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# About Signature-Change Metrics on Manifolds.

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**Abstract.** We provide a one-parameter family of Lorentz-Riemann signature-change models of metric manifolds. This family generalizes the Kossowski´s signature type-changihg stablished in [9].
Simple local expressions are sought around the hypersurface of change.

## 1 Introduction.

The initial idea of signature change is due to Hartle and Hawking [8] which makes that it possible to have both Euclidean and Lorentzian regions in quantum gravity to avoid the initial spacetime singularity predicted by the standard model. Changing signature spaces, were initially studied by Larsen [16] but Kossowski an Kriele in [9] marks the beginning of the transverse type-change model (see definition below) which has used in a considerable number of publications of a physical or geometric nature.

Following Kossowski's paper [9], we establish the following

**Definition 1** *Let $M$ a differentiable manifold $g$ a symmetric $(0,2)$ smooth tensor field in $M$ wich fails to have maximal rank on $\Sigma \subset M$. Is said $g$ is a transverse type-changing metric, if around each point $p \in \Sigma$ there exists a coordinate system $(x_a)$ centred in $p$ such that if $g = g_{ab}dx^a dx^b$ is the local expression of $g$, the differential of the function $\det(g_{ab})$ at $p$, does not vanish. This condition (which is obviously independent of the chosen coordinate system) implies that the set $\Sigma$ of singular points is a hypersurface (called a singular hypersurface), and the signature changes by one unit when passing through $\Sigma$.*

*If, in addition, at each singular point $p$, the radical $Rad(g_p)$ is not contained in $T_p\Sigma$, and $M\backslash\Sigma$ only has Lorentz or Riemann components, we call $(M,g)$ a $\Sigma$-space.*

**Remark 2 (Notational convention)** *The dimension of the manifold $M$ will* *be denoted by $m \geq 2$, and henceforth:*

*The indices $a, b, c, ...$range between 1 and $m$*
*while $i, j, k, ...$ range between 1 and $m-1$*
*differentiable means $C^{\infty}$.*

Paper [9] begins the study of $\Sigma$-spaces, with the idea of determining the geodesic lines that cross $\Sigma$ transversely. While the results presented here are correct, some proofs are incomplete. For example, to prove that around any singular point, there exist (normal) coordinates $(x_i, x_m)$ where $g = \sum g_{ij} dx_i dx_j - x_m dx_m^2$, he assumes without proof that is differentiable the function

$$F(\lambda, t) = sgn(t) \left( \int_0^t \sqrt{|x|} \psi(\lambda, x)\, dx \right)^{2/3}$$

defined in an open $\Lambda \times ]-\varepsilon, \varepsilon[$ of $\mathbb{R}^{m+1}$, where $\psi$ is a smooth function with $\psi(\lambda, 0) > 0$. The aim of this work is to prove a more general result such as, is differentiable the function

$$F(\lambda, t) = sgn(t) \left( \int_0^t |x|^r \psi(\lambda, x)\, dx \right)^{\frac{1}{r+1}}, \text{ for } r \neq -1$$

(see proof in the Appendix). Using of this result we initiated an analogous study for the spaces that we have called $\Sigma^{\alpha}$-spaces (where $\alpha > 0$) which has a similar definition as $\Sigma$-spaces $(M, g)$ but now the $\alpha$-transversality condition is that, around each point $p \in \Sigma$ there exists a coordinate system $(x_a)$ such that if $g = g_{ab} dx^a dx^b$ is the local expression of $g$, the function $sign(\det(g_{ab})) |\det(g_{ab})|^{\alpha}$ extends differentiably, and their differential does not vanish at $p$ (see definition in section 3, for more details). The (called signature change $\alpha$-transversal) metric $g$, must be continous in $M$, and differentiable in $M \backslash \Sigma$.

A $\Sigma^{\alpha}$-space $(M, g)$ we say normal if around any singular point $p$ there exists coordinate system $(x_i, x_m)$, where the metric is written as $g = \sum g_{ij} dx_i dx_j - sign(x_m) |x_m|^{1/\alpha} dx_m^2$. The main result of this paper is that $(M, g)$ is normal if and only if around each singular point there exist a differentiable geodesic line distribution crossing $\Sigma$ in the radical direction.

# 2 Preliminaries.

The following functions are defined:

$$\epsilon(t) = \begin{cases} +1 \text{ if } t > 0 \\ 0 \text{ if } t = 0 \\ -1 \text{ if } t < 0 \end{cases}, \text{ and for } \alpha \neq 0,\ t^{[\alpha]} = \epsilon(t) |t|^{\alpha} \qquad (1)$$

which satisfy the properties:

$$\begin{array}{l} t^{[1]} = t,\ t^{[0]} = \epsilon,\ \epsilon^2 = 1 \\ \left(t^{[\alpha]}\right)^{[\beta]} = t^{[\alpha\beta]} \\ t^{[\alpha+\beta]} = \epsilon(t) t^{[\alpha]} t^{[\beta]},\ t^{[-\alpha]} = \frac{1}{t^{[\alpha]}} \\ (st)^{[\alpha]} = s^{[\alpha]} t^{[\alpha]} \end{array}$$

$M$ denotes a differentiable manifold of dimension $m$, and $\Sigma$ is a hypersurface of $M$.

Henceforth, differentiable means $C^\infty$
and unless explicitly stated otherwise,
all elements of $M$, fields, submanifolds, etc.
shall be assumed to be of class $C^\infty$.

Finally, if $X$ is a differentiable vector field on $M$, we denote by $X_t(p)$ the (local) flow of $X$ on $M$, such that the curve $\gamma_p = X_t(p)$ represents the integral curve of $X$ through the point $p$, i.e., $\gamma_p'(t) = X(\gamma_p(t))$ and $\gamma_p'(0) = p$. The one-manifold defined by $\gamma_p$ is called line of $X$.

Let $\Sigma$ be a hypersurface of $M$

**Definition 3** *A simple open set of $(M, \Sigma)$ is a connected open set $M_0$ of $M$, such that $\Sigma_0 = M_0 \cap \Sigma \neq \emptyset$ is connected, and $M_0 \backslash \Sigma_0$ has exactly two connected components.*

**Remark 4** *Note that for each point $p \in \Sigma$, one can find a neighborhood $M_0$ that is a simple open set. We call it a simple neighborhood of $p$. Simple neighborhoods of $p$ obviously constitute a basis of neighborhoods for $p$.*

**Definition 5** *A* simple *equation for $\Sigma$, around a point $p \in \Sigma$, is a differentiable function $\sigma : M_0 \to \mathbb{R}$ defined on a neighborhood $M_0$ of $p$ in $M$, such that*

$$\begin{cases} \Sigma_0 = M_0 \cap \Sigma = \{x \in M_0 : \sigma(x) = 0\} \\ d\sigma|_x \neq 0,\ \forall x \in \Sigma_0 \end{cases}$$

*we then write $\Sigma :_s \sigma = 0$.*

**Proposition 6** *If $\Sigma :_s \sigma = 0$ and $\varphi : M_0 \to \mathbb{R}$ is another equation around $p$ for $\Sigma$ (i.e. $\Sigma : \varphi = 0$), then there exists a differentiable $h : M_0 \to \mathbb{R}$ such that $\varphi = h\sigma$; furthermore, $\Sigma :_s \varphi = 0 \Leftrightarrow h$ is non-zero at the point.*

**Proof.** We need the following preliminary result:

Let $\mathbb{R}_0^m$ be an open set of $\mathbb{R}^m$ and $I_\varepsilon = ]-\varepsilon, \varepsilon[ \subset \mathbb{R}$.

If $f = f(x,t) : \mathbb{R}_0^m \times I_\varepsilon \to \mathbb{R}$ is $C^k$ differentiable, $k \geq 1$, and $f(x, 0) = 0$, then there exists $g = g(x,t) : \mathbb{R}_0^m \times I_\varepsilon \to \mathbb{R}$ of class $C^{k-1}$ such that $f = tg(x,t)$. Indeed, for a fixed $t$, let $f_t = f_t(x,s) = f(x, st)$, then:

$$f(x,t) = [f_t]_{s=0}^{s=1} = \int_0^1 \frac{\partial f_t}{\partial s} ds = t \int_0^1 \left. \frac{\partial f}{\partial t} \right|_{st} ds$$

and therefore the function

$$g(t) = \int_0^1 \left.\frac{\partial f}{\partial t}\right|_{st} ds$$

is a function of class $C^{k-1}$ that satisfies $f(t) = tg(t)$.

To prove the proposition now, we can construct a chart $(x_i, x_m)$ in the open set $M_0$ with $x_m = \sigma$, and then the function $\varphi = \varphi(x_i, x_m)$ satisfies $\varphi(x_i, 0) = 0$. Using the preliminary result, it follows that there exists a differentiable $h = h(x_i, x_m) : M_0 \to \mathbb{R}$ such that $\varphi = hx_m = h\sigma$. Furthermore:

$$\Sigma :_s \varphi = 0 \Longleftrightarrow 0 \neq \left.\frac{\partial \varphi}{\partial x_m}\right|_{x_m=0} = h(x_i, 0)$$

■

**Remark 7** *Note that if $\Sigma :_s \sigma = 0$ is a simple equation, then $e^{\varphi}\sigma = 0$ is also a simple equation of $\Sigma$ for any differentiable $\varphi : M_0 \to \mathbb{R}$.*

**Remark 8** *In particular, if $\varphi = \varphi(x_i, x_m)$ is a differentiable function on an open set of $\mathbb{R}^m$ and $\varphi(x_i, 0) = 0$, then there exists $\psi = \psi(x_i, x_m)$ such that $\varphi = x_m\psi$.*

**Definition 9** *A $\Sigma$-distribution around $p_0 \in \Sigma$ is an integrable distribution $\mathcal{H}$ defined on a simple neighborhood $M_0$ of $p_0$, such that $\mathcal{H}_p = T_p\Sigma$ for all $p \in \Sigma_0 = \Sigma \cap M_0$.*

**Theorem 10** *Let $X$ be a field on $M$ with $X(x) \notin T_x\Sigma$ for all $x \in \Sigma$. Then for each $p_0 \in \Sigma$, a neighborhood $M_0$ in $M$ and a chart $(u_i)$ on $\Sigma_0 = M_0 \cap \Sigma$ can be taken. Taking $M_0$ sufficiently small, a chart $(x_i, x_m)$ can be constructed such that:*

$$X = \frac{\partial}{\partial x_m}, \ \Sigma : x_m = 0, \quad x_i|_{\Sigma_0} = u_i$$

**Proof.** Given $p_0 \in \Sigma$ and a chart $(u_i)$ in a neighborhood $\Sigma_0$ of $p_0$ in $\Sigma$, one can take an open set $M_0$ of $M$ around $p_0$ and $\varepsilon > 0$ such that the map

$$\Psi : \Sigma_0 \times (-\varepsilon, \varepsilon) \to M_0$$

is a diffeomorphism, where $\Psi = \Psi(p, t) = X_t(p)$ is the local flow of $X$. This is because $d\Psi|_{(0,p_0)} : T_{p_0}\Sigma \times \mathbb{R} \to T_{p_0}M$ is an isomorphism. Indeed, if $v \in T_{p_0}\Sigma_0$ and $\gamma_p = \gamma_p(t) = \Psi(p, t)$ is the integral curve of $X$ through $p$, taking a curve $\alpha : I_\varepsilon \to \Sigma_0$ with $\alpha'(0) = v$, we have:

$$\begin{aligned} d\Psi|_{(0,p_0)}(v, 0) &= \left.\frac{d}{dt}\right|_{t=0} \Psi(\alpha(t), 0) \\ &= \left.\frac{d}{dt}\right|_{t=0} \gamma_{\alpha(t)}(0) = \alpha'(0) = v \end{aligned}$$

$$
\begin{aligned}
d\Psi|_{(0,p_0)}(0,1) &= \left.\frac{d}{dt}\right|_{t=0} \Psi(p_0,t) \\
&= \left.\frac{d}{dt}\right|_{t=0} \gamma_{p_0}(t) = X(p_0)
\end{aligned}
$$

Then the chart $(x_i, x_m)$ is constructed with $x_i = u_i \circ \pi_2 \circ \Psi^{-1}$ and $x_m = \pi_1 \circ \Psi^{-1}$, where $\pi_1, \pi_2$ are the corresponding projections of $(-\varepsilon, \varepsilon) \times \Sigma_0$ onto its factors, following the scheme:

$$
\begin{array}{ccc}
 & & (-\varepsilon,\varepsilon) \\
 & \overset{x_m}{\nearrow} & \pi_1 \uparrow \\
M_0 & \overset{\Psi^{-1}}{\longrightarrow} & (-\varepsilon,\varepsilon)\times\Sigma_0 \\
 & \underset{\searrow}{(x_i)} & \pi_2 \downarrow \\
 & & \Sigma_0
\end{array}
$$

■

# 3 $\Sigma^\alpha$-spaces.

In what follows, $M$ is a differentiable manifold of dimension $m$.

$(M, g)$ is said to be a $\Sigma^\alpha$-space ($\alpha > 0$) if $g$ is a continuous and symmetric tensor field of type $(0, 2)$ and the set $\Sigma$ of singular points is a hypersurface $\Sigma \subset M$. The following properties are also required

- $(\Sigma, g|_\Sigma)$ is a Riemannian manifold.
- $g$ is differentiable on $M \setminus \Sigma$.
- **$\alpha$-transversality**: Around each point $p \in \Sigma$, there exists a coordinate system $(x_i, x_m)$ such that if $g = g_{ab}dx^a dx^b$, then the equation $\det(g_{ab})^{[\alpha]} = 0$ defines by extension[1] a simple equation of $\Sigma$.

**Remark 11** *The property of $\alpha$-transversality is maintained for the matrix of $g$ with respect to any chart or even with respect to any parallelization $(E_i, E_m)$ around a point in $\Sigma$. Since the metric $g|_\Sigma$ is Riemannian, it follows that* $\mathrm{Rad}(g_p) \in T_pM \setminus T_p\Sigma$ *for all $p \in \Sigma$.*

**Definition 12** *Let $(M, g)$ be a $\Sigma^\alpha$. A differentiable field $\rho$ on $M$ is called:*

- ***radical**: If it has no zeros, and for all $p \in \Sigma$, $\rho(p) \in \mathrm{Rad}(g_p)$.*
- ***radical geodesic:** If it is radical, and all its integral lines are geodesic lines*
- ***$\Sigma$-Adapted**: If the distribution $\rho^\perp$ defines (by extension) a $\Sigma$-distribution.*

[1]This means that the function $\det(g_{ab})^{[\alpha]}$ extends differentiably to $\Sigma$

- ***synchronized:*** *If it is $\Sigma$-Adapted and $\Sigma_t = \{\rho_t(p) : p \in \Sigma\}$ is a leaf of $\rho^\perp$ for all $t$.*

**Definition 13** *A coordinate system $(x_i, x_m)$ in a simple neighborhood of a point $p \in \Sigma$ where the matrix $(g_{ab})$ of the metric $g = g_{ab}dx_a dx_b$ is of the form:*

$$= \begin{pmatrix} g_{ij} & 0 \\ 0 & \hbar x_m^{[1/\alpha]} \end{pmatrix}, \quad \hbar \text{ is differentiable and never null.} \tag{2}$$

*is called **special coordinates**. (Note that $\partial/\partial x_m$ is then radical and $\Sigma$-adapted vector field).*

*If $\hbar = 1$, they are called **normal coordinates**.*

**Proposition 14** *If for a differentiable function $\sigma : M \to \mathbb{R}$, $\Sigma :_s \sigma = 0$, the field $G_\sigma$*

$$G^\sigma = \frac{\operatorname{grad}\sigma}{\langle \operatorname{grad}\sigma, \operatorname{grad}\sigma\rangle} \tag{3}$$

*extends differentiably to $M$, then in a simple neighborhood of each point $p \in \Sigma$, special coordinates $(x_i, x_m)$ can be constructed with $G_\sigma = \partial/\partial x_m$. In particular $G_\sigma$ is a radical and $\Sigma$-Adapted vectorfield.*

**Proof.** With $G_\sigma$, a chart $(x_i, x_m)$ can be built where $G_\sigma = \partial/\partial x_m$. For $x_m \neq 0$, the metric matrix verifies $g_{im} = \langle G_\sigma, \partial/\partial x_i\rangle_g = 0$ since $G_\sigma \perp \Sigma_t$. As $g_{im}$ are continuous, $g_{im}(x_i, 0) = 0$. By the $\alpha$-transversality condition, $g_{mm}$ is of the form $\hbar x_m^{[1/\alpha]}$ for some differentiable, non-null $\hbar$. ■

**Proposition 15** *If for a differentiable function $\sigma : M \to \mathbb{R}$, with $\Sigma :_s \sigma = 0$, the field $G^\sigma$ extends differentiably to $M$, then it is a synchronized radical field. Conversely, if $\xi$ is a synchronized radical field, there exists $\sigma : M \to \mathbb{R}$ such that $\xi = G_\sigma$.*

**Proof.** If $G_\sigma$ extends differentiably 0, let $G^\sigma_t(p)$ be the flow of $G_\sigma$; let us see then that $G^\sigma$ is a synchronized radical field. By Proposition 14 $G^\sigma$ is a radical and $\Sigma$-Adapted vectorfield; furthermore, for $p \in \Sigma$:

$$\begin{aligned} \frac{d}{dt}(\sigma(G^\sigma_t(p))) &= d\sigma(\gamma'_p(t)) = \langle \operatorname{grad}\sigma, \gamma'_p(t)\rangle \\ &= \left\langle \operatorname{grad}\sigma, \left.\frac{\operatorname{grad}\sigma}{\langle \operatorname{grad}\sigma, \operatorname{grad}\sigma\rangle}\right|_{\gamma_p(t)} \right\rangle = 1 \end{aligned}$$

Therefore, the map $t \mapsto \sigma(G^\sigma_t(p)) = t + \text{const}$, and since $G^\sigma_0(p) = p \in \Sigma$, we conclude that const $= \sigma(p) = 0$, thus $\sigma(G^\sigma_t(p)) = t$, and $\xi = G_\sigma$ is a synchronized radical field.

If $\xi$ is a synchronized radical field, let us take $M_0$ and $\varepsilon$ small enough so that the following map is a diffeomorphism:

$$\Psi : \Sigma_0 \times I_\varepsilon \to M_0, \quad (p, t) \mapsto \xi_t(p) \tag{4}$$

Then locally we can see the distribution $\xi^\perp$ as $\{\Sigma_t : \sigma = t\}$ for the function $\sigma = \pi_2 \circ \Psi^{-1}$, since for each $p \in \Sigma$:

$$\begin{aligned}\sigma(\xi_t(p)) &= (\pi_2 \circ \Psi^{-1})(\xi_t(p)) \\ &= \pi_2(p,t) = t\end{aligned}$$

And so $\xi = G^\sigma$, since the curves $t \mapsto \xi_t(p)$ and $t \mapsto G^\sigma_t(p)$ coincide as $\sigma(\xi_t(p)) = \sigma(G^\sigma_t(p)) = t$, defining both integral lines of $\xi^\perp$ through each $p \in \Sigma$. ■

**Proposition 16** *If $\rho$ is a $\Sigma$-Adapted field, there exists a synchronized field $\xi = e^\varphi \rho$ (with the same integral lines as $\rho$).*

**Proof.** We take $\rho$ as a radical field in a simple open set $M_0$, with $\rho^\perp = \{\Sigma_t : \sigma = t\}$, with flow $\rho_s(p) = \alpha_p(s)$ we can take $M_0$ small enough so that $\Phi : \Sigma_0 \times I_\varepsilon \to M_0, (p,s) \mapsto \alpha_p(s)$ is a diffeomorphism (see Theorem 10), and for each $p \in \Sigma_0$, let $\gamma_p = \gamma_p(t) = \alpha_p(s_p(t)) \in \Sigma_t$ where $s = s(p,t) = s_p(t)$ is the reparameterization of $\alpha_p = \alpha_p(s)$ such that $\gamma_p(t) \in \Sigma_t$.

Taking $\varepsilon > 0$ and reducing $M_0$, we can assume the map $\Psi : \Sigma_0 \times I_\varepsilon \to M_0, (p,t) \mapsto \gamma_p(t)$ is a diffeomorphism verifying $(\sigma \circ \Psi)|_{(p,t)} = t$ As the curves $\gamma_p$ are reparameterizations of $\alpha_p$ (integral curves of $\rho$), the field $\xi$ that has $\gamma_p(t)$ as integral curves has the same integral lines as $\rho$Thus they are proportional ($\xi = e^\varphi \rho$) and $\xi^\perp = \rho^\perp$, making $\xi$ the desired radical field. ■

**Corollary 17** *A $\Sigma$-Adapted field is necessarily a radical field.*

**Proof.** Let $\rho$ be a $\Sigma$-Adapted field. Around $\Sigma$ we can define $\rho^\perp$ as $\{\Sigma_t : \sigma = t\}$, and by Proposition 16, take $\xi = e^\varphi \rho$ synchronized such that $\xi_t(p) \in \Sigma_t$. Then by Proposition 15, necessarily $\xi = G_\sigma$ which is radical. Therefore $\rho = e^{-\varphi}\xi$ is a radical field. ■

**Proposition 18** *A radical geodesic field is necessarily a $\Sigma$-Adapted field.*

**Proof.** Given the radical geodesic field $\rho$, we will use its flow /as in Theorem 10) to obtain a chart $(x_i, x_m)$ in a simple open set $M_0$ with $\rho = \partial/\partial x_m$ and $\Sigma :_s x_m = 0$. Let be

$$(g_{ab}) = \begin{pmatrix} g_{ij} & g_{im} \\ g_{im} & g_{mm} \end{pmatrix}$$

the metric matrix in these coordinates.

first we work in $M_0^+ : x_m > 0$ and reparameterize each $\gamma_p$ by arc length. We seek a parameter change $t = t(x_i(p), \widetilde{t})$ such that if $\widetilde{\gamma_p}(\widetilde{t}) = \gamma_p(t(x_i(p), \widetilde{t}))$, then:

$$g(\gamma_p'(t), \gamma_p'(t)) = g\left(\frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_m}\right)_{x_m = t} = g_{mm}(\gamma_p(t))$$

We have:

$$1 = \left|\frac{d\widetilde{\gamma_p}}{d\widetilde{t}}\right| = \sqrt{g_{mm}(\gamma_p(t))}\frac{dt}{d\widetilde{t}} \Rightarrow \widetilde{t} = \int_0^t \sqrt{g_{mm}(\gamma_p(t))}dt$$

We then, take coordinate:

$$\left\{ \begin{array}{l} \widetilde{x}_i = x_i \\ \widetilde{x}_m = \int_0^t \sqrt{g_{mm}\left(x_i, x_m\right)} dx_m \end{array} \right. \tag{5}$$

In these coordinates, the matrix of $g$ is $\begin{pmatrix} \widetilde{g}_{ij} & \widetilde{g}_{im} \\ \widetilde{g}_{im} & 1 \end{pmatrix}$ Now the curves $\widetilde{\gamma}_p$ are geodesics. Then $\widetilde{g}_{im} = 0$ because:

$$\frac{\partial \widetilde{g}_{im}}{\partial \widetilde{x}^m} = \frac{\partial}{\partial \widetilde{x}_i} \widetilde{g}\left(\frac{\partial}{\partial \widetilde{x}_i}, \frac{\partial}{\partial \widetilde{x}_m}\right) = g\left(\widetilde{\nabla}_{\frac{\partial}{\partial \widetilde{x}_m}} \frac{\partial}{\partial \widetilde{x}_i}, \frac{\partial}{\partial \widetilde{x}_m}\right) = g\left(\widetilde{\nabla}_{\frac{\partial}{\partial \widetilde{x}_i}} \frac{\partial}{\partial \widetilde{x}_m}, \frac{\partial}{\partial \widetilde{x}_m}\right)$$
$$= \frac{1}{2} \frac{\partial}{\partial \widetilde{x}_m} g\left(\frac{\partial}{\partial \widetilde{x}_m}, \frac{\partial}{\partial \widetilde{x}_m}\right) = \frac{1}{2} \frac{\partial}{\partial \widetilde{x}_m} (1) = 0$$

Thus, the function $\widetilde{x}_m \mapsto \widetilde{g}_{im}\left(x_i, x_m\right)$ is constant. Since the coordinate change 5 is continuous at $x_m \geq 0$ and $\widetilde{g}_{im}(\widetilde{x}_i, 0) = 0$, we conclude $\widetilde{g}_{im} = 0$.

A similar reasoning applies to the zone $x_m < 0$. Denote $\gamma_{p_0}(t) = \rho_t(p_0)$ for $p_0 \in \Sigma_0$, we define $\sigma : M_0 \to \mathbb{R}$ as the length application:

$$\sigma(p) = \begin{cases} \text{len}(\gamma_{p_0}|_{[0, x_m(p)]}) & \text{if } x_m\left(p\right) \geq 0 \\ -\text{len}(\gamma_{p_0}|_{[0, x_m(p)]}) & \text{if } x_m\left(p\right) < 0 \end{cases}$$

This is a continuous map where $\Sigma_0 : \sigma = 0$. The level surfaces $\Sigma_s : \sigma = s$ form a continuous distribution of hyperplanes that coincide for $s \neq 0$ with $\rho^{\perp}$). Consequently, it is concluded that $\Sigma_0 \in \rho^{\perp}$, $\rho$ is $\Sigma$-Adapted. ■

**Corollary 19** *If $\rho$ is a $\Sigma$-Adapted field, then $\langle \rho, \rho \rangle^{[\alpha]} = 0$ is a simple equation for $\Sigma$.*

**Proof.** By Proposition 16 we construct synchronized $\xi = e^{\varphi} \rho$, and by Proposition 15 there exists $\sigma$ such that $\xi = G_{\sigma}$. Using Proposition 14, we construct special coordinates $(x_i, x_m)$ where $\xi = \partial / \partial x_m$. By $\alpha$-transversality, $\det(g_{ab})^{[\alpha]} = 0$ is a simple equation for $\Sigma$. Since $\det(g_{ij})$ is positive, $\det(g_{ab})^{[\alpha]} = \langle \xi, \xi \rangle^{[\alpha]} \det(g_{ij})^{\alpha}$Thus $\langle \xi, \xi \rangle^{[\alpha]} = 0$ is a simple equation for $\Sigma$, and so is $\langle \rho, \rho \rangle^{[\alpha]} = 0$. ■

**Theorem 20 (main)** *If $(x_i, x_m)$ are normal coordinates, then $\partial / \partial x_m$ is a $\Sigma$-Adapted geodesic field Conversely, if $\rho$ is a radical geodesic field, then around each singular point there exist normal coordinates $(x_i, x_m)$ with $\rho = \hbar \partial / \partial x_m$, where $\hbar$ is never null..*

**Proof.** The first statement is evident. For the converse, we first observe that in a normal chart the curves $\gamma_p$ defined by:

$$\gamma_p : \left\{ \begin{array}{l} x = x_i\left(p\right) \\ x_m = s \end{array} \right.$$

for $p \in \Sigma$, satisfy

$$\left\langle \frac{d\gamma_p}{ds}, \frac{d\gamma_p}{ds} \right\rangle \Big|_s = \left\langle \frac{\partial}{\partial x_m}, \frac{\partial}{\partial x_m} \right\rangle \Big|_{x_m = s} = s^{[1/\alpha]}$$

because of this, they are said to be $\alpha$-parameterized.

Given the radical geodesic field $\rho$, we will use its flow (see Theorem 10) to obtain a chart $(x_i, x_m)$ in a simple open set $M_0$ with $\rho = \partial/\partial x_m$ and $\Sigma :_s x_m = 0$. Let be

$$(g_{ab}) = \begin{pmatrix} g_{ij} & g_{im} \\ g_{im} & g_{mm} \end{pmatrix}$$

the metric matrix in these coordinates. By Proposition 18 $\rho$ is $\Sigma$-Adapted, and by corollary 19 $\langle \rho, \rho \rangle^{[\alpha]} = 0$ is a simple equation for $\Sigma$.

We are going to $\alpha$-parameterize the curves $\gamma_p = \rho_t(p)$, which are defined by $\gamma_p : x = x_i(p)$, $x_m = t$.

If $\gamma(t)$ is an integral curve of $\rho$, with $\gamma(0) \in \Sigma$, then since $\Sigma :_s \langle \rho, \rho \rangle^{[\alpha]} = 0$, the function $\varphi = \langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \rangle^{[\alpha]}$ defines a simple equation $\varphi(t) = 0$ of $\{0\} \subset \mathbb{R}$ Thus $\varphi(t) = t\psi(t)^{[2\alpha]}$ for a differentiable function $\psi$ with $\psi(0) > 0$.

$$\varphi(t) = t\psi(t)^{[2\alpha]} \Rightarrow \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle = \left[ t^{\frac{1}{2\alpha}} \psi(t) \right]^2$$

Baldomero's Theorem allows us to $\alpha$-parameterize $\gamma(t)$. We seek $t = t(s)$ such that if $\overline{\gamma}(s) = \gamma(t(s))$ we have

$$\begin{aligned} s^{[1/\alpha]} &= \left\langle \frac{d\overline{\gamma}}{ds}, \frac{d\overline{\gamma}}{ds} \right\rangle = \left( \frac{dt}{ds} \right)^2 \left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt} \right\rangle \\ &= \left( \frac{dt}{ds} \right)^2 \left[ t^{\left[\frac{1}{2\alpha}\right]} \psi(t) \right]^2 \end{aligned}$$

therefore

$$s^{\left[\frac{1}{2\alpha}\right]} = \frac{dt}{ds} t^{\left[\frac{1}{2\alpha}\right]} \psi(t) \Rightarrow \frac{s^{1+\frac{1}{2\alpha}}}{\frac{1}{2\alpha}+1} = \epsilon \int_0^t \tau^{\left[\frac{1}{2\alpha}\right]} \psi(\tau)\, d\tau$$

$$s(t) = \left( \left( 1 + \frac{1}{2\alpha} \right) \int_0^t (\epsilon\tau)^{\left[\frac{1}{2\alpha}\right]} \psi(\tau)\, d\tau \right)^{\left[1+\frac{1}{2\alpha}\right]}$$

which it is smooth by Baldomero´s theorem for $r = \frac{1}{2\alpha}$.

In general taking $\gamma = \gamma_p : \begin{cases} x_i = x_i(p) \\ x_m = t \end{cases}$ $x_m(p) = 0$, we make the functions $\psi(x_i, t)$ such that

$$\left\langle \frac{d\gamma_p}{dt}, \frac{d\gamma_p}{dt} \right\rangle = \left[ t^{\left[\frac{1}{2\alpha}\right]} \psi(x_i(p), t) \right]^2$$

and the smooth function

$$s\left(x_i,t\right)=\left(\left(1+\frac{1}{2\alpha}\right)\int_0^t\left(\epsilon\tau\right)^{\left[\frac{1}{2\alpha}\right]}\psi\left(x_i,\tau\right)d\tau\right)^{\left[1+\frac{1}{2\alpha}\right]}$$

defines the reparametrization $s=s\left(x_i\left(p\right),t\right)$ that $\alpha$-parameterizes $\gamma_p$, that is, if $\overline{\gamma}_p=\overline{\gamma}_p\left(s\right)$ is the reparametrized one such that

$$s^{[1/\alpha]}=\left\langle\frac{d\overline{\gamma}_p}{ds},\frac{d\overline{\gamma}_p}{ds}\right\rangle$$

and we can write $\overline{\gamma}_p=\overline{\gamma}_p\left(s\right):\left\{\begin{array}{l}\overline{x}_i=\overline{x}_i\left(p\right)\\ \overline{x}_m=s\end{array}\right.$ in the coordinates given by

$$\left\{\begin{array}{l}\overline{x}_i=x_i\\ \overline{x}_m=\left(\left(1+\frac{1}{2\alpha}\right)\int_0^{x_m}\left(\epsilon\tau\right)^{\left[\frac{1}{2\alpha}\right]}\psi\left(x_i,\tau\right)d\tau\right)^{\left[1+\frac{1}{2\alpha}\right]}\end{array}\right.$$

At these coordinates $\left(\overline{x}_i,\overline{x}_m\right)$ the metric matrix is

$$\left(\overline{g}_{ab}\right)=\left(\begin{array}{cc}\overline{g}_{ij}&\overline{g}_{im}\\ \overline{g}_{im}&\overline{x}_m^{[1/\alpha]}\end{array}\right)$$

To prove that $\overline{g}_{im}$ is zero , we will proceed as in (5). We define the continuous change of coordinates, $t=t\left(s\right)$ differentiable at $x_m>0$ that makes geodesics $\widetilde{\gamma}_p=\overline{\gamma}_p\left(t\left(s\right)\right)$ to the curves $\overline{\gamma}_p$. We have for $\alpha\neq 1/2$

$$1=\left\langle\frac{d\widetilde{\gamma}_p}{ds},\frac{d\widetilde{\gamma}_p}{ds}\right\rangle=\left(\frac{ds}{dt}\right)^2t^{[1/\alpha]}\Rightarrow$$
$$ds=t^{[-1/(2\alpha)]}dt\Rightarrow s=\epsilon\left(t\right)\left|t\right|^{(2\alpha-1)/2\alpha}$$

This means that in the coordinates

$$\left\{\begin{array}{l}\widetilde{x}_i=\overline{x}_i\\ \widetilde{x}_m=\epsilon\left(\overline{x}_m\right)\left|\overline{x}_m\right|^{(2\alpha-1)/2\alpha}\end{array}\right.$$

the metric matrix is

$$\left(\widetilde{g}_{ab}\right)=\left(\begin{array}{cc}\widetilde{g}_{ij}&0\\ 0&1\end{array}\right)$$

But now it turns out that

$$\frac{\partial}{\partial\overline{x}_i}=\frac{\partial}{\partial\widetilde{x}_i},\frac{\partial}{\partial\overline{x}_m}=\frac{\partial\widetilde{x}_m}{\partial\overline{x}_m}\frac{\partial}{\partial\widetilde{x}_m}=\overline{x}_m^{[-1/2\alpha]}\frac{\partial}{\partial\widetilde{x}_m}$$

therefore

$$\overline{g}_{im}=\left\langle\frac{\partial}{\partial\overline{x}_i},\frac{\partial}{\partial\overline{x}_m}\right\rangle=\overline{x}_m^{[-1/2\alpha]}\left\langle\frac{\partial}{\partial\widetilde{x}_i},\frac{\partial}{\partial\widetilde{x}_m}\right\rangle=0$$

For $\alpha = 1/2$ the change of coordinates would be

$$\begin{cases} \widetilde{x}_i = \overline{x}_i \\ \widetilde{x}_m = \epsilon\left(\overline{x}_m\right) \ln\left|\overline{x}_m\right| \end{cases}$$

which unfortunately is not a continuous change of coordinates, but nevertheless at $x_m \neq 0$ it still happens that

$$\frac{\partial}{\partial \overline{x}_i} = \frac{\partial}{\partial \widetilde{x}_i}, \frac{\partial}{\partial \overline{x}_m} = \frac{\partial \widetilde{x}_m}{\partial \overline{x}_m}\frac{\partial}{\partial \widetilde{x}_m} = \frac{\epsilon\left(\overline{x}_m\right)}{\overline{x}_m}\frac{\partial}{\partial \widetilde{x}_m}$$

therefore

$$\overline{g}_{im} = \left\langle \frac{\partial}{\partial \overline{x}_i}, \frac{\partial}{\partial \overline{x}_m} \right\rangle = \frac{\epsilon\left(\overline{x}_m\right)}{\overline{x}_m}\left\langle \frac{\partial}{\partial \widetilde{x}_i}, \frac{\partial}{\partial \widetilde{x}_m} \right\rangle = 0$$

and by continuity is also $\overline{g}_{im}\left(x_i, 0\right) = 0$. ■

# 4 Conclussion.

We have constructed a one-parameter family, called the $\Sigma^\alpha$-spaces, which are models of metric manifolds $(M, g)$ such that they change the signature from Lorentz to Riemann, when traversing a hypersurface $\Sigma$ called a singular. These spaces are of geometric interest in themselves but they also have applications in cosmology.

Our $\Sigma^\alpha$-spaces spaces $(\alpha > 0)$ are characterized by inducing a Riemannian metric on $\Sigma$, and that around each point $p \in \Sigma$, there exists a local expression of $g = \Sigma g_{ab} dx_a dx_b$ such that the function $sign\left(\det\left(g_{ab}\right)\right)\left|\det\left(g_{ab}\right)\right|^\alpha$ extends differently, and their differential does not vanish at $p$. When $\alpha = 1$ and $g$ is smooth, we obtain the type-change model proposed by Kossowski in the foundational paper [9] where it is shown that these metrics admit a local representation of the form $g = \Sigma g_{ij} dx_i dx_j - x_m dx_m^2$ around each singular point. But it is used without proof, that for $r = 1/2$ the function $F\left(\lambda, t\right) = sgn\left(t\right)\left(\int_0^t |x|^r \psi\left(\lambda, x\right) dx\right)^{\frac{1}{r+1}}$ is differentiable and this is a far from trivial fact. We have proven it for all $r > -1$, and this has motivated the definition of the family of $\Sigma^\alpha$-spaces, and has allowed us to prove that in the presence of a geodesic radical line distribution, there is a local expression $g = \Sigma g_{ij} dx_i dx_j - sign\left(x_m\right)\left|x_m\right|^{1/\alpha} dx_m^2$ around each singular point.

# 5 Discussion.

1. The metric of a $\Sigma^\alpha$-space $(M, g)$ does not have to be differentiable on all of $M$. For example, on$M = \mathbb{R}^2$ with coordinates $(x, y)$ and metric matrix

$$\left(g_{ab}\right) = \begin{pmatrix} 1 & y^{[1/2]} \\ y^{[1/2]} & 2y \end{pmatrix}$$

$\det(g_{ab}) = y$ and it is a $\Sigma^1$-space, but obviously the differentiability of the metric fails at each point of the singular line $\Sigma : y = 0$. Therefore, this $\Sigma^1$–space is not of the Kossowski type [9]. In fact, it cannot admit a distribution of radical geodesic lines, because if it did, by our main theorem 20 there would be a coordinate system $(x_1, x_2)$ where the metric have a matrix as $\begin{pmatrix} 1 & 0 \\ 0 & x_2 \end{pmatrix}$, and would be differentiable at the points of $\Sigma : x_2 = 0$. This proves that not every $\Sigma^\alpha$-espacio admits a distribution of radical geodesic lines.

2. The metric $g$ on $\mathbb{R}^m = \{(x_i, x_m)\}$ with matrix

$$(g_{ab}) = \begin{pmatrix} g_{ij} & 0 \\ 0 & \hbar x_m^{[1/\alpha]} \end{pmatrix}, \quad \begin{array}{l} \hbar \text{ smooth} \\ \text{and nonnull.} \end{array} \tag{6}$$

define a $\Sigma^\alpha$-space, and the field $\rho = \partial/\partial x_m$ turns out to be a $\Sigma$-radical field. It can be shown that conversely, if $(M, g)$ is a $\Sigma^\alpha$-space that admits a $\Sigma$-radical field, it also admits local representations of the metric as in (6) around $\Sigma$. We ask whether, in this type of $\Sigma^\alpha$-spaces, the existence of a distribution of radical geodesic lines around the singular hypersurface can be guaranteed, and we might ask whether this distribution is essentially unique.

3. There is a natural definition of $\Sigma^\alpha$-space for $\alpha < 0$, $\alpha \neq -1/2$; it consists of Lorentz or Riemann metrics $g$ defined on the components of the complement $M \backslash \Sigma$ of the hypersurface $\Sigma$ such that their dual metric $g^*$ extends continuously over all of $M$ and satisfies a property of $\alpha$-transversality on every point of $\Sigma$. Note that in this case the metric $g$ is not defined on $\Sigma$ now its points become poles of the metric and $\Sigma$ is called polar hypersurface of $(M, g)$. This is, in a sense, a dual situation to the case $\alpha > 0$; however, interpreting and dualizing the results is far from trivial.

   See [15] for a study of this topic for the case $\alpha = -1$.

# 6 APPENDIX: Baldomero's Theorem.

Recall the definition of the sign function:

$$\epsilon(t) = \begin{cases} +1 \text{ if } t > 0 \\ 0 \text{ if } t = 0 \\ -1 \text{ if } t < 0 \end{cases}$$

Fixed a real number $\alpha$ with $r \neq -1$, it is verified that

$$\int_0^t x^r dx = \frac{t^{r+1}}{r+1}$$

and therefore the function $F : ]0, \infty) \to \mathbb{R}$ defined by

$$F(t) = \left( \int_0^t x^r dx \right)^{\frac{1}{r+1}} = \frac{t}{r+1}$$

is $C^\infty$ differentiable on $[0, \infty)$ by taking $F(0) = 0$.

Furthermore, for $t < 0$, we have

$$\int_0^t (-x)^r dx = \left[ -\frac{(-x)^{r+1}}{r+1} \right]_0^t$$
$$= -\frac{(-t)^{r+1}}{r+1}$$

hence for $t < 0$

$$\left( \epsilon(t) \int_0^t |x|^r dx \right)^{\frac{1}{r+1}} = \frac{-t}{r+1}$$

so the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(t) = \epsilon(t) \left( \epsilon(t) \int_0^t |x|^r dx \right)^{\frac{1}{r+1}} = \frac{t}{r+1}$$

is $C^\infty$ differentiable. Inspired by this, we conjecture the validity of the following statement:

**Theorem 21 (Baldomero)** *Assume that $\psi = \psi(\lambda, t) : \Lambda \times \mathcal{I} \to \mathbb{R}$ is a $C^q$ function ($q = 1, 2, ...\infty$) defined on an open interval $\mathcal{I}$ of $\mathbb{R}$, with $0 \in \mathcal{I}$ and $\psi(0) > 0$. Fixed a real number $r$ with $r \neq -1$, the function $F : \mathcal{I} \to \mathbb{R}$ defined by*

$$F : t \mapsto F(\lambda, t) = \epsilon(t) \left| \int_0^t |x|^r \psi(\lambda, x) \, dx \right|^{\frac{1}{r+1}} \tag{7}$$

*is a $C^q$ differentiable function, and if $q \geq 1$, then $F'(0) > 0$.*

## 6.1 Preliminaries.

The proof requires some preparations.

### 6.1.1 Notations.

If $\Omega$ is an open set in $\mathbb{R}^n$, $C^q(\Omega)$ denotes the set of functions $f : \Omega \to \mathbb{R}$ that are differentiable of class $C^q$, $q = 0, 1, 2, ..., \infty$. We adopt the following simplifications for functions $f : \Lambda \times \mathcal{I} \to \mathbb{R}$:

$$f \in C^q \Leftrightarrow \exists \varepsilon \text{ such that } f \in C^q(\Lambda \times I_\varepsilon) \tag{8}$$

$$f \in C_*^q \Leftrightarrow \exists \varepsilon \text{ such that } f \in C^q(\Lambda \times I_\varepsilon^*)$$

$$f \in C_*^\infty \Leftrightarrow f \in C_*^q, \forall q \geq 0 \tag{9}$$

**Definition 22** *We say that $f \in \mathcal{C}^\omega$ if $f \in C_*^\infty$ and if there exists $\overline{f} \in C^0$ with $\overline{f}(\lambda, t) = f(\lambda, t)$ if $t \neq 0$.*

$$\textit{We agree to denote } f = \overline{f} \in C^0 \tag{10}$$

**Remark 23** *Using convention (10) and when $f \in C_*^\infty$, we can write*

$$f \in C^q \Leftrightarrow f \in C^\omega \cap C^q$$

*For $f \in C_*^\infty$ we shall write*
*$\lim f$ instead of $\lim_{t\to 0} f$*
*and we denote $f^{(q)} = \frac{\partial^q f}{\partial t^q} \in C_*^\infty$*

Furthermore, we denote $I = I(\lambda, t) = t$ as the identity map and $J = J(\lambda, t) = |t|$ as the absolute value map. Thus, $\epsilon \in C_*^\infty$ is the sign function:

$$\epsilon(\lambda, t) = \begin{cases} 1 \text{ if } t > 0 \\ -1 \text{ if } t < 0 \end{cases}$$

it then follows that

$$J = \epsilon I \in C_*^\infty$$

Observe that

$$(J^q)' = \epsilon q J^{q-1} \tag{11}$$

If $f \in C_*^\infty$, we denote $\int f \in C_*^\infty$ as the function:

$$\int f : t \longmapsto \int_0^t f(x)\, dx$$

Note that by the Fundamental Theorem of Calculus (Barrow's rule):

$$\left(\int f\right)' = f$$

With these notations, function 7 can be written as

$$F = \left(\epsilon \int \psi J^r\right)^{\frac{1}{r+1}} \tag{12}$$

Finally, to avoid overloading the calculations, we define the following relation in $C_*^\infty$:

If $f, g \in C_*^\infty$, we write $f \cong g$ if and only if the following condition holds:

$$f \cong g \Leftrightarrow \exists \varphi \in C^\infty \text{ and } \exists a \neq 0 \text{ such that } g = af + \varphi \tag{13}$$

If $f, g \in C_*^\infty$ we write $f \simeq g$ if $f$ and $g$ are functionally related such that the following equivalence holds:

$$f \in \mathcal{C}^q \Leftrightarrow g \in \mathcal{C}^q, \forall q = 0, 1, 2, ..., \infty$$

It is easy to see that both are equivalence relations, and for our purposes, the relevant fact is that:

$$f \cong g \Rightarrow f \simeq g$$

However, the converse is not true; for example, if $\varphi \in C^\infty$, the functions $f$ and $g = (f + \varphi)^3$ satisfy $f \simeq g$ and yet $f \ncong g$. Note that if $g = \varphi f$ with $\varphi \in C^\infty$ and if $\varphi(0) \neq 0$, then $f \cong g$. The following result is essential:

**Lemma 24** *If $f : I_\varepsilon \to \mathbb{R}$ is a continuous function, of class $C^1$ in $I_\varepsilon^* = ]-\varepsilon, \varepsilon[ \setminus \{0\}$ and there exists $\ell = \lim_{t\to 0} f'(t)$, then*

$$f(t) - f(0) = t \int_0^1 f'(st)\, ds$$

*and in particular $f$ is of class $C^1$ in $I_\varepsilon$, and $f'(0) = \ell$.*

**Proof.** For a fixed $t$, let $f_t = f_t(s) = f(st)$. Then:

$$f(t) - f(0) = [f_t]_{s=0}^{s=1} = \int_0^1 \frac{df_t}{ds} ds = t \int_0^1 f'(st)\, ds$$

therefore:

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = \lim_{t\to 0} \int_0^1 f'(st)\, ds =$$
$$= \int_0^1 \lim_{t\to 0} f'(st)\, ds = \int_0^1 \ell ds = \ell$$

■

### 6.1.2 Calculation Rules.

**Rule C1**

$$\text{If } m > 1, \text{ then } J^m \in C^1 \text{ and } (J^m)' = \epsilon m J^{m-1} \tag{14}$$

**Rule H.** L'Hopital rule states:

If $f, g \in C^1(I_\varepsilon^*)$ satisfy $g'(t) \neq 0 \ \forall t \neq 0$, and $\lim_{t\to 0} f(t) = \lim_{t\to 0} g(t) = 0$, then:

$$\exists \lim_{t\to 0} \frac{f'(t)}{g'(t)} = \ell \Rightarrow \lim_{t\to 0} \frac{f(t)}{g(t)} = \ell$$

Then for $\varphi \in \mathcal{C}^0, m > 1$:

$$\lim_{t\to 0} \frac{\epsilon \int \varphi J^n}{J^m} = \lim_{t\to 0} \frac{\epsilon \varphi J^n}{\epsilon m J^{m-1}} = \lim_{t\to 0} \varphi J^{n-m+1} = \begin{cases} 0, \text{ if } n > m-1 \\ \frac{1}{m}\varphi(\lambda, 0) \ \text{ if } n = m-1 \end{cases} \tag{15}$$

and for $\varphi \in C^\infty$, and $n \geq m-1$:

$$\varphi_{n/m} = \frac{\epsilon \int \varphi J^n}{J^m} \in \mathcal{C}^\omega \text{ and } \varphi_{n/m}(\lambda, 0) = \begin{cases} 0, \text{ if } n > m-1 \\ \frac{1}{m}\varphi(\lambda, 0) \ \text{ if } n = m-1 \end{cases} \tag{16}$$

moreover, it holds that: $f \in C^\omega$, $f(0) = 0 \Rightarrow \epsilon f \in C^\omega$, in particular:

$$\epsilon \varphi_{n/m} = \frac{\int \varphi J^n}{J^m} \in C^\omega \text{ if } n > m-1 \tag{17}$$

**Proof.** It is trivial using observation 15 to verify that for a fixed $\lambda = \lambda_0$:

$$\exists \lim_{t\to 0} \varphi_{n/m}(\lambda_0, t) = \begin{cases} 0, \text{ if } n > m-1 \\ \frac{1}{m}\varphi(\lambda_0, 0) \ \text{ if } n = m-1 \end{cases}$$

but we need to prove that:

$$\exists \lim_{(\lambda,t)\to(\lambda_0,0)} \varphi_{n/m}(\lambda, t) = \begin{cases} 0, \text{ if } n > m-1 \\ \frac{1}{m}\varphi(\lambda_0, 0) \ \text{ if } n = m-1 \end{cases}$$

Let us fix a small neighborhood $I_\delta \times \Lambda_\delta$ of $(\lambda_0, 0)$ and let:

$$M = \sup_{(\lambda,t)\in I_\delta \times \Lambda_\delta} |\varphi(\lambda, t)|, \ N = \sup_{(\lambda,t)\in I_\delta \times \Lambda} \left| \left. \frac{\partial \varphi}{\partial \lambda} \right|_{(\lambda,t)} \right|$$

When $n > m-1$, we have:

$$\left|\varphi_{n/m}(\lambda, t)\right| = \left| \frac{\epsilon \int \varphi J^n}{J^m} \right| \leq \frac{M \int J^n}{J^m} \underset{(\lambda,t)\to(\lambda_0,0)}{\rightarrow} 0$$

If $n = m-1$, then using the mean value theorem, for each $(\lambda, t) \in I_\delta \times \Lambda_\delta$:

$$\varphi(\lambda, t) - \varphi(\lambda_0, t) = (\lambda - \lambda_0) \left. \frac{\partial \varphi}{\partial \lambda} \right|_{(\lambda_1,t)} \text{ for some } (\lambda_1, t) \in I_\delta \times \Lambda_\delta$$

therefore:

$$|\varphi(\lambda, t) - \varphi(\lambda_0, t)| \leq |\lambda - \lambda_0| N$$

and thus:

$$\left|\varphi_{n/m}(\lambda,t)-\frac{1}{m}\varphi(\lambda,0)\right|$$
$$\leq\left|\varphi_{n/m}(\lambda,t)-\varphi_{n/m}(\lambda_0,t)\right|+\left|\varphi_{n/m}(\lambda_0,t)-\frac{1}{m}\varphi(\lambda_0,0)\right|$$
$$=\left|\int(\varphi(\lambda,t)-\varphi(\lambda_0,t))J^{m-1}\right|+\left|\varphi_{n/m}(\lambda_0,t)-\frac{1}{m}\varphi(\lambda_0,0)\right|$$
$$\leq\int|\varphi(\lambda,t)-\varphi(\lambda_0,t)|+\left|\varphi_{n/m}(\lambda_0,t)-\frac{1}{m}\varphi(\lambda_0,0)\right|\underset{(\lambda,t)\to(\lambda_0,0)}{\to}0$$

■

**Rule P.** Using integration by parts:

$$\left.\begin{array}{c}\varphi\in\mathcal{C}^1\\ n+1>0\end{array}\right\}\Rightarrow\int\varphi J^n=\frac{\epsilon}{n+1}\left\{\varphi J^{n+1}-\int\varphi' J^{n+1}\right\}\tag{18}$$

## 6.2 Proof of Theorem 21

We can write $F$ in the form:

$$F=\epsilon\left(\epsilon\int\psi J^r\right)^{\beta},\ \beta=\frac{1}{r+1}$$

| |
|---|
| Since all variables $(\lambda_1,\ldots,\lambda_n)\in\Lambda$ play the same role, we will take $n=1$ that is, we take $\lambda=\lambda_1$. |

We will freely use the notations introduced in Section 6.1.1. Clearly $F\in C^0$ since $\lim_{t\to0}F=\epsilon\lim_{t\to0}\left(\epsilon\int\psi J^r\right)^{\beta}=0$, and $F(\lambda,0)=0$.

To show that $F\in C^1$, we first prove that $F:t\mapsto F(\lambda,t)$ satisfies the hypotheses of Lemma 24. We have:

$$F'=(r+1)^{-\frac{1}{r+1}}(\psi_1+F_1)^{-\frac{r}{r+1}}\ \text{with}\ \left\{\begin{array}{l}\psi_1=\psi^{-1/r}\in C^\infty\\ F_1=-\psi^{-\frac{r+1}{r}}J^{-r-1}\int\psi' J^{(r+1)}\end{array}\right.\tag{19}$$

We note that $\lim F_1=0$ since, using rule H (15):$\lim J^{-r-1}\int\psi' J^{(r+1)}=0$ and since $\psi_1>0$, we have:

$$\lim F'=(r+1)^{-\frac{1}{r+1}}\psi_1(\lambda,0)^{-\frac{1}{r}}$$

by Lemma 24, $F'(\lambda,t)$ exists in $\Lambda\times I_\varepsilon$ and $F'(\lambda,0)=(r+1)^{-\frac{1}{r+1}}\psi(\lambda,0)^{\frac{1}{r+1}}>0$

To proceed, it is essential to remark that $F'\simeq F_1$ since by (19), if $F_1\in\mathcal{C}^k$, then because $\psi_1\in C^\infty$, it follows that $F'\in C^k$; and if $F'\in C^k$, then $F_1\in C^k$ because $F_1=(r+1)^{-\frac{1}{r}}F'^{-\frac{r+1}{r}}-\psi_1$

Therefore:

$$\begin{array}{l} F' \simeq F_1 = B_1 \int \psi' J^{(r+1)} \in C^{\omega} \\ B_1 = \psi_{11} J^{-r-1},\ \psi_{11} = -\psi^{-\frac{r+1}{r}} \in C^{\infty} \end{array} \tag{20}$$

and using rule H (15):$\lim F_1 = \lim \left(\psi_{11} J^{-r-1}\right) = 0$ and by rule H (17), $F' \simeq F_1 \in C^{\omega}$.

$$\boxed{\begin{array}{l}\text{Notation: Henceforth, functions} \\ \text{denoted by } \psi_i,\ \psi_{ij},\ \varphi_i,\ \varphi_{ij} \text{ are in } C^{\infty}.\end{array}} \tag{21}$$

Furthermore:

$$\frac{\partial F_1}{\partial \lambda} = \frac{\partial \psi_{11}}{\partial \lambda} J^{-r-1} \int \psi' J^{(r+1)} + \psi_{11} J^{-r-1} \int \frac{\partial \psi'}{\partial \lambda} J^{r+1}$$

It can then be proved using an inductive argument that for all $q \geq 1$: $\frac{\partial^q F_1}{\partial \lambda^q} = \sum \psi_i J^{-r-1} \int \varphi_i J^{r+1}$ and then by the generalized rule H (16), $\frac{\partial F_1}{\partial \lambda} \in C^{\omega}$. By Lemma 24, it is concluded that:$\exists \left.\frac{\partial^q F_1}{\partial \lambda^q}\right|_{(\lambda,t)} \forall (\lambda, t)$, and $\frac{\partial^q F_1}{\partial \lambda^q} \in C^{\omega}$ thus, since $F' \simeq F_1$, we have proved that $F \in C^1$ and $\exists \left.\frac{\partial^q F'}{\partial \lambda^q}\right|_{(\lambda,t)} \forall (\lambda, t)$ and $\forall q$

Let's see that $F \in C^2$. Indeed:$F_1' = B_1' \int \psi' J^{(r+1)} + B_1 \psi' J^{r+1}$ but the second term is $C^{\infty}$ since $B_1 \psi' J^{(r+1)} = \psi_{11} \psi'$, so it follows:$F_1' \cong B_1' \int \psi' J^{r+1}$ thus, taking $\psi_{21} = \psi_{11}'$ and $\psi_{22} = -r - 1/\psi_{11}$ we get:$B_1' = \psi_{21} J^{-r-1} + \epsilon \psi_{22} J^{-r-2}$ and by rule (18) $\int \psi' J^{r+1} = \frac{\epsilon}{r+2} \left(\psi' J^{r+2} - \int \psi'' J^{r+2}\right)$ we have:

$$\begin{aligned} F_1' &\cong \frac{1}{r+2} \left\{ \epsilon B_1' \psi' J^{r+2} - \epsilon B_1' \int \psi'' J^{r+2} \right\} \\ &\cong \epsilon B_1' \int \psi'' J^{r+2} \end{aligned}$$

since: $\epsilon B_1' \psi' J^{(r+2)} = \left(\epsilon \psi_{11} J^{-r-1} + \psi_{12} J^{-r-2}\right) J^{r+2} = \psi_{11} (\epsilon J) + \psi_{12} = \psi_{11} + \psi_{12} \in C^{\infty}$

We thus have:

$$\begin{array}{c} F_1' \cong F_2 = B_2 \int \psi'' J^{r+2} \\ B_2 = \epsilon B_1' = \epsilon \psi_{21} J^{-r-1} + \psi_{22} J^{-r-2},\ \psi_{ij} \in C^{\infty} \end{array} \tag{22}$$

But now it can be proved, using the rule (16) (17) and Lemma 24 repeatedly, that $F_1' \in C^{\omega}$, and for each $q = 1, 2.., \exists \frac{\partial^q F_2}{\partial \lambda^q} \in C^{\omega}$

And since $F' \simeq F_1 \cong F_2$ and $F_1' \in C^{\omega}$, it is concluded that $F'' \in C^{\omega}$, and for $k = 1, 2..., \exists \frac{\partial^k F'}{\partial \lambda^k}$ Thus $F \in C^2$.

We will prove the following Lemma:

**Lemma 25** *Let* $B_1 = \psi_{11} J^{-r-1}$, $\psi_{11} = -\psi^{-\frac{r+1}{r}}$, *and* $F_0 = F$. *For each* $k \geq 1$, *the functions:*

$$B_{k+1} = \epsilon B_k',\ F_k = B_k \int \psi^{(k)} J^{r+k} \tag{23}$$

*satisfy:*

$$F'_{k-1} \cong F_k \in C^{\omega} \tag{24}$$

$$\exists \left.\frac{\partial^q F_k}{\partial \lambda^q}\right|_{(\lambda,t)} \forall (\lambda, t)\,, \;\; \textit{and} \;\; \frac{\partial^q F_k}{\partial \lambda^q} \in C^{\omega}, \forall q$$

#### 6.2.1 End of the Proof of Theorem 21

Assuming Lemma 25 is proven and knowing that each $F_k$ admits all partial derivatives with respect to $\lambda$, the following implications are valid:

$$\begin{array}{l} F' \simeq F_1 \\ F'_1 \cong F_2 \in C^{\omega} \Rightarrow F' \in C^1 \\ F'_2 \cong F_3 \in C^{\omega} \Rightarrow F' \in C^2 \\ \text{etc...} \end{array}$$

and we arrive at:

$$\exists \frac{\partial^{q+s} F}{\partial \lambda^q \partial t^s} \in C^{\omega} \text{ if } s \geq 1 \tag{25}$$

but the problem now is to prove that:

$$\exists \frac{\partial^q F}{\partial \lambda^q} \in C^{\omega}$$

Indeed, substituting in Lemma 24 the function $f(t)$ by the function $f^{\lambda}(t) = F(t, \lambda)$ for a fixed $\lambda \in \Lambda$, and taking into account that $F(0, \lambda) = 0, \forall \lambda$, we have:

$$F(t, \lambda) = t \int_0^1 \frac{\partial F}{\partial t}(st, \lambda)\, ds$$

using now (25), it is seen that:

$$\frac{\partial^q F}{\partial \lambda^q} = t \int_0^1 \frac{\partial F}{\partial \lambda^q \partial t}(st, \lambda)$$

which belongs to $C^{\omega}$.

#### 6.2.2 Proof of Lemma 25

**Proof.** Note first that the $B_k$ for $k \geq 1$are of the form (for certain $\psi_{ij} \in C^{\infty}$):

$$B_k = \epsilon B'_{k-1} = \sum_{\ell=1}^{k} \epsilon^{k-\ell} \psi_{k,\ell} J^{-r-\ell}$$

Observe first that the $F_k = B_k \int \psi^{(k)} J^{r+k}$ thus constructed satisfy $F_k \in C^{\omega}$ since using the rule H (17) we have:

$$\epsilon^{k-\ell} \psi_{k,\ell} J^{-r-\ell} \int \psi^{(k)} J^{r+k} \in C^{\omega} \text{ for } 1 \leq \ell \leq k$$

furthermore, as $(r+\ell) < (r+k)+1$ for $1 \leq \ell \leq k$, by (17):

$$\lim F_k = \lim \sum_{\ell=1}^{k} \epsilon^{k-\ell} \psi_{k,\ell} J^{-r-\ell} \int \psi^{(k)} J^{r+k} = 0$$

For $k = 1$, the lemma was already proven in (20).

The proof of the lemma for $k \geq 2$ is done by induction on $k$. We have proven the Lemma for $k = 2$ in (22), i.e., $F_1' \cong F_2$.

Assuming the induction hypothesis (24) $F_{k-1}' \cong F_k \in C^\omega$ and (23) $F_k = B_k \int \psi^{(k)} J^{r+k}$ for some $k > 2$, we have:

$$F_k' = B_k' \int \psi^{(k)} J^{r+k} + B_k \psi^{(k)} J^{r+k}$$

but the second term is $C^\infty$ since:

$$\begin{aligned} B_k \psi^{(k)} J^{r+k} &= \left( \sum_{\ell=1}^{k} \epsilon^{k-\ell} \psi_{k,\ell} J^{-r-\ell} \right) \psi^{(k)} J^{r+k} \\ &= \sum_{\ell=1}^{k} \psi_{k,\ell} \psi^{(k)} \left( \epsilon^{k-\ell} J^{k-\ell} \right) \in C^\infty \text{ since } \epsilon J = I \end{aligned}$$

so $F_k = B_k \int \psi^{(k)} J^{(r+k)}$, and $F_k' \cong B_k' \int \psi^{(k)} J^{\frac{2k+1}{2}}$ applying now rule P (18), we have:

$$\int \psi^{(k)} J^{k+r} = \frac{\epsilon}{k+r+1} \left\{ \psi^{(k)} J^{k+r+1} - \int \psi^{(k+1)} J^{k+r+1} \right\}$$

therefore:

$$\begin{aligned} F_k' &= \frac{1}{k+r+1} \left( \epsilon B_k' \right) \left\{ \psi^{(k)} J^{k+r+1} - \int \psi^{(k+1)} J^{k+r+1} \right\} \\ &\cong B_{k+1} \int \psi^{(k+1)} J^{k+r+1} \end{aligned}$$

since $\epsilon B_k' \psi^{(k)} J^{k+r+1} \in C^\infty$, because:

$$\begin{aligned} \epsilon B_k' \psi^{(k)} J^{k+\ell+1} &= B_{k+1} \psi^{(k)} J^{k+r+1} \\ &= \left( \sum_{\ell=1}^{k+1} \epsilon^{k-\ell+1} \psi_{k+1,\ell} J^{-r-\ell} \right) \psi^{(k)} J^{k+r+1} \\ &= \sum_{\ell=1}^{k+1} \psi_{k+1,\ell} \psi^{(k)} \left( \epsilon J \right)^{k-\ell+1} \in C^\infty \end{aligned}$$

where we have used $\epsilon J = I$.

Thus the conditions (24) and (23) of the induction hypothesis hold for $k+1$:

$$F_k' \cong F_{k+1} = B_{k+1} \int \psi^{(k+1)} J^{k+r+1} \in C^\omega \text{ with } B_{k+1} = \epsilon B_k'$$

■

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