The Contact Structure in the Space of Light Rays

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ABSTRACT

The natural structures, topological, differentiable and geometrical, on the space of light rays of a given spacetime are discussed. The relation between the causality properties of the original spacetime and the natural structures on the space of light rays are stressed. Finally, a symplectic geometrical approach to the construction of the canonical contact structure on the space of light rays is offered.

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1. Introduction

In the recent articles [Ba14, Ba15] it was shown that causality relations on spacetimes can be described alternatively in terms of the geometry and topology of the space of light rays and skies. This alternative description of causality, whose origin can be

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traced back to Penrose, was pushed forward by R. Low [Lo88, Lo06] and, as indicate above, largely accomplished in the referred works by Bautista, Ibort and Lafuente.

Shifting the point of view from "events" to "light rays" and "skies" to analyze causality relations has deep and worth discussing implications. Thus, for instance, as Low himself noticed [Lo90, Lo94], in some instances, two events are causally related iff the corresponding skies are topologically linked and, a more precise statement of this fact, constitutes the so called Legendrian Low's conjecture (see for instance [Na04, Ch10]).

The existence of a canonical contact structure on the space of light rays plays a cornerstone role in this picture. Actually it was shown in [Ba15] that two events on a strongly causal spacetime are causally related iff there exists a non-negative sky Legendrian isotopy relating their corresponding skies.

Moreover, and as an extension of Penrose's twistor programme, it would be natural to describe attributes of the conformal class of the Lorentzian metric such as the Weyl tensor, in terms of geometrical structures on the space of light rays.

It is also worth to point out that in dimension 3 this dual approach to causality becomes very special. Actually if the dimension of spacetime is $m = 3$, the dimension of the space of light rays is also $2m - 3 = 3$ and the skies are just Legendrian circles. Even more because the dimension of the contact distribution is 2 the space of 1-dimensional subspaces is the 1-dimensional projective space $\mathbb{RP}^1$, hence the curve defined by the tangent spaces to the skies along the points of a light ray defines a projective segment of it and it is possible to define Low's causal boundary [Lo06] unambiguously. We will not dwell into these matters in the present paper and we will leave it to a detailed discussion elsewhere. (see also A. Bautista Ph. D. Thesis [Ba15b]).

Because of all these reasons we have found relevant to describe in a consistent and uniform way the fundamental structures present on the space of light rays of a given spacetime: that is, its topological, differentiable and geometrical structures, the later one, exemplified by its canonical contact structure. Thus, the present work will address in a sistematic and elementary way the description and construction of the aforementioned structures, highlighting the interplay between the causility properties of the original spacetime and the structures of the corresponding space of light rays.

The paper will be organized as follows. Section 2 will be devoted to review the basic elements of causality theory in Lorentzian manifolds. The space of light rays will be properly introduced in Sect. 3 and in Sect. 4 its natural differentiable structure will be constructed. The tangent bundle of the space of light rays and the canonical contact structure on the space of light rays will be the subject of Sects. 5 and 6 and, finally, the description from a symplectic reduction viewpoint of the light rays canonical contact structure will be addressed in Sect. 7.

2. Differential geometry and causality

2.1. Causality in spacetimes

We will begin by succinctly reviewing the main elements of causality and spacetimes from a differential-geometric viewpoint.

Definition 2.1 *Let M be a differentiable manifold such that* $\dim M \geq 2$ *and* $\mathbf{g} \in \mathfrak{T}_2^0(M)$ *a symmetric tensor. Then* $(M, \mathbf{g})$ *is said to be a* semi–riemannian manifold *if* $\mathbf{g}$ *is non-degenerated at every* $p \in M$. *A* Lorentzian manifold *is a semi–riemannian manifold* $(M, \mathbf{g})$ *such that for every* $p \in M$ *there exists a basis at* T_pM *in which the matrix representing* $\mathbf{g}_p$ *is* $\mathrm{diag}(-1, +1, ..., +1)$.

Equivalently, we will say that M is semi–riemannian or Lorentzian when $(M, \mathbf{g})$ is so and the metric $\mathbf{g}$ is not specified explicitly.

By ∇, we will denote the *Levi–Civita connection* of the Lorentzian manifold M, that is, the unique torsionless symmetric connection on TM verifying $\nabla \mathbf{g} = 0$, that is,

$$X(\mathbf{g}(Y, Z)) = \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z) \tag{2.1}$$

where $X, Y, Z \in \mathfrak{X}(M)$ and its *curvature* or *Riemann tensor* is defined by

$$R(X, Y) Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \, .$$

Consider a smooth curve $\alpha\colon (a, b) \to M$, in M. Let $\mathfrak{X}_\alpha$ denote the vector fields on M along α, that is $X \in \mathfrak{X}_\alpha$ is a smooth map $X\colon (a, b) \to TM$ such that $X(t) \in T_{\alpha(t)}M$. Then we define the *covariant derivative along* α as a the map $\frac{D}{dt} : \mathfrak{X}_\alpha \to \mathfrak{X}_\alpha$ given by:

$$\frac{DX}{dt} = \nabla_{\dot{c}(t)} X \, .$$

Given a function $f \in \mathfrak{F}(M)$, we will say that *gradientf* is the *gradient* of f and it describes the vector field metrically equivalent to the 1–form df, that is, for any $X \in \mathfrak{X}(M)$

$$\mathbf{g}(gradient f, X) = df(X) = X(f) \in \mathfrak{F}(M) \tag{2.2}$$

A detailed exposition of the properties of the previous geometrical objects can be found in, for example, [Ab88], [BE96], [HE73] and [On83].

From now on, we will consider that $(M, \mathbf{g})$ is a Lorentzian manifold. We can classify the tangent vectors depending on their *causal character*, that is, we will say that a vector $v \in \widehat{T}_pM = T_pM - \{\mathbf{0}\}$[1] is:

$$\begin{array}{ll} \textit{spacelike} & \text{if } \mathbf{g}_p(v, v) > 0 \, , \\ \textit{null} \text{ or } \textit{lightlike} & \text{if } \mathbf{g}_p(v, v) = 0 \, , \\ \textit{timelike} & \text{if } \mathbf{g}_p(v, v) < 0 \, . \end{array} \tag{2.3}$$

[1] In what follows, if N is a differentiable manifold, the notation $\widehat{T}N$ will be used to make reference to the bundle resulting of eliminating the zero section of TN, that is $\widehat{T}N = \{v \in TN : v \neq 0\}$.

A nonzero tangent vector v will be said to be *causal* if it is either timelike or null. It is clear that at each $p \in M$, the set of causal vectors has two connected components. A *time–orientation* of M is a continuous function τ on M assigning to every $p \in M$ a connected component τ_p of the set of causal vectors in T_pM. We will say that (M, g) is *time–orientable* if M admits a time–orientation. If a time–orientation τ is provided then we will say that (M, g) is *time–oriented* [On83, p. 145] .

Time–orientability is equivalent to the existence of a *timelike vector field* X (see [On83, Lemma 5.32] for details), that is, a vector field X such that for all $p \in M$, $X_p \in T_pM$ is timelike. In fact, if X exists, then it is possible to assign to every $p \in M$ the connected component of T_pM containing X_p and so we get a time–orientation. On the other hand, if M is furnished with a time–orientation τ then for every $p \in M$ there exists a neighbourhood U_p where a timelike vector field X_{U_p} can be defined, and its image for any $q \in U_p$ is in τ_q. Then using partitions of unity a global timelike vector field X can be easily constructed in M.

In a time–oriented Lorentzian manifold $(M, \mathbf{g})$ both connected components of the set of causal vectors can be distinguished. The τ component will be called the *future causal cone* of p and the $-\tau$ component will be called the *past causal cone* of p. So we will say that a causal vector $v \in T_pM$ is *future* (respectively *past*) if $v \in \tau_p$ (respectively $-v \in \tau_p$). The spaces that we will be considering in what follows are time-oriented Lorentzian manifolds.

Definition 2.2 *A spacetime of dimension $d > 2$ is a Hausdorff smooth time-oriented Lorentzian manifold $(M, \mathbf{g})$.*

Consider now the subset of all null vectors $\mathbb{N}$ of $\widehat{T}M$. Let $L : \widehat{T}M \to \mathbb{R}$ be the differentiable function defined by

$$L(v) = \frac{1}{2}\mathbf{g}(v, v) \,, \tag{2.4}$$

that can be written as $L\left(x^k, v^k\right) = \frac{1}{2} g_{ij} v^i v^j$ in the local natural bundle coordinates $\left(x^k, v^k\right)$ in TM, i.e., $v \in T_pM$ is written as $v = v^i \partial/\partial x^i \mid_p$, and the metric $\mathbf{g}$ is expressed in these coordinates as $\mathbf{g}(u, v) = g_{ij} u^i v^j$.

By definition of $\mathbb{N}$, it is trivial to see that $\mathbb{N} = L^{-1}(0) \subset \widehat{T}M$. The differential of L is given by:

$$dL = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k} v^i v^j dx^k + g_{ik} v^i dv^k \,. \tag{2.5}$$

Since $\mathbf{g}$ is a non–degenerate metric, then for every $v \in \widehat{T}M$ there exists $u \in \widehat{T}M$ such that $\mathbf{g}(v, u) \neq 0$. This implies that the rank of $dL_{(x^k, v^k)}$ is 1 and therefore $0 \in \mathbb{R}$ is a regular value of the function L. By [Br93, Cor. II.7.4], since $\mathbb{N} = L^{-1}(0)$ is the inverse image of a regular value, then it is a regular submanifold of $\widehat{T}M$ and, by restriction, it inherits the structure of a locally trivial bundle over M. So $\mathbb{N}$ is a bundle over M and we will denote by $\pi_M^{\mathbb{N}} : \mathbb{N} \to M$ its canonical projection and by $\mathbb{N}_p$ its fibre at $p \in M$.

The zero section of TM separates both connected components of $\mathbb{N}$ that will be denoted by

$$\mathbb{N}^+ = \{v \in \mathbb{N} : v \text{ future } \} \;, \qquad \mathbb{N}^- = \{v \in \mathbb{N} : v \text{ past } \} \;.$$

We will call the fibres $\mathbb{N}_p$, $\mathbb{N}_p^+$ and $\mathbb{N}_p^-$ *lightcone*, *future lightcone* and *past lightcone* at $p \in M$ respectively.

By the previous classification of tangent vectors, we will say that a curve γ is *timelike* (respectively *null*, *spacelike*, *causal*) if its tangent vector is timelike (respectively null, spacelike, causal) at every of its points. We will say that a causal curve is *future–directed* (respectively *past–directed*) if its tangent vectors are future (respectively past).

Definition 2.3 *Let S be a subset of a Lorentzian manifold M.*

1. *The* chronological future of *S is the set of all points in M that can be connected to S by a future–directed timelike curve. It will be denoted by $I^+(S)$. Analogously, it is possible to define the* chronological past of *S denoted by $I^-(S)$.*

2. *The* causal future of *S is the union of S and the set of all points in M that can be connected to S by a future–directed causal curve. It will be denoted by $J^+(S)$. In the same way, we can define the* causal past of *S denoted by $J^-(S)$.*

3. *A subset $S \subset M$ is* achronal *if any $p \in S$ verifies $I^+(p) \cap S = \varnothing$.*

4. *Let S be an achronal set, we will name* future *(*past*)* Cauchy development of *S to the set of points $p \in M$ such that any causal curve inextensible to the past (future) passing through p intersects S. We will denote it by $D^+(S)$ $(D^-(S))$. And we will say that $D(S) = D^+(S) \cup D^-(S)$ is the* Cauchy development *of S.*

In a equivalent way, we will use the notation

$$p \prec q \tag{2.6}$$

to indicate $q \in I^+(p)$. Also

$$p < q \tag{2.7}$$

can be used to denote the existence of a future–directed causal curve from p to q. The notation

$$p \leq q \tag{2.8}$$

is used to indicate both $p < q$ or $p = q$, that is $q \in J^+(p)$.

The following theorem is a basic result to study the causal structure of spacetimes and it can be found in [On83, Prop. 10.46].

Theorem 2.1 *Let M be a Lorentzian manifold. If α is a causal curve joining the points $p, q \in M$ but not a null pregeodesic, then in any neighbourhood of α there exists a timelike curve β connecting the points p and q.*

As an immediate consequence of theorem 2.1, we get the following corollary.

Corollary 2.1 *If $r \in J^+(q)$ and $q \in I^+(p)$, or also $r \in I^+(q)$ and $q \in J^+(p)$, then we have that $r \in I^+(p)$.*

Proof. In former case, if $q \in I^+(p)$ then there exists a future–directed timelike curve α_1 joining p and q, and if $r \in J^+(q)$ then there exists a future–directed causal curve α_2 connecting q with r (if $q = r$ then α_2 is constant). Then the curve $\alpha = \alpha_1 \cup \alpha_2$ is a future–directed causal curve joining p and r and it is not a null pregeodesic because α_1 is timelike. By theorem 2.1, there exists a timelike curve β joining p and r that, by construction conserves the same time–orientation of α. Therefore $r \in I^+(p)$.

The proof for the latter case can be done in an analogous way. □

The previous corollary is also true when we consider the chronological and causal past, and its proof is similar if we interchange the roles of future and past.

It is possible to classify Lorentzian manifolds according to some conditions on the nature of its causal curves, i.e., on the character of the causal relation defined by its geometry. The classification given below is not exhaustive but it is enough for the purposes of this work. See for instance [On83], [Mi08], [BE96], [HE73] and [Pe72] for further details.

In the following definition we will introduce only conditions that will be used later on, and it should be taken into account that if one of them is verified then all the previous conditions are also verified.

Definition 2.4 *Let M be a time–oriented Lorentzian manifold, then:*

1. *It is said that M verifies the* chronological condition *or that M is* chronological *if there are not closed timelike curves.*

2. *It is said that M verifies the* causal condition *or that M is* causal *if there does not exist closed causal curves.*

3. *We say that M is* strongly causal at $p \in M$ *or verifies the* strong causality condition *if for every neighbourhood U of $p \in M$ there exists a neighbourhood $V \subset U$ of p such that any segment of causal curve with endpoints at V, is wholly contained in U. This means that there is not* almost closed *causal curves at p, that is there exists a neigbourhood V of p such that any causal curve that leaves V does not return to said neigbourhood. We will say that M is strongly causal if it is so for every $p \in M$.*

4. *We say that M is* globally hyperbolic *or verifies the* global hyperbolicity condition *if it causal and $J^{+}(p)\cap J^{-}(q)$ is compact for any $p,q\in M$.*

Definition 2.5 *We will say that a* naked singularity *occurs at the future (resp. past) of a causal curve λ inextensible to the future (resp. past) if there exists a point $p\in M$ such that $I^{-}(\lambda)\subset I^{-}(p)$ (resp. $I^{+}(\lambda)\subset I^{+}(p)$).*

In [Pe79] it was shown by Penrose that a strongly causal spacetime M is globally hyperbolic if naked singularities does not exist in M.

Definition 2.6 *A future–directed causal curve γ inextensible to the future such that it enters and remains into a compact set K is said to be* totally imprisoned to the future *in K. If γ does not remain in K, but continually re–enters into K, then γ is said to be* partially imprisoned to the future *in K.*

These phenomena of imprisonment can not exist under some causality conditions, as it can be observed in the next proposition [HE73, Prop. 6.4.7].

Proposition 2.2 *If there exists a totally or partially imprisoned future–directed causal curve inextensible to the future in some compact set $K\subset M$, then the strong causality condition does not hold on K.*

Definition 2.7 *A* Cauchy surface *is a topological hypersurface $S\subset M$ such that any inextensible timelike curve intersects S exactly once.*

Proposition 2.3 *Let M be a spacetime with a Cauchy surface $S\subset M$ and let X be a timelike vector field on M. If $p\in M$, every maximal integral curve of X passing through p intersects S in a unique point $\sigma(p)$. Then the map $\sigma : M\to S$ is open, continuous and surjective leaving fixed any point of S. Moreover S is connected.*

Proof. We offer the proof of [On83, Prop. 14.31]. It is known that the maximal integral curves of X are inextensible. Let $\tilde{\Psi} : D\longrightarrow M$ be the flow of X where D is open in $M\times\mathbb{R}$. Since S is a topological hypersurface of M, then $D_S=(S\times\mathbb{R})\cap D$ is a topological hypersurface in D and since $\tilde{\Psi}$ is differentiable, then its restriction $\Psi : D_S\longrightarrow M$ is continuous. Moreover S is a Cauchy surface and then $\Psi : D_S\to M$ is bijective. Since the dimensions of D_S and M are the same then Ψ is a homeomorphism. The projection $\pi : S\times\mathbb{R}\to S$ is an open, continuous and surjective map, hence since $\sigma=\pi\circ\Psi^{-1}$, then σ is also open, continuous and surjective and leaves fixed any point of S. Since M is connected then we conclude that $\sigma(M)=S$ is connected. □

An important consequence of proposition 2.3 is the topological equivalence of Cauchy surfaces. It is described in the next corollary.

Corollary 2.2 *All the Cauchy surfaces in a spacetime M are homeomorphic.*

Proof. We sketch the idea of the proof in [On83, Cor. 14.32]. Let S and T be two Cauchy surfaces of M and let X be a timelike vector field. If σ_S and σ_T are the respective retractions built in proposition 2.3 for S and T by means of the flow of X, then the restrictions $\sigma_S : T \to S$ and $\sigma_T : S \to T$ are mutually inverses. □

Theorem [On83, Th. 14.38] states a relation between Cauchy developments and global hyperbolicity. It claims that given a achronal set A, then the interior of the Cauchy development of A, that is $\operatorname{int}(D(A))$, if it is not empty, then it is globally hyperbolic. This result can be applied to a Cauchy surface S, and since $D(S) = M$ then $\operatorname{int}(D(A)) = M$, therefore M is globally hyperbolic. So, the existence of a Cauchy surface implies the global hyperbolicity of M.

The next theorem is an important characterization of globally hyperbolic spacetimes.

Theorem 2.4 ***(Geroch-Bernal-Sánchez)*** *Any globally hyperbolic spacetime M admits a differentiable spacelike Cauchy surface S, and moreover M is diffeomorphic to $S \times \mathbb{R}$.*

Proof. See [Be03, Th. 1]. □

According to [Mi08], we have the following definitions and results.

Definition 2.8 *Let U, V be open sets in a spacetime M such that $V \subset U$. Then V is said to be* causally convex *in U if any causal curve contained in U with endpoints in V is totally contained in V.*

Theorem 2.5 *Let M be a spacetime. For any $p \in M$ and any neighbourhood U of p there exists a neighbourhood U' such that $p \in U' \subset U$ and a sequence of globally hyperbolic nested neighbourhoods $\{V_n\}$ such that $V_{n+1} \subset V_n$ and $\{p\} = \bigcap_n Vn$ all contained in U' and verifying that every V_n is causally convex in U'.*

Proof. See [Mi08, Th. 2.14]. □

By theorem 2.5, it is possible to give a different, but equivalent, definition of strongly causal spacetime.

Definition 2.9 *A spacetime M is said to be* strongly causal *if for all $p \in M$ and all neighbourhood $U \subset M$ of p there exists a causally convex neighbourhood $V \subset U$ of p. This neighbourhood V, according to theorem 2.5 can be considered globally hyperbolic.*

Proposition 2.6 *Let M be a strongly causal spacetime, then for every $p \in M$ there exists a neighbourhood V of p such that if γ is an inextensible causal curve then $\gamma \cap V$ has exactly one connected component.*

Proof. It is a direct consequence of strong causality of M. It is known that for all $p \in M$ there exist a causally convex neighbourhood V of p. Let γ be a causal curve intersecting V, if $\gamma \cap V$ had more that one connected component, then taking two points $q, r \in \gamma$ contained in different connected components, since γ is connected, there would exist a point $s \in \gamma$ between q and r such that $s \notin V$, contradicting that V is causally convex. □

3. The space $\mathcal{N}$ of light rays and its differentiable structure

3.1. Constructions of the space $\mathcal{N}$

Given a spacetime $(M, \mathbf{g})$, we define the *set of light rays* of $(M, \mathbf{g})$ by

$$\mathcal{N}_{\mathbf{g}} = \{\mathrm{Im}\ (\gamma) \subset M : \gamma \text{ is null geodesic in } (M, \mathbf{g})\}$$

where Im (γ) denotes the image of the curve γ. This definition seems to depend on the metric $\mathbf{g}$, however we will show that the space of light rays depends only on the conformal class of the spacetime metric.

We define the *conformal class of metrics in M equivalent to* $\mathbf{g}$ by

$$\mathcal{C}_{\mathbf{g}} = \left\{\overline{\mathbf{g}} \in \mathfrak{T}_0^2(M) : \overline{\mathbf{g}} = e^{2\sigma}\mathbf{g},\ \ 0 < \sigma \in \mathfrak{F}(M)\right\}$$

and we call $(M, \mathcal{C}_{\mathbf{g}})$ the corresponding *conformal class of spacetimes equivalent to* $(M, \mathbf{g})$.

It is known that two metric are conformally equivalent if the lightcones coincide at every point (see [Mi08, Prop. 2.6 and Lem. 2.7] or [HE73, p. 60-61]). This fact can be automatically translated to the spaces of light rays defined by two different metrics on a manifold M. The following proposition brings to light the equivalence among spaces of light rays.

Proposition 3.1 *Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two spacetimes and let $\mathcal{N}_{\mathbf{g}}$ and $\mathcal{N}_{\overline{\mathbf{g}}}$ be their corresponding spaces of light rays. Then $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ are conformally equivalent if and only if $\mathcal{N}_{\mathbf{g}} = \mathcal{N}_{\overline{\mathbf{g}}}$.*

Proposition 3.1 permits to state the next definition.

Definition 3.1 *Let $(M, \mathcal{C}_{\mathbf{g}})$ be a conformal class of spacetimes with* $\dim M = m \geq 3$. *We will call* light ray *the image $\gamma(I)$ in M of a maximal null geodesic $\gamma : I \to M$ related to any metric $\bar{\mathbf{g}} \in \mathcal{C}_{\mathbf{g}}(M)$. It will be denoted by $[\gamma]$ or γ when there is not possibility of confusion, that is $[\gamma] \in \mathcal{N}$, $\gamma \in \mathcal{N}$ or also $\gamma \subset M$. So, every light ray is equivalent to an unparametrized null geodesic. Then, we will say that the* space of light rays $\mathcal{N}$ of a conformal class of spacetimes $(M, \mathcal{C}_{\mathbf{g}})$ *is the set*

$$\mathcal{N} = \{\gamma(I) \subset M \mid \gamma : I \to M \textit{ is a maximal null geodesic for any metric } \bar{\mathbf{g}} \in \mathcal{C}_{\mathbf{g}}\}\ .$$

A more geometric construction of $\mathcal{N}$ is possible as a quotient space of the tangent bundle TM [Lo06]. This construction will allow us to show how $\mathcal{N}$ inherits the topological and differentiable structures of TM.

Let us consider the *geodesic spray* $X_{\mathbf{g}}$ related to the metric $\mathbf{g}$, that is the vector field in TM such that its integral curves define the geodesics in $(M, \mathbf{g})$ and their tangent vectors. So, the canonical projection $\pi_M^{TM} : TM \to M$ maps integral curves of $X_{\mathbf{g}}$ into geodesics of M. Take a coordinate chart $\left(\left(x^k, v^k\right), TU\right)$ in TM such that a vector $v \in TU$ can be written as $v = v^k \frac{\partial}{\partial x^k}$, where x^k with $k = 1, \ldots, m$ are coordinates in M. Then the geodesic spray $X_{\mathbf{g}}$ is written as:

$$X_{\mathbf{g}} = v^k \frac{\partial}{\partial x^k} - \Gamma_{ij}^k v^i v^j \frac{\partial}{\partial v^k}$$

where Γ_{ij}^k with $i, j, k = 1, \ldots, m$ denote the Christoffel symbols of the Levi–Civita connection ∇ for $\mathbf{g}$.

We claim that $X_{\mathbf{g}}$ is tangent to the submanifold $\mathbb{N}$. Indeed, for any geodesic γ, the curve $\gamma'(t) \in T_{\gamma(t)}M$ is an integral curve of $X_{\mathbf{g}}$. Calling $f(v) = \mathbf{g}(v, v)$, then we have $f(\gamma'(t))$ constant, hence $X_{\mathbf{g}}(f) = 0$ and therefore $X_{\mathbf{g}}$ is tangent to any level set of f, in particular it is tangent to $\mathbb{N} = f^{-1}(0)$.

Moreover, the integral curve of $X_{\mathbf{g}}$ passing through $v \in \mathbb{N}$ is projected onto the null geodesic $\gamma \subset M$ such that $\gamma(t_0) = \pi_M^{\mathbb{N}}(v)$ and $\gamma'(t_0) = v$.

On the other hand, we define the *Euler field* Δ *in* TM as the vector field in TM whose flow are scale transformations along the fibres of TM, that is, if $u \in T_pM$ then

$$\Delta(u) = dc\left(\frac{\partial}{\partial t}\right)(0)$$

where $c : \mathbb{R} \to T_pM$ is defined by $c(t) = e^t u$. In case of $u \in \mathbb{N}_p^+$, since for all $t \in \mathbb{R}$ we have that $e^t u \in \mathbb{N}_p^+$, then c is a curve in $\mathbb{N}_p^+$. Moreover, since

$$c'(t) = dc\left(\frac{\partial}{\partial t}\right)(t) = \Delta(c(t))$$

then c is an integral curve of Δ contained in $\mathbb{N}^+$ if $u \in \mathbb{N}^+$, then the Euler field Δ is tangent to $\mathbb{N}^+$. In the previous coordinates $\left(x^k, v^k\right)$, the field Δ can be expressed as:

$$\Delta = v^k \frac{\partial}{\partial v^k}\,.$$

Now, we can define the differentiable distribution in $\mathbb{N}^+$ given by $\mathcal{D} = \text{span}\,\{X_{\mathbf{g}}, \Delta\}$. Since

$$[\Delta, X_{\mathbf{g}}] = X_{\mathbf{g}}\,,$$

then $\mathcal{D}$ is involutive and, by Fröbenius' Theorem [Wa83, Thm. 1.60], it is also integrable. This means that the quotient space $\mathbb{N}^+/\mathcal{D}$ is well defined. Every leaf of $\mathcal{D}$

is the equivalence class consisting of a future–directed null geodesic and all its affine reparametrizations preserving time–orientation, hence the space

$$\mathcal{N}^+ = \mathbb{N}^+/\mathcal{D}\,.$$

is the space of future-oriented light rays of M. In a similar way we may construct the space of past-oriented light rays $\mathcal{N}^- = \mathbb{N}^-/\mathcal{D}$.

The space of light rays $\mathcal{N}$ can be obtained as the quotient $\mathbb{N}/\widetilde{\mathcal{D}}$ where $\widetilde{D}$ denotes the scale transformation group acting on $\mathbb{N}$, that is, $v \mapsto \lambda v$, $\lambda \neq 0$, $v \in \mathbb{N}$. Notice that $\widetilde{\mathcal{D}} \cong \mathbb{R} - \{0\}$ and the orbits of the connected component containing the identity can be identified with the leaves of $\mathcal{D}$.

Because our spacetime M is time-orientable, in what follows we will consider the space of future-oriented light rays $\mathcal{N}^+$ and we will denote it again as $\mathcal{N}$ without risk of confussion. As it will be shown later on, this convention will be handy as the space $\mathcal{N}^+$ carries a co-orientable contact structure (see Sect. 5) whereas $\mathcal{N}$ does not.

Lemma 3.1 *Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two conformally equivalent spacetimes such that $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$, and let $X_{\mathbf{g}} \in \mathfrak{X}(\mathbb{N}^+)$ and $X_{\overline{\mathbf{g}}} \in \mathfrak{X}(\mathbb{N}^+)$ be their respective geodesics sprays. Then we have that*

$$X_{\overline{\mathbf{g}}} = -2d\sigma \cdot \Delta + X_{\mathbf{g}}$$

Proof. Let us consider the chart $\varphi = \left(x^k, v^k\right)$ defined in $W \subset TM$ as above. Let $\overline{\Gamma}_{ij}^k$ and Γ_{ij}^k be the Christoffel symbols related to the metrics $\overline{\mathbf{g}}$ and $\mathbf{g}$ respectively. So, we have

$$\begin{aligned}\overline{\Gamma}_{ij}^k &= \frac{1}{2}\overline{g}^{mk}\left(\frac{\partial \overline{g}_{im}}{\partial x^j} + \frac{\partial \overline{g}_{jm}}{\partial x^i} - \frac{\partial \overline{g}_{ij}}{\partial x^m}\right) = \\ &= \frac{\partial \sigma}{\partial x^j} g^{mk} g_{im} + \frac{\partial \sigma}{\partial x^i} g^{mk} g_{jm} - \frac{\partial \sigma}{\partial x^m} g^{mk} g_{ij} + \Gamma_{ij}^k = \\ &= \frac{\partial \sigma}{\partial x^j}\delta_i^k + \frac{\partial \sigma}{\partial x^i}\delta_j^k - \frac{\partial \sigma}{\partial x^m} g^{mk} g_{ij} + \Gamma_{ij}^k\end{aligned}$$

where δ_i^j denotes the Kronecker's delta. So, the geodesic spray $X_{\overline{\mathbf{g}}}$ can be written as

$$X_{\overline{\mathbf{g}}} = v^k \frac{\partial}{\partial x^k} - \overline{\Gamma}_{ij}^k v^i v^j \frac{\partial}{\partial v^k} = -2\frac{\partial \sigma}{\partial x^j} v^k v^j \frac{\partial}{\partial v^k} + X_{\mathbf{g}} = -2d\sigma \cdot \Delta + X_{\mathbf{g}}\,,$$

as we claimed and where we have used that $g_{ij}v^i v^j = 0$ since $v \in \mathbb{N}^+$. □

Lemma 3.1 allows to prove the next proposition equivalent to the proposition 3.1 above.

Proposition 3.2 *The space of light rays $\mathcal{N}$ of $(M, \mathcal{C}_{\mathbf{g}})$ does not depend on the conformal class representative $\mathbf{g}$ of $\mathcal{C}_{\mathbf{g}}$.*

Proof. Let $(M, \mathbf{g})$ and $(M, \overline{\mathbf{g}})$ be two conformally equivalent spacetimes such that $\overline{\mathbf{g}} = e^{2\sigma}\mathbf{g}$ and let $X_{\mathbf{g}}, X_{\overline{\mathbf{g}}} \in \mathfrak{X}(\mathbb{N}^+)$ be their corresponding geodesic sprays. Consider the distributions $\mathcal{D} = \operatorname{span}\{X_{\mathbf{g}}, \Delta\}$ and $\overline{\mathcal{D}} = \operatorname{span}\{X_{\overline{\mathbf{g}}}, \Delta\}$. Then, by lemma 3.1 we have

$$\overline{\mathcal{D}} = \operatorname{span}\{X_{\overline{\mathbf{g}}}, \Delta\} = \operatorname{span}\{-2d\sigma \cdot \Delta + X_{\mathbf{g}}, \Delta\} = \operatorname{span}\{X_{\mathbf{g}}, \Delta\} = \mathcal{D}$$

and hence the distribution $\mathcal{D}$ does not depends on the metric $\mathbf{g}$ inside the same conformal class $\mathcal{C}_{\mathbf{g}}$. Then $\mathcal{N} = \mathbb{N}^+/\mathcal{D}$ only depends on the conformal class and not on their representatives. □

3.2. Differentiable structure of $\mathcal{N}$

In order to give differentiable structure to a quotient space, as $\mathcal{N}$ is, we will need to define what is a regular distribution and to use the proposition 3.3 for this purpose.

Definition 3.2 *A k–dimensional integrable distribution $\mathcal{D}$ in M is said to be* regular *if for every point in M there exists a coordinate chart (φ, U) adapted to $\mathcal{D}$, that is a chart such that for every leaf $\mathcal{F}$ of the foliation generated by $\mathcal{D}$ there exist $c_{k+1}, \ldots, c_n \in \mathbb{R}$ verifying that $x_j(\mathcal{F} \cap U) = c_j$ for all $j = k+1, \ldots, n$.*

The next proposition and its proof can be found at [Bri70, Prop. 11.4.2].

Proposition 3.3 *Let $\mathcal{D}$ be a regular distribution in a differentiable manifold M. Then, a differentiable structure can be provided to the set $\mathcal{F}$ of leaves of $\mathcal{D}$ in such a way the canonical projection $p : M \to \mathcal{F}$ is a submersion.*

If we require the space of light rays of M to be a differentiable manifold, it is necessary to ensure that the leaves of the distribution that builds $\mathcal{N}$, are regular submanifolds. This is not automatically ture for any spacetime M, as example 3.4 shows, so it will be necessary to impose further conditions to ensure it.

Example 3.4 *Light rays are not always leaves of a regular distribution. An analogous example can be seen in [Lo01, Ex. 1]. Consider the restriction of the two–dimensional Minkowski spacetime to the rectangle $R = [0, \alpha) \times [0, 1)$ with $\alpha \in \mathbb{R} - \mathbb{Q}$ identifying its borders as $(x, 1) \sim (x, 0)$ for all $x \in [0, \alpha)$ and $(\alpha, t) \sim (0, t)$ for all $t \in [0, 1)$. Then any null geodesic is dense in R and therefore the distribution can not be regular. Figure 1 illustrates how the null geodesic γ moves from the point $(0, 0) \in R$ to become dense due to the irrationality of the value α.*

Let us use the proposition 3.3 above to show that the space of light rays $\mathcal{N}$ of a strongly causal spacetime M has a differentiable structure [Lo89, Pr. 2.1].

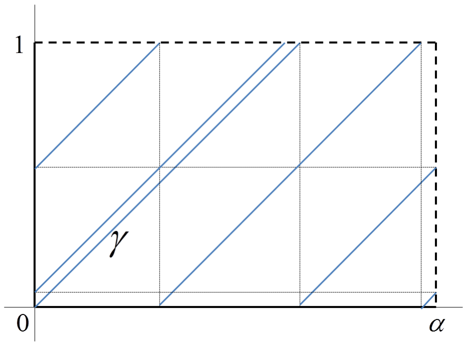


Figure 1: $\mathcal{D}$ is not regular.

Proposition 3.5 *Let M be a strongly causal spacetime, then the distribution above defined by $\mathcal{D} = \text{span}\{X_{\mathbf{g}}, \Delta\}$ is regular and the space of light rays $\mathcal{N}$ inherits from $\mathbb{N}^+$ the structure of differentiable manifold such that $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ defined by $p_{\mathbb{N}^+}(u) = [\gamma_u]$ is a submersion.*

Proof. We have that $\mathbb{N}^+$ is foliated by the elevation of null geodesics from M. Let $\mathcal{D}$ be the distribution generated by this foliation and consider the canonical projection $\pi_M^{\mathbb{N}^+} : \mathbb{N}^+ \to M$. Given $u \in \mathbb{N}^+$, there exists an adapted coordinate chart (ψ, U) to $\mathcal{D}$ in u. Since $\pi_M^{\mathbb{N}^+}$ is a submersion, then $\pi_M^{\mathbb{N}^+}(U)$ is open in M containing $\pi_M^{\mathbb{N}^+}(u) = p \in M$. By proposition 2.6, there exists a neighbourhood V of p such that if γ is a causal curve passing through V, then $\gamma \cap V$ have a unique connected component. So, the elevation of any null geodesic γ to $\mathbb{N}^+$ will intersect $\left(\pi_M^{\mathbb{N}^+}\right)^{-1}(V)$ in exactly one connected component, hence, denoting $W = U \cap \left(\pi_M^{\mathbb{N}^+}\right)^{-1}(V)$, then we have that $\left(\psi|_W, W\right)$ is an adapted chart to $\mathcal{D}$ verifying that each leaf of the generated foliation (that is each null geodesic with its null tangent vector at every point) is regular in W. Now, applying proposition 3.3, we conclude that $\mathcal{N}$ inherits from $\mathbb{N}^+$ the differentiable structure and, moreover $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ is a submersion. □

The space $\mathcal{N}$ can also be constructed as a quotient of the bundle of null directions $\mathbb{PN}$ defined below, that is we can proceed to compute the quotient space $\mathbb{N}^+/\mathcal{D}$ in two steps, we will first quotient with respect to the action dilation field Δ and, secondly we will pass to the quotient defined by the integral curves of the geodesic field.

In order to construct $\mathcal{N}$ in this way, we need to build $\mathbb{PN}$ as the quotient $\mathbb{N}^+/\mathcal{D}_\Delta$ where $\mathcal{D}_\Delta = \text{span}\{\Delta\}$. First, we will study if $\mathcal{D}_\Delta$ is a regular distribution in $\mathbb{N}^+$. Consider a local chart $\left(V, \varphi = \left(x^1, \ldots x^m\right)\right)$ in M and let $\{E_1, \ldots, E_m\}$ be a orthonormal frame in V such that E_1 is a future timelike vector field. A vector $\xi \in T_pV$ can be written as $\xi = \sum_{j=1}^{m} u^j E_j(p)$ then (ϕ, TV) with

$$\phi\colon TV \to \mathbb{R}^{2m}; \quad \xi \mapsto \left(x^1, \ldots, x^m, u^1, \ldots, u^m\right) \tag{3.1}$$

is a coordinate chart in TM. Let us denote by $\mathbb{N}^+(V)$ the restriction of the bundle $\mathbb{N}^+$ to V. For $\xi \in \mathbb{N}^+(V)$ we have that $\left(u^1\right)^2 = \sum_{j=2}^{m} \left(u^j\right)^2$ and hence coordinates in $\mathbb{N}^+(V)$ can be given by the map

$$\phi_{\mathbb{N}^+}\colon \mathbb{N}^+(V) \to \mathbb{R}^{2m-1}; \quad \xi \mapsto \left(x^1, \ldots, x^m, u^2, \ldots, u^m\right) \tag{3.2}$$

We have seen above that the Euler field Δ is tangent to $\mathbb{N}$ and it determines a differentiable distribution, that being one–dimensional, is also involutive. Since for all $\xi_0 \in \mathbb{N}^+$ some of the coordinates $u^k(\xi_0)$ with $k = 2, \ldots, m$ does not vanish, then there exists a neighbourhood $W \subset \mathbb{N}^+$ of ξ_0 such that $u^k(\xi) \neq 0$ for all $\xi \in W$. Assuming, without any lack of generality, that $u^2 \neq 0$ in W, a coordinate chart $\overline{\phi}_{\mathbb{N}^+}$ can be defined in W by

$$\overline{\phi}_{\mathbb{N}^+}\colon \mathbb{N}^+(W) \to \mathbb{R}^{2m-1}; \quad \xi \mapsto \left(x^1, \ldots, x^m, w^2, w^3, \ldots, w^m\right) \in \mathbb{R}^{2m-1} \tag{3.3}$$

where $w^2 = u^2$ and $w^k = \frac{u^k}{u^2}$ for $k = 3, \ldots, m$. If $c(t) = e^t \xi$ is the integral curve of Δ passing through $\xi \in \mathbb{N}^+$, and

$$\phi_{\mathbb{N}^+}(\xi) = \left(x_0^1, \ldots, x_0^m, u_0^2, u_0^3, \ldots, u_0^m\right)$$

then

$$\overline{\phi}_{\mathbb{N}^+}(c(t)) = \left(x_0^1, \ldots, x_0^m, e^t u_0^2, \frac{u_0^3}{u_0^2} \ldots, \frac{u_0^m}{u_0^2}\right) \tag{3.4}$$

hence $\overline{\phi}_{\mathbb{N}^+}$ is a chart adapted to the integral curves of Δ. Moreover, if $\eta \in \mathbb{N}^+$ verifies

$$\begin{cases} x^k(\eta) = x_0^k & \text{for } k = 1, \ldots, m \\ w^k(\eta) = \frac{u_0^k}{u_0^2} & \text{for } k = 3, \ldots, m \end{cases}$$

then, it is clear that $\eta = e^{t_0}\xi$ for some $t_0 \in \mathbb{R}$. This implies that the distribution $\mathcal{D}_\Delta = \operatorname{span}\{\Delta\}$ is regular. By proposition 3.3, the quotient space $\mathbb{N}^+/\mathcal{D}_\Delta$ defined by

$$\mathbb{PN} = \mathbb{N}^+/\mathcal{D}_\Delta = \left\{[\xi] : \eta \in [\xi] \Leftrightarrow \eta = e^t \xi \text{ for some } t \in \mathbb{R} \text{ and } \xi \in \mathbb{N}^+\right\}$$

is a differentiable manifold and, moreover, the canonical projection

$$\begin{array}{rccc} \pi_{\mathbb{PN}}^{\mathbb{N}^+}: & \mathbb{N}^+ & \to & \mathbb{PN} \\ & \xi & \mapsto & [\xi] \end{array}$$

is a submersion.

The next step is to find a regular distribution that allows us to define $\mathcal{N}$ by a quotient. For each vector $u \in \mathbb{N}_p^+$ there exists a null geodesic γ_u such that $\gamma_u(0) = p$ and $\gamma_u'(0) = u$, and given two vectors $u, v \in \mathbb{N}_p^+$ verifying that $v = \lambda u$ with $\lambda > 0$, then the geodesics γ_u and γ_v such that $\gamma_u(0) = \gamma_v(0) = p$ have the property

$$\gamma_v(s) = \gamma_{\lambda u}(s) = \gamma_u(\lambda s)$$

hence they have the same image in M and then $\gamma_v = \gamma_u$ as unparametrized sets in M. This fact implies that the elevations to $\mathbb{PN}$ of the null geodesics of M define a foliation $\mathcal{D}_G$. Two directions $[u], [v] \in \mathbb{PN}$ belong to the same leaf of the foliation $\mathcal{D}_G$ if for the vectors $v \in \mathbb{N}_p^+$ and $u \in \mathbb{N}_q^+$ there exist null geodesics γ_1 and γ_2 and values $t_1, t_2 \in \mathbb{R}$ verifying

$$\begin{cases} \gamma_1(t_1) = p \in M \\ \gamma_1'(t_1) = v \in \mathbb{N}_p^+ \end{cases} \quad \text{and} \quad \begin{cases} \gamma_2(t_2) = q \in M \\ \gamma_2'(t_2) = u \in \mathbb{N}_q^+ \end{cases}$$

such that there is a reparametrization h verifying $\gamma_1 = \gamma_2 \circ h$.

Hence, the space of leaves of $\mathcal{D}_G$ in $\mathbb{PN}$ coincides with $\mathcal{N}$, that is,

$$\mathcal{N} = \mathbb{PN}/\mathcal{D}_G$$

The map

$$\begin{array}{rccc} p_{\mathbb{PN}}: & \mathbb{PN} & \longrightarrow & \mathcal{N} \\ & [u] & \longmapsto & [\gamma_u] \end{array}$$

is well defined, since $\gamma_{\lambda u}(s) = \gamma_u(\lambda s)$ as seen above, and it verifies the identity

$$p_{\mathbb{PN}}([\gamma_u'(s)]) = [\gamma_u] \in \mathcal{N}$$

for all s.

Remark 3.1 *Proposition 3.5 can be formulated for the bundle $\mathbb{PN}$ instead of $\mathbb{N}^+$, because the proof is,* mutatis mutandis, *the same, where in this case $\mathcal{D}_G$ is a regular distribution and a differentiable structure is also inherited from $\mathbb{PN}$ such that $p_{\mathbb{PN}} : \mathbb{PN} \to \mathcal{N}$ is a submersion. In [Lo06, Thm. 1], this result is shown for the subbundle $(\mathbb{N}^*)^+$ of the cotangent bundle T^*M.*

Now, we will describe a generic way to construct coordinate charts in $\mathcal{N}$. First, we define for any subset $W \subset M$

$$\mathbb{N}(W) = \left\{\xi \in \mathbb{N} : \pi_M^{\mathbb{N}}(\xi) \in W \subset M\right\}$$

$$\mathbb{PN}(W) = \left\{[\xi] \in \mathbb{PN} : \pi_M^{\mathbb{PN}}([\xi]) \in W \subset M\right\}.$$

By theorem 2.5, we can take $V \subset M$ as a causally convex, globally hyperbolic and open set. Let $\mathcal{U}$ be the image of the projection $p_{\mathbb{N}} : \mathbb{N}^+(V) \mapsto \mathcal{N}$. Since $\mathbb{N}^+(V)$ is open in $\mathbb{N}^+$ and $p_{\mathbb{N}}$ is a submersion, then $\mathcal{U} \subset \mathcal{N}$ is open. Moreover, since V is globally hyperbolic, then we can fix a smooth spacelike Cauchy surface $C \subset V$. So, each null geodesic passing through V intersects C in a unique point and since $p_{\mathbb{N}^+} = p_{\mathbb{PN}} \circ \pi^{\mathbb{N}^+}_{\mathbb{PN}}$, this ensures that

$$\mathcal{U} = p_{\mathbb{N}}\left(\mathbb{N}^+(V)\right) = p_{\mathbb{N}}\left(\mathbb{N}^+(C)\right) = p_{\mathbb{N}} \circ \pi^{\mathbb{N}^+}_{\mathbb{PN}}\left(\mathbb{N}^+(C)\right) = p_{\mathbb{PN}}\left(\mathbb{PN}(C)\right) = p_{\mathbb{PN}}\left(\mathbb{PN}(V)\right).$$

Since C is a regular differentiable submanifold of V, then the bundles $\mathbb{N}^+(C)$ and $\mathbb{PN}(C)$ are also regular differentiable submanifolds of $\mathbb{N}^+(V)$ and $\mathbb{PN}(V)$ respectively, and moreover the map $\sigma = p_{\mathbb{PN}}|_{\mathbb{PN}(C)} : \mathbb{PN}(C) \mapsto \mathcal{U}$ is a differentiable bijection. The map $p_{\mathbb{PN}}$ is a submersion verifying that for any $[\xi] \in \mathbb{PN}(V)$, the kernel of $(dp_{\mathbb{PN}})_{[\xi]}$ is the one–dimensional subspace generated by the tangent vectors to curves defining light rays, that is, curves $\lambda(s) = [\gamma'(s)] \in \mathbb{PN}_{\gamma(s)}$ where γ is a null geodesic and $[\gamma'(s)] = \{\lambda\gamma'(s) : \lambda \in \mathbb{R}\}$. Being C a spacelike hypersurface, the kernel of $\left(dp_{\mathbb{PN}}|_{\mathbb{PN}(C)}\right)_{[\xi]} = d\sigma_{[\xi]}$ is trivial, hence $d\sigma_{[\xi]}$ is a surjective linear map between vector spaces of the same dimension, then it is also bijective y therefore σ is a diffeomorphism. So, we have the following diagram

$$\begin{array}{ccc} \mathbb{PN}(V) & \xrightarrow{p_{\mathbb{PN}}} & \mathcal{U} \\ \text{inc}\uparrow & \nearrow_{\sigma} & \\ \mathbb{PN}(C) & & \end{array} \tag{3.5}$$

If ϕ is any coordinate chart for $\mathbb{PN}(C)$ then $\phi \circ \sigma^{-1}$ is a coordinate chart for $\mathcal{U} \subset \mathcal{N}$.

Observe that if M is time–orientable, there exists a non–vanishing future timelike vector field $T \in \mathfrak{X}(M)$ everywhere. Then we can define the submanifold $\Omega^T(C) \subset \mathbb{N}^+(C)$ by

$$\Omega^T(C) = \left\{\xi \in \mathbb{N}^+(C) : \mathbf{g}(\xi, T) = -1\right\}$$

We have that $\pi^{\mathbb{N}^+}_{\mathbb{PN}} : \mathbb{N}^+ \to \mathbb{PN}$ is a submersion such that the kernel of the differential $d\pi^{\mathbb{N}^+}_{\mathbb{PN}}$ at any point $\xi \in \mathbb{N}^+$ is generated by $\Delta(\xi)$. If we consider the restriction $\left.\pi^{\mathbb{N}^+}_{\mathbb{PN}}\right|_{\Omega^T(C)} : \Omega^T(C) \to \mathbb{PN}(C)$, it is clear that it is a bijection. Moreover, since

$$\ker\left(\left(\left.d\pi^{\mathbb{N}^+}_{\mathbb{PN}}\right|_{\Omega^T(C)}\right)_{\xi}\right) = \{\mathbf{0}\}$$

at any point ξ, and due to $\dim\left(\Omega^T(C)\right) = \dim\left(\mathbb{PN}(C)\right) = 2m-3$, then $\left.\pi^{\mathbb{N}^+}_{\mathbb{PN}}\right|_{\Omega^T(C)}$ is a diffeomorphism. So, we have the following diagram

$$\mathcal{N} \supset \mathcal{U} \leftrightarrow \mathbb{PN}(C) \leftrightarrow \Omega^T(C) \hookrightarrow \mathbb{N}^+(C) \hookrightarrow \mathbb{N}^+ \hookrightarrow TM \tag{3.6}$$

where $\leftrightarrow$ and $\hookrightarrow$ represent diffeomorphisms and inclusions respectively.

Then, the composition of the diffeomorphism $\mathcal{U} \to \Omega^T(C)$ with the restriction of a coordinate chart in TM to the vectors in $\Omega^T(C)$, can be used to construct a coordinate chart in $\mathcal{N}$.

Remark 3.2 *By construction of the diffeomorphism $\sigma : \mathbb{PN}(C) \to \mathcal{U} \subset \mathcal{N}$, if M is globally hyperbolic then it is possible to choose the causal convex open set as $V = M$ and C a global Cauchy surface. In this case we have that $\sigma : \mathbb{PN}(C) \to \mathcal{N}$ is a global diffeomorphism.*

If there exists a non-vanishing $X \in \mathfrak{X}(C)$, then $\mathbb{PN}(C)$ is a trivial fibre bundle because it is possible to construct a global section taking X and a non-vanishing timelike $T \in \mathfrak{X}(M)$. Since X is spacelike then for any $p \in C$ there exist $\alpha_p > 0$ such that $T_p + \alpha_p X_p \in T_pM$ is a null vector. Then $s : C \to \mathbb{PN}(C)$ defined by $s(p) = [T_p + \alpha_p X_p] \in \mathbb{PN}_p \subset \mathbb{PN}(C)$ is a global section and therefore

$$\mathcal{N} \simeq C \times \mathbb{S}^{m-2}.$$

If we require the space of light rays of M to be a differentiable manifold, it remains to ensure that $\mathcal{N}$ is a Hausdorff topological space. Again, it is not verified for any strongly causal spacetime M as we can check in example 3.6, so we need to state conditions to ensure it.

Example 3.6 *$\mathcal{N}$ is not Hausdorff. Consider the two–dimensional Minkowski spacetime and remove the point $(1,1)$. Clearly, M is strongly causal. Let $\{\tau_n\} \subset \mathbb{R}$ be a sequence such that $\lim_{n\mapsto\infty} \tau_n = 0$. Then the sequence of null geodesic given by $\lambda_n(s) = (s, \tau_n + s)$ with $s \in (-\infty, \infty)$ converges to two different null geodesics, $\mu_1(s) = (s,s)$ with $s \in (-\infty, 1)$ and $\mu_2(s) = (s,s)$ with $s \in (1,\infty)$. Figure 2 illustrates this example.*

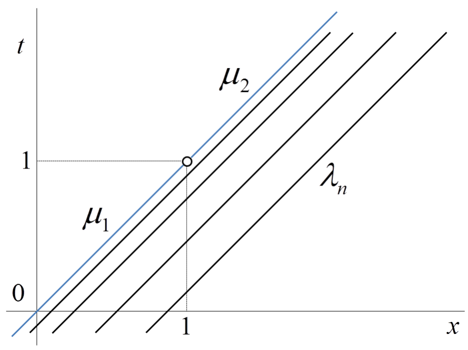


Figure 2: $\mathcal{N}$ is not Hausdorff.

A sufficient condition to ensure that $\mathcal{N}$ is Hasudorff is the absence of naked singularities, as next proposition shows. But we will see in example 3.8 that it is not a necessary condition.

Proposition 3.7 *Let M be a strongly causal spacetime and $\mathcal{N}$ its corresponding space of light rays. If $\mathcal{N}$ is not Hausdorff then M possesses a naked singularity.*

Proof. We will follow the proof of [Lo89, Prop. 2.2]. If $\mathcal{N}$ is not Hausdorff, then there exists two null geodesics $\gamma_1, \gamma_2 \in \mathcal{N}$ such that any pair of neighbourhoods $U_1, U_2 \subset \mathcal{N}$ of γ_1 and γ_2 respectively verifies that $U_1 \cap U_2 \neq \varnothing$. Hence, it is possible to build a sequence $\{\mu_n\} \subset \mathcal{N}$ such that γ_1 and γ_2 are their limits. If we consider the same sequence as curves in M, we can take points $p_1 \in \gamma_1 \subset M$ and $p_2 \in \gamma_2 \subset M$ and corresponding neighbourhoods V_1 and V_2 such that $V_1 \cap V_2 = \varnothing$. This is possible since M is actually Hausdorff. We can assume without any lack of generality that $\mu_n \cap V_i \neq \varnothing$ for all n with $i = 1, 2$. Let us take points $q_n^i \in \mu_n \cap V_i$ with $i = 1, 2$ such that p_i is a limit point of the sequence$\left\{q_n^i\right\}$. Since each geodesic μ_n is causal, we can consider that $q_n^2 \in J^+\left(q_n^1\right)$ for all n. If $r \in I^+(p_2)$ then $I^-(r)$ is a neighbourhood of p_2 and, hence there exists n_0 such that $q_n^2 \in I^-(r)$ for all $n > n_0$. Moreover, since $q_n^2 \in J^+\left(q_n^1\right)$ then $q_n^1 \in I^-(r)$, therefore $p_1 \in \overline{I^-(r)}$. Now if we take $w \in I^-(p_1)$ then $I^+(w)$ is a neighbourhood of p_1 and it must intersect $I^-(r)$, hence $w \in I^-(r)$ but, since it does not depends on the chosen point $p_1 \in \gamma_1$, then any point of $z \in I^-(\gamma_1)$ verifies that $z \in I^-(r)$. Consequently $I^-(\gamma_1) \subset I^-(r)$ and since γ_1 is an inextensible causal curve then there exists a naked singularity en M. $\square$

Example 3.8 *Let $\mathbb{M}$ be the 3–dimensional Minkowski space–time described by coordinates (t, x, y) and equipped with the metric $\mathbf{g} = -dt \otimes dt + dx \otimes dx + dy \otimes dy$. The hypersurface $C \equiv \{t = 0\}$ is a spacelike Cauchy surface. The corresponding space of light rays $\mathcal{N}_{\mathbb{M}}$ is diffeomorphic to the bundle of circumferences on C, that is, $\mathcal{N}_{\mathbb{M}} \simeq C \times \mathbb{S}^1$.*

Now, consider the restriction $\mathbb{B} = \left\{(t, x, y) \in \mathbb{M} : t^2 + x^2 + y^2 < 1\right\}$. It is clear that $\mathbb{B}$ is strongly causal.

First, we will see that $\mathbb{B}$ is not globally hyperbolic. Consider the inextensible null geodesics in $\mathbb{B}$ given by

$$\gamma_1(s) = \left(s, \frac{7}{5} - s, 0\right) \qquad s \in \left(\frac{3}{5}, \frac{4}{5}\right)$$

$$\gamma_2(\tau) = \left(\tau, \frac{7}{5} + \tau, 0\right) \qquad \tau \in \left(-\frac{4}{5}, -\frac{3}{5}\right)$$

It is easy to see that any point of γ_1 is in the chronological future of any point of γ_2. Indeed, the curve $\mu(u) = \gamma_2(\tau) + u \cdot (\gamma_1(s) - \gamma_2(\tau))$ is a future–directed timelike geodesic connecting $\gamma_2(\tau)$ to $\gamma_1(s)$ since

$$\mu'(u) = (s - \tau, s + \tau, 0)$$

and

$$\mathbf{g}\left(\mu', \mu'\right) = 4s\tau < 0$$

for all $s \in \left(\frac{3}{5}, \frac{4}{5}\right)$ *and* $\tau \in \left(-\frac{4}{5}, -\frac{3}{5}\right)$. *If a spacelike Cauchy surface* $\Omega \subset \mathbb{B}$ *exists, then* $\Omega \cap \gamma_i \neq \varnothing$ *for* $i = 1, 2$, *and then* Ω *would have timelike related points, but this is not possible in a Cauchy surface. Therefore* $\mathbb{B}$ *is not globally hyperbolic.*

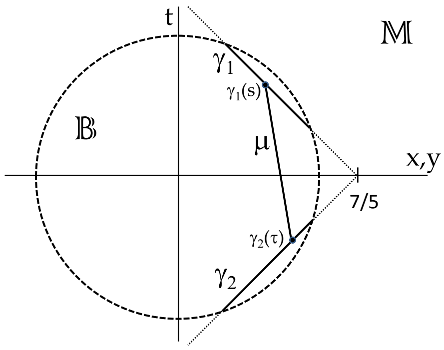


Figure 3: $M = \mathbb{B}$ is naked singular and Hausdorff.

We have already shown that $\mathbb{B}$ *is nakedly singular, because for any* $s \in \left(\frac{3}{5}, \frac{4}{5}\right)$ *we have that*

$$I^{-}\left(\gamma_2\right) \subset I^{-}\left(\gamma_1\left(s\right)\right)$$

Finally, we will show that the space of light rays $\mathcal{N}_{\mathbb{B}}$ *in* $\mathbb{B}$ *is Hausdorff. It is clear that* $\mathcal{N}_{\mathbb{B}} \subset \mathcal{N}_{\mathbb{M}}$. *Consider* $\gamma \in \mathcal{N}_{\mathbb{B}}$. *As a curve in* $\mathcal{N}_{\mathbb{M}} = C \times \mathbb{S}^1$ *we denote*

$$\gamma = (x_0, y_0, \theta_0)$$

We can parametrize γ *as* $\gamma\left(s\right) = \left(t, x_0 + t\cos\theta_0, y_0 + t\sin\theta_0\right)$ *and since* $\gamma \in \mathcal{N}_{\mathbb{B}}$, *then there exists* $t_0 \in \mathbb{R}$ *such that*

$$t_0^2 + \left(x_0 + t_0\cos\theta_0\right)^2 + \left(y_0 + t_0\sin\theta_0\right)^2 < 1 \tag{3.7}$$

Since inequality 3.7 is an open condition, then there exist $\alpha, \beta, \delta, \epsilon \in \mathbb{R}$ *verifying*

$$t^2 + \left(x + t\cos\theta\right)^2 + \left(y + t\sin\theta\right)^2 < 1$$

for any (t, x, y, θ) *with*

$$\begin{array}{l} t \in \left(t_0 - \alpha, t_0 + \alpha\right) \\ x \in \left(x_0 - \beta, x_0 + \beta\right) \\ y \in \left(y_0 - \delta, y_0 + \delta\right) \\ \theta \in \left(\theta_0 - \epsilon, \theta_0 + \epsilon\right) \end{array}$$

Then $\mathcal{N}_{\mathbb{B}}$ *is open in* $\mathcal{N}_{\mathbb{M}}$. *Since* M *is globally hyperbolic, then* $\mathcal{N}_{\mathbb{M}}$ *is Hausdorff and therefore* $\mathcal{N}_{\mathbb{B}}$ *is also Hausdorff.*

Example 3.8 shows that the absence of naked singularities is a condition too strong for a strongly causal spacetime M. Moreover in this case, M becomes globally hyperbolic as Penrose proved in [Pe79].

A suitable condition to avoid the behavior of light rays in the paradigmatic example 3.6 but to permit naked singularities similar to the ones in example 3.8 is the condition of *null pseudo–convexity.*

Definition 3.3 *A spacetime M is said to be* null pseudo–convex *if for any compact $K \subset M$ there exists a compact $K' \subset M$ such that any null geodesic segment γ with endpoints in K is totally contained in K'.*

In [Lo90-2], Low states the equivalence of null pseudo–convexity of M and the Hausdorffness of $\mathcal{N}$ for a strongly causal spacetime M. From now on, we will assume that M is a strongly causal and null pseudo–convex spacetime unless others conditions are pointed out.

4. Tangent bundle of $\mathcal{N}$

To take advantage of the geometry and topology of $\mathcal{N}$ it is needed to have a suitable characterization of the tangent spaces $T_\gamma\mathcal{N}$ for any $\gamma \in \mathcal{N}$. We will proceed as follows: first, we will define *geodesic variations* (in particular, *variations by light rays*) and *Jacobi fields*, explaining the relation between both concepts (in lemmas 4.1, 4.2, 4.3, 4.4 and proposition 4.2). Then, in proposition 4.3, we will characterize tangent vector of TM by Jacobi field. Second, we will keep an eye on how the Jacobi fields changes when we vary the parameters of the corresponding variation by light rays or conformal metric of M (see from lemma 4.5 to 4.10).

Finally, in proposition 4.4, we will get the aim of this section identifying tangent vectors of $\mathcal{N}$ with some equivalence classes of Jacobi fields.

Definition 4.1 *A differentiable map $\mathbf{x} : (a, b) \times (\alpha, \beta) \to M$ is said to be a* variation of a segment of curve $c : (\alpha, \beta) \to M$ *if $c(t) = \mathbf{x}(s_0, t)$ for some $s_0 \in (a, b)$. We will say that $V^{\mathbf{x}}_{s_0}$ is the* initial field of $\mathbf{x}$ *in $s = s_0$ if*

$$V^{\mathbf{x}}_{s_0}(t) = d\mathbf{x}_{(s_0,t)}\left(\frac{\partial}{\partial s}\right)_{(s_0,t)} = \left.\frac{\partial \mathbf{x}(s, t)}{\partial s}\right|_{(s_0,t)}$$

defining a vector field along c.

We will say that $\mathbf{x}$ is a geodesic variation *if any longitudinal curve of $\mathbf{x}$, that is $c^{\mathbf{x}}_s = \mathbf{x}(s, \cdot)$ for $s \in (a, b)$, is a geodesic.*

If the longitudinal curves $c^{\mathbf{x}}_s : (\alpha, \beta) \to M$ are regular curves covering segments of light rays, then $\mathbf{x} : (a, b) \times (\alpha, \beta) \to M$ is said to be a variation by light rays.

Moreover, a variation by light rays $\mathbf{x}$ is said to be a variation by light rays of $\gamma \in \mathcal{N}$ *if γ is a longitudinal curve of $\mathbf{x}$.*

Notation 4.1 *It is possible to identify a given segment of null geodesic $\gamma : (-\delta, \delta) \to M$, with a slight abuse in the notation, to the light ray in $\mathcal{N}$ defined by it. So, if $\mathbf{x} = \mathbf{x}(s,t)$ is a variation by light rays, we can denote by $\gamma_s^{\mathbf{x}} \subset M$ the null pregeodesics of the variation and also by $\gamma_s^{\mathbf{x}} \in \mathcal{N}$ the light rays they define.*

Consider a geodesic curve $\mu(t)$ in a spacetime $(M, \mathbf{g})$. Given $J \in \mathfrak{X}_\mu$, we will abbreviate the notation $J' = \frac{DJ}{dt}$ and $J'' = \frac{D}{dt}\frac{DJ}{dt} = \frac{D^2 J}{dt^2}$. We can define the *Jacobi equation* by

$$J'' + R(J, \mu')\mu' = 0 \tag{4.1}$$

where R is the Riemann tensor. We will name the solutions of the equation 4.1 by *Jacobi field* along μ. So, the set of Jacobi fields along μ is then defined by

$$\mathcal{J}(\mu) = \{J \in \mathfrak{X}_\mu : J'' + R(J, \mu')\mu' = 0\} \tag{4.2}$$

The linearity of $\frac{D}{dt}$ and R provides a vector space structure to $\mathcal{J}(\mu)$. Indeed, for $\alpha, \beta \in \mathbb{R}$ and $J, K \in \mathcal{J}(\gamma)$ we have

$$\frac{D}{dt}\frac{D}{dt}(\alpha J + \beta K) + R((\alpha J + \beta K), \mu')\mu' =$$

$$= \frac{D}{dt}(\alpha J' + \beta K') + \alpha R(J, \mu')\mu' + \beta R(K, \mu')\mu' =$$

$$= \alpha J'' + \beta K'' + \alpha R(J, \mu')\mu' + \beta R(K, \mu')\mu' =$$

$$= \alpha(J'' + R(J, \mu')\mu') + \beta(K'' + R(K, \mu')\mu') =$$

$$= \alpha \cdot 0 + \beta \cdot 0 = 0$$

then $\alpha J + \beta K$ is a Jacobi field and hence $\mathcal{J}(\mu)$ is a vector subspace of $\mathfrak{X}_\mu$.

The relation between geodesic variations and Jacobi fields is expounded in next lemma.

Lemma 4.1 *If $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ is a geodesic variation of a geodesic γ, then the initial field $V^{\mathbf{x}}$ is a Jacobi field along γ.*

Proof. See [On83, Lem. 8.3]. □

A Jacobi field along a geodesic γ is fully defined by its initial values at any point of γ as lemma 4.2 claims, and moreover it also implies that the vector space $\mathcal{J}(\mu)$ is isomorphic to $T_pM \times T_pM$ therefore $\dim(\mathcal{J}(\gamma)) = 2\dim(M) = 2m$.

Lemma 4.2 *Let γ be a geodesic in M such that $\gamma(0) = p$ and $u, v \in T_pM$. Then there exists a only Jacobi field J along γ such that $J(0) = u$ and $\frac{DJ}{dt}(0) = v$.*

Proof. See [On83, Lem. 8.5]. □

Next lemma characterizes the Jacobi fields of a particular type of variation. This type will be the general case for the variations by light rays studied below.

Lemma 4.3 *Let M be a spacetime, $\gamma : (-\delta, \delta) \to M$ a geodesic segment, $\lambda : (-\epsilon, \epsilon) \to M$ a curve verifying $\lambda(0) = \gamma(0)$, and $W(s)$ a vector field along λ such that $W(0) = \gamma'(0)$. Then the Jacobi field J along γ defined by the geodesic variation*

$$\mathbf{x}(s,t) = \exp_{\lambda(s)}(tW(s))$$

verifies that

$$\begin{cases} J(0) = \lambda'(0) \\ \frac{DJ}{dt}(0) = \frac{DW}{ds}(0) \end{cases}$$

Proof. First, the vector $\frac{\partial \mathbf{x}}{\partial s}(0,0)$ is the tangent vector of the curve $\mathbf{x}(s,0)$ in $s=0$, and since $\mathbf{x}(s,0) = \exp_{\lambda(s)}(0 \cdot W(s)) = \exp_{\lambda(s)}(0) = \lambda(s)$, then we have

$$J(0) = \frac{\partial \mathbf{x}}{\partial s}(0,0) = \frac{d\lambda}{ds}(0) = \lambda'(0)$$

On the other hand, $\frac{D}{ds}\frac{\partial \mathbf{x}}{\partial t}(0,0)$ is the covariant derivative of the vector field $\frac{\partial \mathbf{x}}{\partial t}(s,0) = W(s)$ for $s=0$ along the curve $\mathbf{x}(s,0) = \lambda(s)$. Then

$$\frac{DJ}{dt}(0) = \frac{D}{dt}\frac{\partial \mathbf{x}}{\partial s}(0,0) = \frac{D}{ds}\frac{\partial \mathbf{x}}{\partial t}(0,0) = \frac{DW}{ds}(0).$$

as required. □

Remark 4.1 *It can be observed that given a geodesic variation $\mathbf{x} = \mathbf{x}(s,t)$ such that J is the corresponding Jacobi field at $s=0$, if we change the geodesic parameters such that $\overline{\mathbf{x}}(s,\tau) = \mathbf{x}(s, a\tau + b)$ for $a > 0$ and $b \in \mathbb{R}$, then the initial values of the Jacobi field $\overline{J}$ of $\overline{\mathbf{x}}$ at $s=0$ verify*

$$\overline{J}(-b/a) = \frac{\partial \overline{\mathbf{x}}}{\partial s}(0,-b/a) = \frac{\partial \mathbf{x}}{\partial s}(0,0) = J(0)$$

and also

$$\begin{aligned}
\overline{J}'(-b/a) &= \left.\frac{D}{d\tau}\right|_{(0,-b/a)} \frac{\partial \overline{\mathbf{x}}}{\partial s}(s,\tau) = \left.\frac{D}{ds}\right|_{(0,-b/a)} \frac{\partial \overline{\mathbf{x}}}{\partial \tau}(s,\tau) = \\
&= \left.\frac{D}{ds}\right|_{(0,-b/a)} \frac{\partial \mathbf{x}}{\partial \tau}(s, a\tau+b) = \left.\frac{D}{ds}\right|_{(0,0)} a\frac{\partial \mathbf{x}}{\partial t}(s, a\tau+b) = \\
&= a \left.\frac{D}{ds}\right|_{(0,0)} \frac{\partial \mathbf{x}}{\partial t}(s, a\tau+b) = a \left.\frac{D}{dt}\right|_{(0,0)} \frac{\partial \mathbf{x}}{\partial s}(s, a\tau+b) = \\
&= aJ'(0)
\end{aligned}$$

If we denote by $Y(\tau) = J(a\tau + b)$*, then it is trivial to see that* $Y(-b/a) = J(0)$ *and* $Y'(-b/a) = aJ'(0)$*, therefore* $Y = \overline{J}$ *and this implies that changing the geodesic parameter does not modify the Jacobi field as a geometric object.*

Proposition 4.2 *Given a geodesic* γ *in* $(M, \mathbf{g})$ *and a Jacobi field* $J \in \mathcal{J}(\gamma)$ *along* γ*, then* $\mathbf{g}(J(t), \gamma'(t)) = a + bt$ *is verified.*

Proof. Deriving $\mathbf{g}(J(t), \gamma'(t))$, we obtain

$$\left.\frac{d}{dt}\right|_t \mathbf{g}(J, \gamma') = \mathbf{g}\left(\left.\frac{D}{dt}\right|_t J, \gamma'\right) + \mathbf{g}\left(J, \left.\frac{D}{dt}\right|_t \gamma'\right) = \mathbf{g}\left(\left.\frac{D}{dt}\right|_t J, \gamma'\right)$$

and so

$$\begin{aligned}\left.\frac{d^2}{dt^2}\right|_t \mathbf{g}(J, \gamma') &= \mathbf{g}\left(\left.\frac{D^2}{dt^2}\right|_t J, \gamma'\right) + \mathbf{g}\left(\left.\frac{D}{dt}\right|_t J, \left.\frac{D}{dt}\right|_t \gamma'\right) = \\ &= \mathbf{g}\left(\left.\frac{D^2}{dt^2}\right|_t J, \gamma'\right) = \mathbf{g}\left(-R(J, \gamma')\gamma', \gamma'\right) = 0.\end{aligned}$$

where the anti–symmetric property of the curvature tensor R has been used. Then, $\left.\frac{d}{dt}\right|_t \mathbf{g}(J, \gamma') = b$ constant and therefore $\mathbf{g}(J(t), \gamma'(t)) = a + bt$. □

We will need the following technical lemma. It shows that the information contained in the tangent vector of a curve $v \subset TM$ coincides with the one in the covariant derivative of v as vector field along its base curve in M.

Lemma 4.4 *Let* $v_i : I_i \to TM$ *be differentiable curves where* $I_i \subset \mathbb{R}$ *are intervals verifying that* $0 \in I_i$ *for* $i = 1, 2, 3$ *such that* $v_1(0) = v_2(0) = v_3(0) \in TM$*. Denote by* $\alpha_i = \pi_M^{TM} \circ v_i$ *the projections of* v_i *onto* M*, by* $\frac{D}{ds_i}$ *the covariant derivative along* α_i *relative to its parameter* s_i *and by* v_i' *the tangent vector of the curve* v_i *for* $i = 1, 2, 3$*. Then for* $\beta_1, \beta_2 \in \mathbb{R}$ *we have*

$$v_3'(0) = \beta_1 v_2'(0) + \beta_2 v_2'(0) \Longleftrightarrow \begin{cases} \alpha_3'(0) = \beta_1 \alpha_1'(0) + \beta_2 \alpha_2'(0) \\ \\ \frac{Dv_3}{ds_3}(0) = \beta_1 \frac{Dv_1}{ds_1}(0) + \beta_2 \frac{Dv_2}{ds_2}(0) \end{cases}$$

Proof. Denote by $p = \alpha_1(0) = \alpha_2(0) = \alpha_3(0) \in M$ and $v = v_1(0) = v_2(0) = v_3(0) \in TM$ and consider coordinates $(x_1, \ldots, x_m)$ in a neighbourhood of $p \in M$ to build the coordinates $\left(x^1, \ldots, x^m, v^1, \ldots, v^m\right)$ in a neighbourhood $W \subset TM$ containing v in such way that $w \in W$ can be written as $w = \sum_{k=1}^m v^k \left(\frac{\partial}{\partial x^k}\right)_q$. In order to avoid unnecessary complications, we will write the parameters $s_i = s$ and denote $v_i(s) = \sum_{k=1}^m v_i^k(s) \left(\frac{\partial}{\partial x^k}\right)_{\alpha_i(s)}$ and $a_i^k = x^k \circ \alpha_i$ for $i = 1, 2, 3$.

So we have

$$v_3'(0) = \beta_1 v_2'(0) + \beta_2 v_2'(0)$$

$$\Updownarrow$$

$$\frac{da_3^k}{ds}(0)\left(\frac{\partial}{\partial x^k}\right)_v + \frac{dv_3^k}{ds}(0)\left(\frac{\partial}{\partial v^k}\right)_v = \sum_{j=1}^{2}\beta_j\left(\frac{da_j^k}{ds}(0)\left(\frac{\partial}{\partial x^k}\right)_v + \frac{dv_j^k}{ds}(0)\left(\frac{\partial}{\partial v^k}\right)_v\right)$$

$$\Updownarrow$$

$$\begin{cases} \frac{da_3^k}{ds}(0) = \beta_1\frac{da_1^k}{ds}(0) + \beta_2\frac{da_2^k}{ds}(0) \\ \frac{dv_3^k}{ds}(0) = \beta_1\frac{dv_1^k}{ds}(0) + \beta_2\frac{dv_2^k}{ds}(0) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_3'(0) = \beta_1\alpha_1'(0) + \beta_2\alpha_2'(0) \\ \frac{dv_3^k}{ds}(0) + \Gamma_{ij}^k v_3^i(0)\frac{da_3^k}{ds}(0) = \beta_1\frac{dv_1^k}{ds}(0) + \beta_2\frac{dv_2^k}{ds}(0) + \Gamma_{ij}^k v_3^i(0)\frac{da_3^k}{ds}(0) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_3'(0) = \beta_1\alpha_1'(0) + \beta_2\alpha_2'(0) \\ \frac{Dv_3}{ds}(0) = \beta_1\frac{dv_1^k}{ds}(0) + \beta_2\frac{dv_2^k}{ds}(0) + \Gamma_{ij}^k v_3^i(0)\left(\beta_1\frac{da_1^k}{ds}(0) + \beta_2\frac{da_2^k}{ds}(0)\right) \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} \alpha_3'(0) = \beta_1\alpha_1'(0) + \beta_2\alpha_2'(0) \\ \frac{Dv_3}{ds}(0) = \beta_1\frac{Dv_1}{ds}(0) + \beta_2\frac{Dv_2}{ds}(0) \end{cases}$$

since $v_3(0) = v_2(0) = v_1(0)$. Then the proof is complete. $\square$

It is possible to identify any tangent vector $\xi \in TTM$ with a Jacobi field along the geodesic γ defined by the exponential of the vector $u = \pi_{TM}^{TTM}(\xi) \in TM$.

Proposition 4.3 *Given a vector* $u_0 \in T_pM$ *and consider the geodesic* γ_{u_0} *defined by* $\gamma_{u_0}(t) = \exp_p(tu_0)$. *Let* $u : (-\delta, \delta) \to TM$ *be a differentiable curve such that* $u(0) = u_0$ *and* $u'(0) = \xi$. *If* $J \in \mathcal{J}(\gamma_{u_0})$ *is the Jacobi field of the geodesic variation given by* $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tu(s))$ *where* $\alpha = \pi_M^{TM} \circ u$, *then the map*

$$\begin{array}{cccc} \zeta: & T_{u_0}TM & \to & \mathcal{J}(\gamma_{u_0}) \\ & \xi & \mapsto & J \end{array}$$

is a well–defined linear isomorphism.

Proof. First, we will show that ζ is well–defined. Let $u_i : (-\delta_i, \delta_i) \rightarrow TM$ be differentiable curves such that $u_i(0) = u_0$ and $u_i'(0) = \xi$ for $i = 1, 2$, and consider the geodesic variations $\mathbf{x}_i(s_i, t) = \exp_{\alpha(s_i)}(t u_i(s_i))$. By lemma 4.4 with $\beta_1 = 1$ and $\beta_2 = 0$, we have that $\alpha_1'(0) = \alpha_2'(0)$ and $\frac{Du_1}{ds_1}(0) = \frac{Du_2}{ds_2}(0)$, and therefore by lemma 4.3, the Jacobi fields J_1 and J_2 along γ_{u_0} corresponding to $\mathbf{x}_1$ and $\mathbf{x}_2$ respectively verify $J_1 = J_2$. So ζ is well–defined.

In order to prove ζ is linear, consider $\xi_1, \xi_2 \in T_{u_0}TM$ such that $\zeta(\xi_1) = J_1$ and $\zeta(\xi_2) = J_2$ and moreover $\xi = \beta_1 \xi_1 + \beta_2 \xi_2$ for $\beta_1, \beta_2 \in \mathbb{R}$. Let $u : (-\delta, \delta) \rightarrow TM$ be a differentiable curve verifying $u(0) = u_0$ and $u'(0) = \xi$ and let $J \in \mathcal{J}(\gamma_{u_0})$ be the Jacobi field such that $J(0) = \alpha'(0)$ and $J'(0) = \frac{Du}{ds}(0)$ with $\alpha = \pi_M^{TM} \circ u$. By lemma 4.4, we have that $\alpha'(0) = \beta_1 \alpha_1'(0) + \beta_2 \alpha_2'(0)$ and $\frac{Du}{ds}(0) = \beta_1 \frac{Du_1}{ds_1}(0) + \beta_2 \frac{Du_2}{ds_2}(0)$ and since the Jacobi field $Y = \beta_1 J_1 + \beta_2 J_2$ verifies that

$$\begin{cases} Y(0) = \beta_1 J_1(0) + \beta_2 J_2(0) = \beta_1 \alpha_1'(0) + \beta_2 \alpha_2'(0) \\ Y'(0) = \beta_1 J_1'(0) + \beta_2 J_2'(0) = \beta_1 \frac{Du_1}{ds_1}(0) + \beta_2 \frac{Du_2}{ds_2}(0) \end{cases}$$

then $J = Y$ and therefore

$$\zeta(\beta_1 \xi_1 + \beta_2 \xi_2) = \zeta(\xi) = J = Y = \beta_1 J_1 + \beta_2 J_2$$

Then ζ is linear.

Finally, let us see that ζ is an isomorphism. If $\xi \in TM$ such that $\zeta(\xi) = 0$ then, in virtue of lemmas 4.3 and 4.4, we have that $\xi = 0$. This implies that ζ is injective, but since $\dim(\mathcal{J}(\gamma_{u_0})) = \dim(T_{u_0}TM) = 2m$ then we conclude that ζ is an isomorphism. □

Now, we will focus on the variations by light rays and the Jacobi fields they define. Fix a null geodesic $\gamma \in \mathcal{N}$ and assume that $\mathbf{x}(s,t)$ is a variation by light rays of $\gamma = \gamma_0^{\mathbf{x}} \in \mathcal{N}$ in such a way that $J(t) = V_0^{\mathbf{x}}(t)$ is the Jacobi field over γ corresponding to the initial field of $\mathbf{x}$ and $\frac{\partial \mathbf{x}}{dt}(s,t) = (\gamma_s^{\mathbf{x}})'(t)$. Since $\mathbf{x}$ is a variation by light rays, then it provides that $\mathbf{g}\left((\gamma_s^{\mathbf{x}})'(t), (\gamma_s^{\mathbf{x}})'(t)\right) = 0$ for all (s,t) in the domain of $\mathbf{x}$, hence

$$0 = \left.\frac{\partial}{\partial s}\right|_{(0,t)} \mathbf{g}\left((\gamma_s^{\mathbf{x}})'(t), (\gamma_s^{\mathbf{x}})'(t)\right) = 2\mathbf{g}\left(\left.\frac{D}{ds}\right|_{(0,t)} \frac{\partial \mathbf{x}}{dt}(s,t), \frac{\partial \mathbf{x}}{dt}(0,t)\right) =$$

$$= 2\mathbf{g}\left(\left.\frac{D}{dt}\right|_{(0,t)} \frac{\partial \mathbf{x}}{ds}(s,t), \frac{\partial \mathbf{x}}{dt}(0,t)\right) = \left.\frac{\partial}{\partial t}\right|_{(0,t)} \mathbf{g}\left(V_s^{\mathbf{x}}(t), (\gamma_s^{\mathbf{x}})'(t)\right) =$$

$$= \left.\frac{\partial}{\partial t}\right|_{t} \mathbf{g}\left(V^{\mathbf{x}}(t), \gamma'(t)\right) = \left.\frac{\partial}{\partial t}\right|_{t} \mathbf{g}\left(J(t), \gamma'(t)\right)$$

then the variations by light rays of a null geodesic γ verify that their Jacobi fields J fulfil

$$\mathbf{g}(J(t), \gamma'(t)) = c$$

with $c \in \mathbb{R}$ constant. Then, we define the set of Jacobi fields of variations by light rays by

$$\mathcal{J}_L(\gamma) = \{J \in \mathcal{J}(\gamma) : \mathbf{g}(J, \gamma') = c \text{ constant}\}$$

Since $\mathbf{g}(\alpha J + \beta K, \gamma') = \alpha \mathbf{g}(J, \gamma') + \beta \mathbf{g}(K, \gamma')$ for all $\alpha, \beta \in \mathbb{R}$ and every $J, K \in \mathcal{J}_L(\gamma)$ then $\mathcal{J}_L(\gamma)$ is a vector subspace of $\mathcal{J}(\gamma)$, and by proposition 4.2, it verifies that $\dim(\mathcal{J}_L(\gamma)) = 2\dim(M) - 1 = 2m - 1$.

Now, we define subsets of $\mathcal{J}(\gamma)$ given by

$$\widehat{\mathcal{J}}_0(\gamma) = \{J(t) = bt\gamma'(t) : b \in \mathbb{R}\}$$
$$\widehat{\mathcal{J}}_0'(\gamma) = \{J(t) = a\gamma'(t) : a \in \mathbb{R}\}$$

It is trivial to see that $\widehat{\mathcal{J}}_0(\gamma) \subset \mathcal{J}_L(\gamma)$ and $\widehat{\mathcal{J}}_0'(\gamma) \subset \mathcal{J}_L(\gamma)$.

Moreover, observe that for any $\beta_1, \beta_2 \in \mathbb{R}$ and any $J_1, J_2 \in \widehat{\mathcal{J}}_0(\gamma)$, if $J_1(t) = b_1 t\gamma'(t)$ and $J_2(t) = b_2 t\gamma'(t)$ then

$$\beta_1 J_1(t) + \beta_2 J_2(t) = (\beta_1 b_1 + \beta_2 b_2)\, t\gamma'(t) \in \widehat{\mathcal{J}}_0(\gamma)$$

hence $\widehat{\mathcal{J}}_0(\gamma)$ is a vector subspace of $\mathcal{J}_L(\gamma)$ such that $\dim\left(\widehat{\mathcal{J}}_0(\gamma)\right) = 1$. Analogously, for any $\beta_1, \beta_2 \in \mathbb{R}$ and any $J_1, J_2 \in \widehat{\mathcal{J}}_0'(\gamma)$, verifying $J_1(t) = a_1\gamma'(t)$ and $J_2(t) = a_2\gamma'(t)$ then

$$\beta_1 J_1(t) + \beta_2 J_2(t) = (\beta_1 a_1 + \beta_2 a_2)\, \gamma'(t) \in \widehat{\mathcal{J}}_0'(\gamma)$$

hence $\widehat{\mathcal{J}}_0'(\gamma)$ is also a one–dimensional vector subspace of $\mathcal{J}_L(\gamma)$.

If $J \in \widehat{\mathcal{J}}_0(\gamma) \cap \widehat{\mathcal{J}}_0'(\gamma)$, then its initial values must verify

$$\begin{cases} J(0) = 0 \\ J'(0) = b\gamma'(0) \end{cases} \quad \text{and} \quad \begin{cases} J(0) = a\gamma'(0) \\ J'(0) = 0 \end{cases}$$

then $a = b = 0$ and therefore $\widehat{\mathcal{J}}_0(\gamma) \cap \widehat{\mathcal{J}}_0'(\gamma) = \{0\}$. So, we can define the direct product

$$\mathcal{J}_0(\gamma) = \widehat{\mathcal{J}}_0(\gamma) \oplus \widehat{\mathcal{J}}_0'(\gamma) = \{J(t) = (a + bt)\,\gamma'(t) : a, b \in \mathbb{R}\}$$

being the vector subspace of Jacobi fields proportional to γ' and verifying $\dim(\mathcal{J}_0(\gamma)) = 2$.

Now, we can define the quotient vector space

$$\mathcal{L}(\gamma) = \mathcal{J}_L(\gamma) / \mathcal{J}_0(\gamma) = \{[J] : K \in [J] \Leftrightarrow K = J + J_0 \text{ such that } J_0 \in \mathcal{J}_0(\gamma)\}$$

whose dimension is $\dim(\mathcal{L}(\gamma)) = \dim(\mathcal{J}_L(\gamma)) - \dim(\mathcal{J}_0(\gamma)) = 2\dim(M) - 3$. The elements of $\mathcal{L}(\gamma)$ will be denoted by $[J] \equiv J(\text{mod}\gamma')$ and we will say that $K = J(\text{mod}\gamma')$ when $[K] = [J]$.

Next lemma claims that there exist a change of parameter such that any variation by light rays can be transformed in a geodesic variation by light rays. So, lemma 4.1 can be used.

Lemma 4.5 *Let* $\mathbf{x} = \mathbf{x}(s,t)$ *be a variation by light rays in* $(M, \mathcal{C})$ *such that* $\gamma_s(t) = \mathbf{x}(s,t)$ *defines its light rays. Fixed any metric* $\mathbf{g} \in \mathcal{C}$ *then there exists a differentiable function* $h = h(s,\tau)$ *such that the light rays parametrized as* $\overline{\gamma}_s = \gamma_s(h(s,\tau))$ *are null geodesics related to* $\mathbf{g}$.

Proof. Since each γ_s is a segment of light ray then $\gamma_s = \gamma_s(t)$ is a pregeodesic related to $\mathbf{g}$. Hence

$$\frac{D\gamma_s'(t)}{dt} = \frac{D}{dt}\frac{\partial \mathbf{x}}{\partial t}(s,t) = f(s,t)\,\gamma_s'(t)$$

where f is differentiable and $\frac{D}{dt}$ denotes the covariant derivative related to $\mathbf{g}$ along $\gamma_s(t)$. We look for the function $h = h(s,\tau)$ such that $\overline{\gamma}_s = \gamma_s \circ h$ is geodesic. For any s, for convenience, we will call $h_s(\tau) = h(s,\tau)$, $h_s'(\tau) = \frac{\partial h(s,\tau)}{\partial \tau}$ and $h_s''(\tau) = \frac{\partial^2 h(s,\tau)}{\partial \tau^2}$. Since h is a change of parameter for every s, we can assume that $\frac{\partial h}{\partial \tau}(s,t) \neq 0$ for every (s,t). So,

$$0 = \frac{D\overline{\gamma}_s'(\tau)}{d\tau} = \frac{Dh_s'(\tau)\,\gamma'(h_s(\tau))}{d\tau} = h_s''(\tau)\,\gamma_s'(h_s(\tau)) + h_s'(\tau)\,\frac{D\gamma_s'(h_s(\tau))}{d\tau} =$$

$$= h_s''(\tau)\,\gamma_s'(h_s(\tau)) + (h_s'(\tau))^2\,\frac{D\gamma_s'(h_s(\tau))}{dt} = h_s''(\tau)\,\gamma_s'(h_s(\tau)) + (h_s'(\tau))^2 f(s,h_s(\tau))\,\gamma_s'(h_s(\tau))$$

hence

$$h_s''(\tau) + (h_s'(\tau))^2 f(s,h_s(\tau)) = 0$$

and therefore

$$\frac{h_s''(\tau)}{h_s'(\tau)} = -h_s'(\tau)\, f(s,h(s,\tau))$$

With no lack of generality, we assume that $h_s(0) = 0$ and $h_s'(0) = 1$ for any s, and then integrating

$$\log h_s'(\tau) = -\int_0^{h_s^{-1}(\tau)} f(s,y)\,dy$$

$$h_s'(\tau) = e^{-\int_0^{h_s(\tau)} f(s,y)dy}$$

and calling $t = h_s(\tau)$ then

$$h_s'\left(h_s^{-1}(t)\right) = e^{-\int_0^t f(s,y)dy}$$

It is known that $\left(h_s^{-1}\right)'(t) = \frac{1}{h_s'\left(h_s^{-1}(t)\right)}$, then we have

$$\left(h_s^{-1}\right)'(t) = e^{\int_0^t f(s,y)dy}$$

and we conclude that

$$h_s^{-1}(t) = \int_0^t e^{\int_0^x f(s,y)dy}dx \tag{4.3}$$

is the inverse of the change of parameter h_s for each γ_s. Define $k(s,t) = h_s^{-1}(t)$ and the map $T(s,t) = (s, k(s,t))$. By the expression 4.3, T is clearly differentiable, and since the jacobian matrix of T verifies

$$|JT| = \begin{vmatrix} 1 & 0 \\ \frac{\partial k}{\partial s} & \frac{\partial k}{\partial t} \end{vmatrix} = \frac{\partial k}{\partial t} = e^{\int_0^t f(s,y)dy} > 0$$

then T is invertible with T^{-1} differentiable. A trivial computation shows that

$$T^{-1}(s,\tau) = (s, h(s,t))$$

therefore, since T^{-1} is differentiable, then h is also so. □

Lemma 4.6 shows that any differentiable curve $\Gamma \subset \mathcal{N}$ defines a variation by light rays $\mathbf{x}$ such that the longitudinal curves of $\mathbf{x}$ corresponds to points in Γ. This variation is not unique by construction.

Lemma 4.6 *Given a differentiable curve $\Gamma : I \to \mathcal{N}$ such that $0 \in I$ and $\Gamma(s) = \gamma_s \subset M$, then there exists a variation by light rays $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ verifying*

$$\mathbf{x}(s,t) = \gamma_s(t)$$

for all $(s,t) \in (-\epsilon, \epsilon) \times (-\delta, \delta)$. Moreover, the variation $\mathbf{x}$ can be written as

$$\mathbf{x}(s,t) = \exp_{\pi_M^{\mathbb{N}^+}(v(s))}(tv(s))$$

where $v : (-\epsilon, \epsilon) \to \mathbb{N}^+(C)$ is a differentiable curve.

Proof. Consider the restriction $\pi = \pi_{\mathbb{PN}}^{\mathbb{N}^+}\Big|_{\mathbb{N}^+(C)} : \mathbb{N}^+(C) \to \mathbb{PN}(C)$ and the diffeomorphism $\sigma : \mathbb{PN}(C) \to \mathcal{U}$ in the diagram 3.5, where $\mathcal{U} \subset \mathcal{N}$ and $V \subset M$ are open such that V is globally hyperbolic and $C \subset V$ is a Cauchy surface of V and moreover $\gamma_0 \in \mathcal{U}$, in such a way the following diagram arise

$$\begin{array}{ccc} \mathbb{PN}(C) & \xrightarrow{\sigma} & \mathcal{U} \\ \pi \uparrow & \nearrow_{\sigma \circ \pi} & \\ \mathbb{N}^+(C) & & \end{array} \tag{4.4}$$

Also consider the canonical projection $\pi_M^{\mathbb{N}^+} : \mathbb{N}^+ \to M$ and the exponential map $\exp : (-\delta, \delta) \times \mathbb{N}^+ \to M$ defined by $\exp(t,v) = \exp_{\pi_M^{\mathbb{N}^+}(v)}(tv)$. Fix $\epsilon > 0$ such that $\Gamma(s) \in \mathcal{U}$ for all $s \in (-\epsilon, \epsilon)$ and let $z : \mathbb{PN}(C) \to \mathbb{N}^+(C)$ be a section of π that, without restriction of generality, can be considered a global section due to the locality of π. Naming $v(s) = z \circ \sigma^{-1} \circ \Gamma(s)$ for $s \in (-\epsilon, \epsilon)$, then we can define a variation $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ by $\mathbf{x}(s,t) = \exp(t, v(s)) = \exp_{\pi_M^{\mathbb{N}^+}(v(s))}(tv(s))$. By construction as a composition of differentiable maps, $\mathbf{x}$ is differentiable. Moreover, since $v(s)$ is the initial vector of the geodesic $\gamma_s^{\mathbf{x}}$ defined by $\mathbf{x}(s,t) = \gamma_s^{\mathbf{x}}(t)$, then

$$\gamma_s^{\mathbf{x}} = \sigma \circ \pi(v(s)) = \sigma \circ \pi \circ z \circ \sigma^{-1} \circ \Gamma(s) = \sigma \circ \sigma^{-1} \circ \Gamma(s) = \Gamma(s)$$

for all $s \in (-\epsilon, \epsilon)$, and the lemma follows. □

Lemma 4.7 *Given a variation $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ by light rays such that $\mathbf{x}(s,t) = \gamma_s^{\mathbf{x}}(t)$, then the curve $\Gamma^{\mathbf{x}} : I \to \mathcal{N}$ verifying $\Gamma^{\mathbf{x}}(s) = \gamma_s^{\mathbf{x}}$ is differentiable.*

Proof. Let $\mathbf{x} : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M$ be a variation by light rays such that $\gamma_s^{\mathbf{x}}(t) = \mathbf{x}(s,t)$. Then the curve

$$\lambda(s) = d\mathbf{x}_{(s,0)}\left(\frac{\partial}{\partial t}\right)_{(s,0)} \in \mathbb{N}^+$$

is clearly differentiable. If $p_{\mathbb{N}^+} : \mathbb{N}^+ \to \mathcal{N}$ is the submersion of proposition 3.5, then $p_{\mathbb{N}^+} \circ \lambda : I \to \mathcal{N}$ is differentiable in $\mathcal{N}$, by composition of differentiable maps. Since

$$p_{\mathbb{N}^+} \circ \lambda(s) = p_{\mathbb{N}^+}\left((\gamma_s^{\mathbf{x}})'(0)\right) = \gamma_s^{\mathbf{x}} = \Gamma^{\mathbf{x}}(s).$$

then $\Gamma^{\mathbf{x}}$ is also differentiable. □

Let us adopt the notation used in lemma 4.7 and call $\Gamma^{\mathbf{x}}$ the curve in $\mathcal{N}$ defined by the variation $\mathbf{x}$ by light rays such that if $\mathbf{x}(s,t) = \gamma_s^{\mathbf{x}}(t)$ then $\Gamma^{\mathbf{x}}(s) = \gamma_s^{\mathbf{x}} \in \mathcal{N}$.

Although the variations defined in lemma 4.6 are not unique, lemma 4.8 shows that all they define the same Jacobi field except by a term in the direction of γ'.

Lemma 4.8 *Let $\overline{\mathbf{x}} : I \times \overline{H} \to M$ and $\mathbf{x} : I \times H \to M$ be variations by light rays such that $\Gamma^{\overline{\mathbf{x}}}(s) = \gamma_s^{\overline{\mathbf{x}}}$ and $\Gamma^{\mathbf{x}}(s) = \gamma_s^{\mathbf{x}}$ with $\gamma_0^{\overline{\mathbf{x}}} = \gamma_0^{\mathbf{x}} = \gamma \in \mathcal{N}$. Let us denote by $\overline{J}$ and J the Jacobi fields over γ of $\overline{\mathbf{x}}$ and $\mathbf{x}$ respectively. If $\Gamma^{\overline{\mathbf{x}}} = \Gamma^{\mathbf{x}}$ then $\overline{J} = J\,(\mathrm{mod}\gamma')$.*

Proof. We have that $\overline{\mathbf{x}}(s,t) = \gamma_s^{\overline{\mathbf{x}}}(t)$ and $\mathbf{x}(s,\tau) = \gamma_s^{\mathbf{x}}(\tau)$. By lemma 4.5, we can assume without any lack of generality, that $\gamma_s^{\overline{\mathbf{x}}}$ are null geodesics for the metric $\mathbf{g} \in \mathcal{C}$ giving new parameters if necessary. If $\Gamma^{\overline{\mathbf{x}}} = \Gamma^{\mathbf{x}}$ then $\gamma_s^{\overline{\mathbf{x}}} = \gamma_s^{\mathbf{x}}$ for all $s \in I$. Then there exist a differentiable function $h_s(t) = h(s,t)$ such that $\overline{\mathbf{x}}(s,t) = \mathbf{x}(s, h(s,t))$. Hence we have that

$$\frac{\partial \overline{\mathbf{x}}(s,t)}{\partial s} = \frac{\partial \mathbf{x}(s, h(s,t))}{\partial s} + \frac{\partial h(s,t)}{\partial s} \cdot \frac{\partial \mathbf{x}(s, h(s,t))}{\partial \tau}$$

then if $s = 0$

$$\overline{J}(t) = J(h(0,t)) + \frac{\partial h}{\partial s}(0,t) \cdot \gamma'(t)$$

therefore $\overline{J} = J \,(\mathrm{mod}\gamma')$. □

We can wonder how a Jacobi field changes when another metric of the same conformal class is considered in M. The following result shows it with a proof similar to the one of lemma 4.8.

Lemma 4.9 *Let $\mathbf{x} : I \times H \to M$ be a variation by light rays of $\gamma = \mathbf{x}(0, \cdot)$. If $J \in \mathcal{J}_L(\gamma)$ is the Jacobi field of $\mathbf{x}$ along γ related to the metric $\mathbf{g} \in \mathcal{C}$, then the Jacobi field $\overline{J}$ of $\mathbf{x}$ along γ related to another metric $\overline{\mathbf{g}} \in \mathcal{C}$ verifies*

$$\overline{J} = J \,(\mathrm{mod}\gamma') \,.$$

Proof. Let $\mathbf{x} : I \times H \to M$ be a variation by light rays of γ where $\mathbf{x}(s,t) = \gamma_s(t)$ with $\gamma = \gamma_0$. By lemma 4.5, we can assume that γ_s is null geodesic related to the metric $\mathbf{g}$ and there exists changes of parameters $h_s : \overline{H} \to H$ such that $\overline{\gamma_s}(\tau) = \gamma_s(h_s(\tau))$ are null geodesics related to $\overline{\mathbf{g}} \in \mathcal{C}$ for all $s \in I$ and where $h(s,\tau) = h_s(\tau)$ is a differentiable function. So, we consider that $J \in \mathcal{J}_L(\gamma)$ is the Jacobi field of $\mathbf{x}$ along γ and $\overline{J} \in \overline{\mathcal{J}}_L(\gamma)$ is the one of $\overline{\mathbf{x}}$. Then $\overline{\mathbf{x}}(s,\tau) = \mathbf{x}(s, h_s(\tau))$ and we have that

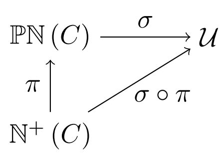
$$\begin{aligned}
\overline{J}(\tau) = \left.\frac{\partial \overline{\mathbf{x}}(s,\tau)}{\partial s}\right|_{(0,\tau)} &= \left.\frac{\partial \mathbf{x}(s, h_s(\tau))}{\partial s}\right|_{(0,\tau)} = \\
&= \frac{\partial \mathbf{x}}{\partial s}(0, h_0(\tau)) + \frac{\partial h}{\partial s}(0,\tau)\frac{\partial \mathbf{x}}{\partial t}(0, h_0(\tau)) = \\
&= J(h_0(\tau)) + \frac{\partial h}{\partial s}(0,\tau)\,\gamma'(h_0(\tau))
\end{aligned}$$

therefore $\overline{J} = J \,(\mathrm{mod}\gamma')$. □

Lemma 4.10 *Given two variations by light rays $\mathbf{x} : I \times H \to M$ and $\overline{\mathbf{x}} : \overline{I} \times \overline{H} \to M$ such that $\Gamma^{\mathbf{x}}(0) = \Gamma^{\overline{\mathbf{x}}}(0) = \gamma$. Let us denote by J and $\overline{J}$ their corresponding Jacobi fields at $0 \in I$ and $0 \in \overline{I}$ of $\mathbf{x}$ and $\overline{\mathbf{x}}$ respectively. If $(\Gamma^{\mathbf{x}})'(0) = \left(\Gamma^{\overline{\mathbf{x}}}\right)'(0)$ then $J = \overline{J}\,(\mathrm{mod}\gamma')$.*

Proof. Due to we want to compare the Jacobi fields J and $\overline{J}$ on γ, we can assume without any lack of generality that $\mathbf{x}$ as well as $\overline{\mathbf{x}}$ provide the same geodesic parameter for γ, then by lemmas 4.6 and 4.8, we can consider that $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tu(s))$ and $\overline{\mathbf{x}}(r,t) = \exp_{\overline{\alpha}(r)}(t\overline{u}(r))$ where $u = u(0) = \overline{u}(0)$ and also $p = \alpha(0) = \overline{\alpha}(0)$.

Moreover, we can assume the diagram 4.4 holds.



Since $(\Gamma^{\mathbf{x}})'(0) = \left(\Gamma^{\overline{\mathbf{x}}}\right)'(0)$ then we have

$$d\sigma_{[v(0)]} \circ d\pi_{v(0)}(v'(0)) = d\sigma_{[\overline{v}(0)]} \circ d\pi_{\overline{v}(0)}(\overline{v}'(0)) \Leftrightarrow d\pi_{v(0)}(v'(0)) = d\pi_{\overline{v}(0)}(\overline{v}'(0))$$

Observe that $[v(0)] = [\overline{v}(0)]$ and thus, $d\pi_{v(0)} = d\pi_{\overline{v}(0)}$, and its kernel is the subspace generated by the tangent vector at $s=0$ of the curve $c(s) = e^s v(0)$, hence

$$v'(0) = \overline{v}'(0) + \mu c'(0) \tag{4.5}$$

with $\mu \in \mathbb{R}$. By lemma 4.4, we have that

$$\begin{cases} \alpha'(0) = \overline{\alpha}'(0) \\ \frac{Du}{ds}(0) = \frac{D\overline{u}}{dr}(0) + \mu \frac{Dc}{ds}(0) \end{cases} \Rightarrow \begin{cases} \alpha'(0) = \overline{\alpha}'(0) \\ \frac{Du}{ds}(0) = \frac{D\overline{u}}{dr}(0) + \mu\gamma'(0) \end{cases}$$

therefore we conclude that $J = \overline{J}\,(\mathrm{mod}\gamma')$. □

The differentiable structure of $\mathcal{N}$ has been built in section 3 from the one in $\mathbb{PN}(C)$ where C is a local spacelike Cauchy surface. So, we will identify the tangent space $T_\gamma\mathcal{N}$ with some quotient space of $\mathcal{J}_L(\gamma)$ via a tangent space of $\mathbb{PN}(C)$.

Proposition 4.4 *Given $\xi \in T_{\gamma_{u_0}}\mathcal{N}$ such that $\Gamma'(0) = \xi$ for some curve $\Gamma \subset \mathcal{N}$. Let $\mathbf{x} = \mathbf{x}(s,t)$ be a variation by light rays of γ_{u_0} verifying that $\Gamma^{\mathbf{x}} = \Gamma$ such that $J \in \mathcal{L}(\gamma_{u_0})$ is the Jacobi field over γ_{u_0} of $\mathbf{x}$. If $\zeta : T_{\gamma_{u_0}}\mathcal{N} \to \mathcal{L}(\gamma_{u_0})$ is the map defined by*

$$\overline{\zeta}(\xi) = J\left(\mathrm{mod}\gamma'_{u_0}\right)$$

then $\overline{\zeta}$ is well–defined and a linear isomorphism.

Proof. By lemma 4.10, $\overline{\zeta}$ is well–defined.

We have seen in section 3 that for a globally hyperbolic open set $V \subset M$ such that $C \subset V$ is a smooth local spacelike Cauchy surface, the diagram 3.6 given by

$$\mathcal{N} \supset \mathcal{U} \simeq \mathbb{PN}(C) \simeq \Omega^X(C) \hookrightarrow \mathbb{N}^+(C) \hookrightarrow \mathbb{N}^+ \hookrightarrow TM$$

holds. Proposition 4.3 shows that $\zeta : T_uTM \to \mathcal{J}(\gamma_u)$ is a linear isomorphism for any $u \in TM$. In order to complete the proof, we will restrict ζ from T_uTM up to $T_{[u]}\mathbb{PN}(C)$ step by step, identifying the corresponding subspace of $\mathcal{J}(\gamma_u)$ image of the map. By definition of $\mathcal{J}_L(\gamma)$, it is clear that $\zeta|_{\mathbb{N}^+} : T_u\mathbb{N}^+ \to \mathcal{J}_L(\gamma_u)$ is a linear isomorphism. Since $\mathbb{N}^+(C)$ is a local submanifold of $\mathbb{N}^+$ of codimension 1 such that for any future-directed null geodesic γ, the curve $c(s) = \gamma'(s) \in \mathbb{N}^+$ intersects transversally to $\mathbb{N}^+(C)$, then the image of the restriction of the isomorphism ζ of proposition 4.3 to $T_{u_0}\mathbb{N}^+(C)$ is a vector subspace $S \subset \mathcal{J}_L(\gamma_{u_0})$ of the same codimension and transverse (that is, linearly independent) to the vector subspace $\widehat{\mathcal{J}}'_0(\gamma_{u_0})$, which is generated by the Jacobi field J of the variation

$$\mathbf{x}(s,t) = \exp_{\gamma_{u_0}(s)}\left(t\gamma'_{u_0}(s)\right)$$

By lemma 4.3, we have that $J(0) = \gamma'_{u_0}(0)$ and $J'(0) = 0$, hence $J(t) = \gamma'_{u_0}(t)$. Observe that it is clear that the linear map

$$\begin{array}{ccc} S & \rightarrow & \mathcal{J}_L(\gamma_{u_0}) / \widehat{\mathcal{J}}'_0(\gamma_{u_0}) \\ J & \mapsto & [J] \end{array}$$

is an isomorphism.

On the other hand, let $v : (-\epsilon, \epsilon) \rightarrow \mathbb{N}^+(C)$ be a differentiable curve such that $v(0) = u_0$ and let us denote by $\alpha = \pi_M^{\mathbb{N}^+} \circ v$ its projection on $C \subset M$. Consider the variation by light rays defined by $\mathbf{x}(s,t) = \exp_{\alpha(s)}(tv(s))$ where J is the Jacobi field of $\mathbf{x}$ along γ_{u_0}. By lemma 4.3, we have that $J(0) = \alpha'(0)$ and $J'(0) = \frac{Dv}{ds}(0)$. If $\lambda : (-\epsilon, \epsilon) \rightarrow \mathbb{R}$ is a non–vanishing differentiable function where $\lambda(0) = 1$, again by lemma 4.3, the Jacobi field $\overline{J}$ corresponding to the variation

$$\overline{\mathbf{x}}(s,t) = \exp_{\alpha(s)}(t\lambda(s)v(s)).$$

verifies

$$\left\{ \begin{array}{l} \overline{J}(0) = \alpha'(0) \\ \overline{J}'(0) = \left.\frac{D\lambda(s)v(s)}{ds}\right|_{s=0} = \lambda(0)\frac{Dv(0)}{ds} + \lambda'(0)v(0) \end{array} \right.$$

then we have

$$\left\{ \begin{array}{l} \overline{J}(0) = J(0) \\ \overline{J}'(0) = J'(0) + \lambda'(0)\gamma'(0) \end{array} \right.$$

This shows that for all the curves $c \subset \mathbb{N}^+(C)$ such that $c(s)$ is proportional to $v(s) \in \mathbb{N}^+(C)$, their tangent vectors are in correspondence with the same equivalence class in $S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right)$, but this implies that

$$\begin{array}{ccc} T_{[u_0]}\mathbb{PN}(C) & \rightarrow & S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right) \\ [v(0)]' & \mapsto & [J] \end{array}$$

is an isomorphism, where we have denoted $[v(0)]' = \left.\frac{d}{ds}\right|_{s=0}[v(s)]$ and $[v(s)] \in \mathbb{PN}(C)$. Since there is a diffeomorphism $\sigma : \mathbb{PN}(C) \rightarrow \mathcal{U} \subset \mathcal{N}$, then $T_{[u_0]}\mathbb{PN}(C)$ is isomorphic to $T_{\gamma_{u_0}}\mathcal{N}$ therefore, since $\mathbf{x}(s,t) = \gamma_{v(s)}(t) = \exp_{\alpha(s)}(tv(s))$ with $\gamma_{v(0)} = \gamma_{u_0}$, and moreover $(\Gamma^{\mathbf{x}})'(0) = (\Gamma^{\mathbf{x}})'(0) = \xi$ then the map

$$\begin{array}{ccc} T_{\gamma_{u_0}}\mathcal{N} & \rightarrow & S/\left(S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})\right) \\ \xi & \mapsto & [J] \end{array}$$

is a linear isomorphism.

Recall that we have denoted $\mathcal{J}_0(\gamma_{u_0}) = \widehat{\mathcal{J}}_0(\gamma_{u_0}) \oplus \widehat{\mathcal{J}}'_0(\gamma_{u_0})$. Observe that the linear map $q : S \rightarrow \mathcal{J}_L(\gamma_{u_0}) / \mathcal{J}_0(\gamma_{u_0})$ defined by $q(J) = [J]$ verifies that

$$q(J) = [0] \Leftrightarrow J(t) = (a + bt)\gamma'_{u_0}(t) \Leftrightarrow J \in S \cap \widehat{\mathcal{J}}_0(\gamma_{u_0})$$

then $S/\left(S\cap\widehat{\mathcal{J}}_0\left(\gamma_{u_0}\right)\right)$ is isomorphic to $\mathcal{L}\left(\gamma_{u_0}\right)=\mathcal{J}_L\left(\gamma_{u_0}\right)/\mathcal{J}_0\left(\gamma_{u_0}\right)$. This shows that

$$\begin{array}{rccc}\overline{\zeta}: & T_{\gamma_{u_0}}\mathcal{N} & \to & \mathcal{L}\left(\gamma_{u_0}\right)=\mathcal{J}_L\left(\gamma_{u_0}\right)/\mathcal{J}_0\left(\gamma_{u_0}\right)\\ & \xi & \mapsto & [J]\end{array}$$

is a linear isomorphism and the proof is complete. □

Proposition 4.4 allows to see the vectors of the tangent space $T_\gamma\mathcal{N}$ as Jacobi fields of variations by light rays. We will use, from now on, this characterization when working with tangent vectors of $\mathcal{N}$.

By propositions 3.1 or 3.2 and proposition 4.4, it is clear that the characterization of $T_\gamma\mathcal{N}$ as $\mathcal{L}\left(\gamma\right)$ does not depends on the representative of the conformal class $\mathcal{C}$.

5. The canonical contact structure in $\mathcal{N}$

In this section, the canonical contact structure on $\mathcal{N}$ will be discussed. Such contact struture is inherited from the kernel of the *canonical 1–form* of T^*M and it will be described by passing the distribution of hyperplanes to TM before pushing it down to $\mathcal{N}$ in virtue of the inclusions of eq. 3.6. In order to carry out this task in a self–contained way, we will introduce some basic elements of symplectic and contact geometry (see for instance [Ab87], [Ar89] and [LM87]) and observe how the construction of $\mathcal{N}$ can be done from T^*M.

5.1. Elements of symplectic geometry in T^*M

Definition 5.1 *A pair (P,ω) is called a* symplectic manifold *whenever P is a differentiable manifold equipped with a non-degenerate and closed 2–form $\omega\in\Lambda^2(P)$. We will say that ω is the* symplectic 2–form *of P.*

Consider a differentiable manifold M. Its cotangent bundle $\pi\colon T^*M\to M$ carries a canonical 1–form θ defined pointwise at every $\alpha\in T^*M$ by

$$\theta_\alpha=\left(d\pi_\alpha\right)^*\alpha\,.$$

Consequently we have

$$\theta_\alpha\left(\xi\right)=\left(\left(d\pi_\alpha\right)^*\alpha\right)\left(\xi\right)=\alpha\left(\left(d\pi_\alpha\right)\xi\right) \tag{5.1}$$

for $\xi\in T_\alpha\left(T^*M\right)$. In local canonical bundle coordinates (x^k,p_k), we can write

$$\theta=\sum_{k=1}^{m}p_k dx^k\,. \tag{5.2}$$

Now, the 2–form ω given by

$$\omega=-d\theta\,,$$

defines a symplectic 2–form in T^*M, that in the previous local coordinates takes the form:

$$\omega = \sum_{k=1}^{m} dx^k \wedge dp_k$$

Definition 5.2 *A vector field $X \in \mathfrak{X}(P)$ of a symplectic manifold (P, ω) is said to be a* Liouville vector field *if it verifies*

$$\mathcal{L}_X \omega = \omega$$

Definition 5.3 *Given a symplectic manifold (P, ω) and a smooth function $H : P \to \mathbb{R}$, then the only vector field $X_H \in \mathfrak{X}(P)$ verifying*

$$i_{X_H}\omega = dH ,$$

is called the Hamiltonian vector field *associated to H. This function H will be called* Hamiltonian function*.*

In the particular case of $P = T^*M$, for any Hamiltonian function $H\colon T^*M \to \mathbb{R}$, it is possible to express the corresponding Hamiltonian vector field $X_H \in \mathfrak{X}(T^*M)$ as:

$$X_H = \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x^i} - \frac{\partial H}{\partial x^i}\frac{\partial}{\partial p_i} .$$

Now, we want to construct $\mathcal{N}$ again, but this time starting from the contangent bundle T^*M. Consider the natural identification provided by the metric $\mathbf{g}$:

$$\begin{array}{rccc} \widehat{\mathbf{g}}: & TM & \to & T^*M \\ & \xi & \mapsto & \mathbf{g}(\xi, \cdot) \end{array}$$

and denote by $\mathbb{N}^{+*}$ the image of the restriction of $\widehat{\mathbf{g}}$ to $\mathbb{N}^+$, that is

$$\mathbb{N}^{+*} = \widehat{\mathbf{g}}\left(\mathbb{N}^+\right) = \left\{\alpha = \widehat{\mathbf{g}}(\xi) \in T^*M : \xi \in \mathbb{N}^+\right\}$$

In an analogous manner as in Sect. 3, define the *Euler field* $\mathcal{E} \in \mathfrak{X}(T^*M)$ by

$$\mathcal{E}(\alpha) = dc\left(\frac{\partial}{\partial t}\right)(0) ,$$

where $\alpha \in T_p^*M$ and $c : \mathbb{R} \to T_p^*M$ verifies that $c(t) = e^t\alpha$. The curve c is an integral curve of $\mathcal{E}$ because

$$c'(t) = dc\left(\frac{\partial}{\partial t}\right)(t) = \mathcal{E}(c(t)) .$$

In the previous coordinates, $\mathcal{E}$ can be written as $\mathcal{E} = p_k\partial/\partial p_k$. So, for every $\alpha \in \mathbb{N}^{+*}$ the integral curve $c(t) = e^t\alpha$ is contained in $\mathbb{N}^{+*}$, therefore $\mathcal{E}$ is tangent to $\mathbb{N}^{+*}$.

Moreover, if ω is the symplectic 2–form of T^*M it is trivial to see that

$$i_{\mathcal{E}}\omega = -\theta \tag{5.3}$$

where θ is the tautological 1–form in T^*M, hence:

$$\mathcal{L}_{\mathcal{E}}\omega = i_{\mathcal{E}}d\omega + d\left(i_{\mathcal{E}}\omega\right) = d\left(-\theta\right) = -d\theta = \omega\,, \tag{5.4}$$

therefore $\mathcal{E}$ is a Liouville vector field. In fact, $\mathcal{E}$ sometimes is called the *Liouville* or *Euler–Liouville vector field*.

Consider now the Hamiltonian function (again just the kinetic energy) defined by

$$\begin{array}{rccl} H: & T^*M & \rightarrow & \mathbb{R} \\ & \alpha & \mapsto & \frac{1}{2}\mathbf{g}\left(\widehat{\mathbf{g}}^{-1}\left(\alpha\right), \widehat{\mathbf{g}}^{-1}\left(\alpha\right)\right) \end{array}$$

defining the Hamiltonian vector field given by

$$X_H = g^{ki}p_i\frac{\partial}{\partial x^k} - \frac{1}{2}\frac{\partial g^{ij}}{\partial x^k}p_ip_j\frac{\partial}{\partial p_i}$$

Lemma 5.1 *Let $X_{\mathbf{g}}, \Delta \in \mathfrak{X}\left(TM\right)$ be the the geodesic spray and Euler field of TM and $X_H, \mathcal{E} \in \mathfrak{X}\left(T^*M\right)$ the Hamiltonian vector field and Euler field of T^*M respectively. Then we have that $\widehat{\mathbf{g}}_*\left(\Delta\right) = \mathcal{E}$ and $\widehat{\mathbf{g}}_*\left(X_{\mathbf{g}}\right) = X_H$.*

Proof. If we take any $\xi \in T^*M$ and $\alpha = \widehat{\mathbf{g}}\left(\xi\right)$, then the integral curve $c\left(t\right) = e^t\xi$ of Euler field Δ in TM is transformed by $\widehat{\mathbf{g}}$ as

$$\widehat{\mathbf{g}}\left(c\left(t\right)\right) = \mathbf{g}\left(c\left(t\right), \cdot\right) = \mathbf{g}\left(e^t\xi, \cdot\right) = e^t\mathbf{g}\left(\xi, \cdot\right) = e^t\widehat{\mathbf{g}}\left(\xi\right) = e^t\alpha \in T^*M$$

being an integral curve of Euler field $\mathcal{E}$ in T^*M. Then, for any $\xi \in T^*M$ we have that

$$\widehat{\mathbf{g}}_*\left(\Delta\left(\xi\right)\right) = \mathcal{E}\left(\widehat{\mathbf{g}}\left(\xi\right)\right)$$

is verified, therefore this implies $\widehat{\mathbf{g}}_*\left(\Delta\right) = \mathcal{E}$.

The second relation is obtained easily by taking the pull-back of the identity $i_{X_H}\omega = dH$ along the map $\hat{\mathbf{g}}$. □

The following corollary is an immediate consequence of Lemma 5.1 and the construction of $\mathcal{N}$ done in section 3.

Corollary 5.1 *The space of light rays $\mathcal{N}$ of M can be built by the quotient*

$$\mathcal{N} = \mathbb{N}^{+*}/\mathcal{D}^*$$

where $\mathcal{D}^*$ is the distribution generated by the vector fields $\mathcal{E}$ and X_H, that is $\mathcal{D}^* =$ span$\{\mathcal{E}, X_H\}$.

Lemma 5.1 also shows that the null geodesic defined by $\alpha \in \mathbb{N}^*$ coincides to the null geodesic defined by $v \in \mathbb{N}$ if and only if $\widehat{\mathbf{g}}(v) = \alpha$, because the first equation has to be verified. Then we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{N}^* & \xrightarrow{p_{\mathbb{N}^*}} & \mathcal{N} \\ \widehat{\mathbf{g}} \uparrow & \nearrow_{p_{\mathbb{N}}} & \\ \mathbb{N} & & \end{array} \tag{5.5}$$

Next, we will introduce some basic definitions and results in contact geometry that we will need later. See [Ar89, Appx. 4] and [LM87, Ch. 5] for more details.

Definition 5.4 *Given a n–dimensional differentiable manifold P, a* contact element *in P is a $(n-1)$–dimensional subspace $\mathcal{H}_q \subset T_qP$. The point $q \in P$ is called the* contact point *of $\mathcal{H}_q$.*

We will say that a distribution of hyperplanes *$\mathcal{H}$ in a differentiable manifold M is a map $\mathcal{H}$ defined in M such that for every $q \in M$ we have that $\mathcal{H}(q) = \mathcal{H}_q$ is a contact element at q.*

Lemma 5.2 *Every differentiable distribution of hyperplanes $\mathcal{H}$ can be written locally as the kernel of 1–form.*

Proof. We will follow the proof in [G08, Lem. 1.1.1]. Consider the quotient bundles $\pi : TP \to TP/\mathcal{H}$ and $\overline{\pi} : T^*P \to (TP/\mathcal{H})^*$ and observe that $\pi^*(\overline{\pi}(\beta)) = \beta$ for any $\beta \in T^*P$. Recall that every bundle is locally trivial, this means there exists local sections. Take a non–zero local section $\alpha : U \subset (TP/\mathcal{H})^* \to T^*P$ of $\overline{\pi}$. For any $\eta \in (TP/\mathcal{H})^*$ we have that $\alpha(\eta)$ is a 1–form in TP such that $\overline{\pi} \circ \alpha(\eta) = \eta$. Thus, for $X \in T(TP)$ we have

$$\pi^*\eta(X) = \eta(\pi_* X) = \overline{\pi} \circ \alpha(\eta)(\pi_* X) = \pi^*(\overline{\pi} \circ \alpha(\eta))(X) = \alpha(\eta)(X)$$

Then,

$$X \in \mathcal{H} \Leftrightarrow \eta(\pi_* X) = 0 \Leftrightarrow \pi_* X = 0 \Leftrightarrow \alpha(\eta)(X) = 0$$

for all $\eta \in (TP/\mathcal{H})^*$, therefore $\ker(\alpha|_U) = \mathcal{H}$. □

It is clear that if a differentiable distribution of hyperplanes $\mathcal{H}$ is defined locally by the 1–form $\alpha \in \mathfrak{X}^*(P)$ then, for every $f \in \mathfrak{F}(P)$ the 1–form $f\alpha$ also defines $\mathcal{H}$ since α and $f\alpha$ have the same kernel.

Given a distribution of hyperplanes $\mathcal{H}$ we will say that it is maximally non-integrable if for any locally defined 1-form η such that $\mathcal{H} = \ker \eta$, then $d\eta$ is non-degenerate when restricted to $\mathcal{H}$.

Definition 5.5 *A* contact structure $\mathcal{H}$ *in a* $(2n+1)$*–dimensional differentiable manifold* P *is a maximally non–integrable distribution of hyperplanes. The hyperplanes* $\mathcal{H}_x \subset T_xP$ *are called contact elements. If there exists a globally defined 1-form* η *defining* $\mathcal{H}$*, i.e.,* $\mathcal{H} = \ker \eta$*, we will say that* $\mathcal{H}$ *is a* cooriented contact structure *and we will say that* η *is a contact form.*

An equivalent way to determine if a distribution of hyperplanes $\mathcal{H}$ determines a contact structure is provided by the following result (see also [Ar89] and [Ca01]).

Lemma 5.3 *Let* $\mathcal{H}$ *be a distribution of hyperplanes in* P *locally defined as* $\mathcal{H} = \ker(\eta)$*, then* $d\eta|_{\mathcal{H}}$ *is non–degenerated if and only if* $\eta \wedge (d\eta)^n \neq 0$*.*

Proof. Since $\dim(\mathcal{H}_q) = 2n$, then we can take $v \in T_qP$ such that $T_qP = \operatorname{span}\{v\} \oplus \mathcal{H}_q$. Take a basis $\{\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2n}\}$ in T_qP such that $\mathbf{e}_0 \in \operatorname{span}\{v\}$ and $\mathbf{e}_j \in \mathcal{H}_q$ for $j = 1, \ldots, 2n$. Due to $\eta(\mathbf{e}_j) = 0$ for $j = 1, \ldots, 2n$, then we have

$$\eta \wedge (d\eta)^n (\mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{2n}) = \eta(\mathbf{e}_0) (d\eta)^n (\mathbf{e}_1, \ldots, \mathbf{e}_{2n})$$

and since $\eta(\mathbf{e}_0) \neq 0$, then

$$\eta \wedge (d\eta)^n \neq 0 \Leftrightarrow (d\eta)^n|_{\mathcal{H}} \neq 0$$

being equivalent to $d\eta|_{\mathcal{H}}$ is non–degenerated. □

Lemma 5.4 *If* α *is a contact form in* P*, then* $f\alpha$ *is also a contact form for every non–vanishing differentiable function* $f \in \mathfrak{F}(P)$*.*

Proof. Observe that α and $f\alpha$ have the same kernel. In order to show that $f\alpha$ is maximally non–integrable, we will proceed by induction. First, observe that

$$f\alpha \wedge d(f\alpha) = f\alpha \wedge (df \wedge \alpha + f d\alpha) = f\alpha \wedge df \wedge \alpha + f\alpha \wedge f d\alpha =$$

$$= -f\alpha \wedge \alpha \wedge df + f^2 \alpha \wedge d\alpha = f^2 \alpha \wedge d\alpha$$

Assume that

$$f\alpha \wedge (d(f\alpha))^{k-1} = f^k \alpha \wedge (d\alpha)^{k-1}$$

Then we have

$$\begin{aligned} f\alpha \wedge (d(f\alpha))^k &= f\alpha \wedge (d(f\alpha))^{k-1} \wedge d(f\alpha) = \\ &= f^k \alpha \wedge (d\alpha)^{k-1} \wedge d(f\alpha) = \\ &= f^{k+1} \alpha \wedge (d\alpha)^k \ . \end{aligned}$$

Therefore, for non–vanishing f, if $\alpha \wedge (d\alpha)^n \neq 0$ then $f\alpha \wedge (d(f\alpha))^n \neq 0$. □

5.2. Constructing the contact structure of $\mathcal{N}$

Consider the tautological 1–form $\theta \in \mathfrak{X}^* (T^*M)$. The diffeomorphism $\widehat{\mathbf{g}} : TM \to T^*M$ allows to carry away θ to TM by pull–back. Let $\pi_M^{TM} : TM \to M$ and $\pi_M^{T^*M} : T^*M \to M$ be the canonical projections, since $\pi_M^{TM} = \pi_M^{T^*M} \circ \widehat{\mathbf{g}}$, then it is verified

$$\left(d\pi_M^{TM}\right)_v (\xi) = \left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}(v)} \left(\widehat{\mathbf{g}}_* (\xi)\right)$$

for all $\xi \in T_v TM$. If we define

$$\theta_{\mathbf{g}} = \widehat{\mathbf{g}}^* \theta \tag{5.6}$$

then, using the expression 5.1, if $\xi \in T_v TM$ we have

$$(\theta_{\mathbf{g}})_v (\xi) = \widehat{\mathbf{g}} (v) \left(\left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}(v)} \left(\widehat{\mathbf{g}}_* (\xi)\right) \right) = \mathbf{g} \left(v, \left(d\pi_M^{TM}\right)_v (\xi)\right) .$$

For a given globally hyperbolic open set $V \subset M$ equipped with coordinates $\left(x^1, \ldots, x^m\right)$ such that $v \in TV$ is written as $v = v^i \frac{\partial}{\partial x^i}$, then $\left(x^i, v^i\right)$ are coordinates in TV. By expression 5.2, we can write

$$\theta_{\mathbf{g}} = g_{ij} v^i dx^j .$$

Let us denote by $\mathcal{H}^{TV} = \ker (\theta_{\mathbf{g}})$, that is a distribution of hyperplanes in $TV \subset TM$. This implies that $\dim \left(\mathcal{H}_v^{TV}\right) = 2m - 1$ for every $v \in TV$.

As we seen in Sect. 3, we have the chain of inclusions 3.6:

$$\Omega \hookrightarrow \mathbb{N}^+ (C) \hookrightarrow \mathbb{N}^+ (V) \hookrightarrow TV \tag{5.7}$$

where $\Omega = \Omega^T(C) = \{v \in \mathbb{N}^+ \mid g(v, T) = -1\}$ for a non-vanishing timelike vector field T. Observer taht if $v \in \Omega$ is the representative of the class of equivalence $[v] \in \mathbb{PN}(C)$, then clearly the following maps

$$\begin{array}{ccccc} \Omega & \longrightarrow & \mathbb{PN}(C) & \longrightarrow & \mathcal{U} \subset \mathcal{N} \\ v & \mapsto & [v] & \mapsto & \gamma_v \end{array} \tag{5.8}$$

are diffeomorphisms.

Then, we will see that the pullback of $\theta_{\mathbf{g}}$ by the inclusion $\Omega \hookrightarrow TV$ defines a 1–form $\theta_{\mathbf{g}}|_{\Omega^X(C)}$, and therefore a distribution of hyperplanes, in Ω. This 1–form and its kernel can be extended from $\mathcal{U} \subset \mathcal{N}$ obtaining the 1–form θ_0 looked for.

To obtain a suitable formula of θ_0 we will proceed projecting the distribution of hyperplanes in TM up to $\Omega^X (C)$ step by step.

First, observe that the restriction of $\mathcal{H}^{TV}$ to $T\mathbb{N}^+ (V)$, denoted by $\mathcal{H}^{\mathbb{N}^+(V)}$, is again a distribution of hyperplanes. Indeed, if $c : (-\epsilon, \epsilon) \to \mathbb{N}^+ (V)$ is a differentiable curve such that

$$\begin{cases} \alpha (s) = \pi_M^{\mathbb{N}^+} (c (s)) \text{ is a timelike curve} \\ v = c (0) \in \mathbb{N}^+ (V) \\ \xi = c' (0) \in T_v \mathbb{N}^+ (V) \end{cases}$$

then

$$\theta_{\mathbf{g}}(\xi) = \mathbf{g}(v, \alpha'(0)) \neq 0$$

since v is null and $\alpha'(0)$ timelike. This implies that $\xi \notin \mathcal{H}_v^{TV}$. So, we have that $T_vTV = span\{\xi\} \oplus \mathcal{H}_v^{TV}$ and since $span\{\xi\} \subset T_v\mathbb{N}^+(V)$ and $\mathcal{H}_v^{\mathbb{N}^+(V)} = \mathcal{H}_v^{TV} \cap T_v\mathbb{N}^+(V)$ then we have that

$$\dim\left(\mathcal{H}_v^{\mathbb{N}^+(V)}\right) = 2m - 2$$

therefore $\mathcal{H}^{\mathbb{N}^+(V)}$ is a distribution of hyperplanes in $\mathbb{N}^+(V)$.

The next step is to restrict $\mathcal{H}^{\mathbb{N}^+(V)}$ to $T\mathbb{N}^+(C)$, where C is a Cauchy surface of V. Again, as done above, if $\gamma : I \to M$ is a null geodesic verifying $\gamma(0) \in C$ and $\gamma'(0) = v \in \mathbb{N}^+(C)$, since the vector subspace $Z = \{u \in T_vM : \mathbf{g}(v,u) = 0\}$ is $m-1$–dimensional and $v = \gamma'(0) \in Z$, then $\dim\left(Z \cap T_{\gamma(0)}C\right) = m - 2$. Hence, we can pick up a vector $\eta \in T_{\gamma(0)}C$ such that $T_{\gamma(0)}C = span\{\eta\} \oplus \left(Z \cap T_{\gamma(0)}C\right)$. Now, we can choose a differentiable curve $c : (-\epsilon, \epsilon) \to \mathbb{N}^+(C)$ verifying

$$\begin{cases} c(0) = v \in \mathbb{N}^+(C) \\ c'(0) = \kappa \in T_v\mathbb{N}^+(C) \\ \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa) = \lambda\eta \text{ for } \lambda \neq 0 \end{cases}$$

then

$$\theta_{\mathbf{g}}(\kappa) = \mathbf{g}\left(v, \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa)\right) = \mathbf{g}(v, \lambda\eta) \neq 0$$

because $\eta \notin Z$, and this shows that $\kappa \notin \mathcal{H}_v^{\mathbb{N}^+(V)}$. Then $T_v\mathbb{N}^+(V) = span\{\kappa\} \oplus \mathcal{H}_v^{\mathbb{N}^+(V)}$ and since $span\{\kappa\} \subset T_v\mathbb{N}^+(C)$ and $\mathcal{H}_v^{\mathbb{N}^+(C)} = \mathcal{H}_v^{\mathbb{N}^+(V)} \cap T_v\mathbb{N}^+(C)$, then it follows

$$\dim\left(\mathcal{H}_v^{\mathbb{N}^+(C)}\right) = \dim\left(T_v\mathbb{N}^+(C)\right) - 1 = 2m - 3$$

thus $\mathcal{H}^{\mathbb{N}^+(C)}$ is a distribution of hyperplanes in $\mathbb{N}^+(C)$.

It is possible to repeat the previous argument to show that the restriction of $\mathcal{H}^{\mathbb{N}^+(C)}$ to $T\Omega$ defines a distribution of hyperplanes. In fact, consider some $\eta \in T_{\gamma(0)}C$ in the same condition as before and take a differentiable curve $c : (-\epsilon, \epsilon) \to \Omega$ verifying

$$\begin{cases} c(0) = v \in \Omega \\ c'(0) = \kappa \in T_v\Omega \\ \left(d\pi_M^{\mathbb{N}^+}\right)_v(\kappa) = \lambda\eta \text{ for } \lambda \neq 0 \end{cases}$$

then again

$$\theta_{\mathbf{g}}(\kappa) = \mathbf{g}(v, \lambda\eta) \neq 0$$

showing that $\kappa \notin \mathcal{H}_v^{\mathbb{N}^+(C)}$. Then $T_v\mathbb{N}^+(C) = \mathrm{span}\{\kappa\} \oplus \mathcal{H}_v^{\mathbb{N}^+(C)}$ and since $\mathrm{span}\{\kappa\} \subset T_v\Omega$ then we have that

$$\dim\left(\mathcal{H}_v^{\Omega}\right) = \dim(T_v\Omega) - 1 = 2m-4$$

thus $\mathcal{H}^\Omega$ is a distribution of hyperplanes in $\Omega \subset \mathbb{N}^+(C)$.

By this process of restriction from TV to Ω we have passed $\mathcal{H}^{TV} \subset TTV$ as a distribution of hyperplanes $\mathcal{H}^\Omega \subset T\Omega \subset TTV$. Moreover since $\mathcal{H}^{TV} = \ker(\theta_{\mathbf{g}})$ and $\mathcal{H}^\Omega = T\Omega \cap \mathcal{H}^{TV}$ then

$$\mathcal{H}^\Omega = \ker\left(\theta_{\mathbf{g}}|_\Omega\right)$$

where $\theta_{\mathbf{g}}|_\Omega$ denotes the restriction of $\theta_{\mathbf{g}}$ to Ω. This fact is important in order to show that $\mathcal{H}^\Omega$ is a contact structure.

Then, using the diffeomorphisms in (5.8), $\mathcal{H}^\Omega$ passes to $\mathcal{U} \subset \mathcal{N}$ as a distribution of hyperplanes of dimension $2m-4$. Let us denote by $\mathcal{H} \subset T\mathcal{N}$ said distribution.

Proposition 5.1 *If $\mathcal{U} \subset \mathcal{N}$ and $T \in \mathfrak{X}(M)$ is a given global non-vanishing timelike vector field as above, then the distribution of hyperplanes*

$$\mathcal{H}(\mathcal{U}) = \{[J] \in T_\gamma\mathcal{U} : \mathbf{g}(\gamma'(0), J(0)) = 0 \text{ with } \mathbf{g}(\gamma'(0), T) = -1\} \tag{5.9}$$

is a contact structure.

Proof. Since $\omega = -d\theta$, then taking the exterior derivative on $\theta_{\mathbf{g}}$ we obtain

$$\omega_{\mathbf{g}} = -d\theta_{\mathbf{g}}\,, \tag{5.10}$$

therefore we have

$$\omega_{\mathbf{g}} = -d\left(g_{ij}v^i dx^j\right) = -g_{ij}dv^i \wedge dx^j - \frac{\partial g_{ij}}{\partial x^k} v^i dx^k \wedge dx^j$$

then it can be written by

$$\omega_{\mathbf{g}} = g_{ij}dx^j \wedge dv^i + \frac{\partial g_{ij}}{\partial x^k} v^i dx^j \wedge dx^k \tag{5.11}$$

that clearly shows that $\omega_{\mathbf{g}}$ is a symplectic 2–form in TM (notice that $\omega_{\mathbf{g}}^n = \det(g_{ij})\, dx^1 \wedge \cdots \wedge dx^n \wedge dv^1 \wedge \cdots \wedge dv^n \neq 0$).

Consider two curves $u_n(s) = u_n^i(s)\left(\frac{\partial}{\partial x^i}\right)_{\alpha_n(s)} \in TM$ where $n = 1, 2$ such that

$$\alpha_n'(s) = a_n^i(s)\left(\tfrac{\partial}{\partial x^i}\right)_{\alpha_n(s)}$$
$$u_n'(s) = a_n^i(s)\left(\tfrac{\partial}{\partial x^i}\right)_{u_n(s)} + \tfrac{du_n^i}{ds}(s)\left(\tfrac{\partial}{\partial v^i}\right)_{u_n(s)}$$

and recall that

$$\frac{Du_n}{ds} = \left(\frac{du_n^k}{ds} + \Gamma_{ij}^k a_n^i u_n^j\right)\left(\frac{\partial}{\partial x^k}\right)_{\alpha_n}$$

calling $\frac{D^k u_n}{ds} = \frac{du_n^k}{ds} + \Gamma_{ij}^k a_n^i u_n^j$ to the k–th component of $\frac{Du_n}{ds}$. If $u = u_1(0) = u_2(0)$ and $\xi_n = u_n'(0)$ for $n = 1, 2$, then we have that

$$\begin{aligned}
\omega_{\mathbf{g}}(\xi_1, \xi_2) &= g_{ij} a_1^i \frac{du_2^j}{ds} - g_{ij} a_2^j \frac{du_1^i}{ds} + \frac{\partial g_{ij}}{\partial x^k} u^i a_1^j a_2^k - \frac{\partial g_{ij}}{\partial x^k} u^i a_1^k a_2^j = \\
&= g_{ij} a_1^i \left(\frac{D^j u_2}{ds} - \Gamma_{lr}^j a_2^l u^r \right) - g_{ij} a_2^j \left(\frac{D^i u_1}{ds} - \Gamma_{lr}^i a_1^l u^r \right) + \left(\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right) u^i a_1^j a_2^k = \\
&= g_{ij} a_1^i \frac{D^j u_2}{ds} - g_{ij} a_2^j \frac{D^i u_1}{ds} + \left(g_{kl} \Gamma_{ji}^l - g_{jl} \Gamma_{ki}^l + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^j} \right) u^i a_1^j a_2^k = \\
&= g_{ij} a_1^i \frac{D^j u_2}{ds} - g_{ij} a_2^j \frac{D^i u_1}{ds} = \\
&= \mathbf{g}\left(\alpha_1'(0), \frac{Du_2}{ds}(0) \right) - \mathbf{g}\left(\alpha_2'(0), \frac{Du_1}{ds}(0) \right) \qquad (5.12)
\end{aligned}$$

where we have used that $g_{kl} \Gamma_{ji}^l = \frac{1}{2} \left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ji}}{\partial x^k} \right)$.

Since the exterior derivative commutes with the restriction to submanifolds, then

$$\omega_{\mathbf{g}}|_{\Omega} = -\left(d\theta_{\mathbf{g}}\right)|_{\Omega} = -\, d\left(\theta_{\mathbf{g}}|_{\Omega}\right)$$

Proposition 4.3 permit to transmit $\theta_{\mathbf{g}}|_{\Omega}, \omega_{\mathbf{g}}|_{\Omega}$ to $\mathcal{L}(\gamma_u)$ pointwise. Calling θ_0 and ω_0 the resulting forms, then for $[J], [J_1], [J_2] \in \mathcal{L}(\gamma_u)$ we have

$$\theta_0([J]) = \mathbf{g}\left(\gamma_u'(0), J(0)\right)$$

where γ_u is parametrized such that $\gamma_u'(0) \in \Omega$, and

$$\omega_0([J_1], [J_2]) = \mathbf{g}\left(J_1(0), J_2'(0)\right) - \mathbf{g}\left(J_2(0), J_1'(0)\right) \qquad (5.13)$$

In order to prove that $\mathcal{H}$ is a contact structure, we will show that $\omega_0|_{\mathcal{H}}$ is non–degenerated. Consider $[J_1], [J_2] \in \mathcal{H}$, then the initial values of J_1 and J_2 in expression 5.13 verify

$$\left\{ \begin{array}{l} \mathbf{g}\left(J_i(0), \gamma_u'(0)\right) = 0 \\ \mathbf{g}\left(J_i'(0), \gamma_u'(0)\right) = 0 \end{array} \right. \qquad (5.14)$$

for $i = 1, 2$, that is $J_i(0), J_i'(0) \in \{\gamma_u'\}^{\perp} = \left\{ v \in T_{\gamma_u(0)} M : \mathbf{g}\left(v, \gamma_u'(0)\right) = 0 \right\}$.

For a given $[J_1] \in \mathcal{H}$, if $\omega_0([J_1], [J_2]) = 0$ for all $[J_2] \in \mathcal{L}(\gamma_u)$, then in particular, also for $[J_2]$ verifying $J_2'(0) = 0$, we have

$$\omega_0([J_1], [J_2]) = 0 \Rightarrow \mathbf{g}\left(J_2(0), J_1'(0)\right) = 0$$

Since $J_1'(0) \in \{\gamma_u'\}^{\perp}$, the only vector $J_1'(0)$ such that $\mathbf{g}\left(J_2(0), J_1'(0)\right) = 0$ for all $J_2(0) \in \{\gamma_u'\}^{\perp}$ is, by definition of $\{\gamma_u'\}^{\perp}$, the vector $J_1'(0) = 0 \, (\mathrm{mod} \gamma_u')$.

On the other hand, for $[J_2]$ verifying $J_2(0) = 0$ we have

$$\omega_0([J_1], [J_2]) = 0 \Rightarrow \mathbf{g}(J_1(0), J_2'(0)) = 0$$

and again, since $J_1(0) \in \{\gamma_u'\}^\perp$ then the only vector $J_1(0)$ such that $\mathbf{g}(J_1(0), J_2'(0)) = 0$ for all $J_2'(0) \in \{\gamma_u'\}^\perp$ is $J_1(0) = 0\,(\mathrm{mod}\gamma_u')$.

Thus, the only $[J_1] \in \mathcal{H}$ such that $\omega_0([J_1], [J_2]) = 0$ for all $[J_2] \in \mathcal{H}$ is $J_1 = 0\,(\mathrm{mod}\gamma_u')$, therefore $\omega_0|_{\mathcal{H}}$ is non–degenerated. This shows that $\mathcal{H}$ is a contact structure in $\mathcal{N}$. □

Let us take $\gamma \in \mathcal{U} \cap \mathcal{V}$, since in general $\frac{d}{dt}\mathbf{g}(\gamma'(t), T(\gamma(t))) \neq 0$, then there are different parameter for γ in order to write $\mathcal{H}(\mathcal{U})$ and $\mathcal{H}(\mathcal{V})$ as in expression 5.9. If we consider that $\gamma = \gamma(t)$ and $\overline{\gamma} = \overline{\gamma}(\tau)$ are the parametrizations of $\gamma \in \mathcal{U} \cap \mathcal{V}$ such that $\overline{\gamma}(\tau) = \gamma(a\tau + b)$ verifying

$$\begin{cases} \mathbf{g}(\gamma'(0), T) = -1 \\ \mathbf{g}(\overline{\gamma}'(0), T) = -1 \end{cases}$$

By definition of $\mathcal{J}_L(\overline{\gamma})$, we have that $\mathbf{g}\left(\overline{J}(\tau), \overline{\gamma}'(\tau)\right)$ is constant, therefore

$$\mathbf{g}\left(\overline{J}(0), \overline{\gamma}'(0)\right) = \mathbf{g}\left(\overline{J}(-b/a), \overline{\gamma}'(-b/a)\right) = a\mathbf{g}(J(0), \gamma'(0))$$

as we have seen in remark 4.1, whence since $\overline{\gamma}(-b/a) = \gamma(0)$ we have

$$\mathbf{g}\left(\overline{J}(-b/a), \overline{\gamma}'(-b/a)\right) = 0 \Leftrightarrow \mathbf{g}(J(0), \gamma'(0)) = 0$$

The same argument above is valid to prove that $\mathcal{H}_\gamma$ does not depends on the timelike vector field used to define Ω, because it only affects to the parametrization of γ. This shows that $\mathcal{H}_\gamma$ is well defined and does not depends on the neighbourhood used in its construction.

At this point, we may consider a covering $\{\mathcal{U}_\delta\}_{\delta \in I} \subset \mathcal{N}$ and, for any $\delta \in I$, consider the local 1–form θ_0^δ defining the contact structure $\mathcal{H}$ as before. If we take a partition of unity $\{\chi_\delta\}_{\delta \in I}$ subordinated to the covering $\{\mathcal{U}_\delta\}_{\delta \in I}$ then we can define a global 1–form by:

$$\theta_0([J]) = \sum_{\delta \in I} \chi_\delta([J]) \cdot \theta_0^\delta([J])$$

then the contact structure $\mathcal{H}$ is cooriented since θ_0 is globally defined and, by Lemma 5.3, remains maximally non-integrable.

In the following section we will provide a slightly more intrinsic construction of the canonical contact structure on the space of light rays based on symplectic reduction techniques.

6. The contact structure in $\mathcal{N}$ and symplectic reduction

Finally, in this section, we will illustrate the construction of the contact structure in $\mathcal{N}$ by using symplectic reduction in two different ways.

The first is performed in TM by using the chain of inclusions 3.6. In fact it is equivalent, but more elegant, to obtain $\mathcal{H}$ as in section 5.2. The second, carried out in section 6.2, is the standard symplectic reduction as expounded by Keshin and Tabachnikov in [KT09].

6.1. The coisotropic reduction of $\mathbb{N}^+$ and the symplectic structure on the space of scaled null geodesics $\mathcal{N}_s$

The celebrated Theorem of Marsden–Weinstein [MW74] claims that a $2m$–dimensional symplectic manifold P, in which a Lie group G acts preserving the symplectic form ω and possessing an equivariant momentum map, can be reduced into another $2(m-r)$–dimensional symplectic manifold P_μ, called the Marsden-Weinstein reduction of P with respect to μ, under the appropriate conditions where μ is an element in the dual of the Lie algebra of the group G and r is the dimension of the coadjoint orbit passing through μ.

The purpose of this section is to show that it is possible to derive the canonical contact structure on the space of light rays by a judiciously use of Marsden-Weinstein reduction when the geodesic flow defines an action of the Abelian group $\mathbb{R}$ in TM. However we will choose a different, simpler, however more general path here. Simpler in the sense that we will not need the full extent of MW reduction theorem, but a simplified version of it obtained when restricted to scalar momentum maps, but more general in the sense that it will not be necessary to assume the existence of a group action. Actually the setting we will be using is a particular instance of the scheme called generalized symplectic reduction (see for instance [Ca14, Ch.7.3] and references therein).

The result we are going to obtain is based on the following elementary algebraic fact. Let (E, ω) be a linear symplectic space. Let $W \subset E$ be a linear subspace. We denote by $W^\perp$ the symplectic orthogonal to W, i.e., $W^\perp = \{u \in E \mid \omega(u,w) = 0\,, \forall w \in W\}$. A subspace W is called coisotropic if $W^\perp \subset W$. It is easy to show that for any subspace W, $\dim W + \dim W^\perp = \dim E$. Hence it is obvious that if H is a linear hyperplane, that is a linear subspace of codimension 1, then H is coistropic (clearly because ω_H is degenerate, then $H \cap H^\perp \neq \{\mathbf{0}\}$ and because $H^\perp$ is one-dimensional, then $H^\perp \subset H$). Moreover the quotient space $H/H^\perp$ inherits a canonical symplectic form $\bar{\omega}$ defined by the expression:

$$\bar{\omega}(u_1 + H^\perp, u_2 + H^\perp) = \omega(u_1, u_2)\,, \qquad \forall u_1 . u_2 \in H\,.$$

The linear result above has a natural geometrical extension:

Theorem 6.1 *Let (P, ω) be symplectic manifold and $i\colon S \to P$ be a hypersurface, i.e., a codimension 1 immersed manifold. Then:*

i. The symplectic form ω induces a 1-dimensional distribution K on S, called the characteristic distribution of ω, defined as $K_x = \ker i^*\omega_x = T_xS^\perp \subset T_xS$.

ii. If we denote by $\mathcal{K}$ the 1-dimensional foliation defined by the distribution K and $\overline{S} = S/\mathcal{K}$ has the structure of a quotient manifold, i.e., the canonical projection map $\rho\colon S \to S/\mathcal{K}$ is a submersion, then there exists a unique symplectic form $\bar{\omega}$ on $\overline{S}$ such that $\rho^*\bar{\omega} = i^*\omega$.

iii. If $\omega = -d\theta$ and there exists $\bar{\theta}$ a 1-form on $\overline{S}$ such that $\rho^*\bar{\theta} = i^*\theta$, then $\bar{\omega} = -d\bar{\theta}$.

Proof. The proof of (i) is just the restriction of the algebraic statements above to $W = T_xS \subset E = T_xP$.

To proof (ii), notice that a vector tangent to the leaves of $\mathcal{K}$ is in the kernel of $i * \omega$, then for any vector field X on S tangent to the leaves of $\mathcal{K}$, i.e, projectable to 0 under ρ, we have $i_X(i^*\omega) = 0$, and $\mathcal{L}_X(i^*\omega) = 0$, then the 2-form $i^*\omega$ is projectable under ρ.

The statement (iii) is trivial because $\rho^*\bar{\omega} = i^*\omega = i^*(-d\theta) = -di^*\theta = -d\rho^*\bar{\theta} = \rho^*(-d\bar{\theta})$ and ρ is a submersion. □

The previous theorem states that any hypersurface on a symplectic manifold is coisotropic and that, provided that the quotient space is a manifold, the space of leaves of its characteristic foliation, inherits a symplectic structure. Such space of leaves is thus the reduced symplectic manifold we are seeking for and it will be called the coisotropic reduction of the hypersurface S. In addition to the previous reduction mechanism, we will also use the following passing to the quotient mechanism for hyperplane distributions.

Theorem 6.2 *Let $(P, \omega = d\theta)$ be an exact symplectic manifold and $\pi\colon P \to N$ be a submersion on a manifold of dimension $\dim P - 1$ and such that it projects the hyperplane distribution $H = \ker\theta$, that is there exists a hyperplane distribution H^N in N such that for any $x \in P$, $\pi_*(x)H_x = H^N_{\pi(x)}$. Then H^N defines a contact structure on N.*

Proof. Notice that necessarily, $\ker \pi_*(x) = H_x^\perp$ and ω induces a symplectic form $\bar{\omega}_x$ in $H/H^\perp$ because Thm. 6.1. Moreover $H_x/H_x^\perp \cong H^N_{\pi(x)}$ and it inherits a symplectic form $\bar{\omega}_x$. Finally, if we pick up a local section σ of the submersion π; then the 1-form $\sigma^*\theta$ is such that $H^N = \ker \sigma^*\theta$ and $d(\sigma^*\theta)$ coincides with the symplectic form $\bar{\omega}_x$ when restricted to H^N_x. □

The two previous results, Thm. 6.1 and 6.2, hold the key to understand how the quotient space $\mathcal{N}$ inherits a canonical contact structure. Consider again a spacetime

$(M, \mathbf{g})$ and the canonical identification provided by the metric $\hat{\mathbf{g}}\colon \widehat{T}M \to \widehat{T}^*M$ (which is just the Legendre transform corresponding to the Lagrangian function $L_{\mathbf{g}}(x,v) = \frac{1}{2}\mathbf{g}_x(v,v)$ on TM). As we discussed at the beginning of Sect. 5, Eqs. (5.6), (5.10), we can pull-back the canonical 1-form θ on T^*M along $\hat{\mathbf{g}}$ as well the canonical symplectic structure ω (Sect. 5.1), that is, we obtain:

$$\theta_g = \hat{\mathbf{g}}^*\theta\,, \qquad \omega_g = \hat{\mathbf{g}}^*\omega = -d\theta_g\,,$$

and $(\widehat{T}M, \omega_{\mathbf{g}})$ becomes a symplectic manifold. Moreover $\mathbb{N}^+ \subset \widehat{T}M$ defines an hypersurface, hence by Thm. 6.1 we can construct its coisotropic reduction.

We will denote by $\mathcal{N}_s$ the space of equivalence classes of future-oriented null geodesics that differ by a translation of the parameter. Thus two parametrized null geodesics $\gamma_1(t)$, $\gamma_2(t')$ are equivalent if there exists a real number s such that $\gamma_2(t') = \gamma_1(t+s)$. The equivalence class of null geodesics containing the parametrized geodesic $\gamma(t)$ such that $\gamma'(0) = v$ will be denoted by γ_v.

Clearly there is a natural projection $\pi\colon \mathcal{N}_s \to \mathcal{N}$ (see below, Sect. 6.2) defined as $\pi(\gamma_v) = [\gamma]$. The space $\mathcal{N}_s$ is sometimes called the space of *scaled null geodesic* and describes equivalence classes of null geodesics distinguishing different scale parametrizations.

Theorem 6.3 *Let $(M, \mathbf{g})$ be a spacetime, then:*

i. The characteristic distribution $K = \ker \omega_{\mathbf{g}} \mid_{\mathbb{N}^+}$ is generated by the restriction of the geodesic spray $X_{\mathbf{g}}$ to $\mathbb{N}^+$ and $\mathbb{N}^+/K$ can be identified naturally with the space of scaled null geodesics $\mathcal{N}_s$.

ii. If M is strongly causal, $\mathcal{N}_s$ is a quotient manifold of $\mathbb{N}^+$, and it becomes a symplectic manifold with the canonical reduced symplectic structure obtained by coisotropic reduction of $\omega_{\mathbf{g}}$.

Proof. To prove [i] we just check that $\omega_{\mathbf{g}_v}(X_{\mathbf{g}}, Y) = dL_v(Y) = Y_v(L) = 0$ for all $Y \in T_v(\mathbb{N}^+)$ because $\mathbb{N}^+ = L^{-1}(\mathbf{0})$ (see Eq. (2.4)).

Notice that the flow φ_t of the geodesic spray $X_{\mathbf{g}}$ is such that $\varphi_s(\gamma(t)) = \gamma(t+s)$ where $\gamma(t)$ is a parametrized geodesic. Then the quotient $\mathbb{N}^+/K$ corresponds exactly to the notion of scaled null geodesic before. We will denote, as before, by $\rho\colon \mathbb{N}^+ \to \mathcal{N}_s$ the canonical projection and, with the notations above, we get simply that $\rho(v) = \gamma_v$.

As M is strongly causal, the proof of [ii] mimics the proof of Prop. 3.5 (see also Remark 3.1). Hence because [ii] in Thm. 6.1, we conclude that the quotient manifold inherits a canonical symplectic structure by coisotropic reduction of $\omega_{\mathbf{g}}$. □

6.2. The contact structure of $\mathcal{N}$ and the reduction of the space of scaled null geodesics $\mathcal{N}_s$

We will prove first that the space of light rays $\mathcal{N}$ is the base manifold of a principal bundle with structural group $\mathbb{R}^+$ whose total space is the space of scaled null geodesics

$\mathcal{N}_s$. Notice that the Euler vector field Δ on $\mathbb{N}^+$ is projectable under the map $\rho\colon \mathbb{N}^+ \to \mathcal{N}_s$. The projected vector field will be denoted by Δ_s and its flow is defined by $\Phi_t(\gamma_v) = \gamma_{tv}$. Clearly $\rho(tv) = \gamma_{tv}$, $\rho \circ \varphi_t = \Phi_t \circ \rho$, hence $\rho_* \Delta = \Delta_s$.

Lemma 6.1 *The map $\pi : \mathcal{N}_s \to \mathcal{N}$ is a principal bundle with structural group the multiplicative group $\mathbb{R}^+$.*

Proof. We will show that the flow $\Phi : \mathbb{R}^+ \times \mathcal{N}_s \to \mathcal{N}_s$ defined by the vector field $\Delta_s \in \mathfrak{X}(\mathcal{N}_s)$ defines a free and proper right action. It is clear that Φ is a right action and it can be written by

$$\Phi(t, \gamma_v) = \gamma_{tv}$$

where γ_u denotes the null geodesic defined by the null vector $u \in \mathbb{N}^+$. It is well-known that for any non-zero $\lambda \in \mathbb{R}$ it is verified that $\gamma_{\lambda u}(s) = \gamma_u(\lambda s)$. Then, the equality $\Phi(t, \gamma_v) = \gamma_v$ implies that $tv = v$, whence $t = 1$, and the action is free.

Now, consider two sequences $\{\gamma_{v_n}\}$ and $\{\Phi_{t_n}(\gamma_{v_n})\}$ converging to γ_v and γ_u respectively. Since $\Phi_{t_n}(\gamma_{v_n}) = \gamma_{t_n v_n}$ and again $\gamma_{t_n v_n}(s) = \gamma_{v_n}(t_n s)$ then γ_u and γ_v have the same image as a parametrized curve in M, then for every s we have $\gamma_u(s) = \gamma_v(\bar{t}s)$ for some $\bar{t} \in \mathbb{R}^+$. So we have that $u = \bar{t}v$ and hence $t_n v_n \mapsto \bar{t}v$. Since $v \neq 0$ and $\begin{cases} v_n \mapsto v \\ t_n v_n \mapsto \bar{t}v \end{cases}$ then we have that $t_n \mapsto \bar{t}$. This shows that the action Φ is proper. Then we have that $\pi : \mathcal{N}_s \to \mathcal{N}$, $\pi(\gamma_v) = [\gamma]$, is a principal bundle with structural group $\mathbb{R}^+$. □

The following theorem shows that the distribution of hyperplanes defined by the canonical 1-form $\overline{\theta}$ actually projects down to $\mathcal{N}$ defining a canonical contact structure. First notice that since $\pi^*(\mathcal{L}_{\Delta_s}\overline{\omega}) = \mathcal{L}_\Delta \pi^* \overline{\omega} = i^* \mathcal{L}_\Delta \omega = \pi^* \overline{\omega}$, then

$$\mathcal{L}_{\Delta_s}\overline{\omega} = \overline{\omega}\,,$$

and Δ_s is a Liouville vector field. Then consider the 1–form

$$\overline{\theta} = -i_{\Delta_s}\overline{\omega}\,, \tag{6.1}$$

in $\mathcal{N}_s$. Clearly $i_{\Delta_s}\bar{\theta} = 0$ and $d\bar{\theta} = -\bar{\omega}$.

Theorem 6.4 *The 1-form $\bar{\theta}$ in $\mathcal{N}_s$ induces a distribution of hyperplanes $\overline{\mathcal{H}} = \ker \bar{\theta}$. Moreover there exists a distribution of hyperplanes $\mathcal{H}$ in $\mathcal{N}$ such that $\pi_*(\overline{H}) = \mathcal{H}$ that defines a contact structure. Hence $\mathcal{N}$ is equipped with a canonical contact structure.*

Proof. Because of Eq. (6.1), $\mathcal{L}_{\Delta_s}\bar{\theta} = i_{\Delta_s} d\bar{\theta} = -\bar{\theta}$, then $(\Phi_t)^*\bar{\theta} = t\bar{\theta}$. Then at each point $[\gamma] \in \mathcal{N}$, the hyperplanes obtained by projecting the hyperplanes $\mathcal{H}_{\gamma_{tv}}$ are the projection of the kernels of the family of proportional covectors $t\bar{\theta}_v$, hence they are the same. Then because of Thm. 6.2, this implies that $\mathcal{H}$ is a contact structure in $\mathcal{N}$. □

Proposition 6.5 *Let $\pi : \mathcal{N}_s \to \mathcal{N}$ be the principal bundle of lemma 6.1, then there exists a 1–form θ_0 in $\mathcal{N}$ such that $\mathcal{H} = \ker \theta_0$ and the contact structure is coorientable.*

Proof. Because the structure group of the principal bundle $\mathcal{N}_s \to \mathcal{N}$ is contractible, then is trivial. Hence there exists a smooth global section σ, then we can define the 1-form $\theta_0 = \sigma^* \overline{\theta}$ and, clearly, $\ker \theta_0 = \mathcal{H}$. □

Notice that the bundle $\mathcal{N}_s \to \mathcal{N}$ is trivial because the group $\mathbb{R}^+$ is contractible, however if we were considering the space of non-oriented (future or past) unparametrized null geodesics instead, such space would be obtained by quotienting $\mathcal{N}_s$ with respect to the group $\mathbb{R}_* = \mathbb{R} - \{\}$ which is not contractible and the corresponding contact structure will not be coorientable.

It is also noticeable, that like in the construction at the end of Sect. 5, the 1-form θ_0 describing the canonical contact structure on $\mathcal{N}$ is not canonically determined (even if the hyperplane distribution is), in former case the 1-form θ_0 depends on the choice of a partition of the unity, in the present case, it depends on a section of a principal bundle.

Next, we will look for an expression for the local contact forms defining the contact structure $\mathcal{H} \subset T\mathcal{N}$. Recall that a coordinate chart $\psi : \mathcal{U} \subset \mathcal{N} \to \mathbb{R}^{2m-3}$ can be defined via the diffeomorphism $\mathcal{U} \to \Omega^T(C)$ of diagram 3.6, where $\Omega^T(C)$ is an embedded submanifold of $TV \subset TM$ with $V \subset M$ a globally hyperbolic and causally convex open set with Cauchy surface C. Then we have the following diagram

$$\begin{array}{ccc} \mathcal{N} \supset \mathcal{U} & \xrightarrow{\ \overline{z}\ } & TV \\ \psi \downarrow & & \downarrow \phi \\ \mathbb{R}^{2m-3} \supset B_0 & \xrightarrow[\ z\ ]{} & B \subset \mathbb{R}^{2m} \end{array} \tag{6.2}$$

where $\overline{z} = \phi^{-1} \circ z \circ \psi$. The image of the embedding $\overline{z}$ is contained in $\mathbb{N}^+(C)$, and moreover if $p_{\mathbb{N}} : \mathbb{N} \to \mathcal{N}$ is the canonical projection, then $p_{\mathbb{N}} \circ \overline{z}([\gamma]) = [\gamma]$ for all $[\gamma] \in \mathcal{U} \subset \mathcal{N}$. Then $\overline{z}$ is a local section of $p_{\mathbb{N}}$.

By proposition 6.5 and theorem 6.4, we have that for $\xi \in T\mathbb{N}$

$$\theta_\alpha(\xi) = (\theta_0)_{p_{\mathbb{N}^*}(\alpha)}\left((dp_{\mathbb{N}^*})_\alpha(\xi)\right)$$

then, by diagram in 5.5, $p_{\mathbb{N}} = p_{\mathbb{N}^*} \circ \widehat{\mathbf{g}}$, and hence we can write for $J \in T_{[\gamma]}\mathcal{N} \subset T\mathcal{U}$

$$\theta_{\widehat{\mathbf{g}} \circ \overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}} \circ \overline{z}\right)_{[\gamma]}(J)\right) = (\theta_0)_{[\gamma]}(J)$$

On the other hand, by definition of the tautological 1–form θ and since $\pi_M^{TM} = \pi_M^{T^*M} \circ \widehat{\mathbf{g}}$, we have

$$\begin{aligned}
\theta_{\widehat{\mathbf{g}}\circ\overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}}\circ\overline{z}\right)_{[\gamma]}(J)\right) &= \widehat{\mathbf{g}}\circ\overline{z}\left([\gamma]\right)\left(d\left(\widehat{\mathbf{g}}\circ\overline{z}\right)_{[\gamma]}(J)\right) = \\
&= \mathbf{g}\left(\left(d\pi_M^{T^*M}\right)_{\widehat{\mathbf{g}}\circ\overline{z}([\gamma])}\left(d\left(\widehat{\mathbf{g}}\circ\overline{z}\right)_{[\gamma]}(J)\right), \overline{z}\left([\gamma]\right)\right) = \\
&= \mathbf{g}\left(\left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}(J)\right), \overline{z}\left([\gamma]\right)\right) = \\
&= \mathbf{g}\left(J\left(0\right), \gamma'\left(0\right)\right)
\end{aligned}$$

where $\overline{z}\left([\gamma]\right) \in \mathbb{N}^+\left(C\right)$ is a vector defining the light ray $[\gamma] \in \mathcal{U}$, so we have considered the null geodesic γ such that $\gamma'\left(0\right) = \overline{z}\left([\gamma]\right)$. On the other hand, observe that if $J = \Gamma'\left(0\right) \in T_{[\gamma]}\mathcal{N}$ where Γ is a smooth curve in $\mathcal{N}$ with $\Gamma\left(0\right) = [\gamma]$, then

$$\left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}(J)\right) = \left(d\pi_M^{TM}\right)_{\overline{z}([\gamma])}\left(d\overline{z}_{[\gamma]}\left(\Gamma'\left(0\right)\right)\right) = \left(\pi_M^{TM}\circ\overline{z}\circ\Gamma\right)'\left(0\right)$$

where $\pi_M^{TM}\circ\overline{z}\circ\Gamma$ is the curve in M where $\overline{z}\circ\Gamma \subset \mathbb{N}\left(C\right)$ rest. By lemma 4.3, we have that $\left(\pi_M^{TM}\circ\overline{z}\circ\Gamma\right)'\left(0\right) = J\left(0\right)$ and therefore we claim that

$$\left(\theta_0\right)_{[\gamma]}(J) = \mathbf{g}\left(J\left(0\right), \gamma'\left(0\right)\right)$$

and then

$$J \in \mathcal{H} \Longleftrightarrow \mathbf{g}\left(J\left(0\right), \gamma'\left(0\right)\right) = 0.$$

It is clear that this characterization does not depends neither on the representative metric of the conformal class $\mathcal{C}$ nor on the parametrization of γ in virtue of lemma 4.8.

Again, since the expression of the local 1–form θ_0 defining the contact structure $\mathcal{H}$ coincides with the one constructed in section 5.2, then the same used argument to show that $\mathcal{H}$ is cooriented remains valid.

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