

RICCI CURVATURE AND METRIC IN CAUSAL SPACETIMES.

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ABSTRACT. A viable spacetime is one that admits a complete timelike geodesic. It is shown that a causal diffeomorphism preserving the Ricci tensor between two spacetimes is necessarily a homothety, if one of them is viable.

1. INTRODUCTION.

A metric $g \in \mathcal{C}$ of a causal spacetime (M, \mathcal{C}) determines an energy tensor

$$(1.1) \quad T = \frac{1}{4\pi G} \left(Ric - \frac{1}{2} Scg \right)$$

where $Ric = Ric_g$ is the Ricci tensor and $Sc = Sc_g$ is the scalar curvature tensor of g . We will then say that T is in the causal structure \mathcal{C} . It is well known that in general an (energy) tensor T does not automatically determine the existence of the metric g that satisfies equation (1.1), and in the event that it exists, uniqueness is not guaranteed unless certain initial conditions are imposed. We now ask whether the field equations (1.1) given by the energy tensor T in the causal structure \mathcal{C} , have a unique solution (up to homotheties) within their conformal class. The main result of this article is to give an affirmative answer to this question, when the energy tensor comes from a viable spacetime, that is, with an inertial observer of infinite life according to their proper time.

There is a theorem by Hawking and Penrose which ensures that if certain physically reasonable conditions are imposed on the energy tensor T , the metrics g that are solutions of (1.1) necessarily contain an inertial observer of finite life ([12][5][6]). It would be wonderful to be able to ensure that there is also another with infinite life capable of bypassing all possible singularities. For example, this happens in the Schwarzschild spacetime, which is therefore viable. Our result therefore ensures that the Schwarzschild metric is the only solution (up to homothety) of equation (1.1) with $T = 0$, within its causal structure.

From another point of view, we have also proved that a causal diffeomorphism preserving the Ricci tensor between two spacetimes is necessarily a homothety, if we assume that one of them is viable. Note that the preservation of the Ricci tensor by a conformal diffeomorphism between two Ricci-flat spaces is automatic, and in particular a conformal diffeomorphism in Schwarzschild space is necessarily a homothety. There is abundant bibliography regarding all this, in the context of general semi-Riemannian manifolds. Kulkarni in [8][7] raises the problem of determining conditions for the curvature to determine the metric in the Riemannian case. In [14] De Turk investigates whether the Ricci tensor uniquely determines (except homothety) the Riemannian structure, and proves that it is true for certain Einstein metrics. On the other hand, Xingwang [16], analyzes this uniqueness within a conformal class, and proves that, for connected, compact and oriented

Riemannian manifold of dimension ≥ 2 . Finally Kühnel [9] proves this same theorem when (M, g) is geodesically complete. We have also proved this same result here using a different method (see Remark 2 on page 7).

A semi-Riemannian manifold (M, g) has been fixed. This means that, M is a real connected differentiable manifold with finite dimension $m > 2$ and g is a metric. When g is a Lorentz metric we will say that (M, g) is a spacetime. Normally the signature is $(-, +, \dots, +)$, but we will make no distinction between a metric g and another homothetic one cg with $c \in \mathbb{R} \setminus \{0\}$, since both define the same geometry (and in particular the same connection) with different units of measurement. Thus we will avoid the tag 'up to homotheties' when we claim uniqueness.

1.1. General assumptions. We denote by $\nabla : (X, Y) \mapsto \nabla_X Y$ the Levi-Civita connection of g , by $\langle X, Y \rangle$ to $g(X, Y)$, and grad is its gradient operator. All connections will be assumed symmetric.

We adopt the following definition of the curvature tensor R :

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_{[X, Y]} Z \quad \text{for } X, Y, Z \in \mathfrak{X}(M)$$

and the Ricci tensor is defined by:

$$\text{Ric}(X, Y) = \text{Trace}(\mathfrak{X}(M) \ni V \longrightarrow R(X, V)Y) \quad \text{for } X, Y \in \mathfrak{X}(M)$$

If $(X, Y) \rightarrow T(X, Y)$ is a covariant tensor we denote by $X \mapsto \tilde{T}(X)$ the unique $(1, 1)$ tensor such that

$$\langle \tilde{T}(X), Y \rangle = T(X, Y)$$

2. CONFORMAL STRUCTURE.

The metrics g, \bar{g} of M are said to be conformally equivalent if there exists a differentiable function $\sigma : M \rightarrow \mathbb{R}$, with $\bar{g} = e^{2\sigma}g$. This is an equivalence relation, and we will denote by $\mathcal{C} = [g]$ the equivalence class defined by the metric g . \mathcal{C} is called a conformal structure on M .

Note that two Lorentzian metrics on M define the same causal structure if and only if they define the same light cones in each tangent space or equivalently define the same causality relation on M . For this reason we also call (M, \mathcal{C}) a causal space and we will assume it is time-oriented.

Definition 1. *The causal space (M, \mathcal{C}) is said to be viable, if there exists a metric $g \in \mathcal{C}$, such that (M, g) is viable, in the sense that it has a timelike geodesic $\gamma = \gamma(t)$ defined $\forall t > 0$.*

2.1. Conformal connections. A connection $\bar{\nabla}$ on (M, \mathcal{C}) , is said to be conformal if it preserves the conformal structure under parallel transport. That is, if parallel transport induces conformal (linear) transformations between the tangent spaces. This happens in the Lorentz case, if and only if $\bar{\nabla}$ preserves the light cones under parallel transport, and we have the following result:

Theorem 1 ([8]). *The necessary and sufficient condition for a connection $\bar{\nabla}$ on M to be conformal is that there exists a vector field $A \in \mathfrak{X}(M)$ such that:*

$$(2.1) \quad \bar{\nabla}_X Y - \nabla_X Y = \langle A, X \rangle Y + \langle A, Y \rangle X - \langle X, Y \rangle A \quad \text{for all } X, Y \in \mathfrak{X}(M)$$

In particular, $A = (\text{grad})_g \sigma$, if and only if ∇ and $\bar{\nabla}$ are the Levi-Civita connections associated with g and $\bar{g} = e^{2\sigma}g$ respectively.

Remark 1. The condition $A = \text{grad}\sigma$ is equivalent to $\alpha = d\sigma$, where α is the 1-form g -metrically equivalent to A (i.e., $\alpha : \mathfrak{X}(M) \ni X \rightarrow \langle A, X \rangle \in \mathbb{R}$). In particular the condition $d\alpha = 0$, means that $\bar{\nabla}$ is locally metric, that is: In a neighborhood U of each point of M , there exists a differentiable $\sigma : U \rightarrow \mathbb{R}$ such that $\bar{\nabla}$ is the Levi-Civita connection of $e^{2\sigma}g$.

Corollary 1. The null geodesic lines of two conformal connections necessarily coincide.

2.2. Comparison of Ricci tensors.

Notation 1. In what follows, the symbols ∇ , $\bar{\nabla}$ and A have the same meaning as in theorem 1. The reference metric is g with Ricci tensor Ric , while $\bar{\text{Ric}}$ will be the corresponding tensor of $\bar{\nabla}$. If α is the unique 1-form such that $\tilde{\alpha} = A$ (that is $\alpha(X) = \langle A, X \rangle$), we define $Q = \nabla\alpha - \alpha \otimes \alpha$ therefore

$$(2.2) \quad \tilde{Q}(X) = \nabla_X A - \langle A, X \rangle A$$

and we then have the following result (see [8]):

Theorem 2. Assuming $\bar{\text{Ric}}$ is symmetric, the difference tensor $E = \bar{\text{Ric}} - \text{Ric}$ satisfies the equation

$$(2.3) \quad -\frac{1}{m-2}E + \frac{1}{2(m-2)(m-1)}\text{tr}(\tilde{E})g = Q + \frac{1}{2}\langle A, A \rangle g$$

Proof. The difference tensor

$$E(X, Z) = \bar{\text{Ric}}(X, Z) - \text{Ric}(X, Z) = \text{tr}\{Y \rightarrow D(X, Y)Z\}$$

is given by the formula

$$E(X, Z) = (2-m)Q(X, Z) - \{q + (m-1)\langle A, A \rangle\}\langle X, Z \rangle$$

where $q = \text{tr}(\tilde{Q})$ that is to say

$$(2.4) \quad E = (2-m)Q - \{q + (m-1)\langle A, A \rangle\}g$$

Using (2.4), the tensor $\tilde{E}(X) = \widetilde{\bar{\text{Ric}}}(X) - \widetilde{\text{Ric}}(X)$ has the following expression:

$$\tilde{E}(X) = (2-m)\tilde{Q}(X) - \{q + (m-1)\langle A, A \rangle\}X$$

contracting \tilde{E} yields

$$(2.5) \quad \begin{aligned} \text{tr}(\tilde{E}) &= (2-m)q - \{q + (m-1)\langle A, A \rangle\}m \\ &= (2-2m)q - (m-1)m\langle A, A \rangle \end{aligned}$$

solving for q in (2.5)

$$(2.6) \quad q = \frac{1}{2-2m}\text{tr}(\tilde{E}) - \frac{m}{2}\langle A, A \rangle$$

and solving for Q in (2.4)

$$(2.7) \quad Q = \frac{1}{2-m}E + \frac{1}{2-m}\{q + (m-1)\langle A, A \rangle\}g$$

finally substituting (2.6) into (2.7) yields (2.3) □

With notation 1, we have the following result

Proposition 1. *The condition $\overline{\text{Ric}} = \text{Ric}$ is equivalent to saying that the field A satisfies*

$$(2.8) \quad \nabla_X A = \langle A, X \rangle A - \frac{1}{2} \langle A, A \rangle X, \quad \forall X \in \mathfrak{X}(M)$$

Proof. If $E = \overline{\text{Ric}} - \text{Ric} = 0$, by (2.3) it follows that $Q + \frac{1}{2} \langle A, A \rangle g = 0$, that is

$$\tilde{Q} + \frac{1}{2} \langle A, A \rangle \tilde{g} = 0$$

since $\tilde{g} = id$, by (2.2) we have

$$\underbrace{\nabla_X A - \langle A, X \rangle A}_{\tilde{Q}(X)} + \frac{1}{2} \langle A, A \rangle X = 0$$

and thus A satisfies (2.8). Conversely, if A satisfies (2.8) then $\tilde{Q}(X) = -\frac{1}{2} \langle A, A \rangle X$ and therefore $Q + \frac{1}{2} \langle A, A \rangle g = 0$. By (2.3) it is seen that

$$(2.9) \quad -\frac{1}{m-2} E + \frac{1}{2(m-2)(m-1)} \text{tr}(\tilde{E}) g = 0$$

but then

$$(2.10) \quad E = \frac{1}{2(m-1)} \text{Trace}(\tilde{E}) g$$

and therefore since $\tilde{g} = Id$

$$\begin{aligned} \tilde{E} &= \frac{1}{2(m-1)} \text{Trace}(\tilde{E}) Id \Rightarrow \\ \text{Trace}(\tilde{E}) &= \frac{m}{2(m-1)} \text{Trace}(\tilde{E}) \\ &\Rightarrow 0 = -\frac{m-2}{2(m-1)} \text{Trace}(\tilde{E}) \end{aligned}$$

If there exists p such that $\text{Trace}(\tilde{E})|_p \neq 0$, then $m = 2$, but $m \geq 3$, by hypothesis. By (2.9) it is $E = 0$. \square

2.3. Atypical fields. Motivated by proposition 1 we establish the following definition

Definition 2. *A field $A \in \mathfrak{X}(M)$ will be called atypical, briefly ATP, if it satisfies:*

$$(2.11) \quad \nabla_X A = \langle A, X \rangle A - \frac{1}{2} \langle A, A \rangle X, \quad \forall X \in \mathfrak{X}(M)$$

Note that the identically zero field A is atypical. On the other hand, if (X_1, \dots, X_m) is a parallelization then A is atypical if and only if (2.11) is satisfied for each $X = X_i$.

Now, in the context of Notation 1 Theorem 1 the condition $\overline{\text{Ric}} = \text{Ric}$ is equivalent to the field A being atypical. Atypical fields are generally difficult to find, we show an example:

Example 1. with $M = \{(x, y) \in \mathbb{R}^2 : y > x > 0\} \setminus$ and the Minkowski metric $g = -dx^2 + dy^2$, which in hyperbolic polar coordinates (ρ, θ) $x = \rho \sinh \theta$, $y = \rho \cosh \theta$ is written $g = d\rho^2 - \rho^2 d\theta^2$, we observe that there is a conformal metric

$$\bar{g} = \frac{1}{\rho^4} g$$

with $\text{Ric} = \overline{\text{Ric}} = 0$, as a consequence of the conformal character of hyperbolic inversions. Indeed

The inversion ι of the unit hyperboloid is written in polar coordinates

$$(\bar{\rho}, \bar{\theta}) \xleftarrow{\iota} (\rho, \theta) : \begin{cases} \rho = 1/\bar{\rho} \\ \theta = \bar{\theta} \end{cases}, \begin{cases} d\rho^2 = -(1/\bar{\rho}^4) d\bar{\rho}^2 \\ d\theta^2 = d\bar{\theta}^2 \end{cases}$$

and

$$\begin{aligned} \iota^* g &= (1/\bar{\rho}^4) d\bar{\rho}^2 - (1/\bar{\rho}^2) d\bar{\theta}^2 \\ &= \frac{1}{\bar{\rho}^4} (d\bar{\rho}^2 - \bar{\rho}^2 d\bar{\theta}^2) \end{aligned}$$

so indeed ι is conformal. From another point of view, in reality we have proved that $\frac{1}{\bar{\rho}^4} (d\bar{\rho}^2 + \bar{\rho}^2 d\bar{\theta}^2)$ is the expression of the metric g in the coordinates $(\bar{\rho} = 1/\rho, \bar{\theta} = \theta)$ so formally \bar{g} has (like g) zero Ricci tensor. But

$$\frac{1}{\rho^4} = e^{2\sigma}, \text{ with } \sigma = -2 \ln \rho$$

therefore

$$\alpha = d\sigma = \frac{-2}{\rho} d\rho \Rightarrow \tilde{\alpha} = A = \frac{-2}{\rho} \frac{\partial}{\partial \rho}$$

although the exposed theory is not consistent for dimension $m = 2$, it can be verified that A is the atypical field and that $\text{div} A = 0$. If we define $\phi(x_0, \dots, x_m) = -x_0^2 + x_1^2 + \dots + x_m^2$ it can be proved that there is a diffeomorphism Φ of

$$M = \{x = (x_0, \dots, x_m) : \phi(x) < 0\}$$

in $\mathbb{R}^+ \times \mathbb{H} = \{(\rho, \theta) : \rho > 0, \theta \in \mathbb{H}\}$ where \mathbb{H} is the hyperboloid of equation

$$\mathbb{H} : \phi(x) = -1$$

which has the equations

$$\Phi : \begin{cases} \rho = \sqrt{-\phi(x)} \\ \theta = x/\sqrt{-\phi(x)} \end{cases}$$

If we endow M with the Minkowski metric $\eta = -dx_0^2 + dx_1^2 + \dots + dx_m^2$ and \mathbb{H} with the Riemannian metric it can be proved that

$$\Phi : (M, \eta) \rightarrow (\mathbb{R}^+ \times \mathbb{H}, h = d\rho^2 - \rho^2 d\theta^2)$$

is an isometry. Note that h is Lorentzian but with signature $(+, - \dots -)$. Therefore $(\mathbb{R}^+ \times \mathbb{H}, h)$ is Ricci-flat, and the map $\phi : \mathbb{R}^+ \times \mathbb{H} \rightarrow \mathbb{R}^+ \times \mathbb{H}$, $(\rho, \theta) \mapsto (\bar{\rho} = 1/\rho, \bar{\theta} = \theta)$ is conformal with factor $\frac{1}{\rho^4}$ that is

$$\phi^* h = \frac{1}{\rho^4} h$$

and since $\frac{1}{\rho^4} = e^{2\sigma}$, with $\sigma = -2 \ln \rho$, $\alpha = d\sigma = \frac{-2}{\rho} d\rho$ it is concluded that the field

$$\tilde{\alpha} = A = \frac{-2}{\rho} \frac{\partial}{\partial \rho}$$

is atypical.

On the other hand these atypical fields show certain global properties that will have their consequences in the following section.

Proposition 2. *Let A be an atypical field. If there exists $p \in M$ with $\langle A, A \rangle(p) \neq 0$, then it is $\langle A, A \rangle(x) \neq 0$ for all $x \in M$.*

Proof. Indeed, let $x \in M$, and $\gamma : (a, b) \rightarrow M$ a differentiable curve (which can be assumed an integral curve of a field X) with $a < 0 < 1 < b$ and $\gamma(0) = p$, $\gamma(1) = x$. Calling $f(t) = \langle A, A \rangle \circ \gamma(t)$ and $F(t) = \langle A(\gamma(t)), \gamma'(t) \rangle$ we verify:

$$\begin{aligned} X(\langle A, A \rangle)(\gamma(t)) &= f'(t) = \langle A, X \rangle(\gamma(t)) \langle A, A \rangle(\gamma(t)) \\ &= F(t)f(t) \quad \forall t \in (a, b), \text{ and } f(0) \neq 0 \end{aligned}$$

therefore the hypotheses of the following lemma are satisfied, and it is concluded then that $F(1) = \langle A, A \rangle(\gamma(1)) = \langle A, A \rangle(x) \neq 0$. \square

Lemma 1. *Let $f : (a, b) \rightarrow \mathbb{R}$ be a differentiable function not identically zero. Suppose there exists a continuous $F : (a, b) \rightarrow \mathbb{R}$ such that $f'(t) = F(t)f(t) \forall t \in (a, b)$. Then f is never zero (that is $f(t) \neq 0 \forall t \in (a, b)$).*

Proof. If the function f is not identically zero there exists $t_0 \in (a, b)$ with $f(t_0) \neq 0$ and satisfies the differential equation

$$\begin{aligned} \frac{dy}{dt} = yF(t) &\Rightarrow \int \frac{dy}{y} = \int F \\ \Rightarrow y = Ce^{\varphi(t)} &\text{ with } \begin{cases} \varphi' = F \\ C \in \mathbb{R} \end{cases} \end{aligned}$$

since the function f is not identically zero it is $C \neq 0$, so f is never zero. \square

Proposition 3. *If $\text{Ric} = \overline{\text{Ric}}$, then $\overline{\nabla}$ is locally metric.*

Proof. If $\langle A, A \rangle$ is identically zero, then since A is atypical:

$$\nabla_X A = \langle A, X \rangle A - \frac{1}{2} \langle A, A \rangle X = \langle A, X \rangle A, \quad \forall X \in \mathfrak{X}(M)$$

Thus for all $X, Y \in \mathfrak{X}(M)$ if $\alpha(X) = \langle A, X \rangle$ it is verified:

$$\begin{aligned} X(\alpha(Y)) &= X(\langle A, Y \rangle) = \langle \nabla_X A, Y \rangle + \langle A, \nabla_X Y \rangle \\ &= \langle A, X \rangle \langle A, Y \rangle + \langle A, \nabla_X Y \rangle \end{aligned}$$

Therefore $d\alpha = 0$, (and by 0.3.2 $\overline{\nabla}$ is locally metric) since:

$$X(\alpha(Y)) - Y(\alpha(X)) = \langle A, \nabla_X Y - \nabla_Y X \rangle = \langle A, [X, Y] \rangle$$

If $\langle A, A \rangle|_p \neq 0$ for some p then by proposition 2 $\langle A, A \rangle$ is never zero. Suppose now for example $\langle A, A \rangle > 0$ on M . We have:

$$\begin{aligned} X(\langle A, A \rangle) &= 2\langle \nabla_X A, A \rangle = 2(\langle A, X \rangle \langle A, A \rangle - \frac{1}{2} \langle A, A \rangle \langle X, A \rangle) \\ &= \alpha(X) \langle A, A \rangle \quad \forall X \in \mathfrak{X}(M) \end{aligned}$$

Thus $X(\log(\langle A, A \rangle)) = \alpha(X)$, so $\alpha = d(\log(\langle A, A \rangle))$. Analogously if $\langle A, A \rangle < 0$ on M it is $\alpha = d(\log(-\langle A, A \rangle))$, and the theorem is true on the non-empty open submanifold $M' = \{x \in M : \langle A, A \rangle(x) \neq 0\}$. \square

We have therefore proved the following corollary

Corollary 2. *If A is atypical and for some $p \in M$ it is $\langle A, A \rangle(p) \neq 0$, then it is $\langle A, A \rangle(x) \neq 0$ for all $x \in M$. and if $\sigma = \log(|\langle A, A \rangle|)$ it is concluded that $\bar{\nabla}$ is the Levi-Civita connection for the metric $\bar{g} = e^{2\sigma}g \in \mathcal{C}$. In particular $A = \text{grad}\sigma$.*

3. MAIN RESULTS.

3.1. Assumptions. Recall: ∇ is the Levi-Civita connection of the metric g of M , $\mathcal{C} = [g]$. $\bar{\nabla}$ is a \mathcal{C} -conformal symmetric connection, with the same Ricci tensor $\bar{\text{Ric}} = \text{Ric}$ as g , and $A \in \mathfrak{X}(M)$ is the atypical field of 1 that relates ∇ and $\bar{\nabla}$.

Since M is connected, from proposition 2 it follows that the following possibilities is verified:

$$\begin{aligned} \langle A, A \rangle(x) &> 0 \text{ for all } x \in M. & \langle A, A \rangle(x) < 0 \text{ for all } x \in M. \\ \langle A, A \rangle(x) &= 0 \text{ for all } x \in M. \end{aligned}$$

Proposition 4. *Suppose $\bar{\text{Ric}} = \text{Ric}$. Then A defines a field of ∇ -pregeodesics. Furthermore, if $\langle A, A \rangle \neq 0$ on M , all integral curves of A define incomplete ∇ -geodesics. Consequently: If A is spacelike (respectively timelike) then there exist spacelike (respectively, timelike) ∇ -geodesics that are incomplete.*

Proof. There exists a field $U \in \mathfrak{X}(M)$ and a differentiable $f : M \rightarrow \mathbb{R}^+$, such that $\langle U, U \rangle = \epsilon$ and $A = fU$, where $\epsilon = 1$, if A is spacelike, and $\epsilon = -1$, if A is timelike. On the other hand, since $\nabla_A A = \frac{1}{2}\langle A, A \rangle A$ it is concluded that A is a field of ∇ -pregeodesics, therefore the integral curves of U are ∇ -geodesics. But

$$\frac{\epsilon}{2}f^3U = \frac{1}{2}\langle A, A \rangle A = \nabla_A A = \nabla_{fU}(fU) = f\{U(f)U + f\nabla_U U\} = fU(f)U$$

dividing both members by f^3 we obtain: $\frac{U(f)}{f^2} = \frac{\epsilon}{2}$. Under these conditions, all (∇ -geodesics) integral curves of U are incomplete. Indeed, suppose there exists an integral curve $\gamma : \mathbb{R} \rightarrow M$ of U . The function $\varphi = f \circ \gamma$ then satisfies the differential equation: $\frac{\varphi'}{\varphi^2} = \frac{\epsilon}{2}$ with solution:

$$f(\gamma(t)) = \frac{2\epsilon f(\gamma(0))}{2 - \epsilon f(\gamma(0))t}$$

in particular it is concluded that the function f would not be differentially defined in a neighborhood of $\gamma(\frac{2}{\epsilon f(\gamma(0))})$. \square

Remark 2 (Theorem). *In the Riemannian case all geodesics are spacelike, and proposition 4 guarantees that if (M, g) is complete then it does not admit atypical fields, and therefore the Ricci tensor determines the metric. This result has also been proved in [9], which generalizes [16].*

Lemma 2. *If $\bar{\text{Ric}} = \text{Ric}$, and $\langle A, A \rangle = 0$ on M . Then, if A is not identically zero, then A is nowhere zero and (M, g) is timelike and spacelike incomplete, and furthermore all timelike ∇ -geodesics are incomplete.*

Proof. Since $\langle A, A \rangle = 0$ it is $\nabla_A A = 0$, and A defines a field of null ∇ -geodesics. Let γ be a maximal ∇ -geodesic, $\langle \gamma', \gamma' \rangle = \epsilon = \pm 1$, and let $E_0 = \gamma', E_1, \dots, E_n$ be a parallel orthonormal basis along γ with $\epsilon_i = \langle E_i, E_i \rangle = \pm 1$. It is then verified that:

$$A(\gamma(t)) = \sum a_i(t) \epsilon_i E_i(t) \quad \text{with } a_i = \langle A, E_i \rangle$$

Since A is atypical, and $\langle A, A \rangle = 0$ it is $\nabla_X A = \langle A, X \rangle A$ and we have:

$$\frac{\nabla A}{dt} = \sum \epsilon_i \frac{da_i}{dt} E_i = \langle A, \gamma' \rangle A = a_0 \sum a_i \epsilon_i E_i$$

in particular, $\frac{da_0}{dt} = a_0^2$. If $\langle A(\gamma(0)), \gamma'(0) \rangle = \epsilon a_0(0) \neq 0$, taking $\alpha = a_0(0) \neq 0$, it is concluded $a_0(t) = \frac{\epsilon \alpha}{-\epsilon \alpha t + 1}$, γ cannot be complete, since $a_0(t)$ would become infinite for $t = \frac{1}{\epsilon \alpha}$. If $\langle A(\gamma(0)), \gamma'(0) \rangle = \epsilon a_0(0) = 0$, it is concluded that $\epsilon a_0(t) = \langle A, \gamma' \rangle(t) = 0$, and since A is atypical and $\langle A, A \rangle = 0$, $\frac{\nabla A}{dt} = \langle A, \gamma' \rangle A = 0$. Thus A is parallel along γ . This proves that if for a point $p \in M$, it is $A(p) = 0$, then $A \circ \gamma$ is identically zero for all non-null geodesics γ starting from p . Therefore the open set $N = \{x \in M : A(x) \neq 0\}$, is also closed, and since it is non-empty, it coincides with the complete manifold M . In the Lorentz case, any maximal timelike ∇ -geodesic γ verifies $\langle A(\gamma(0)), \gamma'(0) \rangle \neq 0$, and is therefore incomplete. \square

Theorem 3 (Main.). *If (M, g) is a viable spacetime (Definition 1) and $\bar{\nabla}$ is a symmetric connection with the same Ricci tensor as g that preserves the light cones by parallel transport, then $\bar{\nabla}$ is the metric connection of g .*

Proof. If ∇ is the metric connection of g , since $\bar{\text{Ric}} = \text{Ric}$, by proposition 1, there exists A atypic vectorfield verifying (2.11)

If $\langle A, A \rangle = 0$, and there exists $p \in M$ with $A(p) \neq 0$, by Lemma 2 A does not vanish at any point, all timelike geodesics are incomplete which contradicts the hypothesis. Suppose then that there exists $p \in M$ with $\langle A, A \rangle(p) \neq 0$. By 2 we have $\langle A, A \rangle \neq 0$ in all of M . Let $\gamma : \mathbb{R} \rightarrow M$ be the timelike geodesic which by hypothesis is complete, and let $(E_0 = \gamma', E_1, \dots, E_n)$ be a parallel orthonormal basis along γ where $A = \sum a_i E_i$ with $a_i : \mathbb{R} \rightarrow \mathbb{R}$ differentiable. Note that by Lemma 1.4 A defines a field of incomplete pregeodesics, so $A \circ \gamma$ is never tangent to γ , that is, the function:

$$f(t) = \frac{1}{2}(a_1(t)^2 + \dots + a_n(t)^2)$$

verifies $f(t) > 0$ for all $t \in \mathbb{R}$. Since $\frac{\nabla A}{dt} = \sum \frac{da_i}{dt} E_i = \langle A, \gamma' \rangle A - \frac{1}{2} \langle A, A \rangle \gamma' = -a_0[\sum a_i E_i] - \frac{1}{2}(-a_0^2 + a_1^2 + \dots + a_n^2)E_0$, and comparing the coefficients multiplying E_0 we get:

$$\frac{da_0}{dt} = -\frac{1}{2}a_0^2 - f(t)$$

Calling $y(t) = -a_0(t)$, we conclude that $y(t)$ is defined on all of \mathbb{R} and satisfies the differential equation:

$$(3.1) \quad \frac{dy}{dt} = \frac{1}{2}y^2 + f(t)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is differentiable, with $f(t) > 0$.

We can choose the direction of traversal of γ , such that $a_0(0) = -\langle A, E_0 \rangle|_{t=0} > 0$

This contradicts the following Lemma. \square

Lemma 3. *Any solution $y = y(t)$ of the differential equation:*

$$\frac{dy}{dt} = \frac{1}{2}y^2 + f, \quad \text{where } f : \mathbb{R} \rightarrow \mathbb{R} \text{ is differentiable, with } f(t) > 0$$

with $y(0) > 0$, cannot be defined on all of \mathbb{R} .

Proof. Suppose there exists $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ a solution of the differential equation defined on all of \mathbb{R} that satisfies $\beta = \psi(0) > 0$. Let

$$\varphi(t) = \frac{2\beta}{-\beta t + 2}$$

which is the solution to $\frac{dy}{dt} = \frac{1}{2}y^2$ such that $\varphi(0) = \beta$. Consider the interval $I =]0, b = \frac{2}{\beta}[$. Note that $\varphi > 0$ on I and $\lim_{t \rightarrow b} \varphi(t) = +\infty$. Let us prove that $\varphi \leq \psi$ on I , which prevents the existence of $\psi(b)$. For this it is sufficient to prove that the set $J = \{s \in I : \varphi(t) \leq \psi(t) \text{ for } 0 < t < s\}$ is non-empty, as well as open and closed in I :

Since $\varphi(0) = \psi(0) = \beta$, $\varphi'(0) = \frac{1}{2}\beta^2 < \frac{1}{2}\beta^2 + f(0) = \psi'(0)$, it is concluded by (*) that for a certain $\epsilon > 0$, and all t with $0 < t < \epsilon$, we have $\varphi(t) < \psi(t)$ and $(0, \epsilon) \subset J$, and $J \neq \emptyset$. On the other hand, if $s_1 \in I - J$ there exists t_1 with $\tau < t_1 < s_1$ and $\varphi(t_1) > \psi(t_1)$, and this remains true for all values $s \in (s_1 - \delta, s_1 + \delta)$ with $\delta < s_1 - t_1$, thus $I - J$ is closed. Finally, let us prove that J is open:

Indeed, if $s \in J$ we have $0 < \varphi(s) \leq \psi(s)$. Then:

$$\varphi'(s) = \frac{1}{2}\varphi(s)^2 \leq \frac{1}{2}\psi(s)^2 < \frac{1}{2}\psi(s)^2 + f(s) = \psi'(s)$$

by (*) there exists ϵ such that $\varphi(t) < \psi(t)$ for $s < t < s + \epsilon$, thus $(s - \epsilon, s + \epsilon) \subset J$. (*) Let $f = f(t)$, $g = g(t)$ be real differentiable functions defined around $\tau \in \mathbb{R}$, with $f(\tau) = g(\tau)$. Suppose that $f'(\tau) < g'(\tau)$. There then exists an $\epsilon > 0$ such that $f(t) \leq g(t)$ for all t with $\tau < t < \tau + \epsilon$. \square

Theorem 1.8 admits other equivalent statements, which we include here as corollaries.

Corollary 3. *A conformal diffeomorphism between spacetimes that preserves the Ricci tensor is necessarily a homothety, if at least one of them is viable.*

Corollary 4. *In a viable spacetime, the Ricci tensor determines the metric up to a multiplicative constant.*

4. CONCLUSIONS.

It is a strange phenomenon that two distinct conformal non homothetic metrics g and $\bar{g} = e^{2\sigma}g$, share the same Ricci tensor, as this is equivalent to the existence of an atypical field $A = \text{grad}_g \sigma$ verifying the exotic property $\nabla_X A = \langle A, X \rangle A - \frac{1}{2} \langle A, A \rangle X$ (2.11) which relates elements that depend on the first-order derivatives of the metric with others that depend only on the point value of the metric. In general the search for atypical fields turns out to be a formidable problem.

But at a theoretical level, if the existence of such a field is admitted, (that is $\text{Ric}_g = \text{Ric}_{\bar{g}}$) then we prove that it A is a pregeodesic field with incomplete integral curves in the Riemannian case (Proposition 4), and this implies that the space should be geodesically incomplete. Therefore, a complete Riemannian space does not admit atypical fields (see note 2)

In the Lorentz case, it is proven that an atypical field A maintains the same causal character at every point. In each of the cases (timelike, lightlike or spacelike) it is proven that if A were to coexist with a timelike geodesic $\gamma = \gamma(t)$ defined for all t , the function $\langle A, \gamma' \rangle$ would be a solution to a certain ordinary differential equation that does not admit a solution for all t (Lemma 2 and Theorem 3). Thus all its timelike geodesics are incomplete and the spacetime could not be viable. The *viable* condition has a global character.

The analysis of the existence of local atypical fields is in general an inaccessible problem. Nevertheless, there are local obstructions to their existence. For example, it is a matter of simple calculation to verify that if A is an atypical field then $R(X, Y)A = 0$ for all X, Y and from here it follows that $Ric(X, A) = 0$ for all X , therefore the Ricci tensor is degenerate. So if Ric_p is non-degenerate for a certain point $p \in M$, then an atypical field cannot exist around p .

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