THE RICCI TENSOR IN CONFORMAL SEMI–RIEMANNIAN STRUCTURES

JAVIER LAFUENTE(*)

ABSTRACT. *We show that in the general case, two semi–riemannian conformal metrics that have the same Ricci tensor must be homothetic.*

§1 INTRODUCTION.

We consider a Semi–riemannian connected manifold (M,g) of dimension $n\geq 3$. We denote by ∇, Ric, and Sc, the Levi–Civita connection, the Ricci tensor, and the scalar curvature respectively. We say that a symmetric connection $\bar{\nabla}$ is conformally related with ∇ if there is a vector field $A\in\mathfrak{X}(M)$ such that:

$$\bar{\nabla}_X Y-\nabla_X Y=g(A,X)Y+g(A,Y)X-g(X,Y)A \text{ for all } X,\ Y\in\mathfrak{X}(M)$$

It is well known that $\bar{\nabla}$ is conformally related with ∇ iff $\bar{\nabla}$ preserves the conformal structure of g by parallel transport. Also, A=grad σ iff $\bar{\nabla}$ is the Levi–Civita connection of $\bar{g}=e^{2\sigma}g$.

In order to give the relation between the Ricci tensor $\overline{Ric}$ of $\bar{\nabla}$ and Ric, we need the 1–form $\alpha(X)=g(A,X)$, and the tensor $Q=\nabla\alpha-\alpha\otimes\alpha$.

If $\overline{Ric}$ is symmetric the following formulas are well know ([Ku], [Ei]):

$$\tilde{Q}(X)=\nabla_X A-g(A,X)A \qquad [1]$$

$$\overline{Ric}-Ric=(2-n)Q+\frac{(2-n)}{2}g(A,A)g+\frac{\overline{SC}-SC}{2(n-1)} \qquad [2]$$

In [1] $\tilde{Q}$ is the (1,1) tensor defined by $g(\tilde{Q}(X),Y)=Q(X,Y)$. Indeed in [2], SC=Sc g, and $\overline{SC}=Tr(\overline{Ric})$ g where $Tr(\overline{Ric})$, is the trace of $\overline{Ric}$ computed with the metric g. Thus we have the following theorem:

THEOREM 1

A necessary and sufficient condition for $\overline{Ric}=Ric$ is that for all $X\in\mathfrak{X}(M)$ we have the following strange property for A:

$$\nabla_X A=g(A,X)A-\frac{1}{2}g(A,A)X \qquad \textit{[strange property]} \qquad [3]$$

Proof:

If $\overline{Ric}$=Ric, then $\overline{SC}$=SC. Using [2] we have, $Q=-\frac{1}{2}g(A,A)g$ and by [1] we obtain:

(*) Partially supported by DGICYT (Spain) PB-0220.

1980 Mathematics Subject Classification (1985 Revision) Primary 53A53 53C20 35G20.

Key words. Ricci Tensor, Conformal structure, Linear Connection, geodesic, completeness.

$$\tilde{Q}(X)=\nabla_X A-g(A,X)A=-\frac{1}{2}g(A,A)X$$

Reciprocally if the A has the strange property [3], then by [2] we have:

$$2(n-1)(\overline{Ric}-Ric)=\overline{SC}-SC=tr(\overline{Ric}-Ric)g$$

Computing the trace of the two members we have $2(n-1)(\overline{SC}-SC)=n(\overline{SC}-SC)$. Thus $\overline{SC}=SC$ and $\overline{Ric}=Ric$.■

§2 THE MAIN RESULTS

The main aim of this work is to prove that in general case, the Ricci tensor determine the connection in a conformal structure. To see that we will prove that the condition [3] of theorem 1 is a very estrange property for a non null field A. In fact, if such field exist, then almost every geodesics for ∇ are incomplete.

In §3 we will prove the following key theorem:

(KEY) THEOREM 2

If $A\in\mathfrak{X}(M)$ is a not identically null field in M, having the property [3], then $A(x)\neq 0$ for all $x\in M$, and if the function $g(A,A):M\longrightarrow\mathbb{R}$ is not identically null, then $g(A,A)(x)\neq 0$ for all $x\in M$. Moreover we have:

a) Any ∇–geodesic which is not orthogonal to A in some point, is incomplete.

b) In the case $g(A,A)\neq 0$, the function $\sigma=Ln(|g(A,A)|)$ verifies $grad(\sigma)=A$. Moreover, for all ∇–geodesics γ with $g(\gamma',\gamma')\neq 0$, we have that $g(A,\gamma')$ is not a null constant. In particular γ is incomplete.■

We recall that a semi–riemanian manifold is timelike, lightlik or spacelike complete, if all geodesics of such character are complete.

The next main theorem is a easy consequence of the key theorem 2.

(MAIN) THEOREM 3

Suppose (M,g) semi–riemannian, $\bar{g}=e^{2\sigma}g$ and $Ric(g)=Ric(\bar{g})$. Then σ is necessary constant, if any of the following hypotheses is verified:

1) (M,g) is riemannian, and there is a complete geodesic.

2) (M,g) is Lorentz, and there is a complete timelike geodesic.

3) (M,g) is not riemannian, and spacelike or timelike or lightlike complete.

4) (M,g) is compact not riemannian, and all the lightlike geodesics are incomplete.

Proof:

Let $A=grad\ \sigma$ be in (M,g). ∇ and $\bar{\nabla}$ are as in §1. Suppose that σ is non constant, therefore the hypothesis of theorem 2 is verified. Thus if g(A,A) is not identically null, then $g(A,A)(x)\neq 0$ for all $x\in M$. By a) and b) of theorem 2 we conclude that all the geodesics of (M,g) are incomplete. This proves that under the hypothesis 1), A=0 and σ is a constant.

In order to prove the same, under the hypothesis 2) or 3) for (M,g) we can suppose that g(A,A) is a null constant:

If (M,g) is not riemannian then there are spacelike ligthlike and timelike geodesics which are not orthogonal to A, hence by a) of theorem 2, all of these geodesics are incomplete.

On the other hand, if (M,g) is Lorentz, then there are not timelike geodesics which are orthogonal to A (because A is lightlike and is never zero) hence all the timelike geodesics are incomplete.

Finally, if (M,g) is compact non Riemannian then also is g(A,A)=0, since in other case (using (b) of theorem 2) the function $\sigma=\mathrm{Ln}(|g(A,A)|)$ has not critical points on the compact space M. The strange condition [3] says that $\nabla_A A=0$, and A is a geodesic field. Since M is compact, the integral curves de A are complete lightlike geodesics. This proves 4).■

REMARK 1

The main theorem generalizes [Xu]. Here it is proved that under the hypothesis of theorem 3, σ is constant when (M,g) is *riemannian, compact,* and *oriented.*

REMARK 2

Note that the key theorem 2 also show a slight modification of theorem 3:

THEOREM 4

Let ∇ be the Levi_Civita connection for the semi–riemannian space (M,g). Suppose that (M,g) verifies some of the hypotheses 1) 2) 3) or 4) of the theorem 3. Let $\bar{\nabla}$ be a symmetric conection conformally related with ∇. If $\bar{\nabla}$ and ∇ are the same Ricci tensor, then $\bar{\nabla}=\nabla$.■

§3 PROOF OF THE KEY THEOREM 3.

From now onwards we suppose that A is a not identically null field in the semi–riemannian connected space (M,g), that has the strange property [3]:

$$\nabla_X A=\langle A,X\rangle A-\frac{1}{2}\langle A,A\rangle X \text{ for all } X\in\mathfrak{X}(M)$$

where ∇ denote the Levi–Civita connection of g, and $\langle X,Y\rangle=g(X,Y)$. By geodesic we mean that ∇–geodesic.

Also, let α be the 1–form defined by $\alpha(X)=\langle A,X\rangle$ for all $X\in\mathfrak{X}(M)$.

LEMMA 1

Let $c:I\longrightarrow M$ be a differentiable curve. If $F=\langle A,c'\rangle:I\longrightarrow\mathbb{R}$, then the *function $f=\langle A,A\rangle\circ c$, verifies the differential equation:*

$$\frac{dy}{dt}=yF$$

Thus there is a constant k such that $f(t)=k\,\exp(\varphi(t))$ where $\varphi'=F$.

In particular, if $f(t)\neq 0$ for some t, then $f(t)\neq 0$ for all t.

Proof:

We can suppose without loss of generality that c is an integral curve of a differentiable field $X\in\mathfrak{X}(M)$. We have, $f'(t)=X(\langle A,A\rangle)(c(t))$. Using now the *strange* property for A we get:

$$f'(t)=2\langle\nabla_X A,A\rangle\circ c(t)=2\langle\langle A,X\rangle A-\frac{1}{2}\langle A,A\rangle X,A\rangle\circ c(t)=(\langle A,X\rangle\langle A,A\rangle)\circ c(t)=F(t)f(t).\blacksquare$$

Next we can easily prove a first part of the key theorem:

COROLLARY 1.

If there is a point $p\in M$ such that $\langle A,A\rangle(p)\neq 0$ then $\langle A,A\rangle(x)\neq 0$ for all $x\in M$. Also $A=grad(\sigma)$, where $\sigma=log(|\langle A,A\rangle|):M\longrightarrow\mathbb{R}$

Proof:

Since M is (pathwise) connected, we can join p to a fix point $x\in M$ by a differentiable curve $c:I\longrightarrow M$ with $c(0)=p$, $c(1)=x$. Since $\langle A,A\rangle\circ c(0)\neq 0$ we conclude by lemma 1 that $\langle A,A\rangle\circ c(1)=\langle A,A\rangle(x)\neq 0$.

Suppose for example $\langle A,A\rangle>0$. Then using the same argument of Lemma 1, we have for all $X\in\mathfrak{X}(M)$, $X(\langle A,A\rangle)= \langle A,X\rangle\langle A,A\rangle=\alpha(X)\langle A,A\rangle$. This means that

$$\alpha=d\sigma,\ \text{where } \sigma=Ln(\langle A,A\rangle)$$

■

Fix any differentiable curve $c:I\longrightarrow M$, there always exist a differentiable function (determined up a constant) $u:I\longrightarrow\mathbb{R}$, such that

$$u'(t)=\alpha(c'(t))=\langle A,c'(t)\rangle \text{ for all } t\in I$$

we call u, a primitive of α along γ.

In order to end the proof of the key theorem 2 we establish:

LEMMA 2

Let $\gamma:I\longrightarrow M$ be a geodesic with $\langle\gamma',\gamma'\rangle=\varepsilon$, ($\varepsilon$ is the sign in $\{-1,0,1\}$) and let $u(t)$, $t\in I$ be a primitive of α along γ. Then $u(t)$ verifies the second order differential equation for some $k\in\mathbb{R}$:

$$\frac{d^2u}{dt^2}=\left(\frac{du}{dt}\right)^2-\frac{k\varepsilon}{2}e^u$$

Also the sign of k and the sign off $\langle A,A\rangle$ coincide.

Proof:

As in Lemma 1 we denote $F=\langle A,\gamma'\rangle=\alpha(\gamma'):I\longrightarrow\mathbb{R}$, and $f=\langle A,A\rangle\circ\gamma:I\longrightarrow\mathbb{R}$. Thus there are $\varphi:I\longrightarrow\mathbb{R}$ and $k'\in\mathbb{R}$ such that $\varphi'=F$ and $f(t)=k'\exp(\varphi(t))$. Note that $k'=0$ if $\langle A,A\rangle=0$.

Since u(t) and $\varphi(t)$ are primitives of α along γ we conclude that $\varphi=u+k''$, and

$$f=k.e^{u}$$

where $k=k'.\exp(k'')$. (Note that $k=k'=0$ if $<A,A>=0$).

Moreover, using [3] we have

$$\frac{\nabla(A\circ\gamma)}{dt}=<A,\gamma'>A-\frac{1}{2}<A,A>\circ\gamma'=FA-\frac{1}{2}f\gamma'=u'A-\frac{1}{2}ke^{u}\gamma'$$

by scalar multiplication for γ' and using that γ is geodesic we get:

$$u''=F'=\frac{d}{dt}<A,\gamma'>=<\frac{\nabla(A\circ\gamma)}{dt},\gamma'>=(u')^2-\frac{1}{2}k\varepsilon e^{u}$$

■

We prove now the assert a) of key theorem 2

COROLLARY 2

Any geodesic γ, such that $<A,\gamma'(t)>\neq 0$ for some t, is incomplete.

Proof:

Let $\gamma:I\longrightarrow M$ be a maximal geodesic, as in Lemma 2, with $<A,\gamma'(0)>\neq 0$.

Suppose first $<A,A>=0$ and let $u:I\longrightarrow\mathbb{R}$ be a primitive of α along γ.

By Lemma 2, u verifies the second order differential equation

$$\frac{d^2u}{dt^2}=\left(\frac{du}{dt}\right)^2 \qquad [4]$$

since $u'=<A,\gamma'>$, is not a null constant, u is not the trivial solution.The nontrivial solution is

$$u(t)=\mathrm{Ln}\left(\frac{e^{a}}{1-bt}\right) \text{ where } u(0)=a,\ u'(0)=b\neq 0 \qquad [5]$$

which is not defined on the whole real line as it should be if γ were complete.

Suppose now $<A,A>\neq 0$. By the Corollary 1 is $\alpha=d\sigma$, and by Lemma 2 $v(t)=\sigma(\gamma(t))$ verifies the second order differential equation :

$$\frac{d^2v}{dt^2}=\left(\frac{dv}{dt}\right)^2-\frac{k\varepsilon}{2}e^{v}$$

where ε is the sign of γ, and $k\in\mathbb{R}$ have the same sign of $<A,A>$.

If $k\varepsilon=0$ we argue as before. Else ($k\varepsilon\neq 0$) we compare the solution [5] of [4] for $a=u(0)=v(0)$, $b=u'(0)=v'(0)\neq 0$.

Taylor formula gives:

$$u(t)-v(t)=t^2\frac{k\varepsilon}{4}e^{v(s)} \text{ for some s between 0 and t} \qquad [6]$$

Since $v'(t)=<A,\gamma'(t)>$ and sign(k)=sign(<A,A>), we can suppose (reversing if necessary the orientation of γ) that $v'(0)=b$ has the same sign of $-k\varepsilon$. We get now:

1) If $b>0$ and $k\varepsilon<0$ then $u(t)\longrightarrow+\infty$ for $t\longrightarrow 1/b$, and $v(t)\longrightarrow+\infty$ for $t\longrightarrow 1/b$ because $u(t)>v(t)$.

2) if $b<0$ and $k\varepsilon>0$ then $u(t)\longrightarrow-\infty$ for $t\longrightarrow 1/b$, and $v(t)\longrightarrow-\infty$ for $t\longrightarrow 1/b$ because $u(t)<v(t)$.

Thus $v(t)$ is not defined in the whole real line, and γ is incomplete.∎

Moreover we have:

COROLLARY 3

The vector field A verifies $A(x)\neq 0$ for all $x\in M$.

Proof:

By Corollary 1, we can suppose that $\langle A,A\rangle=0$. If there is $p\in M$, with $A(p)=0$ we will prove that A is zero in the whole M. In fact, fixing any geodesic $\gamma:I\longrightarrow M$ with $\gamma(0)=p$, let $u:I\longrightarrow\mathbb{R}$ be a primitive u of α along γ. By Lemma 2, $u(t)$ verifies the differential equation [4] with solution [5]. But $b=u'(0)=\langle A(p),\gamma'(0)\rangle=0$, thus $u(t)=u(0)=a$, and $u'=\langle A,\gamma'\rangle=0$. Finally, using the strange property [3] for A, we have:

$$\frac{\nabla(A\circ\gamma)}{dt}=\langle A,\gamma'\rangle A=0$$

Since $A(\gamma(0))=0$ is $A=0$ along γ. Using now the exponential function in p we see that $A=0$ in a neighbourhood of p. Thus $\{x\in M:A(x)=0\}$ is open and closed set, which coincides with the whole M. ∎

To end the proof of the key theorem it is sufficient to prove:

COROLLARY 4

If $\langle A,A\rangle\neq 0$, then for any geodesic $\gamma:I\longrightarrow M$ such that $\langle\gamma',\gamma'\rangle=\varepsilon\in\{-1,1\}$, we have that the function $\langle A,\gamma'\rangle:I\longrightarrow\mathbb{R}$ is not a null constant.

Proof:

Let $u:I\longrightarrow\mathbb{R}$ a primitive of α along γ. By Lemma 2, u verifies:

$$\frac{d^2u}{dt^2}=\left(\frac{du}{dt}\right)^2-\frac{k\varepsilon}{2}e^u$$

where $k\varepsilon\neq 0$. If we suppose $u'=\langle A,\gamma'\rangle=0$ then $u''=0$, and $k\varepsilon e^u=0$. This is a contradiction. ∎

This end the proof of the key Theorem.

REMARK 3

Note that if $\langle A,A\rangle\neq 0$ then the form α is closed (since is exact).

It is easy to prove that α is also closed if $\langle A,A\rangle=0$. This means that if $\overline{Ric}=Ric$, then $\bar\nabla$ is always *locally* the Levi_Civita connection for some $\bar g=e^{2\sigma}g$.

Moreover, the orthogonal distribution of A is integrable and the integral surfaces are totally geodesics. We do not know if such geodesic are complete.

ACKNOWLEDGMENT

The author is indebted to Professor M.A.Amores for his interest in this paper, and some valuable suggestions for the simplification of the proof of the key theorem.

REFERENCES

[Ei] Einsenhart. L. P. *Riemannian Geometry.* Princ. Univ. Press (1949)

[Ku] R.S. Kulkarni. *Curvature and metric.* Ann. of Math. (2) **91** (1970) 311-331

[Xi] Xingwang Xu *Prescribing a Ricci Tensor in a Conformal class of Riemannian Metrics.* Proc. of Am. Math Soc. Vol.115 N.2 (1992) 455-459.