We will start this section by introducing a class of curves that are going to play a fundamental role in characterizing when the spaces of light rays and skies of a given strongly causal space–time are "isomorphic" regarding the reconstruction problem.

Let us recall that a curve $\mu : [a,b] \to M$ is a *null curve* if it is differentiable and $\mathbf{g}(\mu', \mu') = 0$. Notice that this is a conformal property and μ does not have to be a regular curve.

Definition 3.7.1. *The set of all null curves $\mu : I \to M$ will be denoted by $\mathfrak{L}(M)$. The subset of $\mathfrak{L}(M)$ consisting of all time–orientable (future or past)–directed null curves μ will be denoted by $\mathfrak{L}_c(M)$, i.e., $\mu \in \mathfrak{L}_c(M)$ if μ is differentiable, $\mathbf{g}(\mu', \mu') = 0$ and either $\mu'(s) \in \mathbb{N}^+$ or $\mu'(s) \in \mathbb{N}^-$ wherever μ is regular.*

Observe that example 3.3.3 shows the existence of dust curves $\mu \in \mathfrak{L}(M)$ such that they are not time–oriented. So, it motivates the following definition.

Definition 3.7.2. *A differentiable curve $\Gamma : I \to \mathcal{N}$ such that $\Gamma \subset X$ for some sky $X \in \Sigma$ is called a* sky curve. *We will denote the set of all sky curves by $\mathfrak{C}_s(\mathcal{N})$.*

It is clear that any sky curve $\Gamma \subset X \in \Sigma$ verifies $\Gamma'(s) \in T_{\Gamma(s)}X$ for any $s \in I$, then we have Γ is celestial, thus

$$\mathfrak{C}_s(\mathcal{N}) \subset \mathfrak{C}(\mathcal{N})$$

Recall, that any basic neighbourhood $V \subset M$ is null non–conjugate, and similarly, a neighbourhood "small enough" of any closed spacelike hypersurface has this property too.

By convention, we can consider $M \subset \mathfrak{L}(M)$ since any point $p \in M$ can be identified with a constant curve. Moreover, if M is null non–conjugate, then the *dust map* $\pi_{CL} : \mathfrak{C}(\mathcal{N}) \to \mathfrak{L}(M)$ given by $\pi_{CL}(\Gamma) = \mu$ is well defined and μ is characterized by $\Gamma'(s) \in \widehat{T}_{\Gamma(s)} S(\mu(s))$ for every s. In general $\Gamma \in \mathfrak{C}(\mathcal{N})$ can be defined by several curves μ_i with $i = 1, 2, \ldots$, and so $\pi_{CL}(\Gamma)$ should be interpreted as the family $\{\mu_i\}$. We call $\{S(\mu(s))\}$ the Legendrian isotopy of Γ.

Definition 3.7.3. *Let $(\mathcal{N}, \Sigma)$ be the spaces of light rays and skies of a null non–conjugate strongly causal space–time M. We define the set of* causal celestial curves *as*

$$\mathfrak{C}_c(\mathcal{N}) = \{\Gamma \in \mathfrak{C}(\mathcal{N}) : \mu = \pi_{CL}(\Gamma) \in \mathfrak{L}_c(M)\}$$

The previous definition of the class of causal celestial curves in $\mathcal{N}$ uses explicitly the space M, however because of the results of section 3.4 we can provide a characterization of $\mathfrak{C}_c(\mathcal{N})$ without making any reference to M. In fact, using corolary 3.4.8 and propositions 3.4.7 and 3.3.2, we see that $\mu \in \mathfrak{L}_c(M)$ if and only if μ is a null curve defining a non–positive (or non–negative) legendrian isotopy and we get the following corollary that could be used as an alternative definition of $\mathfrak{C}_c(\mathcal{N})$.

Corollary 3.7.4. *A celestial curve $\Gamma \in \mathfrak{C}(\mathcal{N})$ is a past (future) causal celestial curve if and only if Γ defines a non-negative (non-positive) legendrian isotopy of skies.*

Definition 3.7.5. *Let M_1 and M_2 be two strongly causal spacetimes and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be their corresponding spaces of light rays. A diffeomorphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ will be called a celestial map if it preserves celestial vectors, i.e. $\phi_*\left(\widehat{\Sigma}_1\right) \subset \widehat{\Sigma}_2$.*

The following lemma is a direct consequence of the definitions.

Lemma 3.7.6. *Any celestial map $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ preserves celestial curves.*

Proof. If $\Gamma : I \to \mathcal{N}_1$ is a celestial curve, then $\Gamma'(s) \in \widehat{\Sigma}_1$ for every $s \in I$. Since ϕ is celestial then $(\phi \circ \Gamma)'(s) = \phi_*(\Gamma'(s)) \in \widehat{\Sigma}_2$ and hence, $\phi \circ \Gamma : I \to \mathcal{N}_2$ is a celestial curve. Moreover ϕ induces a map $\phi : \mathfrak{C}(\mathcal{N}_1) \to \mathfrak{C}(\mathcal{N}_2)$. □

Finally we have the following definition:

Definition 3.7.7. *Let M_1 and M_2 be two strongly causal spacetimes and let $\mathcal{N}_1$ and $\mathcal{N}_2$ be their corresponding spaces of light rays. A celestial map $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ will be called a causal celestial map if ϕ preserves causal celestial curves, that is*

$$\phi : \mathfrak{C}_c(\mathcal{N}_1) \to \mathfrak{C}_c(\mathcal{N}_2)$$

Theorem 3.7.8. *Let M_1 and M_2 be two strongly causal spacetimes, suppose that M_2 is null non–conjugate, and let $(\mathcal{N}_1, \Sigma_1)$ and $(\mathcal{N}_2, \Sigma_2)$ be their corresponding pairs of spaces of light rays and skies. Let $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ be a celestial map. Then the following conditions are equivalent:*

1. *ϕ is a causal celestial map, that is $\phi \circ \Gamma_1 \in \mathfrak{C}_c(\mathcal{N}_2)$, for all $\Gamma_1 \in \mathfrak{C}_c(\mathcal{N}_1)$*
2. *ϕ is a celestial sky map, that is $\phi \circ \Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_2)$, for all $\Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_1)$.*

3. *There exists a conformal immersion* $\Phi : M_1 \to M_2$ *such that* $\phi(\gamma) = \Phi \circ \gamma$ *for every* $\gamma \in \mathcal{N}_1$.

Proof. (1) $\Rightarrow$ (2) Consider $X_1 \in \Sigma_1$ and a closed sky curve $\Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_1)$ such that $\Gamma_1 : [0,1] \to X_1 \subset \mathcal{N}_1$. Since ϕ is a diffeomorphism and by lemma 3.7.6, then $\Gamma_2 = \phi \circ \Gamma_1$ is a closed celestial curve. Let μ_2 be the dust of Γ_2. Then, its endpoints verify

$$\mu_2(0), \mu_2(1) \in \Gamma_2(0) = \Gamma_2(1) = \gamma_2 \in \mathcal{N}_2$$

By hypothesis, we have that $\Gamma_2 \in \mathfrak{C}_c(\mathcal{N}_2)$ and therefore $\mu_2 \in \mathfrak{L}_c(M)$. Since M_2 is strongly causal, then $\mu_2(0) \neq \mu_2(1)$ and since μ_2 can not be a geodesic, by [53, Prop. 10.46], $\mu_2(0)$ and $\mu_2(1)$ are timelikely related. Now, applying [53, Prop. 10.51] to γ_2, then there exists a conjugate point of $\mu_2(0)$ in γ_2 before $\mu_2(1)$ contradicting that M_2 is null non–conjugate. Hence μ_2 must be constant and therefore $\Gamma_2 \in \mathfrak{C}_s(\mathcal{N}_2)$.

(2) $\Rightarrow$ (3) It is trivial to see that ϕ preserves skies, then by the reconstruction theorem 3.6.3, the statement (3) follows.

(3) $\Rightarrow$ (1) Consider $\Gamma_1 \in \mathfrak{C}_s(\mathcal{N}_1)$ and let us denote $\Gamma_2 = \phi \circ \Gamma_1$. Then there is $X \in \Sigma_1$ such that

$$\Gamma_1'(s) \in T_{\Gamma_1(s)}X \Rightarrow \phi_*(\Gamma_1'(s)) \in T_{\phi\circ\Gamma_1(s)}\phi(X) \Rightarrow \Gamma_2'(s) \in T_{\Gamma_2(s)}\phi(X)$$

for all $s \in I$. Hence, since Φ is conformal, then it also preserves skies and we have that $\phi(X) \in \Sigma_2$. Therefore $\Gamma_2 \in \mathfrak{C}_s(\mathcal{N}_2)$. $\square$

The following example illustrates that the existence of a contactomorphism preserving celestial vectors between the spaces of light rays of two spacetimes is not sufficient to induce a conformal diffeomorphism (on its image) between them, showing that condition (1) in theorem 3.7.8 cannot be weakened.

Example 3.7.9. *Let* $M = \mathbb{M}^3$ *be the 3–dimensional Minkowski spacetime with coordinates given by* $(t, x, y) \in \mathbb{R}^3$ *and let* $\mathcal{N}$ *be its space of light rays. The hypersurface* $C \equiv \{t = 0\}$ *is a Cauchy surface, then* $(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1$ *are coordinates in* $\mathcal{N}$ *for any null geodesic* $\gamma(s) = (s, x + s\cos\theta, y + s\sin\theta)$. *Then* $\left\{ \left(\frac{\partial}{\partial x}\right)_\gamma, \left(\frac{\partial}{\partial y}\right)_\gamma, \left(\frac{\partial}{\partial \theta}\right)_\gamma \right\}$ *is a basis of* $T_\gamma\mathcal{N}$. *The contact hyperplane* $\mathcal{H}_\gamma$ *is generated by the tangent spaces of two different skies containing* γ, *therefore*

$$\mathcal{H}_\gamma = \operatorname{span}\left\{ \left(\frac{\partial}{\partial \theta}\right)_\gamma, \sin\theta\left(\frac{\partial}{\partial x}\right)_\gamma - \cos\theta\left(\frac{\partial}{\partial y}\right)_\gamma \right\}$$

and a contact form α *can be written as*

$$\alpha = \cos\theta dx + \sin\theta dy$$

For this γ, *we have that* $T_\gamma S(\gamma(s)) = \operatorname{span}\left\{ s\left(\sin\theta\left(\frac{\partial}{\partial x}\right)_\gamma - \cos\theta\left(\frac{\partial}{\partial y}\right)_\gamma\right) + \left(\frac{\partial}{\partial \theta}\right)_\gamma \right\}$ *with* $s \in \mathbb{R}$ *and hence the celestial vectors at* γ *are given by* $\widetilde{\gamma} = \bigcup_{s\in\mathbb{R}} T_\gamma S(\gamma(s))$. *It can be easily observed that the whole* $\mathcal{H}_\gamma$ *is covered by* $\widetilde{\gamma}$ *except the subspace defined by*

$$\operatorname{span}\left\{ \sin\theta\left(\frac{\partial}{\partial x}\right)_\gamma - \cos\theta\left(\frac{\partial}{\partial y}\right)_\gamma \right\}.$$

We can restrict this spacetime to $M_0 = \left\{(t,x,y) \in \mathbb{M}^3 : t < 0\right\}$ *denoting* $\mathcal{N}_0$ *its corresponding space of light rays. By global hyperbolicity of* M *and* M_0*, every null geodesic* $\gamma_0 \in \mathcal{N}_0$ *can be written as* $\gamma_0 = \gamma \cap M_0$ *for a unique null geodesic* $\gamma \in \mathcal{N}$*, then we can define the restriction map*

$$\begin{array}{rccc} \rho : & \mathcal{N} & \longrightarrow & \mathcal{N}_0 \\ & \gamma & \longmapsto & \gamma_0 = \gamma \cap M_0 \end{array}$$

and the extension map

$$\begin{array}{rccc} \varepsilon : & \mathcal{N}_0 & \longrightarrow & \mathcal{N} \\ & \gamma_0 & \longmapsto & \gamma \end{array}$$

Both ρ *and* ε *are contactomorphisms and they verify* $\varepsilon = \rho^{-1}$ *and hence we have that* $\mathcal{N} \simeq \mathcal{N}_0$.

Now, let us consider $M_\epsilon = \left\{(t,x,y) \in \mathbb{R}^3 : t < \epsilon\right\}$ *for* $\epsilon > 0$*, equipped with the metric*

$$\mathbf{g}_\epsilon = -\left(1 + f\left(t\right)\right) dt \otimes dt + 2f\left(t\right) dt \otimes dx + \left(1 - f\left(t\right)\right) dx \otimes dx + dy \otimes dy$$

where f *is a smooth function verifying* $f\left(t\right) = 0$ *for every* $t \leq 0$*. We can see* $\mathbf{g}_\epsilon$ *as a small perturbation of the metric* $\mathbf{g}$ *of* M *for* $0 < t < \epsilon$*. Trivially, we observe that* M *and* M_ϵ *are two space–times extending* M_0*. By [52], the value of* ϵ *can be chosen small enough such that* M_ϵ *remains globally hyperbolic, then we can consider* $\mathcal{N}_\epsilon \simeq \mathcal{N}$ *and therefore* $\mathcal{H}_\gamma \simeq \mathcal{H}_{\gamma_0} \simeq \mathcal{H}_{\gamma_\epsilon}$ *for* $\gamma_0 = \gamma \cap M_0$ *and* $\gamma_\epsilon = \gamma \cap M_\epsilon$*. This extension is independent from the coordinates* x *and* y*. Denoting by* $\widetilde{\gamma}_\epsilon$*,* $\widetilde{\gamma}_0$ *the celestial vectors at the corresponding curve, and working at* $\mathcal{N}$ *with certain abuse of notation we have that* $\widetilde{\gamma}_0 = \bigcup_{s\in(-\infty,0)} T_\gamma S\left(\gamma\left(s\right)\right) \subset \widetilde{\gamma} \cap \widetilde{\gamma}_\epsilon$ *then the value* ϵ *also can be selected small enough such that* $\widetilde{\gamma}_\epsilon \subset \widetilde{\gamma}$ *and therefore the contactomorphism* $\Phi : \mathcal{N}_\epsilon \to \mathcal{N}$ *preserves celestial vectors. In spite of the existence of* Φ *preserving celestial vectors, the space–times* M *and* M_ϵ *can not be conformally equivalent. Observe that 3–dimensional Minkowski space–time* M *is flat. Denoting as* R_{ij}*,* R *and* g^ϵ_{ij} *the Ricci curvature, the scalar curvature and the metric in* M_ϵ *respectively, then the components of the Cotton tensor* $\mathbf{C}_\epsilon$ *in* M_ϵ *are given by* $C_{ijk} = \nabla_k R_{ij} - \nabla_j R_{ik} + \frac{1}{4}\left(\nabla_j R g^\epsilon_{ik} - \nabla_k R g^\epsilon_{ij}\right)$*. It is widely known (see [33, Th. 9]) that a 3–dimensional manifold is locally conformally flat if its Cotton tensor vanishes. A straightforward calculation shows that* $\mathbf{C}_\epsilon \neq 0$*, then* M_ϵ *is not conformally flat and therefore it can not be conformal to* M.

Chapter 4

Miscellanea

The following sections compound a catchall chapter. First, as application of the concepts previously developed, we will deal with the boundary proposed by Low in [45]. This author suggests the construction of a new boundary, invariant by conformal diffeomorphisms, for spacetimes of any dimension $m \geq 3$. In section 4.1, we will accomplish the construction of Low's boundary for $\dim M = 3$ and then, in section 4.1.2 we will check if, in some very simple conditions, it has good properties.

In order to illustrate the geometric structures contained in spaces of light rays of specific spacetimes, we will collect some examples in section 4.2. Mainly, we will focus our study on dimension $m = 3$, even though the first offered example will be the 4–dimensional Minkowski spacetime $\mathbb{M}^4$. All the calculations done for $\mathbb{M}^4$ can be generalized to describe a general Minkowski spacetime $\mathbb{M}^m$ for $m \geq 3$. This example will also help us to obtain, by restriction, the structures of Minkowski $\mathbb{M}^3$ and de Sitter S_1^3 since they are embedded in $\mathbb{M}^4$, and where null geodesics in the embedded manifold are also null geodesics in the ambient one. In section 4.2.1 is justified how to achieve said restriction.

Finally, in section 4.3, we will list some open problem that could be studied in future researches.

Section 4.1

Low's boundary in the 3–dimensional case

In [45], the author introduces the following new idea for a causal boundary in M. Given a null geodesic $\gamma : (a, b) \to M$, we can consider the curve $\widetilde{\gamma} : (a, b) \to Gr^{m-2}(\mathcal{H}_\gamma)$ defined by $\widetilde{\gamma}(s) = T_\gamma S(\gamma(s))$ contained in the grassmannian manifold $Gr^{m-2}(\mathcal{H}_\gamma)$ of $(m-2)$–dimensional subspaces of $\mathcal{H}_\gamma \subset T_\gamma \mathcal{N}$. Defining

$$\begin{aligned} \ominus_\gamma &= \lim_{s \mapsto a^+} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \\ \oplus_\gamma &= \lim_{s \mapsto b^-} \widetilde{\gamma}(s) \in Gr^{m-2}(\mathcal{H}_\gamma) \end{aligned} \tag{4.1.1}$$

if the previous limits exist, then it is possible to assign endpoints to $\widetilde{\gamma}$. The compactness of $Gr^{m-2}(\mathcal{H}_\gamma)$ assures the existence of accumulation points when $s \mapsto a^+, b^-$. In case of $\ominus_\gamma$ and $\oplus_\gamma$ exist for any $\gamma \in \mathcal{N}$, they define subsets in $Gr^{m-2}(\mathcal{H})$ but, *a priori*, they do not constitute any distribution. Low defines the points in this new future causal boundary as the classes of equivalence of light rays that can be connected by a curve tangent to some $\oplus_\gamma$ at any point. Analogously, the new past causal boundary is defined by $\ominus_\gamma$.

Now, we will show that, in case of M being 3–dimensional, Low's causal boundaries can have fair topological and differentiable structures. Observe that when the dimension of the spacetime is $\dim M = m = 3$ then $\mathcal{N}$ is also 3–dimensional since $\dim \mathcal{N} = 2m - 3 = 3$, and moreover the grassmannian manifold $Gr^{m-2}(\mathcal{H})$ is $Gr^1(\mathcal{H}) = \mathbb{P}(\mathcal{H})$.

4.1.1 *Construction of Low's boundary*

Let us consider a conformal manifold $(M, \mathcal{C})$ where M is 3–dimensional, strongly causal and null pseudo–convex. We will use $\mathbf{g} \in \mathcal{C}$ as an auxiliary metric.

In order to obtain the Low's boundary, we will construct a manifold $\widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$ equipped with a regular distribution $\widetilde{\mathcal{D}}$ generated by the tangent spaces of the skies. The quotient space $\Sigma^\sim = \widetilde{\mathcal{N}}/\widetilde{\mathcal{D}}$ will be diffeomorphic to M. Then, assigning endpoints to any $\widetilde{\gamma} \subset \widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$ provides us two fields of directions $\ominus$ and $\oplus$ in $\mathcal{N}$ whose orbits, under some conditions, will be identified to points at the boundary of $\widetilde{\mathcal{N}}$ in $\mathbb{P}(\mathcal{H})$. Finally, this boundary can be propagated to M via an extension of the diffeomorphism $\Sigma^\sim \simeq M$. In this way, Low's boundary can be seen as the orbits of the fields $\ominus$ and $\oplus$.

Notice that the projection

$$\begin{array}{rccc} \pi^{T\mathcal{N}}_{\mathbb{P}(T\mathcal{N})}: & T\mathcal{N} & \rightarrow & \mathbb{P}(T\mathcal{N}) \\ & J & \mapsto & \operatorname{span}\{J\} \end{array}$$

is a submersion, then the restriction

$$\pi = \left.\pi^{T\mathcal{N}}_{\mathbb{P}(T\mathcal{N})}\right|_{\mathcal{H}} : \mathcal{H} \rightarrow \mathbb{P}(\mathcal{H})$$

also is so.

Observe that for $X \in \Sigma$ and $J \in T_\gamma X$, we have that $\lambda J \in T_\gamma X$ and $\pi(\lambda J) = \pi(J)$ for any $\lambda \in \mathbb{R}$.

For each sky $X \in U \subset \Sigma$, we define the map

$$\begin{array}{rccc} \rho_X: & X & \rightarrow & \mathbb{P}(\mathcal{H}) \\ & \gamma & \mapsto & T_\gamma X \end{array} \tag{4.1.2}$$

Let us see that ρ_X is differentiable. Restrict the canonical projection $\pi^{T\mathcal{N}}_{\mathcal{N}}$ to the regular submanifold $\widehat{T}X \subset \mathcal{H}(\mathcal{U})$ and consider a differentiable local section $\sigma : W \subset X \rightarrow \widehat{T}X$ of $\pi^{T\mathcal{N}}_{\mathcal{N}}|_{\widehat{T}X}$. Since any $T_\gamma X$ is 1–dimensional, then $\rho_X|_W = \pi|_{\widehat{T}X} \circ \sigma$ independently of the section σ. By composition of differentiable maps, ρ_X is differentiable.

Now, we will show that ρ_X is an immersion proving that it maps regular curves into regular curves. So, consider any regular curve $\Gamma : I \rightarrow X$. The composition of Γ with the map in (4.1.2) gives us the differentiable curve $c = \rho_X \circ \Gamma : I \to \mathbb{P}(\mathcal{H})$ defined by $c(s) = T_{\Gamma(s)}X$ and since the base curve $\Gamma = \pi_{\mathcal{N}}^{\mathbb{P}(\mathcal{H})} \circ c$ is regular then the curve c in the fibre bundle $\mathbb{P}(\mathcal{H})$ is also regular.

The image of ρ_X will be denoted as

$$X^{\sim} = \{T_\gamma X : \gamma \in X\}$$

Next lemma shows that the union of images $X^{\sim}$ where X lives in any open $U_0 \subset \Sigma$ is also open in $\mathbb{P}(\mathcal{H})$.

Lemma 4.1.1. *Let $V_0 \subset M$ be an open set and $U_0 = S(V_0) \subset \Sigma$. Then $U_0^{\sim} = \bigcup_{X \in U_0} X^{\sim}$ is open in $\mathbb{P}(\mathcal{H})$.*

Proof. Given any $P \in U_0^{\sim}$ there exist $X \in U_0$ and $\gamma \in X$ such that $P = T_\gamma X$. Then for this $X \in U_0$, by corollary 3.5.6, there exists a regular open neighbourhood $U \subset U_0$ of X. It means that the set of celestial vectors $\widehat{U} = \bigcup_{X \in U} \widehat{T}X$ is a regular submanifold in $T\mathcal{U} \subset T\mathcal{N}$ where $\mathcal{U} = \left\{\gamma \in \mathcal{N} : \gamma \cap S^{-1}(U) \neq \varnothing\right\}$. Also observe that, since $\mathcal{H}(\mathcal{U}) = \mathcal{H} \cap T\mathcal{U}$ is a submanifold of $T\mathcal{U}$ then $\widehat{U}$ is also a regular submanifold of $\mathcal{H}(\mathcal{U})$. Due to $\dim \widehat{U} = \dim \mathcal{H}(\mathcal{U}) = 5$ and $\mathcal{H}(\mathcal{U})$ is open in $\mathcal{H}$ then $\widehat{U}$ is open in $\mathcal{H}(\mathcal{U})$ as well as in $\mathcal{H}$. Since the restriction of the projection $\pi : \mathcal{H}(\mathcal{U}) \to \mathbb{P}(\mathcal{H}(\mathcal{U}))$ is a submersion then $\pi(\mathcal{H}(\mathcal{U}))$ is open in $\mathbb{P}(\mathcal{H}(\mathcal{U}))$. Observe that for $\xi \in T_\gamma X$ we have

$$\pi(\xi) = T_\gamma X \Longrightarrow \pi\left(\widehat{T}X\right) = X^{\sim} \Longrightarrow \pi\left(\widehat{U}\right) = U^{\sim}$$

and since $\widehat{U} \subset \mathcal{H}(\mathcal{U})$ is open, then $U^{\sim} = \pi\left(\widehat{U}\right) \subset \mathbb{P}(\mathcal{H}(\mathcal{U}))$ is also open, therefore $U^{\sim}$ is open in $\mathbb{P}(\mathcal{H})$. This shows that $U_0^{\sim}$ is open in $\mathbb{P}(\mathcal{H})$. $\square$

The next step is to define the space

$$\widetilde{\mathcal{N}} = \{T_\gamma X \in \mathbb{P}(\mathcal{H}) : \gamma \in X \in \Sigma\} = \bigcup_{X \in \Sigma} X^{\sim}$$

Lemma 4.1.2. *$\widetilde{\mathcal{N}}$ is open in $\mathbb{P}(\mathcal{H})$.*

Proof. If $\{U_\alpha\}_{\alpha \in \Omega}$ is a open covering of Σ, then

$$\widetilde{\mathcal{N}} = \bigcup_{X \in \Sigma} X^{\sim} = \bigcup_{X \in \bigcup_{\alpha \in \Omega} U_\alpha} X^{\sim} = \bigcup_{\alpha \in \Omega} \left(\bigcup_{X \in U_\alpha} X^{\sim}\right)$$

and, by lemma 4.1.1, $\widetilde{\mathcal{N}}$ is union of the open sets $U_\alpha^{\sim} = \bigcup_{X \in U_\alpha} X^{\sim}$, then $\widetilde{\mathcal{N}}$ is open in $\mathbb{P}(\mathcal{H})$. $\square$

If we would want to do the present construction for a higher dimensional M, it would be necessary that $\widetilde{\mathcal{N}}$ were a regular submanifold of $\mathbb{P}(\mathcal{H})$. This is trivially implied by lemma 4.1.2 in case of a 3–dimensional M.

Corollary 4.1.3. *$\widetilde{\mathcal{N}}$ is a regular submanifold of $\mathbb{P}(\mathcal{H})$.*

We are going to express $\widetilde{\mathcal{N}}$ in a different way. Let $\gamma : I \to M$ be a future–directed parametrized light ray, then we define the curve $\widetilde{\gamma} : I \to \mathbb{P}(\mathcal{H}_\gamma)$ given by

$$\widetilde{\gamma}(s) = T_\gamma S(\gamma(s)) \in \mathbb{P}(\mathcal{H}_\gamma)$$

and we denote its image by

$$\widetilde{\gamma} = \{T_\gamma S(\gamma(s)) \in \mathbb{P}(\mathcal{H}_\gamma) : s \in I\}$$

Applying the previous definition, it is clear that we can see $\widetilde{\mathcal{N}}$ in two different ways:

$$\widetilde{\mathcal{N}} = \bigcup_{X \in \Sigma} X^\sim = \bigcup_{\gamma \in \mathcal{N}} \widetilde{\gamma}$$

It is important to observe that the curve $\widetilde{\gamma}$ is locally injective. Indeed, for any $s \in I$ there exists a basic neighbourhood $V \subset M$ of $\gamma(s)$. This implies that there is no conjugate points in V along γ, but this also means that for any $t_1, t_2 \in I$ such that $\gamma(t_i) \in V$ with $i = 1, 2$ we have that

$$T_\gamma S(\gamma(t_1)) \cap T_\gamma S(\gamma(t_2)) = \{\mathbf{0}\}.$$

Therefore it is clear that $T_\gamma S(\gamma(t_1)) \neq T_\gamma S(\gamma(t_2))$.

Definition 4.1.4. *Given a conformal manifold $(M, \mathcal{C})$, we will say that M has* tangent skies *if there exist skies $X, Y \in \Sigma$ and $\gamma \in X \cap Y \subset \mathcal{N}$ verifying $T_\gamma X = T_\gamma Y$.*

It is obvious that null non–conjugation condition automatically implies absence of tangent skies for M of any dimension. In the 3–dimensional case, the converse is also true, as it is shown in the following lemma.

Lemma 4.1.5. *If M is a 3–dimensional spacetime without tangent skies at M then it is also null non–conjugate.*

Proof. Given $X \neq Y \in \Sigma$ with $\gamma \in X \cap Y$ verifying $\widehat{T}_\gamma X \cap \widehat{T}_\gamma Y \neq \varnothing$, since $\dim T_\gamma X = \dim T_\gamma Y = 1$ then we have $T_\gamma X = T_\gamma Y$ and therefore X and Y are tangent skies at M. □

We have seen that in the 3–dimensional case, $\widetilde{\mathcal{N}}$ is a regular submanifold of $\mathbb{P}(\mathcal{H})$ and it is foliated by the leaves $X^\sim = \{T_\gamma X : \gamma \in X\}$. Since each $X^\sim$ is compact, this foliation $\mathcal{D}^\sim$ is regular and defines the quotient manifold

$$\Sigma^\sim = \widetilde{\mathcal{N}} / \mathcal{D}^\sim$$

We will use the following technical result.

Proposition 4.1.6. *Let $f : M_1 \to M_2$ be a submersion. If $g : M_2 \to M_3$ verifies that $g \circ f$ is differentiable, then g is also differentiable.*

Proof. See [11, Prop. 6.1.2]. □

Next proposition gives us the geometric equivalence between $\Sigma^\sim$ and its corresponding conformal manifold.

Proposition 4.1.7. *If M is such that there are not tangent skies, then the map $S^{\sim} : M \to \Sigma^{\sim}$ defined by $S^{\sim}(p) = S(p)^{\sim}$ is a diffeomorphism.*

Proof. Given a basic open set $V \subset M$, we consider the set of skies $U = S(V) \subset \Sigma$, the set of celestial vectors $\widehat{U} = \bigcup_{X \in U} \widehat{T}X$ and the set $U^{\sim} = \bigcup_{X \in U} X^{\sim}$. Recall that the inclusion $\widehat{U} \hookrightarrow T\mathcal{N}$ is an embedding, and consider the submersion $\pi : \mathcal{H} \to \mathbb{P}(\mathcal{H})$. For $\xi \in T_\gamma X \subset \widehat{U}$ then we have that $\pi(\xi) = T_\gamma X$, and then

$$\pi\left(\widehat{T}X\right) = X^{\sim} \tag{4.1.3}$$

hence

$$\pi\left(\widehat{U}\right) = U^{\sim} \tag{4.1.4}$$

So, since $\widehat{U} \subset \mathcal{H}$ and $U^{\sim} \subset \widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$ are open sets, it is clear that the restriction $\pi : \widehat{U} \to U^{\sim}$ is submersion. We also know from proposition 3.5.2 that there exists a regular distribution $\widehat{\mathcal{D}}$ in $\widehat{U}$ whose leaves are $\widehat{T}X = \bigcup_{\gamma \in X} T_\gamma X$ with $X \in U$.

The equation (4.1.3) implies that there exist a bijection

$$\begin{array}{rccc} \widehat{\pi} : & \widehat{U}/\widehat{\mathcal{D}} & \to & U^{\sim}/\mathcal{D}^{\sim} \\ & \widehat{T}X & \mapsto & X^{\sim} \end{array}$$

and we obtain the following diagram

$$\begin{tikzcd} \widehat{U} \arrow[r, "\pi"] \arrow[d, "p_1"'] & U^{\sim} \arrow[d, "p_2"] \\ \widehat{U}/\widehat{\mathcal{D}} \arrow[r, "\widehat{\pi}"'] & U^{\sim}/\mathcal{D}^{\sim} \end{tikzcd}$$

where p_1 and p_2 are the corresponding quotient maps. Since $\widehat{\mathcal{D}}$ and $\mathcal{D}^{\sim}$ are regular distributions, by proposition 2.2.4, there exists differentiable structures in $\widehat{U}/\widehat{\mathcal{D}}$ and $U^{\sim}/\mathcal{D}^{\sim}$ such that p_1 and p_2 are submersions. In this case, $p_2 \circ \pi$ is another submersion, then since both p_1 and $p_2 \circ \pi$ are open and continuous, it is clear that the bijection $\widehat{\pi}$ is a homeomorphism.

On the other hand, since p_1 is a submersion and $p_2 \circ \pi$ is differentiable, by proposition 4.1.6, we have that $\widehat{\pi}$ is differentiable. Analogously, since $p_2 \circ \pi$ is a submersion and p_1 is differentiable, then $\widehat{\pi}^{-1}$ is differentiable, therefore $\widehat{\pi}$ is a diffeomorphism.

It is known by proposition 3.5.2 that the quotient $\widehat{U}/\widehat{\mathcal{D}}$ is diffeomorphic to $V \subset M$ by mean of the sky map S. So, we have shown that

$$\begin{array}{rccc} S^{\sim} : & V & \to & U^{\sim}/\mathcal{D}^{\sim} \\ & p & \mapsto & S^{\sim}(p) = S(p)^{\sim} \end{array}$$

is a diffeomorphism.

Under the hypothesis of absence of tangent skies, then given $x \neq y \in M$ and $X = S(x)$, $Y = S(y)$, we have that $T_\gamma X \neq T_\gamma Y$, hence $X^{\sim} = S^{\sim}(x) \neq S^{\sim}(y) = Y^{\sim}$ implying the injectiveness of the map $S^{\sim} : M \to \Sigma^{\sim}$. The surjectiveness of $S^{\sim}$ is obtained by definition, hence it is also a bijection. Finally, since $S^{\sim}$ is a bijection and a local difeomorphism at every point then it is a global diffeomorphism. $\square$

For a parametrized light ray $\gamma : (a,b) \to M$ we define

$$\begin{aligned} \ominus_\gamma &= \lim_{s\mapsto a^+} \widetilde{\gamma}(s) \\ \oplus_\gamma &= \lim_{s\mapsto b^-} \widetilde{\gamma}(s) \end{aligned} \tag{4.1.5}$$

when the limits exist.

Under the hypotheses that M is 3–dimensional null non–conjugate spacetime, by lemma 4.1.5, there are no tangent skies in M, and by the compactness of $\mathbb{P}(\mathcal{H}_\gamma) \simeq \mathbb{S}^1$ and the local injectivity of $\widetilde{\gamma}$, we have ensured the existence of the limits in (4.1.5). Then it is possible to define the maps

$$\begin{array}{rccc} \ominus: & \mathcal{N} & \to & \mathbb{P}(\mathcal{H}) \\ & \gamma & \mapsto & \ominus(\gamma) = \ominus_\gamma \end{array} \quad \text{and} \quad \begin{array}{rccc} \oplus: & \mathcal{N} & \to & \mathbb{P}(\mathcal{H}) \\ & \gamma & \mapsto & \oplus(\gamma) = \oplus_\gamma \end{array}$$

and the set

$$\overline{\widetilde{\mathcal{N}}} = \bigcup_{\gamma\in\mathcal{N}} \left(\widetilde{\gamma} \cup \{\ominus_\gamma, \oplus_\gamma\}\right).$$

First, we will construct local coordinates in $\mathcal{H}$ and $\mathbb{P}(\mathcal{H})$ using the ones in $T\mathcal{N}$ defined by the initial values of Jacobi fields at a local Cauchy surface as done in section 3.1.

Indeed, given a set $V \subset M$ we define $U = S(V) \subset \Sigma$ and $\mathcal{U} = \bigcup_{X\in U} X \subset \mathcal{N}$. Let us assume that V is a basic open set in such a way $(V, \varphi = (t,x,y))$ is a coordinate chart such that the local hypersurface $C \subset V$ defined by $t = 0$ is a spacelike (local) Cauchy surface. Let $\{E_1, E_2, E_3\}$ be an orthonormal frame in V such that E_1 is a future oriented timelike vector field in V. Normalizing the timelike component along E_1 and considering tangent vectors of null geodesics at C as $\gamma'(0) = E_1 + u^2E_2 + u^3E_3$ and since γ is lightlike, then $\left(u^2\right)^2 + \left(u^3\right)^2 = 1$. So, we can parametrize all the light rays passing through $\gamma(0)$ by $u^2 = \cos\theta$ and $u^3 = \sin\theta$. This permit us to define coordinates in $\mathcal{U}$ by

$$\psi : \mathcal{U} \to \mathbb{R}^3; \qquad \psi = (x, y, \theta)$$

Moreover, in this case we have that $U \subset \Sigma$ is a regular set in the sense of definition 3.5.1, hence $\widehat{U} = \bigcup_{X\in U} \widehat{T}X$ is a regular submanifold of $T\mathcal{U} \subset T\mathcal{N}$ and the inclusion $\widehat{U} \hookrightarrow T\mathcal{N}$ is an embedding.

Consider $\gamma \in \mathcal{U}$ and $J \in T_\gamma\mathcal{U}$, since J can be identified with a Jacobi field along the stated parametrization of γ, we can write $J(0) = w^1E_1 + w^2E_2 + w^3E_3$ and $J'(0) = v^1E_1 + v^2E_2 + v^3E_3$. Since $\mathbf{g}(\gamma', J') = 0$ and considering the equivalence modγ', then denoting $\overline{w}^k = w^k - w^1u^k$ and $\overline{v}^k = v^k - v^1u^k$ we have that $\overline{v}^2u^2 + \overline{v}^3u^3 = 0$. Supposing without lack of generality that $u^2 \neq 0$ since $\left(u^2, u^3\right) \neq (0,0)$, we can have $v = \overline{v}^3$, $\overline{w}^2$ and $\overline{w}^3$ as coordinates in $T\mathcal{U}$. So, we obtain the chart

$$\overline{\psi} : T\mathcal{U} \to \mathbb{R}^6; \qquad \overline{\psi} = \left(x, y, \theta, \overline{w}^2, \overline{w}^3, v\right)$$

Let us denote $\mathcal{H}(\mathcal{U}) = \mathcal{H} \cap T\mathcal{U} = \bigcup_{\gamma\in\mathcal{U}} \mathcal{H}_\gamma$ and now we can construct coordinates in $\mathcal{H}(\mathcal{U}) \subset T\mathcal{U}$ from $\overline{\psi}$. If $J \in \mathcal{H}_\gamma$ then $\mathbf{g}(\gamma', J) = 0$ and therefore

$$\overline{w}^2u^2 + \overline{w}^3u^3 = 0$$

Again, since $u^2 \neq 0$, then we have $\overline{w}^2 = \frac{-1}{u^2}\overline{w}^3 u^3$ and we can consider $w = \overline{w}^3$ as a coordinate for $\mathcal{H}(\mathcal{U})$, then

$$\varphi : \mathcal{H}(\mathcal{U}) \to \mathbb{R}^5; \qquad \varphi = (x, y, \theta, w, v)$$

is a coordinate chart.

The projection $\pi = \pi^{T\mathcal{N}}_{\mathbb{P}(T\mathcal{N})}\Big|_{\mathcal{H}} : \mathcal{H} \to \mathbb{P}(\mathcal{H})$ allows to define coordinates in $\mathbb{P}(\mathcal{H})$ as follows. From the coordinates $\varphi = (x, y, \theta, w, v)$, if we consider $J \in \mathcal{H}_\gamma$ and $\overline{J} = \lambda J$ for some $\lambda \in \mathbb{R}$, then

$$\begin{cases} \overline{J}(0) = \lambda J(0) = \lambda w^1 E_1 + \cdots + \lambda w^m E_m \\ \overline{J}'(0) = \lambda J'(0) = \lambda v^1 E_1 + \cdots + \lambda v^m E_m \end{cases}$$

thus the coordinates w and v verify

$$\begin{cases} w\left(\overline{J}\right) = \lambda w(J) \\ v\left(\overline{J}\right) = \lambda v(J) \end{cases}$$

then the homogeneous coordinate $\phi = [w : v]$ verifies

$$\phi\left(\overline{J}\right) = \left[w\left(\overline{J}\right) : v\left(\overline{J}\right)\right] = [w(J) : v(J)] = \phi(J)$$

and defines the element $\operatorname{span}\{J\} \in \mathbb{P}(\mathcal{H}_\gamma)$. Therefore, we obtain that

$$\widetilde{\varphi} : \mathbb{P}(\mathcal{H}(\mathcal{U})) \to \mathbb{R}^4; \qquad \widetilde{\varphi} = (x, y, \theta, \phi) \tag{4.1.6}$$

is a coordinate chart in $\mathbb{P}(\mathcal{H})$. Observe that, equivalently, we can also consider ϕ as the polar coordinate $\phi = \arctan \frac{w}{v}$.

We will use a coordinate chart $(\mathbb{P}(\mathcal{H}(\mathcal{U})), \widetilde{\varphi} = (x, y, \theta, \phi))$ as in (4.1.6), where $\mathcal{U} = \{\gamma \in \mathcal{N} : \gamma \cap V \neq \varnothing\}$ is open in $\mathcal{N}$, to describe $\overline{\widetilde{\mathcal{N}}}$ as a manifold with boundary. In this chart, the coordinate ϕ describes the entire $\widetilde{\gamma}$ as well as its limit points. Also observe that a light ray γ is defined by a fixed $(x, y, \theta) = (x_0, y_0, \theta_0)$.

Every $\mathbb{P}(\mathcal{H}_\gamma)$ can be represented by a circumference as shown in figure 4.1, where $\widetilde{\gamma}$ is a connected segment of it with endpoints $\ominus_\gamma$ and $\oplus_\gamma$.

Proposition 4.1.8. *Let M be a 3–dimensional null non–conjugate space–time. Consider that $\ominus$ and $\oplus$ are differentiable distributions,*

1. *If $\ominus = \oplus$ then $\overline{\widetilde{\mathcal{N}}}$ is a manifold without boundary.*
2. *If $\ominus_\gamma \neq \oplus_\gamma$ for all $\gamma \in \mathcal{N}$, then $\overline{\widetilde{\mathcal{N}}}$ is a manifold with boundary $\partial\widetilde{\mathcal{N}} = \bigcup_{\gamma \in \mathcal{N}} \{\ominus_\gamma, \oplus_\gamma\}$.*

Proof. Since $\ominus_\gamma$ and $\oplus_\gamma$ are defined by the limit of $\widetilde{\gamma}(s)$ at the endpoints, $\widetilde{\gamma}$ is locally injective and, by lemma 4.1.5, there is no tangent skies in M, then $\widetilde{\gamma}$ must be a connected open set in $\mathbb{P}(\mathcal{H}_\gamma) \simeq \mathbb{S}^1$ with boundary $\{\ominus_\gamma, \oplus_\gamma\}$. Now, consider $P \in \mathbb{P}(\mathcal{H})$ such that there exist $\gamma \in \mathcal{N}$ verifying $\ominus_\gamma = P$ and a coordinate chart $\widetilde{\varphi} = (x, y, \theta, \phi)$ at P as in (4.1.6). Since $\ominus$ is a distribution, for any $\gamma \in \mathcal{N}$ there exists a point $\ominus_\gamma \in \mathbb{P}(\mathcal{H}_\gamma) \subset \mathbb{P}(\mathcal{H})$ which smoothly depends on the light ray γ. In this case, the coordinates (x, y, θ) define the light rays in $\mathcal{N}$, and hence the function $\phi \circ \ominus : \mathcal{N} \to [0, 2\pi) \simeq \mathbb{S}^1$ has to depend differentially on the coordinates (x, y, θ). Analogously, the same rules for $\oplus$. Let us

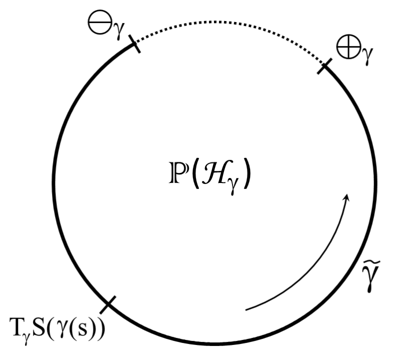

Figure 4.1: Representation of $\mathbb{P}(\mathcal{H}_\gamma)$.

denote by $\phi_\ominus = \phi_\ominus(x,y,\theta)$ and $\phi_\oplus = \phi_\oplus(x,y,\theta)$ the functions $\phi \circ \ominus$ and $\phi \circ \oplus$ in coordinates respectively.

If $\ominus = \oplus$, then for any $\gamma \in \mathcal{U}$ we have that $\widetilde{\gamma} \cup \{\ominus_\gamma\} = \mathbb{P}(\mathcal{H}_\gamma)$ therefore $\mathbb{P}(\mathcal{H}(\mathcal{U})) = \overline{\widetilde{\mathcal{U}}} \subset \overline{\widetilde{\mathcal{N}}}$ and since $\mathbb{P}(\mathcal{H}(\mathcal{U})) \subset \mathbb{P}(\mathcal{H})$ is open, then $\overline{\widetilde{\mathcal{N}}} = \mathbb{P}(\mathcal{H})$, and therefore $\overline{\widetilde{\mathcal{N}}}$ is a manifold without boundary.

In case of $\ominus_\gamma \neq \oplus_\gamma$, without any lack of generality, we can restrict the domain of $\phi_\ominus$ and $\phi_\oplus$, and choose a diffeomorphism $[0,2\pi) \simeq \mathbb{S}^1$ such that

$$0 < \phi_\ominus(x,y,\theta) < \phi_\oplus(x,y,\theta) < 2\pi$$

for all (x,y,θ). Then, for all $\gamma \in \mathcal{U}$, the points in $\overline{\widetilde{\mathcal{N}}}$ restricted to the chart can be written as

$$\{(x,y,\theta,\phi) : \phi_\ominus(x,y,\theta) \leq \phi \leq \phi_\oplus(x,y,\theta)\}$$

describing a manifold with boundary. $\square$

Notice that the previous result is also true if $\ominus$ and $\oplus$ are continuous distributions. In this case, the functions $\phi_\ominus$ and $\phi_\oplus$ depends continuously of the coordinates (x,y,θ) and the proof is still valid.

Now, we will see how Low's boundary can be assigned to M. We will split the boundary $\partial\widetilde{\mathcal{N}}$ into the past boundary $\partial^-\widetilde{\mathcal{N}} = \{\ominus_\gamma : \gamma \in \mathcal{N}\}$ and the future boundary $\partial^+\widetilde{\mathcal{N}} = \{\oplus_\gamma : \gamma \in \mathcal{N}\}$.

Let us define the sets of orbits of $\ominus$ and $\oplus$ as

$$\partial^-\Sigma = \mathcal{N}/\ominus \qquad \partial^+\Sigma = \mathcal{N}/\oplus$$

Since $\ominus$ and $\oplus$ are 1–dimensional distributions, their orbits are 1–dimensional differentiable submanifolds of $\mathcal{N}$. So, for an orbit $X^+ \in \partial^+\Sigma$ and for any $\gamma \in X^+$ we have that $T_\gamma X^+ = \oplus_\gamma \in \mathbb{P}(\mathcal{H})$, and analogously $T_\gamma X^- = \ominus_\gamma \in \mathbb{P}(\mathcal{H})$. This fact implies that the maps

$$\begin{array}{rcl} X^- & \to & \partial^-\widetilde{\mathcal{N}} \\ \gamma & \mapsto & T_\gamma X^- \end{array} \quad \text{and} \quad \begin{array}{rcl} X^+ & \to & \partial^+\widetilde{\mathcal{N}} \\ \gamma & \mapsto & T_\gamma X^+ \end{array} \tag{4.1.7}$$

are differentiable because they coincide with the restriction $\ominus|_{X^-}$ and $\oplus|_{X^+}$ respectively.

Analogously, we can denote by

$$\left(X^-\right)^\sim = \left\{T_\gamma X^- : \gamma \in X^-\right\}$$

$$\left(X^+\right)^\sim = \left\{T_\gamma X^+ : \gamma \in X^+\right\}$$

the corresponding images of the previous maps in (4.1.7).

If $(X^-)^\sim \cap (Y^-)^\sim \neq \varnothing$ then there exists $\gamma \in X^- \cap Y^-$ but since both X^- and Y^- are orbits of the field of directions $\ominus$ then we have that $X^- = Y^-$. Analogously for orbits of $\oplus$. So, we have that the images in $\mathbb{P}(\mathcal{H})$ of the orbits of $\ominus$ and $\oplus$ are separate, this means

$$\left(X^-\right)^\sim \cap \left(Y^-\right)^\sim \neq \varnothing \quad \Longrightarrow \quad X^- = Y^-$$

$$\left(X^+\right)^\sim \cap \left(Y^+\right)^\sim \neq \varnothing \quad \Longrightarrow \quad X^+ = Y^+$$

This property of separation permits us to define

$$\left(\partial^-\Sigma\right)^\sim = \left\{\left(X^-\right)^\sim : X^- \in \partial^-\Sigma\right\}$$

$$\left(\partial^+\Sigma\right)^\sim = \left\{\left(X^+\right)^\sim : X^+ \in \partial^+\Sigma\right\}$$

and also

$$\left(\overline{\Sigma}\right)^\sim = \Sigma^\sim \cup \left(\partial^-\Sigma\right)^\sim \cup \left(\partial^+\Sigma\right)^\sim$$

Now, observe that the map $S^\sim : M \to \Sigma^\sim$ can be naturally extended to

$$\overline{S^\sim} : \overline{M} \to \left(\overline{\Sigma}\right)^\sim$$

by $\overline{S^\sim}\left(X^\pm\right) = \left(X^\pm\right)^\sim$, where $\overline{M} = M \cup \partial^-\Sigma \cup \partial^+\Sigma$.

Lemma 4.1.9. *The maps*

$$\begin{array}{ccc} \mathcal{N} & \to & \partial^-\widetilde{\mathcal{N}} \\ \gamma & \mapsto & \ominus_\gamma \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathcal{N} & \to & \partial^+\widetilde{\mathcal{N}} \\ \gamma & \mapsto & \oplus_\gamma \end{array}$$

are diffeomorphisms.

Proof. We can see trivially that the map $\mathcal{N} \to \partial^-\widetilde{\mathcal{N}}$ is bijective. Observe that the image of the map $\ominus : \mathcal{N} \to \mathbb{P}(\mathcal{H})$ is $\partial^-\widetilde{\mathcal{N}}$. Since its expression in coordinates is

$$(x, y, \theta) \mapsto (x, y, \theta, \phi_\ominus(x, y, \theta))$$

and $\phi_\ominus$ is differentiable, it is clear that $\mathcal{N}$ is locally diffeomorphic to the graph of $\phi_\ominus$ and moreover this graph is locally diffeomorphic to the image of $\ominus$, that is $\partial^-\widetilde{\mathcal{N}}$. So, the map $\mathcal{N} \to \partial^-\widetilde{\mathcal{N}}$ is a bijection and a local diffeomorphism, therefore it is a global diffeomorphism. The proof for $\mathcal{N} \to \partial^+\widetilde{\mathcal{N}}$ can be done in the same way. $\square$

Since $\ominus$ and $\oplus$ define regular distributions in $\mathcal{N}$, we can propagate them, respectively to $\partial^-\widetilde{\mathcal{N}}$ and $\partial^+\widetilde{\mathcal{N}}$ using the difeomorphisms of lemma 4.1.9. Then we obtain the regular distributions $(\mathcal{D}^-)^\sim$ and $(\mathcal{D}^+)^\sim$ whose leaves are the elements of $(\partial^-\Sigma)^\sim$ and $(\partial^+\Sigma)^\sim$ respectively. These distributions, together with $\mathcal{D}^\sim$, give rise to a new distribution $\overline{\mathcal{D}^\sim}$ whose leaves are disjoint in $\overline{\widetilde{\mathcal{N}}}$ and they can be seen as elements of $\left(\overline{\Sigma}\right)^\sim$. Since all the distributions $\mathcal{D}^{\sim}$, $(\mathcal{D}^{-})^{\sim}$ and $(\mathcal{D}^{+})^{\sim}$ are regular, then $\overline{\mathcal{D}^{\sim}}$ is also a regular distribution. Therefore we can consider the quotient

$$\overline{\widetilde{\mathcal{N}}}/\overline{\mathcal{D}^{\sim}} = \widetilde{\mathcal{N}}/\mathcal{D}^{\sim} \cup \partial^{-}\widetilde{\mathcal{N}}/\left(\mathcal{D}^{-}\right)^{\sim} \cup \partial^{+}\widetilde{\mathcal{N}}/\left(\mathcal{D}^{+}\right)^{\sim} \tag{4.1.8}$$

as a differential manifold identified, in virtue of lemma 4.1.9, with

$$\left(\overline{\Sigma}\right)^{\sim} = \Sigma^{\sim} \cup \left(\partial^{-}\Sigma\right)^{\sim} \cup \left(\partial^{+}\Sigma\right)^{\sim} \simeq \overline{\widetilde{\mathcal{N}}}/\overline{\mathcal{D}^{\sim}}$$

with boundary $\partial\left(\overline{\Sigma}\right)^{\sim} = (\partial^{-}\Sigma)^{\sim} \cup (\partial^{+}\Sigma)^{\sim}$.

Then we can identify $\left(\overline{\Sigma}\right)^{\sim}$ with $\overline{M}$ via the map $\overline{S^{\sim}} : \overline{M} \to \left(\overline{\Sigma}\right)^{\sim}$ obtaining that $\overline{M}$ is the causal Low's completion. We state that Low's boundary of M is

$$\partial M = \overline{M} - M = \partial^{-}\Sigma \cup \partial^{+}\Sigma$$

In case of $\ominus = \oplus$ then $\partial^{+}\widetilde{\mathcal{N}} = \partial^{-}\widetilde{\mathcal{N}}$ and $(\partial^{+}\Sigma)^{\sim} = (\partial^{-}\Sigma)^{\sim}$. Hence $(\mathcal{D}^{+})^{\sim} = (\mathcal{D}^{-})^{\sim}$ and $\partial^{-}\Sigma = \partial^{+}\Sigma$ and therefore, the Low's completion of M is

$$\partial M = \overline{M} - M = \partial\Sigma$$

where $\partial\Sigma = \partial^{-}\Sigma = \partial^{+}\Sigma$.

4.1.2 *Low's boundary and c-boundary*

In order to study a spacetime M at large, the attachment of a boundary can be useful. There are several boundaries defined in the literature (*Geroch's g–boundary* [21], *Schmidt's b–boundary* [60], *GKP c–boundary*[1] [22],...) and their interest depend on the properties we want to study. In [45], the author wonders if Low's and GKP boundaries are the same, so we will focus on it. We will see that, unfortunately, they are not equal as sets of points in the general case, but it is easy to find examples in which they are fairly related. The classical definition of c–boundary has been re–defined along the years to avoid problems arising in the study of its topology. For our purposes, we will recall and deal with this classical definition, but [19], [59] and references therein can be consulted to get a wider understanding on the subject.

Definition 4.1.10. *A set $W \subset M$ is said to be an* indecomposable past set *or an* IP *if it verifies the following conditions:*

1. *W is open and non–empty.*
2. *W is a* past set*, that is $I^{-}(W) = W$.*
3. *W can not be expressed as the union of two proper subsets verifying conditions 1 and 2.*

[1] It is also called Geroch–Kronheimer–Penrose's boundary, causal boundary or just c–boundary.

We will say that an IP W is a proper IP *or* PIP *if there is $p \in M$ such that $W = I^-(p)$. In other case, W will be called a* terminal IP *or* TIP*. In an analogous manner, considering the chronological future, we can define* indecomposable future sets *or* IF*, then we obtain* proper IFs *and* terminal IFs*, that is,* PIFs *and* TIFs

In the figure 4.2, as shown in [8, Fig. 6.4], we offer a trivial example about how IPs and IFs can be identified with the boundary of M. We consider M as a cropped rectangle of the 2–dimensional Minkowski spacetime equipped with the metric $\mathbf{g} = -dy \otimes dy + dx \otimes dx$. Points at the boundary of M such as p are related to TIPs like A, such as q corresponds to TIPs like B and such as r can be related to TIPs like C as well as TIFs like D.

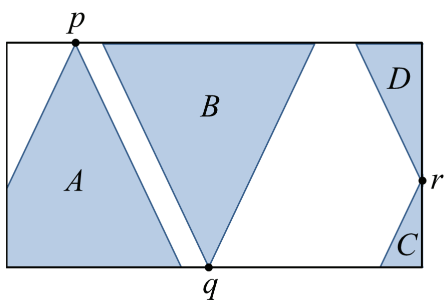

Figure 4.2: TIPs and TIFs.

The following proposition provide us a characterization of all TIPs in a strongly causal spacetime.

Proposition 4.1.11. *For any strongly causal spacetime M, $A \subset M$ is a TIP if and only if there exists a timelike curve μ inextensible to the future such that $A = I^-(\mu)$.*

Proof. See [25, Prop. 6.8.1.]. □

Light rays also define terminal ideal points as next proposition shows.

Proposition 4.1.12. *Let γ be a future–directed inextensible causal curve in a strongly causal spacetime M, then $I^-(\gamma)$ is a TIP.*

Proof. See [19, Prop. 3.32]. □

Now, we are ready for the classical definition of GKP c–boundary.

Definition 4.1.13. *We define the* future (past) causal boundary *or* future (past) c–boundary *of M as the set of all TIPs (TIFs).*

Observe that any point $p \in M$ can be identified with the PIP $I^-(p)$ as well as the PIF $I^+(p)$, moreover it is possible the existence of TIP and TIF identified with the same point at the boundary, as TIP C and TIF D seen in figure 4.2. Then, in order to define the causal completion of M, a suitable identification between sets of IPs and IFs is needed.

This is out of the scope of this work, but [19] and its references can be consulted to get a feedback on that subject.

The question arising now is if all the TIPs in the future c–boundary can be defined by the chronological past of a light ray. Unfortunately, this is not always true as the following example shows because there are TIPs that only can be defined by timelike curves. We will denote by $I^{\pm}(\cdot, V)$ the chronological relations $I^{\pm}(\cdot)$ restricted to V. It is clear that $I^{\pm}(\cdot, V) \subset I^{\pm}(\cdot) \cap V$, but the equality is not always true.

Example 4.1.14. *Let $\mathbb{M}^3$ be the 3–dimensional Minkowski spacetime and $\mathcal{N}$ its space of light rays. Let us choose any point $\omega \in \mathbb{M}^3$ and consider the spacetime M as the restriction of $\mathbb{M}^3$ to any open half $K \subset \mathbb{M}^3$ of a solid cone with vertex in ω such that $K \subset I^-(\omega)$, as figure 4.3 shows. Notice that $M = I^-(\omega)$ can also be considered. Observe that there exists a light ray γ arriving at points like p^*, so a point $X^+_\gamma \in \partial^+\Sigma_V$ can be defined by γ, and notice that p^* can be identified with the TIP $I^-(\gamma, V)$. But also observe that the point ω is not accessible by any light ray in $M = K$ so there is no point in future Low's boundary corresponding to the TIP $M = I^-(\mu)$ defined by the future–inextensible timelike curve μ.*

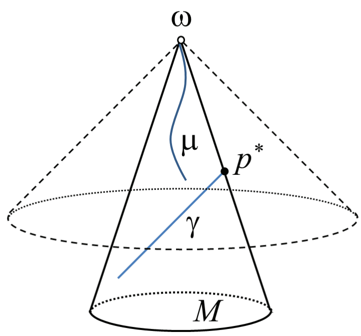

Figure 4.3: Low's boundary is not GKP.

Anyway, Low's boundary can look alike to GKP boundary when we include some topological constraints to the spacetime.

As a first step, it is possible to study Low's boundary corresponding to the restriction of a spacetime M to a suitable open set $V \subset M$. The aim of it is to know how to identify $\partial\Sigma$ under naïve conditions. The study of the future Low's boundary $\partial^+\Sigma$ is enough, because the past one is analogous.

Consider $V \subset M$ an relatively compact basic open set and $\mathcal{U} = \{\gamma \in \mathcal{N} : \gamma \cap V \neq \varnothing\}$. We denote by $\oplus^V$ the field of limiting subspaces tangent to the skies of points in a light ray when they tends to the boundary of V future–directed. So, given $\gamma \in \mathcal{U} \subset \mathcal{N}$ we can parametrize future–directed the segment of γ in V by $\gamma : (a, b) \to V$, then

$$\oplus^V_\gamma = \oplus^V(\gamma) = \lim_{s \mapsto b^-} T_\gamma S(\gamma(s))$$

Observe that a curve $c : I \to \mathcal{U}$ is the integral curve of $\oplus^V$ passing through γ at $\tau = 0$ if

$$\begin{cases} c'(\tau) \in \oplus^V(c(\tau)) \\ c(0) = \gamma \end{cases}$$

Now, consider $x \in \partial V \subset M$ such that $\lim_{s \mapsto b^-} \gamma(s) = x$ and let $\Gamma : I \to X \cap \mathcal{U}$ be a curve travelling along the light rays of the sky $X = S(x)$ in $\mathcal{U}$ such that $\Gamma(\tau) = \gamma_\tau$ with $\gamma_0 = \gamma$ and $\gamma_\tau \cap \overline{V}$ has a future endpoint at x for all $\tau \in I$. Then it is possible to construct a variation of light rays $\mathbf{f} : I \times [0,1] \to \overline{V} \subset M$ such that $\mathbf{f}(\tau, \cdot) \subset \gamma_\tau \in X \cap \mathcal{U}$ and $\mathbf{f}(\tau, 1) = x$ for all $\tau \in I$. It is clear that for all $\tau \in I$ we have

$$\Gamma'(\tau) \in T_{\gamma_\tau} X$$

and using the definition of $\oplus^V$, then

$$\oplus^V_{\Gamma(\tau)} = \oplus^V_{\gamma_\tau} = \lim_{s \mapsto 1^-} T_{\gamma_\tau} S(\gamma_\tau(s)) = T_{\gamma_\tau} S(\gamma_\tau(1)) = T_{\gamma_\tau} S(\mathbf{f}(\tau, 1)) = T_{\gamma_\tau} X$$

and therefore, for all $\tau \in I$

$$\Gamma'(\tau) \in \oplus^V_{\Gamma(\tau)}.$$

This implies that the orbit $X^+ \in \partial^+ \Sigma_V$ of $\oplus^V$ going across γ is just the set of light rays of the sky X coming out of V. So, for any of such extendible spacetime V, Low's boundary is made up of skies of points at the boundary of V.

Let us denote by $\gamma_V = \gamma \cap V$ the segment of the light ray γ contained in V. Consider any $\gamma, \mu \in X^+ \in \partial^+ \Sigma_V$ and any $q \in I^-(\gamma_V, V)$. Since $x \in I^+(q)$ then $\mu_V \cap I^+(q) \neq \varnothing$ and hence there is a timelike curve $\lambda : [0,1] \to M$ such that $\lambda(0) = q \in V$ and $\lambda(1) \in \mu_V \subset V$. But this implies that $\lambda \subset V$ because its endpoints are in a causally convex open set, therefore $q \in I^-(\mu_V, V)$. This shows that $I^-(\gamma_V, V) = I^-(\mu_V, V)$ for any $\gamma, \mu \in X^+$ and therefore there is a well defined map between Low's and GKP boundaries given by

$$X^+ \mapsto I^-(\gamma_V, V)$$

because it is independent from the chosen light ray $\gamma \in X^+$

Since there is no imprisoned causal curve in V, every light ray $\gamma_V \subset V$ has endpoints in the boundary $\partial V \subset M$, then

$$\widetilde{\mathcal{U}} \subset \widetilde{\mathcal{N}} \subset \mathbb{P}(\mathcal{H})$$

is an open manifold with boundary and therefore

$$\partial^+ \widetilde{\mathcal{U}} \hookrightarrow \widetilde{\mathcal{N}}.$$

is a homeomorphism onto its image.

We have proven above that any orbit X^+ of $\oplus^V$ is contained in the sky $X = S(x)$ where $x \in \partial V$, then the set of leaves in the foliation $\left(\mathcal{D}^+_V\right)^\sim$ of tangent spaces to the orbits coincide with the set of leaves in the foliation $(\mathcal{D})^\sim$ of tangent spaces to the skies of points of M restricted to $\partial^+ \widetilde{\mathcal{U}}$, then using equation (4.1.8) we have

$$\left(\partial^+ \Sigma_V\right)^\sim \simeq \partial^+ \widetilde{\mathcal{U}} / \left(\mathcal{D}^+_V\right)^\sim = \partial^+ \widetilde{\mathcal{U}} / \mathcal{D}^\sim \subset \widetilde{\mathcal{N}} / \mathcal{D}^\sim = \Sigma^\sim$$

Using now the inverse of the diffeomorphism $S^\sim : M \to \Sigma^\sim$ of lemma 4.1.7, we obtain that $(S^\sim)^{-1} \left(\partial^+ \widetilde{\mathcal{U}} / \mathcal{D}^\sim\right)$ is contained in ∂V, then the topology of $\left(\partial^+ \Sigma_V\right)^\sim \simeq (S^\sim)^{-1} \left(\partial^+ \widetilde{\mathcal{U}} / \mathcal{D}^\sim\right)$, and therefore also of $\partial^+ \Sigma_V$, is induced by the ambient manifold M. Observe that $(S^\sim)^{-1} \left(\partial^+ \widetilde{\mathcal{U}} / \mathcal{D}^\sim\right)$ is formed by all points in ∂V accessible by a light ray.

In case of any open segment of light ray passing through V is not contained in ∂V, that is any segment of light ray $\gamma : [a,b] \rightarrow M$ with $\gamma(a) \in V$ and $\gamma(b) \notin V$ verifies that $\gamma \cap \partial V$ is just an only point. This is clearly verified for $V = I^+(x) \cap I^-(y)$ such that $J^+(x) \cap J^-(y)$ is closed. Then, it is possible to show that for any $\overline{p} \in \partial V$ accessible by light rays from V there is a neighbourhood $W \subset \partial V$ such that $\overline{q} \in W$ is accessible by light rays from V.

So, let us assume that there is a light ray γ passing by a given $\overline{p} \in \partial V$. We can take a relatively compact, differentiable, spacelike local hypersurface C such that $\overline{p} \in C - \partial C$. If γ is parametrized as the future–directed null geodesic verifying $\gamma(0) = \overline{p}$, then we can construct a non–zero differentiable null vector field $\widetilde{Z} \in \mathfrak{X}_C$ on C such that $\widetilde{Z}_{\overline{p}} = \gamma'(0)$. In this conditions, we will apply the following result.

Lemma 4.1.15. *Let C be a relatively compact, differentiable, spacelike (local) hypersurface and $\widetilde{Z} \in \mathfrak{X}_C$ a non-zero differentiable vector field defined at C and transverse to C, then there exists $\epsilon > 0$ such that*

$$\begin{array}{rccl} F: & C \times (-\epsilon, \epsilon) & \rightarrow & M \\ & (p,s) & \mapsto & F(p,s) = \exp_p\left(s\widetilde{Z}_p\right) \end{array}$$

is a diffeomorphism onto its image.

Proof. First, let us extend $\widetilde{Z}$ to a vector field Z in a neighbourhood $U \subset M$ of C. For every $p \in C$ there are a neighbourhood $U^p \subset C$ and $\delta_p > 0$ such that for all $x \in U^p$ the geodesic $\gamma_x(s) \equiv \exp_x\left(s\widetilde{Z}_x\right)$ is defined for all $s < |\delta_p|$ without conjugated points. Since C is relatively compact, there exists a finite subcovering $\{U^{p_i}\}$ of C. Fixing $\delta = \min\{\delta_{p_i}\}$ then for all $p \in C$ the null geodesic $\gamma_p(s)$ is defined for $s < |\delta|$. Then we can define

$$\begin{array}{rccl} F: & C \times (-\delta, \delta) & \rightarrow & M \\ & (p,s) & \mapsto & F(p,s) = \exp_p\left(s\widetilde{Z}_p\right) \end{array}$$

and if $q = F(p,s) = \gamma_p(s)$ then $Z_q \equiv \gamma_p'(s)$ is an extension of $\widetilde{Z}$ to the open neighbourhood of C given by $\overline{W} = F(C \times (-\delta, \delta)) \subset M$. By the locality of C, we can choose an orthonormal frame $\left\{\widetilde{E}_j\right\}$ on C and propagate it to the whole $\overline{W}$ by parallel transport along every γ_p for all $p \in C$. For every $(p,0) \in C \times (-\delta,\delta)$ we have

$$dF_{(p,0)}\left(\left(\mathbf{0}_p, \tfrac{\partial}{\partial s}\big|_0\right)\right) = \widetilde{Z}_p \in T_pM$$

$$dF_{(p,0)}\left(\left(\left(\widetilde{E}_j\right)_p, \mathbf{0}_0\right)\right) = \left(\widetilde{E}_j\right)_p \in T_pM$$

where $\frac{\partial}{\partial s}$ is the tangent vector field of the curves $\alpha_q(s) = (q,s) \in C \times (-\delta,\delta)$. Since $dF_{(p,0)}$ maps a basis of $T_{(p,0)}(C \times \mathbb{R}) \approx T_pC \times T_0\mathbb{R}$ into a basis of T_pM, then it is an isomorphism and hence F is a local diffeomorphism. So, there exists a neighbourhood $H^p \times (-\epsilon_p, \epsilon_p)$ of $(p,0) \in C \times (-\delta,\delta)$ with $0 < \epsilon_p < \delta$ such that the restriction of F is a diffeomorphism. Again, since C is relatively compact, then from the covering $\{H^p\}$ we can extract a finite subcovering $\left\{H^k\right\}$ of C, then taking $\epsilon = \min\{\epsilon_k\}$ we have

$$C \times (-\epsilon, \epsilon) = \bigcup_k H^k \times (-\epsilon, \epsilon)$$

Calling $W = F(C \times (-\epsilon, \epsilon))$ then for any $(p, s) \in C \times (-\epsilon, \epsilon)$, the map $F : C \times (-\epsilon, \epsilon) \to W$ is a local diffeomorphism. By construction, this restriction of F is surjective, and since there are not conjugated points in the null geodesics γ_q, then we get the injectivity. Therefore we conclude that $F : C \times (-\epsilon, \epsilon) \to W$ is a global diffeomorphism. □

If we apply now lemma 4.1.15 to the proposed hypersurface C, then the image of the map F is an open neighbourhood of $\overline{p} \in M$. We can take a nested sequence $\{C_n\} \subset C$ of neighbourhoods of $\overline{p}$ in C converging to $\{\overline{p}\}$ and restrict F to $C_n \times (-\epsilon, \epsilon)$. Let us assume that for every C_n there exists a null geodesic segment $\gamma_n = F(q_n, (0, \epsilon))$ fully contained in V, then for any $0 < s < \epsilon$ the sequence $F(q_n, s) \mapsto \gamma(s)$ as n increases. Hence $\gamma((0, \epsilon)) \subset \partial V$ since $\gamma((0, \epsilon)) \cap V = \varnothing$, therefore $\gamma|_{(0,\epsilon)}$ is contained in ∂V contradicting that there is no segment of a light ray contained in ∂V.

On the other hand, if for every C_n there is a null geodesic segment $\gamma_n = F(q_n, (-\epsilon, 0))$ without points in V, then as done before, we have that $\gamma((-\epsilon, 0)) \subset \partial V$ but this contradicts that $\gamma((-\epsilon, 0)) \subset V$.

Therefore, there exist $C_k \subset C$ such that for all $q \in C_k$ the null geodesic segment $\gamma_q = F(q, \cdot)$ has endpoints $\gamma_q(s_1) \in V$ and $\gamma_q(s_2) \in M - V$ with $-\epsilon < s_1 < s_2 < \epsilon$. Since ∂V is a topological hypersurface then $B = F(C_k, (-\epsilon, \epsilon)) \cap \partial V$ is an open set of ∂V such that all points in B are accessible by future–directed null geodesic.

Then $\partial^+ \Sigma_V$ is topologically equivalent to an open set relative to ∂V with the induced topology of M. It is also known that the future c–boundary of V is also topologically equivalent to $\partial V \subset M$, so future Low's boundary is equivalent to future c–boundary in the set they shared.

The previous procedure can be carried out for more general spacetimes V, we only need to ensure that any null geodesic γ_q defined by the diffeomorphism F intersects ∂V "transversally" even if ∂V is not smooth, in the sense of crossing ∂V but not remaining in for any interval of the parameter of γ_q.

Now, how can we deal with a general case in order to calculate points in the Low's boundary when there is not any larger spacetime containing M? We can use the previous calculations. Consider any light ray $\gamma \in \mathcal{N}$, then we can parametrize a inextensible future–directed segment of it by $\gamma : [0, b) \to M$. We can cover this segment by means of a numerable collection $\{V_n\}$ formed by relatively compact basic neighbourhoods V_n. Without any lack of generality, we can assume that $V_n \cap V_k \neq \varnothing$ if and only if $n = k \pm 1$ and n increases when $\gamma(s)$ moves to the future. If we denote by $x_n \in \partial V_n$ the future endpoint of $\gamma \cap V_n$, then the orbit of $\oplus^{V_n}$ passing through γ is $X_n \cap \mathcal{U}_n \subset \mathcal{N}$, or in other words, it is defined by $X_n \in \Sigma$. In this way, the orbit $X^+ \in \partial^+\Sigma$ of $\oplus : \mathcal{N} \to \mathbb{P}(\mathcal{H})$ can be constructed by the limit in $\mathcal{N}$ of the sequence $\{X_n\}$, because that limit must exist as we saw in section 4.1.

Section 4.2

Examples of spaces of light rays

In the present section, we offer some examples in which we show explicitly the previously studied structures of their corresponding spaces of light rays. Although we will focus on 3–dimensional spacetimes, we will also deal with 4–dimensional Minkowski spacetime that will help us in the study of two embedded 3–dimensional examples: Minkowski and de Sitter spacetimes. In these two examples, we will proceed restricting them from the 4–dimensional Minkowski example as section 4.2.1 suggests.

4.2.1

Embedded spaces of light rays

Now, we will deal with some particular cases of embedded spacetimes. Let $\overline{M}$ be a $(m+\nu)$–dimensional, strongly causal and null pseudo–convex spacetime with metric $\overline{\mathbf{g}}$ where $m \geq 3$. We will denote overlined its structures $\overline{\mathcal{N}}$, $\overline{\mathcal{H}}$,... Consider $M \subset \overline{M}$ an embedded m–dimensional, strongly causal and null pseudo–convex spacetime equipped with the metric $\mathbf{g} = \overline{\mathbf{g}}|_{TM\times TM}$ such that any maximal null geodesic in M is a maximal null geodesic in $\overline{M}$. Since M is embedded in $\overline{M}$, then trivially TM is embedded in $T\overline{M}$.

Given a basic open set $\overline{V} \subset \overline{M}$ such that $\overline{C} \subset \overline{V}$ is a smooth spacelike Cauchy surface, then clearly $V = \overline{V} \cap M$ is causally convex and contained in a convex normal neighbourhood. Moreover, if $\lambda \subset V$ is a inextensible timelike curve, since $\lambda \subset \overline{V}$ then λ intersects exactly once to $\overline{C}$, hence the intersection point must be in $C = \overline{C} \cap M$ and therefore $C \subset V$ is a smooth spacelike Cauchy surface in V. This implies that V is a basic open set in M.

Observe that the inclusion $TV \hookrightarrow T\overline{V}$ is an embedding, and we can use the chain of manifolds (2.2.8) to ensure that the restriction $\mathbb{N}(C) \hookrightarrow \mathbb{N}\left(\overline{C}\right)$ is also an embedding. Fixed a timelike vector field $Z \in \mathfrak{X}(V)$, since $\overline{V}$ is an arbitrary basic open set, without any lack of generality, we can choose any timelike extension $\overline{Z} \in \mathfrak{X}\left(\overline{V}\right)$ of Z, that is $X = \overline{X}\big|_V$. For all $v \in \mathbb{N}(C) \subset \mathbb{N}\left(\overline{C}\right)$ we have

$$\mathbf{g}(v, Z) = \mathbf{g}\left(v, \overline{Z}\right)$$

Then,

$$\Omega^{Z}(C) = \{v \in \mathbb{N}(C) : \mathbf{g}(v,Z) = -1\} \hookrightarrow \Omega^{\overline{Z}}\left(\overline{C}\right) = \left\{v \in \mathbb{N}\left(\overline{C}\right) : \mathbf{g}\left(v,\overline{Z}\right) = -1\right\}$$

is an embedding. Again, by equation (2.2.8) $\mathcal{U} \simeq \Omega^{Z}(C)$ and $\overline{\mathcal{U}} \simeq \Omega^{\overline{Z}}\left(\overline{C}\right)$, then we have that the inclusion

$$\mathcal{N} \supset \mathcal{U} \hookrightarrow \overline{\mathcal{U}} \subset \overline{\mathcal{N}}$$

is an embedding. Since $\mathcal{N} \hookrightarrow \overline{\mathcal{N}}$ is an inclusion, then it is injective and thus a global embedding. Therefore also

$$T\mathcal{N} \hookrightarrow T\overline{\mathcal{N}}$$

is another global embedding.

Given a point $x \in M \subset \overline{M}$, its sky $X \in \Sigma$ is the set of all light rays contained in $\mathcal{N}$ passing through x, but since every light ray in $\mathcal{N}$ is a light ray in $\overline{\mathcal{N}}$, then calling $\overline{X} \in \overline{\Sigma}$ the sky of x relative to $\overline{\mathcal{N}}$ we have

$$X = \overline{X} \cap \mathcal{N}$$

Since the metric in M is just the restriction to TM of the metric in $\overline{M}$, then the contact structure $\mathcal{H}$ of $\mathcal{N}$ is the restriction of the contact structure $\overline{\mathcal{H}}$ of $\overline{\mathcal{N}}$ to the tangent bundle $T\mathcal{N}$, that is

$$\mathcal{H}_\gamma = \overline{\mathcal{H}}_\gamma \cap T_\gamma \mathcal{N}$$

for all $\gamma \in \mathcal{N}$.

So, for any $\gamma \in X \subset \mathcal{N}$, now it is clear that

$$T_\gamma X = T_\gamma \overline{X} \cap T_\gamma \mathcal{N} = T_\gamma \overline{X} \cap \mathcal{H}_\gamma$$

due to $T_\gamma X \subset \mathcal{H}_\gamma$. For a regular parametrization $\gamma : (a, b) \to M$, then we can write

$$T_\gamma S\left(\gamma\left(s\right)\right) = T_\gamma \overline{S\left(\gamma\left(s\right)\right)} \cap \mathcal{H}_\gamma$$

and hence, the future Low's distribution is

$$\oplus_\gamma = \lim_{s\mapsto b^-} T_\gamma S\left(\gamma\left(s\right)\right) = \lim_{s\mapsto b^-} T_\gamma \overline{S\left(\gamma\left(s\right)\right)} \cap \mathcal{H}_\gamma = \overline{\oplus}_\gamma \cap \mathcal{H}_\gamma$$

If the distribution defined by $\overline{\oplus}$ in $\overline{\mathcal{N}}$ is integrable, then the orbits of $\oplus$ becomes the orbits of $\overline{\oplus}$ restricted to $\mathcal{N}$, that is

$$X^+ = \overline{X}^+ \cap \mathcal{N}$$

Now, we can use the contents of the current section to study of 3–dimensional Minkowski and de Sitter spacetimes as embedded in a 4–dimensional Minkowski spacetime.

4.2.2

4–*dimensional Minkowski*

Consider Minkowski spacetime given by $\mathbb{M}^4 = \left(\mathbb{R}^4, \mathbf{g}\right)$ where the metric is given by $\mathbf{g} = -dt\otimes dt + dx\otimes dx + dy\otimes dy + dz\otimes dz$ in the standard coordinate system $\varphi = (t, x, y, z)$. We will use the notation $\overline{\mathcal{N}}$, $\overline{\mathcal{H}}$, ... for the structures related to $\mathbb{M}^4$.

It is known that the hypersurface $\overline{C} \equiv \{t = 0\}$ is a global Cauchy surface, then by remark 2.2.9, $\overline{\mathcal{N}}$ is diffeomorphic to $\overline{C} \times \mathbb{S}^2$. We can describe points at the sphere $\mathbb{S}^2$ using the angles θ, ϕ of the spherical coordinates. Then, we can use $\psi = (x, y, z, \theta, \phi)$ as a system of coordinates in $\overline{\mathcal{N}}$, where $\psi^{-1}\left(x_0, y_0, z_0, \theta_0, \phi_0\right) = \gamma \in \overline{\mathcal{N}}$ corresponds to the light ray given by

$$\gamma\left(s\right) = \left(s\ ,\ \ x_0 + s\cdot\cos\theta_0 \sin\phi_0\ ,\ \ y_0 + s\cdot\sin\theta_0\sin\phi_0\ ,\ \ z_0 + s\cdot\cos\phi_0\right)$$

with $s \in \mathbb{R}$.

In general, it is possible to calculate the contact hyperplane at $\gamma \in \overline{\mathcal{N}}$ as the vector subspace in $T_\gamma\overline{\mathcal{N}}$ generated by tangent spaces to two different non–conjugated points in γ as done in (3.0.4), or in other words, if $\gamma\left(s_1\right)$ and $\gamma\left(s_2\right)$ are not conjugated along γ then $T_\gamma S\left(\gamma\left(s_1\right)\right) \cap T_\gamma S\left(\gamma\left(s_2\right)\right) = \{\mathbf{0}\}$ and since the suitable dimension is reached, then

$$\overline{\mathcal{H}}_\gamma = T_\gamma S\left(\gamma\left(s_1\right)\right) \oplus T_\gamma S\left(\gamma\left(s_2\right)\right)$$

In case of Minkowski spacetime, there are not conjugate points along any geodesics, so we will use the points $\gamma(0)$ and any $\gamma(s)$.

For any (θ,ϕ) the curve

$$\mu_{(\theta,\phi)}(\tau)=\gamma(s)+\tau\left(1\ ,\ \ \cos\theta\sin\phi\ ,\ \ \sin\theta\sin\phi\ ,\ \ \cos\phi\right)$$

describes a null geodesic passing by $\gamma(s)$ which is in $\overline{C}$ at $\tau=-s$. So, the sky of $\gamma(s)$ can be written in coordinates by

$$\psi\left(S\left(\gamma(s)\right)\right)\equiv\begin{cases} x(\theta,\phi)=x_0+s\left(\cos\theta_0\sin\phi_0-\cos\theta\sin\phi\right)\\ y(\theta,\phi)=y_0+s\left(\sin\theta_0\sin\phi_0-\sin\theta\sin\phi\right)\\ z(\theta,\phi)=z_0+s\left(\cos\phi_0-\cos\phi\right)\\ \theta(\theta,\phi)=\theta\\ \phi(\theta,\phi)=\phi\end{cases}$$

then the derivatives of these expressions with respect to θ and ϕ at $(\theta,\phi)=(\theta_0,\phi_0)$ give us the generators of the tangent space of the sky $S(\gamma(s))$ at γ, so

$$\begin{aligned} T_\gamma S\left(\gamma(s)\right)=\operatorname{span}\Bigg\{ & s\left(\sin\theta_0\sin\phi_0\left(\tfrac{\partial}{\partial x}\right)_\gamma-\cos\theta_0\sin\phi_0\left(\tfrac{\partial}{\partial y}\right)_\gamma\right)+\left(\tfrac{\partial}{\partial\theta}\right)_\gamma,\\ & s\left(-\cos\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial x}\right)_\gamma-\sin\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial y}\right)_\gamma+\sin\phi_0\left(\tfrac{\partial}{\partial z}\right)_\gamma\right)+\left(\tfrac{\partial}{\partial\phi}\right)_\gamma\Bigg\}\end{aligned}$$

and trivially

$$T_\gamma S\left(\gamma(0)\right)=\operatorname{span}\left\{\left(\tfrac{\partial}{\partial\theta}\right)_\gamma,\left(\tfrac{\partial}{\partial\phi}\right)_\gamma\right\}$$

Therefore the contact hyperplane at γ is

$$\begin{aligned}\overline{\mathcal{H}}_\gamma=\operatorname{span}\Bigg\{ & \left(\tfrac{\partial}{\partial\theta}\right)_\gamma,\left(\tfrac{\partial}{\partial\phi}\right)_\gamma,\sin\theta_0\left(\tfrac{\partial}{\partial x}\right)_\gamma-\cos\theta_0\left(\tfrac{\partial}{\partial y}\right)_\gamma,\\ & \cos\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial x}\right)_\gamma+\sin\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial y}\right)_\gamma-\sin\phi_0\left(\tfrac{\partial}{\partial z}\right)_\gamma\Bigg\}\end{aligned}$$

and a contact form is

$$\overline{\alpha}=\cos\theta\sin\phi\cdot dx+\sin\theta\sin\phi\cdot dy+\cos\phi\cdot dz$$

For this spacetime it is possible to calculate $\oplus$ and $\ominus$. We will proceed only for $\oplus$ because the case of $\ominus$ is analogous. Using the definition (4.1.1), we have

$$\begin{aligned}\oplus_\gamma &=\lim_{s\mapsto+\infty}T_\gamma S\left(\gamma(s)\right)=\\ &=\operatorname{span}\Bigg\{\sin\theta_0\sin\phi_0\left(\tfrac{\partial}{\partial x}\right)_\gamma-\cos\theta_0\sin\phi_0\left(\tfrac{\partial}{\partial y}\right)_\gamma,\\ &\quad -\cos\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial x}\right)_\gamma-\sin\theta_0\cos\phi_0\left(\tfrac{\partial}{\partial y}\right)_\gamma+\sin\phi_0\left(\tfrac{\partial}{\partial z}\right)_\gamma\Bigg\}\end{aligned}$$

and therefore $\oplus$ defines a integrable distribution whose partial differential equations are

$$\left\{\begin{array}{l}\frac{\partial x}{\partial \alpha}(\alpha,\beta)=\sin\theta\sin\phi\\ \frac{\partial y}{\partial \alpha}(\alpha,\beta)=-\cos\theta\sin\phi\\ \frac{\partial z}{\partial \alpha}(\alpha,\beta)=0\\ \frac{\partial \theta}{\partial \alpha}(\alpha,\beta)=0\\ \frac{\partial \phi}{\partial \alpha}(\alpha,\beta)=0\end{array}\right. \qquad \left\{\begin{array}{l}\frac{\partial x}{\partial \beta}(\alpha,\beta)=-\cos\theta\cos\phi\\ \frac{\partial y}{\partial \beta}(\alpha,\beta)=-\sin\theta\cos\phi\\ \frac{\partial z}{\partial \beta}(\alpha,\beta)=\sin\phi\\ \frac{\partial \theta}{\partial \beta}(\alpha,\beta)=0\\ \frac{\partial \phi}{\partial \beta}(\alpha,\beta)=0\end{array}\right.$$

and its solution with initial values $(x_0, y_0, z_0, \theta_0, \phi_0)$ is given by

$$\left\{\begin{array}{l}x(\alpha,\beta)=x_0+\alpha\sin\theta_0\sin\phi_0-\beta\cos\theta_0\cos\phi_0\\ y(\alpha,\beta)=y_0-\alpha\cos\theta_0\sin\phi_0-\beta\sin\theta_0\cos\phi_0\\ z(\alpha,\beta)=z_0+\beta\sin\phi_0\\ \theta(\alpha,\beta)=\theta_0\\ \phi(\alpha,\beta)=\phi_0\end{array}\right. \tag{4.2.1}$$

This solution corresponds to the 2–plane

$$\cos\theta_0\sin\phi_0\cdot(x-x_0)+\sin\theta_0\sin\phi_0\cdot(y-y_0)+\cos\phi_0\cdot(z-z_0)=0 \tag{4.2.2}$$

in the Cauchy surface $\overline{C}$ and it defines the orbit $\overline{X}^+_\gamma$ of $\oplus$ passing through γ. The image in M of all the light rays in $\overline{X}^+_\gamma$ is precisely the 3–plane in $\mathbb{M}^4$ given by

$$\cos\theta_0\sin\phi_0\cdot(x-x_0)+\sin\theta_0\sin\phi_0\cdot(y-y_0)+\cos\phi_0\cdot(z-z_0)-t=0$$

and it is easy to show, using straightforward calculations, that any light ray $\mu\in\overline{X}^+_\gamma$ in the same orbit of $\oplus$ than γ determines the TIP

$$I^-(\mu)=I^-(\gamma)=\{t<\cos\theta_0\sin\phi_0\cdot(x-x_0)+\sin\theta_0\sin\phi_0\cdot(y-y_0)+\cos\phi_0\cdot(z-z_0)\}$$

so the future Low's boundary coincides with c-boundary one except for the TIP $I^-(\lambda)=\mathbb{M}^4$ defined by a timelike geodesic λ, because it can not be defined by light rays.

Moreover, [19, Thm. 4.16] ensures that, for this spacetime, c–boundary is the same than conformal boundary.

The Low's boundary corresponds to the set of all orbits of $\oplus$, that is, all existent 2-planes (4.2.2). Observe that the map

$$\begin{array}{ccc}\mathbb{R}^3\times\mathbb{S}^2\simeq\overline{\mathcal{N}} & \to & \partial^+\overline{\Sigma}\simeq\mathbb{R}^1\times\mathbb{S}^2\\ \gamma & \mapsto & \overline{X}^+_\gamma\end{array} \tag{4.2.3}$$

such that every light ray $\gamma\in\mathcal{N}$ is mapped to the point of Low's boundary corresponding to the orbit of $\oplus$ passing through γ can be written in coordinates by

$$(x,y,z,\theta,\phi)\mapsto(\cos\theta\sin\phi\cdot x+\sin\theta\sin\phi\cdot y+\cos\phi\cdot z,\theta,\phi)$$

therefore future Low's boundary is $\partial^+\overline{\Sigma}\simeq\mathbb{R}^1\times\mathbb{S}^2$.

4.2.3

3–*dimensional Minkowski*

Let us proceed now with 3–dimensional Minkowski spacetime given by $\mathbb{M}^3 = \left(\mathbb{R}^3, \mathbf{g}\right)$ with metric $\mathbf{g} = -dt \otimes dt + dx \otimes dx + dy \otimes dy$ in coordinates $\varphi = (t, x, y)$. We will use the notation $\mathcal{N}$, $\mathcal{H}$, ... for the structures related to $\mathbb{M}^3$.

It is possible to see $\mathbb{M}^3$ as the restriction of $\mathbb{M}^4$ to its hyperplane $z = 0$. So, in order to obtain the description of the space of light rays of $\mathbb{M}^3$, we can restrict the results obtained in section 4.2.2 to $z = 0$ and therefore, also $\phi = \pi/2$.

Then, $C \equiv \{t = 0\}$ is still a Cauchy surface and $\mathcal{N} \simeq C \times \mathbb{S}^1$ and we can use $\psi = (x, y, \theta)$ as a system of coordinates in $\mathcal{N}$, where $\psi^{-1}(x_0, y_0, \theta_0) = \gamma \in \mathcal{N}$ describes the light ray given by

$$\gamma(s) = (s\ ,\ \ x_0 + s \cdot \cos\theta_0\ ,\ \ y_0 + s \cdot \sin\theta_0)$$

with $s \in \mathbb{R}$.

So, the tangent space of the skies $S(\gamma(s))$ and $S(\gamma(0))$ at γ can be written as

$$T_\gamma S(\gamma(s)) = \operatorname{span}\left\{ s\left(\sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma - \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma\right) + \left(\tfrac{\partial}{\partial \theta}\right)_\gamma \right\} \tag{4.2.4}$$

and

$$T_\gamma S(\gamma(0)) = \operatorname{span}\left\{ \left(\tfrac{\partial}{\partial \theta}\right)_\gamma \right\}$$

Therefore the contact hyperplane at γ is

$$\mathcal{H}_\gamma = \operatorname{span}\left\{ \sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma - \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma , \left(\tfrac{\partial}{\partial \theta}\right)_\gamma \right\}$$

and any contact form will be proportional to

$$\alpha = \cos\theta \cdot dx + \sin\theta \cdot dy$$

Using expression 4.2.4 it is possible to calculate easily the point in Low's boundary passing by γ , then

$$\oplus_\gamma = \lim_{s \mapsto +\infty} T_\gamma S(\gamma(s)) = \operatorname{span}\left\{ \sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma - \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma \right\}$$

and therefore we can obtain the integral curve $c(\tau) = (x(\tau), y(\tau), \theta(\tau))$ defining the orbit $X_\gamma^+ \subset \mathcal{N}$ of $\oplus$ containing γ solving the initial value problem

$$\left\{ \begin{array}{l} x'(\tau) = \sin\theta \\ y'(\tau) = -\cos\theta \\ \theta'(\tau) = 0 \\ c(0) = (x_0, y_0, \theta_0) \end{array} \right.$$

Its solution is $c(\tau) = (x_0 + \tau\sin\theta_0\ , y_0 - \tau\cos\theta_0\ , \theta_0)$ and it corresponds to the family of null geodesics with tangent vector $v = (1, \cos\theta_0, \sin\theta_0)$ and initial value in the straight line contained in C given by

$$\left\{ \begin{array}{l} \cos\theta_0 (x - x_0) + \sin\theta_0 (y - y_0) = 0 \\ t = 0 \end{array} \right. .$$

Again, by straightforward calculations, it is possible to show that given $\mu_1, \mu_2 \in X^+_\gamma$ then $I^-(\mu_1) = I^-(\mu_2)$, therefore any light ray in X^+_γ defines the same TIP

$$I^-(\gamma) = \left\{(t,x,y) \in \mathbb{M}^3 : t < \cos\theta_0 (x - x_0) + \sin\theta_0 (y - y_0)\right\}.$$

then, again future Low's boundary coincides with the future part of the c–boundary accessible by light rays.

In an analogous way, the orbit X^-_γ of $\ominus$ verifies $X^-_\gamma = X^+_\gamma$ and thus it corresponds to the TIF $I^+(\gamma)$.

The restriction of the map (4.2.3) to $\mathcal{N} \simeq \mathbb{R}^2 \times \mathbb{S}^1$ results

$$\begin{array}{rcl} \mathbb{R}^2 \times \mathbb{S}^1 \simeq \mathcal{N} & \to & \partial^+\Sigma \simeq \mathbb{R}^1 \times \mathbb{S}^1 \\ \gamma & \mapsto & X^+_\gamma \end{array}$$

that, in coordinates, can be written by

$$(x,y,\theta) \mapsto (\cos\theta \cdot x + \sin\theta \cdot y, \theta)$$

therefore, $\partial^+\Sigma \simeq \mathbb{R}^1 \times \mathbb{S}^1$.

We can use the previous calculations to describe a globally hyperbolic block embedded in $\mathbb{M}^3$. Let us call $M_* = \left\{(t,x,y) \in \mathbb{M}^3 : t > -1\right\}$ with the same metric $\mathbf{g}$ restricted to M_*, and denote by $\mathcal{N}_*$, $\mathcal{H}_*$,... the corresponding structures for M_*. Since $M_* \subset \mathbb{M}^3$ is open and they share the same Cauchy surface $C \equiv \{t = 0\}$, then trivially $\mathcal{N}_* \simeq \mathcal{N}$ and $\mathcal{H}_* \simeq \mathcal{H}$. To calculate $\ominus_*$, we can consider the limit of the expression (4.2.4) when s tends to -1, then

$$(\ominus_*)_\gamma = \lim_{s \mapsto -1} T_\gamma S(\gamma(s)) = \operatorname{span}\left\{-\sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma + \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma + \left(\tfrac{\partial}{\partial \theta}\right)_\gamma\right\}$$

Thus, the orbit $X^-_\gamma \subset \mathcal{N}_*$ of $\ominus_*$ passing by γ is the solution $c(\tau) = (x(\tau), y(\tau), \theta(\tau))$ of

$$\left\{\begin{array}{l} x'(\tau) = -\sin\theta \\ y'(\tau) = \cos\theta \\ \theta'(\tau) = 1 \\ c(0) = (x_0, y_0, \theta_0) \end{array}\right.$$

and it is given by $c(\tau) = (x_0 + \cos(\tau + \theta_0)\ , y_0 + \sin\theta_0 (\tau + \theta_0)\ , \tau + \theta_0)$. The light ray in X^-_γ defined by $c(\tau)$ can be parametrized (as a null geodesic) by

$$\gamma_\tau(s) = (s\, , x(\tau) + s\cos\theta(\tau)\ , y(\tau) + s\sin\theta(\tau)) =$$

$$= (s\, , x_0 + (s+1)\cos(\tau + \theta_0)\ , y_0 + (s+1)\sin(\tau + \theta_0))$$

verifying $\lim_{s \mapsto -1} \gamma_\tau(s) = (-1, x_0, y_0)$ for all τ. This clearly shows that $X^-_\gamma \subset \mathcal{N}_*$ can be identified with $S((-1, x_0, y_0)) \subset \mathcal{N}$ and therefore past Low's completion $M_* \cup \partial^-\Sigma_* \simeq \left\{(t,x,y) \in \mathbb{M}^3 : t \geq -1\right\}$ diffeomorphically.

4.2.4

3–*dimensional de Sitter spacetime*

We will continue using the notation of section 4.2.2.

We can define the *de Sitter spacetime* S_1^3 as the set in $\mathbb{M}^4$ verifying

$$-t^2 + x^2 + y^2 + z^2 = 1 \tag{4.2.5}$$

We will denote the structures related to S_1^3 by $\mathcal{N}_S$, $\mathcal{H}_S$,... By [53, Prop. 4.28], light rays in $\mathcal{N}_S$ are straight lines in $\mathbb{M}^4$ contained in S_1^3, that is, light rays in $\mathbb{M}^4$ too.

Let us consider the Cauchy surface in S_1^3 given by $C_S = \overline{C} \cap S_1^3$, that is, the 2-surface verifying

$$\begin{cases} t = 0 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

so we can parametrize C_S by

$$\begin{cases} x = \cos u \sin w \\ y = \sin u \sin w \\ z = \cos w \end{cases} \tag{4.2.6}$$

Obviously, the null geodesic $\gamma \in \overline{\mathcal{N}}$ will entirely lie in S_1^3 if it verifies the equation (4.2.5), so for every s we have

$$-s^2 + (x + s\cos\theta\sin\phi)^2 + (y + s\sin\theta\sin\phi)^2 + (z + s\cos\phi)^2 = 1$$

that can be simplified into

$$2s\left((x\cos\theta + y\sin\theta)\sin\phi + z\cos\phi\right) = 0$$

therefore

$$(x\cos\theta + y\sin\theta)\sin\phi + z\cos\phi = 0 \tag{4.2.7}$$

and hence, we solve

$$\cot\phi = -\frac{x\cos\theta + y\sin\theta}{z}.$$

By the relation (4.2.6) we can write

$$\cot\phi = -\cos(\theta - u)\tan w$$

so ϕ only depends on the variables u, w, θ. We will abbreviate it as

$$\cot\phi = f(u, w, \theta)$$

Let us restrict the contact form α to $\mathcal{N}_S$ considering

$$\begin{cases} x = \cos u \sin w \\ y = \sin u \sin w \\ z = \cos w \\ \theta = \theta \\ \phi = \mathrm{arccot} f(u, w, \theta) \end{cases} \tag{4.2.8}$$

Changing the differentials

$$\begin{cases} dx = -\sin u \sin w du + \cos u \cos w dw \\ dy = \cos u \sin w du + \sin u \cos w dw \\ dz = -\sin w dw \end{cases}$$

into $\overline{\alpha}$ we obtain

$$\alpha_S = \overline{\alpha}|_{\mathcal{N}_S} = \cos\theta \sin\phi \left(-\sin u \sin w du + \cos u \cos w dw\right) +$$
$$+ \sin\theta \sin\phi \left(\cos u \sin w du + \sin u \cos w dw\right) + \cos\phi \left(-\sin w dw\right) =$$
$$= \frac{-\cos w \sin w \sin(\theta-u)}{\sqrt{\cos^2(\theta-u)\sin^2 w + \cos^2 w}} du - \frac{\cos(\theta-u)}{\sqrt{\cos^2(\theta-u)\sin^2 w + \cos^2 w}} dw$$

where we have used the relations, obtained from (4.2.7), given by

$$\begin{cases} \sin\phi = \frac{-\cos w}{\sqrt{\cos^2(\theta-u)\sin^2 w + \cos^2 w}} \\ \\ \cos\phi = \frac{\sin w \cos(\theta-u)}{\sqrt{\cos^2(\theta-u)\sin^2 w + \cos^2 w}} \end{cases} . \tag{4.2.9}$$

We can choose the following contact form in $\mathcal{N}_S$

$$\alpha_S = \cos w \sin w \sin(\theta - u)\, du + \cos(\theta - u)\, dw$$

Then, the 2-plane that annihilates α_S is

$$(\mathcal{H}_S)_\gamma = \operatorname{span}\left\{ -\cos(\theta - u) \left(\tfrac{\partial}{\partial u}\right)_\gamma + \cos w \sin w \sin(\theta - u) \left(\tfrac{\partial}{\partial w}\right)_\gamma , \left(\tfrac{\partial}{\partial \theta}\right)_\gamma \right\}$$

In order to find the future Low's boundary of 3–dimensional de Sitter spacetime, in virtue of section 4.2.1, we will restrict the results obtained in section 4.2.2 for $\mathbb{M}^4$ to the embedded S_1^3. So, using the expression (4.2.8) for the values (u_0, w_0, θ_0) we have

$$(x_0, y_0, z_0, \theta_0, \phi_0) = (\cos u_0 \sin w_0,\ \sin u_0 \sin w_0,\ \cos w_0,\ \theta_0,\ \operatorname{arccot} f(u_0, w_0, \theta_0))$$

and substituting it, together with (4.2.9), into the equation (4.2.2), we obtain the equation of the orbit $\left(X_S^+\right)_\gamma = \overline{X}_\gamma^+ \cap \mathcal{N}_S$ of $\oplus^S$ through γ as a curve in the Cauchy surface C_S given by

$$\cos(\theta_0 - u)\tan w = \cos(\theta_0 - u_0)\tan w_0 \tag{4.2.10}$$

or equivalently

$$f(u, w, \theta_0) = f(u_0, w_0, \theta_0)\,. \tag{4.2.11}$$

If we consider the inclusion in coordinates

$$\begin{array}{rrcl} i: & \mathcal{N}_S \simeq \mathbb{S}^2 \times \mathbb{S}^1 & \to & \overline{\mathcal{N}} \simeq \mathbb{R}^3 \times \mathbb{S}^2 \\ & (u, w, \theta) & \mapsto & (\cos u \sin w,\ \sin u \sin w,\ \cos w,\ \theta,\ \operatorname{arccot} f(u, w, \theta)) \end{array} \tag{4.2.12}$$

then its composition with the map (4.2.3) is

$$\begin{array}{rcl} \mathcal{N}_S \simeq \mathbb{S}^2 \times \mathbb{S}^1 & \to & \partial^+\Sigma_S \subset \mathbb{R}^1 \times \mathbb{S}^2 \\ (u, w, \theta) & \mapsto & (0, \theta, \operatorname{arccot} f(u, w, \theta)) \end{array} \tag{4.2.13}$$

For a fixed $\theta = \theta_0$, because (4.2.11), every level set $U_k = \{(u,w) \in C_S : f(u,v,\theta_0) = k\}$ corresponds to an orbit of $\oplus^S$. Since the image of

$$F(u,w) = f(u,v,\theta_0) = -\cos(\theta_0 - u)\tan w$$

is $(-\infty,\infty)$ then the image of

$$G(u,w) = \text{arccot} f(u,v,\theta_0)$$

is $(0,\pi)$, therefore the image of the map (4.2.13) is $\partial^+\Sigma_S = \{0\} \times \mathbb{S}^2 \simeq \mathbb{S}^2$.

By [53, Prop. 4.28], it can be easily observed that $I^-(p) \cap S_1^3 = I^-\left(p, S_1^3\right)$ and hence, for any light ray $\gamma \in \mathcal{N}_S$

$$I^-(\gamma) \cap S_1^3 = I^-\left(\gamma, S_1^3\right)$$

Thus, the restriction of TIPs of $\mathbb{M}^4$ to de Sitter spacetime are TIPs of S_1^3, and therefore future Low's boundary of de Sitter spacetime coincides with the part of future c–boundary accessible by null geodesics.

4.2.5

A family of 3–dimensional spacetimes

In this section we will study the family of spacetimes given by $M_\alpha = \left\{(t,x,y) \in \mathbb{R}^3 : t > 0\right\}$ with metric tensor $\mathbf{g}_\alpha = -t^{2\alpha} dt \otimes dt + dx \otimes dx + dy \otimes dy$.

It is trivial to see that the transformations given by

$$\underline{\text{For } \alpha < -1}: \quad \left\{ \begin{array}{l} \overline{t} = \frac{t^{\alpha+1}}{\alpha+1} \\ \overline{x} = x \\ \overline{y} = y \end{array} \right. \qquad \underline{\text{For } \alpha = -1}: \quad \left\{ \begin{array}{l} \overline{t} = \log t \\ \overline{x} = x \\ \overline{y} = y \end{array} \right. \qquad \underline{\text{For } \alpha > -1}: \quad \left\{ \begin{array}{l} \overline{t} = \frac{t^{\alpha+1}}{\alpha+1} - 1 \\ \overline{x} = x \\ \overline{y} = y \end{array} \right. \tag{4.2.14}$$

are conformal diffeomorphisms such that

$$\underline{\text{For } \alpha < -1}: \ M_\alpha \simeq \mathbb{M}^3 \qquad \underline{\text{For } \alpha = -1}: \ M_{-1} \simeq \mathbb{M}^3 \qquad \underline{\text{For } \alpha > -1}: \ M_\alpha \simeq M_*$$

where the last spacetime M_* denotes the 3–dimensional Minkowski block studied in section 4.2.3. So, the space of light rays, its contact structure and Low's boundary of these spacetimes are already calculated in section 4.2.3.

Anyway, we will take a closer look at Low's boundary for $\alpha > -1$.

Observe that the null vectors in T_pM_α are proportional to $v = (1, t^\alpha \cos\theta, t^\alpha \sin\theta)$ for $\theta \in [0, 2\pi]$ at $p = (t,x,y)$, and the only non–zero Christoffel symbol is $\Gamma_{00}^0 = \alpha t^{-1}$. Hence, since the equations of geodesics are

$$\left\{ \begin{array}{l} t'' + \frac{\alpha}{t}(t')^2 = 0 \\ x'' = 0 \\ y'' = 0 \end{array} \right.$$

then the null geodesic γ such that $\gamma(0) = (t_0, x_0, y_0)$ and $\gamma'(0) = (1, t_0^\alpha \cos\theta_0, t_0^\alpha \sin\theta_0)$ for a given $\theta_0 \in [0, 2\pi]$ for $\alpha > -1$ can be written as

$$\gamma(s) = \left(\left((\alpha+1)\, t_0^\alpha s + t_0^{\alpha+1} \right)^{1/(\alpha+1)} ,\ x_0 + s t_0^\alpha \cos\theta_0 ,\ y_0 + s t_0^\alpha \sin\theta_0 \right)$$

defined for $s \in \left(\frac{-t_0}{\alpha+1}, \infty \right)$.

Observe that, when $-1 < \alpha < 0$, lightcones open wider as t approaches to 0, becoming a plane at the limit $t = 0$. On the other hand, when $\alpha > 0$, they close narrower when t gets close to 0, degenerating into a line when $t = 0$. The case $\alpha = 0$ corresponds to a Minkowski block isometric to M_*.

Let us consider $C \equiv \{t = 1\}$ as the global Cauchy surface we will use as origin of any given null geodesic

$$\gamma(s) = \left(\left((\alpha+1)\, s + 1 \right)^{1/(\alpha+1)} ,\ x_0 + s\cos\theta_0 ,\ y_0 + s\sin\theta_0 \right) = (t_s, x_s, y_s)$$

Then the curve

$$\mu_\theta(\tau) = \left(\left((\alpha+1)\, t_s^\alpha \tau + t_s^{\alpha+1} \right)^{1/(\alpha+1)} ,\ x_s + \tau t_s^\alpha \cos\theta ,\ y_s + \tau t_s^\alpha \sin\theta \right)$$

describes a null geodesic starting at $\gamma(s)$. So, for $\tau = \frac{-s}{t_s^\alpha}$, we have

$$\mu_\theta\left(-s/t_s^\alpha\right) = \left(0,\ x_0 + s\left(\cos\theta_0 - \cos\theta\right), y_0 + s\left(\sin\theta_0 - \sin\theta\right)\right) \in C.$$

Therefore, the coordinates of the sky of $\gamma(s)$ can be written by

$$\psi\left(S\left(\gamma(s)\right)\right) \equiv \begin{cases} x(\theta) = x_0 + s\left(\cos\theta_0 - \cos\theta\right) \\ y(\theta) = y_0 + s\left(\sin\theta_0 - \sin\theta\right) \\ \theta(\theta) = \theta \end{cases}$$

Deriving with respect to θ at $\theta = \theta_0$, we obtain a generator of the tangent space of the sky $S(\gamma(s))$ at γ, so

$$T_\gamma S\left(\gamma(s)\right) = \operatorname{span}\left\{ s\left(\sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma - \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma \right) + \left(\tfrac{\partial}{\partial\theta}\right)_\gamma \right\}$$

and then

$$\left(\ominus_\alpha\right)_\gamma = \lim_{s \mapsto \frac{-1}{\alpha+1}} T_\gamma S\left(\gamma(s)\right) = \operatorname{span}\left\{ -\sin\theta_0 \left(\tfrac{\partial}{\partial x}\right)_\gamma + \cos\theta_0 \left(\tfrac{\partial}{\partial y}\right)_\gamma + (\alpha+1)\left(\tfrac{\partial}{\partial\theta}\right)_\gamma \right\}$$

The solution $c(\tau) = \left(x(\tau), y(\tau), \theta(\tau)\right)$ of the initial value problem

$$\begin{cases} x'(\tau) = -\sin\theta \\ y'(\tau) = \cos\theta \\ \theta'(\tau) = \alpha + 1 \\ c(0) = (x_0, y_0, \theta_0) \end{cases}$$

describes the orbit $X_\gamma^- \subset \mathcal{N}_\alpha$ of $\ominus_\alpha$ passing by γ. Then

$$c(\tau) = \left(x_0 + \tfrac{\cos((\alpha+1)\tau + \theta_0) - \cos\theta_0}{\alpha+1} ,\ y_0 + \tfrac{\sin((\alpha+1)\tau + \theta_0) - \sin\theta_0}{\alpha+1} ,\ (\alpha+1)\,\tau + \theta_0 \right)$$

It is easy to realize that the points in M_α in the orbit X_γ^- verify

$$t^{2\alpha+2} = (\alpha+1)^2 \left[\left(x - \left(x_0 - \tfrac{\cos\theta_0}{\alpha+1} \right) \right)^2 + \left(y - \left(y_0 - \tfrac{\sin\theta_0}{\alpha+1} \right) \right)^2 \right] \tag{4.2.15}$$

A schematic picture of X_γ^- can be seen in figure 4.4.

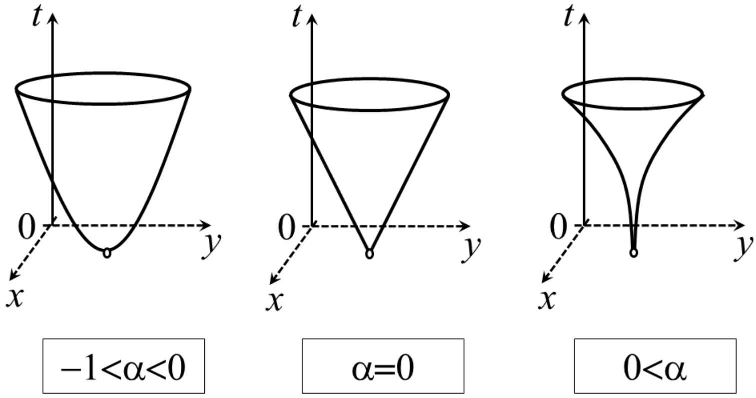

Figure 4.4: The α-family of spacetimes.

Observe that each orbit X_γ^- is determined by the vertex of the surface (4.2.15), therefore past Low's boundary can be identified with $\mathbb{R}^2$ such that any $(u,v) \in \mathbb{R}^2$ corresponds to the orbit of $\ominus_\alpha$ whose light rays emerges from the point $(t,x,y) = (0,u,v)$.

The differentiable structure of $\overline{M_\alpha} = M_\alpha \cup \partial^- \Sigma_\alpha$ can not be the standard one induced from $\overline{M_*} = M_* \cup \partial^- \Sigma_* = \left\{ (t,x,y) \in \mathbb{R}^3 : t \geq -1 \right\}$ by the corresponding conformal mapping (4.2.14), because it would be needed that

$$\begin{array}{ccc} \overline{M_\alpha} & \to & \overline{M_*} \\ (t,x,y) & \mapsto & \left(\frac{t^{\alpha+1}}{\alpha+1} - 1, x, y \right) \end{array}$$

were differentiable, but it is not the case with the standard differentiable structure when $-1 < \alpha < 0$.

Section 4.3

Future lines of research

Throughout this work, we have assumed the strong causality condition for the conformal manifold M in order to construct its space of light rays rays $\mathcal{N}$ and its structures therein. Strongly causal spacetimes is the largest class for which is possible to build the space of light rays but it is just a sufficient condition. In fact, we have not needed all the power of strong causality because we only have used that property for light rays and not for other causal curves.

In virtue of theorem 1.2.15, any spacetime M has a neighbourhood basis consisting of globally hyperbolic and normal neighbourhoods. If the open set of this basis are causally convex, then M is strongly causal. Then, we can replace the property of causal convexity of the basis neighbourhoods with the condition of *light convexity*, that is, $U \subset M$ is said to be *lightly convex* if $x \neq y \in U$ are connected by a future–directed inextensible null geodesic $\gamma : I \to M$ with $x = \gamma(a) < \gamma(b) = y \in U$, then $\gamma([a, b]) \subset U$.

Observe that the neighbourhoods U of this new basis remain globally hyperbolic and normal, then any null geodesic passing through U intersects the local Cauchy surface in an only point and there is not conjugated points along null geodesics in U. Moreover, the definition of light convexity property trivially implies that $\gamma \cap U$ has a unique connected component for $\gamma \in \mathcal{N}$ and U a lightly convex neighbourhood. This implies that the distribution $\mathcal{D} = \text{span}\{X_{\mathbf{g}}, \Delta\}$ used in section 2.2 to define $\mathcal{N} = \mathbb{N}^+/\mathcal{D}$ is still regular, therefore $\mathcal{N}$ becomes a quotient manifold. Moreover, if M is null pseudo–convex, and since there is not any null geodesic imprisoned in any compact, then the proof of proposition 2.2.14 can be achieved in identical way and $\mathcal{N}$ becomes Hausdorff.

The neighbourhoods of this lightly convex basis can be used, as we did in sections 3.1 and 3.2, to define coordinates in $T\mathcal{N}$ as well as the Low's topology in Σ. Theorem 3.2.8 is still true because in its proof we only use the absence of conjugated points and the existence of a local Cauchy surface. Thus, corollary 3.2.9 can be proved with the same proof.

Unfortunately, in this work, it is not possible to replace globally the hypothesis of strong causality for light convexity property. This is because the proof of proposition 3.5.2 uses, in a decisive way, the causal convexity condition in order to prove that property 4 in the definition of regular sets is verified.

The importance of weakening strong causality condition is due to the convenience of studying non–strongly causal spacetimes like Anti–de Sitter among others. This spacetime is chronological but non–causal, and it is lightly convex. It is possible to check that its space of light rays can be well described from the point of view of this work.

As an easy and illustrative example, consider the spacetime given by

$$M = \left\{(t, x, y) \in \mathbb{M}^3 : 0 \leq t < 1, (t + 1, x, y) \sim (t, x, y)\right\}$$

where $\mathbb{M}^3$ is the Minkowski spacetime and $\sim$ denotes the identification of the planes $t = 0$ and $t = 1$. Observe that M is not even chronological since the curve $\lambda(s) = (s, x_0, y_0)$ is a closed timelike geodesic for any fixed (x_0, y_0). But $\mathcal{N}$ can be fairly defined for M since it coincides with Minkowski spacetime locally, and since null geodesics do not accumulates outside themselves, it is equipped with good topological properties.

So, the following open questions arise:

- Is light convexity condition enough to ensure the equivalence between reconstructive and regular sets topology?

- Can the conditions in the definition of regular sets be weaken but still defining the same topology?

- Can weak refocusing exist in a sky–separating non-strongly causal spacetime?

Another subject with many open questions is Low's boundary. We have study some aspects of the 3–dimensional case and shown an example (section 4.2.2) of its existence and good properties in higher dimension. Also, we have checked in section 4.1.2 that this new boundary can be comparable with c-boundary, in spite of they do not coincide in general, because the existence of points at c-boundary that can not be defined by light rays. Moreover, we have seen examples in which both boundaries are essentially the same. In proposition 4.1.8, we have assumed that $\ominus$ and $\oplus$ are distribution, but this fact seems to depend on the global geometry of the spacetime M. So, some pending question are:

- What conditions ensure that $\ominus$ and $\oplus$ are distribution?
- Can the conformal structure of M be extended to Low's completion $\overline{M}$? This means that any $X^{\pm} \in \partial^{\pm}\Sigma$ is, in fact, a sky of $\overline{M}$.
- Is really Low's boundary an open subset of c-boundary as examples suggest?
- Are the results in section 4.1 still true for any dimension $m \geq 3$?

In section 2.4, we studied the existence of a canonical contact structure $\mathcal{H}$ turning $\mathcal{N}$ into a contact manifold. Contact structures of 3–dimensional contact manifolds can be divided in two types: *overtwisted* and *tight* (not overtwisted) [20, Sect. 4.5]. A complete classification of overtwisted and tight contact structures in 3–dimensional contact manifolds is known, mainly due to the work of Y. Eliashberg [18] and K. Honda [28], [29] among others. Thus, the classification of $\mathcal{H}$ is an open question.

- Is $\mathcal{H}$ tight or overtwisted?
- In case of tight, is $\mathcal{H}$ fillable by the symplectic structure defined by the manifold of timelike geodesic of M?

Maybe further but already in the horizon, the converse problem arises.

- Given a contact manifold $\mathcal{N}_0$ and a family of legendrian spheres Σ_0, is there any conformal manifold $(M, \mathcal{C})$ with space of light rays $\mathcal{N}$ and space of skies Σ such that $\mathcal{N} = \mathcal{N}_0$ and $\Sigma = \Sigma_0$ isomorphically?

The point of view in this work is closer to the study of the relationship between M and $\mathcal{N}$ than to set up $\mathcal{N}$ as the manifold where to study conformal properties of specific spacetimes. We have characterized the causal structure of M in terms of $\mathcal{N}$, but many others conformal invariants of M could be described in its space of light rays in order to use $\mathcal{N}$ as working manifold. So, we believe that, in this new scope, an interesting road is laid to be travelled by future researches. Time and new efforts will say if this outlook is fruitful.

Bibliography

[1] R. Abraham and J.E. Marsden. *Foundations of Mechanics.* Addison-Wesley. Redwood City., 1987.

[2] R. Abraham, J.E. Marsden, and T. Ratiu. *Manifolds, tensor analysis, and applications.* Springer Verlag. New York, 1988.

[3] V.I. Arnold. *Mathematical methods of classical mechanics.* Springer Verlag. New York, 1989.

[4] M.F. Atiyah and I.G. Macdonald. *Introduction to conmutative algebra.* Addison–Wesley, London, 1969.

[5] A. Bautista, A. Ibort, and J. Lafuente. *On the space of light rays of a spacetime and a reconstruction theorem by Low.* Class. Quantum Grav. 31, 7, 075020, 2014.

[6] A. Bautista, A. Ibort, and J. Lafuente. *The canonical contact structure in the space of light rays.* Libro homenaje a José María Montesinos. Universidad Complutense of Madrid, 2015.

[7] A. Bautista, A. Ibort, and J. Lafuente. *Causality and skies: is non-refocussing necessary?* Class. Quantum Grav. 32, 10, 105002, 2015.

[8] J.K. Beem, Ehrlich P.E., and K.L. Easley. *Global lorentzian geometry.* Marcel Dekker. New York, 1996.

[9] A.N. Bernal and M. Sánchez. *On smooth Cauchy hypersurfaces and Geroch's splitting theorem.* Commun. Math. Phys. 243, 461-470, 2003.

[10] G.E. Bredon. *Topology and Geometry.* Springer-Verlag. New York, 1993.

[11] F. Brickell and R. S. Clark. *Differentiable manifolds. An Introduction.* Van Nostrand Reinhold. London, 1970.

[12] A. Cannas da Silva. *Lectures on symplectic geometry.* Springer-Verlag. Berlin, 2001.

[13] J.F. Cariñena, A. Ibort, G. Marmo, and G. Morandi. *Geometry from dynamics: classical and quantum.* Springer-Verlag, 2014.

[14] V. Chernov, P.A. Kinlaw, and R. Sadykov. *Topological properties of manifolds admitting a Y^x–riemannian metric.* J.Geom.Phys. 60, 1530-1538, 2010.

[15] V. Chernov and S. Nemirovski. *Legendrian links causality and the Low conjecture.* Geom. Funct. Anal. 19, 1320-1333, 2010.

[16] V. Chernov and S. Nemirovski. *Non–negative legendrian isotopy in STM.* Geometry and Topology. 14, (1), 611–626, 2010.

[17] V. Chernov and Y.B. Rudyak. *Linking and causality in globally hyperbolic space–times.* Comm. Math. Phys. 279:2, 309-354, 2008.

[18] Y. Eliashberg. *Classification of overtwisted contact structures on* 3*–manifolds.* Invent. Math. 98, 623-637, 1989.

[19] J.L. Flores, J. Herrera, and M. Sánchez. *On the final definition of the causal boundary and its relation with the conformal boundary.* Adv. Theor. Math. Phys. 15, (4) 991-1057., 2011.

[20] H. Geiges. *An introduction to contact topology.* Cambridge University Press, London, 2008.

[21] R.P. Geroch. *Local characterization of singularities in General Relativity.* J. Math. Phys. 9, 450–465, 1968.

[22] R.P. Geroch, E.H. Kronheimer, and R. Penrose. *Ideal points in Space–Time.* Proc. Roy. Soc. London. A327, 545–567, 1968.

[23] P. Hartman. *Ordinary differential equations.* Wiley, New York, 1964.

[24] R. Hartshorne. *Algebraic geometry.* Springer Verlag, New York, 1977.

[25] S.W. Hawking and G.F.R. Ellis. *The large scale structure of space-time.* Cambridge University Press, London, 1973.

[26] S.W. Hawking, A.R. King, and P.J. McCarthy. *A new topology for curved space-time which incorporates the causal, differential, and conformal structures.* J. Math. Phys. 17, 174-181, 1976.

[27] N.J. Hicks. *Notes on differential geometry.* D. Van Nostrand, Princeton, New Jersey, 1965.

[28] K. Honda. *On the classification of tight contact structures I.* Geom. Topol. 4, 309368., 2000.

[29] K. Honda. *On the classification of tight contact structures II.* J. Differential Geometry. 55, 83-143, 2000.

[30] B. Khesin and S. Tabachnikov. *Pseudo-riemannian geodesics and billiards.* Adv. Math. 221, 1364-1396, 2009.

[31] P.A. Kinlaw. *Refocusing of light rays in space-time.* J. Math. Phys. 52, 052505, 2011.

[32] R.S. Kulkarni. *Conformal structures and M obius structures.* Conformal geometry (R.S. Kulkarni & U. Pinkall eds.). Friedrich Vieweg & sohn, Braunshweig/Wiesbaden, 1988.

[33] J. Lafontaine. *Conformal geometry from the Riemannian viewpoint.* Conformal geometry (R.S. Kulkarni & U. Pinkall eds.). Friedrich Vieweg & sohn, Braunshweig/Wiesbaden, 1988.

[34] L.D. Landau and E.M. Lifshitz. *The classical theory of fields, vol. 2, 4th edition.* Butterworth–Heinemann, Oxford, 2003.

[35] J.M. Lee. *Introduction to smooth manifolds.* Springer Verlag, New York, 2003.

[36] A.V. Levichev. *Prescribing the conformal geometry of a lorentz manifold by means of its causal structure.* Soviet Math. Dokl. 35, 452455, 1987.

[37] P. Libermann and C.M. Marle. *Symplectic geometry and analytical mechanic.* D. Reidel Publishing Company, Dordrecht, 1987.

[38] R.J. Low. *Causal relations and spaces of null geodesics.* PHD Thesis, Oxford University, 1988.

[39] R.J. Low. *The geometry of the space of null geodesics.* J. Math. Phys. 30, 809-811, 1989.

[40] R.J. Low. *Spaces of causal paths and naked singularities.* Class. Quantum Grav. 7, 943–954, 1990.

[41] R.J. Low. *Twistor linking and causal relations.* Class. Quantum Grav. 7, 17787, 1990.

[42] R.J. Low. *Celestial spheres, light cones, and cuts.* J. Math. Phys. 34 (1), 315–319, 1993.

[43] R.J. Low. *Twistor linking and causal relations in exterior Schwarzschild space.* Clas. Quantum Grav. 11, 453–456, 1994.

[44] R.J. Low. *The space of null geodesics.* Nonlinear Analysis 47 3005–3017, 2001.

[45] R.J. Low. *The space of null geodesics (and a new causal boundary).* Lecture Notes in Physics 692 35–50, 2006.

[46] D.B. Malament. *The class of continuous timeline curves determines the topology of spacetime.* J. Math. Phys. 18, 1399-1404, 1977.

[47] J.E Marsden and A. Weinstein. *Reduction of symplectic manifolds with symmetry.* Rep. Mathematical Phys., 5, 1, 121130., 1974.

[48] E. Minguzzi and M. Sánchez. *The causal hierarchy of spacetimes. Recent developments in pseudo-Riemannian geometry.* ESI Lect. Math. Phys, 299-358, Eur. Math. Soc., Zürich, 2008.

[49] S. Morita. *Geometry of differential forms.* Translations of mathematical monographs, vol. 201. American Mathematical Society, Providence, Rhode Island, 2001.

[50] J. Natário. *Linking and causality in* $(2+1)$*–dimensional static spacetimes.* Class. Quantum Grav. 19, 3115-3126, 2002.

[51] J. Natário and P. Tod. *Linking, Legendrian linking and causality.* Proc. London Math. Soc. (3), 88, 251-272, 2004.

[52] J. Navarro, J. Benavides, and E. Minguzzi. *Global hyperbolicity is stable in the interval topology.* J. Math. Phys. 52, 112504, 2011.

[53] B. O'Neill. *Semi-riemannian geometry with applications to relativity.* Academic Press Inc., 1983.

[54] O. Parrikar and S. Surya. *Causal topology in future and past distinguishing spacetimes.* Clas. Quantum Grav. 28, 155020, 2011.

[55] R. Penrose. *Techniques of differential topology in relativity.* Conference series in Appied Mathematics. Conference board of the Mathematical Sciences. University of London, London, 1972.

[56] R. Penrose. *The twistor programme.* Reports on Mathematical Physics. 12, 6576, 1977.

[57] R. Penrose. *Singularities and time-asymmetry.* General Relativity: An Einstein Centenary (S.W.Hawking & W. Israel, eds.). Cambridge University Press, Cambridge, 1979.

[58] R. Penrose and W. Rindler. *Spinors and space-time: Vol. 2, Spinor and twistor methods in space-time geometry.* Cambridge University Press, Cambridge, 1988.

[59] M. Sánchez. *Causal boundaries and holography on the wave type spacetimes.* Nonlinear Anal. 71, e1744–e1764, 2009.

[60] B.G. Schmidt. *A new definition of singular points in General Relativity.* Gen. Rel. and Grav. 1, 269–280, 1971.

[61] F.W. Warner. *Foundations of differentiable manifolds and Lie groups.* Springer-Verlag, New York, 1983.

[62] J.H.C. Whitehead. *Convex regions in the geometry of paths.* Quart. J. Math. Oxford Ser. 3, 33–42, 1932.

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