CHARACTERIZATION OF A BANACH-FINSLER MANIFOLD IN TERMS OF THE ALGEBRAS OF SMOOTH FUNCTIONS

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Abstract. In this note we give sufficient conditions to ensure that the weak Finsler structure of a complete C^k Finsler manifold M is determined by the normed algebra $C_b^k(M)$ of all real-valued, bounded and C^k smooth functions with bounded derivative defined on M. As a consequence, we obtain: (i) the Finsler structure of a finite-dimensional and complete C^k Finsler manifold M is determined by the algebra $C_b^k(M)$; (ii) the weak Finsler structure of a separable and complete C^k Finsler manifold M modeled on a Banach space with a Lipschitz and C^k smooth bump function is determined by the algebra $C_b^k(M)$; (iii) the weak Finsler structure of a C^1 uniformly bumpable and complete C^1 Finsler manifold M modeled on a Weakly Compactly Generated (WCG) Banach space is determined by the algebra $C_b^1(M)$; and (iv) the isometric structure of a WCG Banach space X with an C^1 smooth bump function is determined by the algebra $C_b^1(X)$.

1. Introduction and Preliminaries

In this note, we are interested in characterizing the Finsler structure of a Finsler manifold M in terms of the space of real-valued, bounded and C^k smooth functions with bounded derivative defined on M. The problem of the interrelation of the topological, metric and smooth structure of a space X and the algebraic and topological structure of the space $C(X)$ (the set of real-valued continuous functions defined on X) has been largely studied. These results are usually referred to as *Banach-Stone type theorems*. Recall the celebrated Banach-Stone theorem, asserting that the compact spaces K and L are homeomorphic if and only if the Banach spaces $C(K)$ and $C(L)$ endowed with the sup-norm are isometric. For more information on Banach-Stone type theorems see the survey [10] and references therein.

The Myers-Nakai theorem states that the structure of a complete Riemannian manifold M is characterized in terms of the Banach algebra $C_b^1(M)$ of all real-valued, bounded and C^1 smooth functions with bounded derivative defined on M endowed with the sup-norm of the function and its derivative. More specifically, two complete Riemannian manifolds M and N are equivalent as Riemannian manifolds, i.e. there is a C^1 diffeomorphism $h : M \to N$ such that

$$\langle dh(x)(v), dh(x)(w)\rangle_{h(x)} = \langle v, w\rangle_x$$

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for every $x \in M$ and $v, w \in T_xM$ if and only if the Banach algebras $C_b^1(M)$ and $C_b^1(N)$ are isometric. This result was first proved by S. B. Myers [22] for a compact and Riemannian manifold and by M. Nakai [23] for a finite-dimensional Riemannian manifold. Very recently, I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] gave an extension of the Myers-Nakai theorem for every infinite-dimensional, complete Riemannian manifold. A similar result for the so-called finite-dimensional Riemannian-Finsler manifolds is given in [14] (see also [26]).

Our aim in this work is to extend the Myers-Nakai theorem to the context of Finsler manifolds. On the one hand, we obtain the Myers-Nakai theorem for (i) finite-dimensional and complete Finsler manifolds, and (ii) WCG Banach spaces with a C^1 smooth bump function. On the other hand, we study for $k \geq 1$ the algebra $C_b^k(M)$ of all real-valued, bounded and C^k smooth functions with bounded first derivative defined on a complete Finsler manifold M. We prove that these algebras determine the weak Finsler structure of a complete Finsler manifold when $k = 1$ and the Finsler structure when $k \geq 2$. In particular, we obtain a weaker version of the Myers-Nakai theorem for (i) separable and complete Finsler manifolds modeled on a Banach space with a Lipschitz and C^k smooth bump function, and (ii) C^1 uniformly bumpable and complete Finsler manifolds modeled on WCG Banach spaces. In the proof of these results we will use the ideas of the Riemannian case [12].

The notation we use is standard. The norm in a Banach space X is denoted by $||\cdot||$. The dual space of X is denoted by X^* and its dual norm by $||\cdot||^*$. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. A C^k smooth bump function $b : X \to \mathbb{R}$ is a C^k smooth function on X with bounded, non-empty support, where $\operatorname{supp}(b) = \overline{\{x \in X : b(x) \neq 0\}}$. If M is a Banach manifold, we denote by T_xM the tangent space of M at x. Recall that the tangent bundle of M is $TM = \{(x, v) : x \in M \text{ and } v \in T_xM\}$. We refer to [6], [8], [19] and [7] for additional definitions. We will say that the norms $||\cdot||_1$ and $||\cdot||_2$ defined on a Banach space X are K-equivalent ($K \geq 1$) whether $\frac{1}{K}||v||_1 \leq ||v||_2 \leq K||v||_1$, for every $v \in X$.

Let us begin by recalling the definition of a C^k Finsler manifold in the sense of Palais as well as some basic properties (for more information about these manifolds see [25], [7], [27], [24], [13] and [18]).

Definition 1.1. *Let M be a (paracompact) C^k Banach manifold modeled on a Banach space $(X, ||\cdot||)$, where $k \in \mathbb{N} \cup \{\infty\}$. Let us consider the tangent bundle TM of M and a continuous map $||\cdot||_M : TM \to [0, \infty)$. We say that $(M, ||\cdot||_M)$ is a C^k* ***Finsler manifold in the sense of Palais*** *if $||\cdot||_M$ satisfies the following conditions:*

(P1) *For every $x \in M$, the map $||\cdot||_x := ||\cdot||_{M|_{T_xM}} : T_xM \to [0, \infty)$ is a norm on the tangent space T_xM such that for every chart $\varphi : U \to X$ with $x \in U$, the norm $v \in X \mapsto ||d\varphi^{-1}(\varphi(x))(v)||_x$ is equivalent to $||\cdot||$ on X.*

(P2) *For every $x_0 \in M$, every $\varepsilon > 0$ and every chart $\varphi : U \to X$ with $x_0 \in U$, there is an open neighborhood W of x_0 such that if $x \in W$ and $v \in X$, then*

$$\frac{1}{1+\varepsilon}||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0} \leq ||d\varphi^{-1}(\varphi(x))(v)||_x \leq (1+\varepsilon)||d\varphi^{-1}(\varphi(x_0))(v)||_{x_0}. \tag{1.1}$$

In terms of equivalence of norms, the above inequalities yield the fact that the norms $||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ *and* $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ *are* $(1+\varepsilon)$*-equivalent.*

Let us recall that Banach spaces and Riemannian manifolds are C^∞ Finsler manifolds in the sense of Palais [25].

Let M be a Finsler manifold, we denote by T_xM^* the dual space of the tangent space T_xM. Let $f : M \to \mathbb{R}$ be a differentiable function at $p \in M$. The norm of $df(p) \in T_pM^*$ is given by

$$||df(p)||_p = \sup\{|df(p)(v)| : v \in T_pM, ||v||_p \leq 1\}.$$

Let us consider a differentiable function $f : M \to N$ between Finsler manifolds M and N. The norm of the derivative at the point $p \in M$ is defined as

$$\begin{aligned} ||df(p)||_p &= \sup\{||df(p)(v)||_{f(p)} : v \in T_pM, ||v||_p \leq 1\} = \\ &= \sup\{\xi(df(p)(v)) : \xi \in T_{f(p)}N^*,\ v \in T_pM \text{ and } ||v||_p = 1 = ||\xi||^*_{f(p)}\}, \end{aligned}$$

where $||\cdot||^*_{f(p)}$ is the dual norm of $||\cdot||_{f(p)}$. Recall that if $(M, ||\cdot||_M)$ is a Finsler manifold, the *length* of a piecewise C^1 smooth path $c : [a, b] \to M$ is defined as $\ell(c) := \int_a^b ||c'(t)||_{c(t)}\, dt$. Besides, if M is connected, then it is connected by piecewise C^1 smooth paths, and the associated *Finsler metric* d_M on M is defined as

$$d_M(p, q) = \inf\{\ell(c) : c \text{ is a piecewise } C^1 \text{ smooth path connecting } p \text{ and } q\}.$$

It was shown in [25] that the Finsler metric is consistent with the topology given in M. The open ball of center $p \in M$ and radius $r > 0$ is denoted by $B_M(p, r) := \{q \in M : d_M(p, q) < r\}$. The Lipschitz constant $\operatorname{Lip}(f)$ of a Lipschitz function $f : M \to N$, where M and N are Finsler manifolds, is defined as $\operatorname{Lip}(f) = \sup\{\frac{d_N(f(x),f(y))}{d_M(x,y)} : x, y \in M, x \neq y\}$. We shall only consider connected manifolds. Let us recall the following "mean value inequality" for Finsler manifolds [1, 18].

Lemma 1.2. *Let M and N be C^1 Finsler manifolds (in the sense of Palais) and $f : M \to N$ a C^1 smooth function. Then, f is Lipschitz if and only if* $||df||_\infty := \sup\{||df(x)||_x : x \in M\} < \infty$. *Furthermore,* $\operatorname{Lip}(f) = ||df||_\infty$.

We will also need the following result related to the $(1+\varepsilon)$-bi-Lipschitz local behavior of the charts of a C^1 Finsler manifold in the sense of Palais [18, Lemma 2.4].

Lemma 1.3. *Let us consider a C^1 Finsler manifold M (in the sense of Palais). Then, for every $x_0 \in M$ and every chart (U, φ) with $x_0 \in U$ satisfying inequality* (1.1)*, there exists an open neighborhood $V \subset U$ of x_0 satisfying*

$$\frac{1}{1+\varepsilon} d_M(p, q) \leq |||\varphi(p) - \varphi(q)||| \leq (1+\varepsilon) d_M(p, q), \qquad \text{for every } p, q \in V, \tag{1.2}$$

where $|||\cdot|||$ is the (equivalent) norm $||d\varphi^{-1}(\varphi(x_0))(\cdot)||_{x_0}$ defined on X.

Now, let us recall the concept of *uniformly bumpable manifold*, introduced by D. Azagra, J. Ferrera and F. López-Mesas [1] for Riemannian manifolds. A natural extension to Finsler manifolds is defined in the same way [18].

Definition 1.4. *A C^k Finsler manifold in the sense of Palais M is C^k* ***uniformly bumpable*** *whenever there are $R > 1$ and $r > 0$ such that for every $p \in M$ and $\delta \in (0, r)$ there exists a C^k smooth function $b : M \to [0, 1]$ such that:*

(1) $b(p) = 1$,
(2) $b(q) = 0$ *whenever* $d_M(p,q) \geq \delta$,
(3) $\sup_{q\in M} ||db(q)||_q \leq R/\delta$.

Note that this is not a restrictive definition: D. Azagra, J. Ferrera, F. López-Mesas and Y. Rangel [3] proved that every separable Riemannian manifold is C^∞ uniformly bumpable. This result was generalized in [18], where it was proved that every C^1 Finsler manifold (in the sense of Palais) modeled on a certain class of Banach spaces (such as Hilbert spaces, Banach spaces with separable dual, closed subspaces of $c_0(\Gamma)$ for every set $\Gamma \neq \emptyset$) is C^1 uniformly bumpable. In particular, every Riemannian manifold (either separable or non-separable) is C^1 uniformly bumpable.

It is straightforward to verify that if a C^k Finsler manifold M is modeled on a Banach space X and M is C^k uniformly bumpable, then X admits a Lipschitz C^k smooth bump function. Besides, a *separable* C^k Finsler manifold M is modeled on a Banach space with a Lipschitz, C^k smooth bump function if and only if M is C^k uniformly bumpable [18]. Nevertheless, we do not know whether this equivalence holds in the non-separable case.

From now on, we shall refer to C^k Finsler manifolds in the sense of Palais as C^k Finsler manifolds, and $k \in \mathbb{N} \cup \{\infty\}$. We shall use the standard notation of $C^k(U,Y)$ for the set of all k-times continuously differentiable functions defined on an open subset U of a Banach space (Finsler manifold) taking values into a Banach space (Finsler manifold) Y. We shall write $C^k(U)$ whenever $Y = \mathbb{R}$.

Now, let us recall the concept of weakly C^k smooth function.

Definition 1.5. *Let X and Y be Banach spaces and consider a function $f : U \to Y$, where U is an open subset of X. The function f is said to be **weakly C^k smooth** at the point x_0 whenever there is an open neighborhood U_{x_0} of x_0 such that $y^* \circ f$ is C^k smooth at U_{x_0}, for every $y^* \in Y^*$. The function f is said to be **weakly C^k smooth** on U whenever f is weakly C^k smooth at every point $x \in U$.*

On the one hand, J. M. Gutiérrez and J.L. G. Llavona [15] proved that if $f : U \to Y$ is weakly C^k smooth on U, then $g \circ f \in C^k(U)$ for all $g \in C^k(Y)$. They also proved that if $f : U \to Y$ is weakly C^k smooth on U, then $f \in C^{k-1}(U)$. For $k = 1$, the above yields that every weakly C^1 smooth function on U is continuous on U. Also, for $k = \infty$, every weakly C^∞ smooth function on U is C^∞ smooth on U. M. Bachir and G. Lancien [4] proved that, if the Banach space Y has the Schur property, then the concept of weakly C^k smoothness coincides with the concept of C^k smoothness. On the other hand, there are examples of weakly C^1 smooth functions that are not C^1 smooth (see [15] and [4]).

Definition 1.6. *Let M and N be C^k Finsler manifolds and $U \subset M$, $O \subset N$ open subsets of M and N, respectively. A function $f : U \to N$ is said to be **weakly C^k smooth** at the point x_0 of U if there exist charts (W,φ) of M at x_0 and (V,ψ) of N at $f(x_0)$ such that $\psi \circ f \circ \varphi^{-1}$ is weakly C^k smooth at $\varphi(W)$. We say that $f : U \to N$ is **weakly C^k smooth** in U if f is weakly C^k smooth at every point $x \in U$. We say that a bijection $f : U \to O$ is a weakly C^k diffeomorphism if f and f^{-1} are weakly C^k smooth on U and O, respectively. Notice that these definitions do not depend on the chosen charts.*

Let us note that there are homeomorphisms which are weakly C^1 smooth but not differentiable. Indeed, we follow [15, Example 3.9] and define $g:\mathbb{R}\to c_0(\mathbb{N})$ and $h:c_0(\mathbb{N})\to c_0(\mathbb{N})$ by $g(t)=(0,\frac{1}{2}\sin(2t),\dots,\frac{1}{n}\sin(nt),\dots)$ and $h(x)=x+g(x_1)$ for every $t\in\mathbb{R}$ and $x=(x_1,\dots,x_n,\dots)\in c_0$. The function h is an homeomorphism, $h^{-1}(y)=y-g(y_1)$ for every $y\in c_0$, and h is weakly C^1 smooth on $c_0(\mathbb{N})$. Notice that if h were differentiable at a point $x\in c_0$ with $x_1=0$, then

$$h'(x)(1,0,0,\dots)=(1,1,1,\dots)\in\ell_\infty\setminus c_0,$$

which is a contradiction.

Now, let us consider different definitions of isometries between C^k Finsler manifolds.

Definition 1.7. *Let $(M,||\cdot||_M)$ and $(N,||\cdot||_N)$ be C^k Finsler manifolds and a bijection $h:M\to N$.*

(**MI**) *We say that h is a* ***metric isometry*** *for the Finsler metrics, if*

$$d_N(h(x),h(y))=d_M(x,y),\qquad \text{for every } x,y\in M.$$

(**FI**) *We say that h is a C^k* ***Finsler isometry*** *if it is a C^k diffeomorphism satisfying*

$$||dh(x)(v)||_{h(x)}=||(h(x),dh(x)(v))||_N=||(x,v)||_M=||v||_x,$$

for every $x\in M$ and $v\in T_xM$. We say that the Finsler manifolds M and N are C^k ***equivalent as Finsler manifolds*** *if there is a C^k Finsler isometry between M and N.*

(ω-**FI**) *We say that h is a* ***weak C^k Finsler isometry*** *if it is a weakly C^k diffeomorphism and a metric isometry for the Finsler metrics. We say that the Finsler manifolds M and N are* ***weakly C^k equivalent as Finsler manifolds*** *if there is a weak C^k Finsler isometry between M and N.*

Proposition 1.8. *Let M and N be C^k Finsler manifolds. Let us assume that there is a C^k diffeomorphism and metric isometry (for the Finsler metrics) $h:M\to N$. Then h is a C^k Finsler isometry.*

Proof. Let us fix $x\in M$ and $y=h(x)\in N$. For every $\varepsilon>0$, there are $r>0$ and charts $\varphi:B_M(x,r)\subset M\to X$ and $\psi:B_N(y,r)\subset N\to Y$ satisfying inequalities (1.1) and (1.2). Since $h:M\to N$ is a metric isometry, h is a bijection from $B_M(x,r)$ onto $B_N(y,r)$.

Let us consider the equivalent norms on X and Y defined as $|||\cdot|||_x:=||d\varphi^{-1}(\varphi(x))(\cdot)||_x$ and $|||\cdot|||_y=||d\psi^{-1}(\psi(y))(\cdot)||_y$, respectively.

Since h is a metric isometry, we obtain from Lemma 1.3, for p,q in an open neighborhood of $\varphi(x)$,

$$\begin{aligned}|||\psi\circ h\circ\varphi^{-1}(p)-\psi\circ h\circ\varphi^{-1}(q)|||_y &\le (1+\varepsilon)d_N(h\circ\varphi^{-1}(p),h\circ\varphi^{-1}(q))=\\ &=(1+\varepsilon)d_M(\varphi^{-1}(p),\varphi^{-1}(q))\le(1+\varepsilon)^2|||p-q|||_x.\end{aligned}$$

Thus, $\sup\{|||d(\psi\circ h\circ\varphi^{-1})(\varphi(x))(w)|||_y : |||w|||_x \le 1\} \le (1+\varepsilon)^2$. Now, for every $v\in T_xM$ with $v\neq 0$, let us write $w = d\varphi(x)(v)\in X$. We have

$$\begin{aligned} ||dh(x)(v)||_y &= ||d\psi^{-1}(\psi(y))d\psi(y)dh(x)(v)||_y = |||d(\psi\circ h)(x)(v)|||_y = \\ &= |||d(\psi\circ h)(x)d\varphi^{-1}(\varphi(x))(w)|||_y = |||d(\psi\circ h\circ\varphi^{-1})(\varphi(x))(w)|||_y \le \\ &\le (1+\varepsilon)^2|||w|||_x = (1+\varepsilon)^2||v||_x. \end{aligned}$$

Since this inequality holds for every $\varepsilon>0$ and the same argument works for h^{-1}, we conclude that $||dh(x)(v)||_y = ||v||_x$ for all $v\in T_xM$. Thus, h is a C^k Finsler isometry. □

Let us now turn our attention to the *Banach algebra* $C_b^1(M)$, the algebra of all real-valued, C^1 smooth and bounded functions with bounded derivative defined on a C^1 Finsler manifold M, i.e.

$$C_b^1(M) = \{f : M\to\mathbb{R} : f\in C^1(M),\ ||f||_\infty<\infty \text{ and } ||df||_\infty<\infty\},$$

where $||f||_\infty := \sup\{|f(x)| : x\in M\}$ and $||df||_\infty := \sup\{||df(x)||_x : x\in M\}$. The usual norm considered on $C_b^1(M)$ is $||f||_{C_b^1} = \max\{||f||_\infty, ||df||_\infty\}$ for every $f\in C_b^1(M)$ and $(C_b^1(M), ||\cdot||_{C_b^1(M)})$ is a Banach space. Let us notice that, by Lemma 1.2, we have $||df||_\infty = \mathrm{Lip}(f)$. Recall that $(C_b^1(M), 2||\cdot||_{C_b^1(M)})$ is a Banach algebra.

For $2\le k\le\infty$ and a C^k Finsler manifold M, let us consider the algebra $C_b^k(M)$ of all real-valued, C^k smooth and bounded functions that have bounded first derivative, i.e.

$$C_b^k(M) = \{f : M\to\mathbb{R} : f\in C^k(M),\ ||f||_\infty<\infty \text{ and } ||df||_\infty<\infty\} = C^k(M)\cap C_b^1(M).$$

with the norm $||\cdot||_{C_b^1}$. Thus, $C_b^k(M)$ is a subalgebra of $C_b^1(M)$. Nevertheless, it is not a Banach algebra.

A function $\varphi : C_b^k(M)\to\mathbb{R}$ $(1\le k\le\infty)$ is said to be an *algebra homomorphism* whether for all $f,g\in C_b^k(M)$ and $\lambda,\eta\in\mathbb{R}$,

(i) $\varphi(\lambda f+\eta g) = \lambda\varphi(f)+\eta\varphi(g)$, and
(ii) $\varphi(f\cdot g) = \varphi(f)\varphi(g)$.

Let us denote by $H(C_b^k(M))$ the set of all nonzero algebra homomorphisms, i.e.

$$H(C_b^k(M)) = \{\varphi : C_b^k(M)\to\mathbb{R} : \varphi \text{ is an algebra homomorphism and } \varphi(1)=1\}.$$

Let us list some of the basic properties of the algebra $C_b^k(M)$ and the algebra homomorphisms $H(C_b^k(M))$. They can be checked as in the Riemannian case (see [11], [12] and [17]).

(a) If $\varphi\in H(C_b^k(M))$, then $\varphi\neq 0$ if and only if $\varphi(1)=1$.
(b) If $\varphi\in H(C_b^k(M))$, then φ is positive, i.e. $\varphi(f)\ge 0$ for every $f\ge 0$.
(c) If the C^k Finsler manifold M is modeled on a Banach space that admits a Lipschitz and C^k smooth bump function, then $C_b^k(M)$ is a *unital algebra that separates points and closed sets* of M. Let us briefly give the proof for completeness. Let us take $x\in M$, and $C\subset M$ a closed subset of M with $x\notin C$. Let us take $r>0$ small enough so that $C\cap B_M(x,r)=\emptyset$ and a chart $\varphi : B_M(x,r)\to X$ satisfying inequality (1.1). Let us take $s>0$ small enough so that $\varphi(x)\in B(\varphi(x),s)\subset\varphi(B(x,r/2))\subset X$ and a Lipschitz and

C^k smooth bump function $b : X \to \mathbb{R}$ with $b(\varphi(x)) = 1$ and $b(z) = 0$ for every $z \notin B(\varphi(x), s)$. Let us define $h : M \to \mathbb{R}$ as $h(p) = b(\varphi(p))$ for every $p \in B_M(x, r)$ and $h(p) = 0$ otherwise. Then $h \in C_b^k(M)$, $h(x) = 1$ and $h(c) = 0$ for every $c \in C$.

(d) The space $H(C_b^k(M))$ is closed as a topological subspace of $\mathbb{R}^{C_b^k(M)}$ with the product topology. Moreover, since every function in $C_b^k(M)$ is bounded, it can be checked that $H(C_b^k(M))$ is compact in $\mathbb{R}^{C_b^k(M)}$.

(e) If $C_b^k(M)$ separates points and closed subsets, then M can be embedded as a topological subspace of $H(C_b^k(M))$ by identifying every $x \in M$ with the *point evaluation homomorphism* δ_x given by $\delta_x(f) = f(x)$ for every $f \in C_b^k(M)$. Also, it can be checked that the subset $\delta(M) = \{\delta_x : x \in M\}$ is dense in $H(C_b^k(M))$. Therefore, it follows that $H(C_b^k(M))$ is a compactification of M.

(f) Every $f \in C_b^k(M)$ admits a continuous extension $\widehat{f}$ to $H(C_b^k(M))$, where $\widehat{f}(\varphi) = \varphi(f)$ for every $\varphi \in H(C_b^k(M))$. Notice that this extension $\widehat{f}$ coincides in $H(C_b^k(M))$ with the projection $\pi_f : \mathbb{R}^{C_b^k(M)} \to \mathbb{R}$, given by $\pi_f(\varphi) = \varphi(f)$, i.e. ${\pi_f}|_{H(C_b^k(M))} = \widehat{f}$. In the following, we shall identify M with $\delta(M)$ in $H(C_b^k(M))$.

The next proposition can be proved in a similar way to the Riemannian case [12].

Proposition 1.9. *Let M be a complete C^k Finsler manifold that is C^k uniformly bumpable. Then, $\varphi \in H(C_b^k(M))$ has a countable neighborhood basis in $H(C_b^k(M))$ if and only if $\varphi \in M$.*

2. A Myers-Nakai Theorem

Our main result is the following Banach-Stone type theorem for a certain class of Finsler manifolds. It states that the algebra structure of $C_b^k(M)$ determines the C^k Finsler manifold. Recall that two normed algebras $(A, ||\cdot||_A)$ and $(B, ||\cdot||_B)$ are *equivalent as normed algebras* whenever there exists an algebra isomorphism $T : A \to B$ satisfying $||T(a)||_B = ||a||_A$ for every $a \in A$. Let us begin by defining the class of Banach spaces where the Finsler manifolds shall be modeled.

Definition 2.1. *A Banach space $(X, ||\cdot||)$ is said to be **k-admissible** if for every equivalent norm $|\cdot|$ and $\varepsilon > 0$, there are an open subset $B \supset \{x \in X : |x| \le 1\}$ of X and a C^k smooth function $g : B \to \mathbb{R}$ such that*

(i) $|g(x) - |x|| < \varepsilon$ *for $x \in B$, and*

(ii) $\mathrm{Lip}(g) \le (1 + \varepsilon)$ *for the norm $|\cdot|$.*

It is easy to prove the following lemma.

Lemma 2.2. *Let X be a Banach space with one of the following properties:*

(A.1) *Density of the set of equivalent C^k smooth norms: every equivalent norm on X can be approximated in the Hausdorff metric by equivalent C^k smooth norms* [6].

(A.2) *C^k-fine approximation property ($k \ge 2$) and density of the set of equivalent C^1 smooth norms: For every C^1 smooth function $f : X \to \mathbb{R}$ and every*

$\varepsilon > 0$, there is a C^k smooth function $g : X \to \mathbb{R}$ satisfying $|f(x) - g(x)| < \varepsilon$ and $||f'(x) - g'(x)|| < \varepsilon$ for all $x \in X$ (see [16], [2] and [20]); also, every equivalent norm defined on X can be approximated in the Hausdorff metric by equivalent C^1 smooth norms (see [6, Theorem II 4.1]).

Then X is k-admissible.

Banach spaces satisfying condition (A.2) are, for instance, separable Banach spaces with a Lipschitz C^k smooth bump function. Banach spaces satisfying condition (A.1) for $k = 1$ are, for instance, Weakly Compactly Generated (WCG) Banach spaces with a C^1 smooth bump function.

Theorem 2.3. *Let M and N be complete C^k Finsler manifolds that are C^k uniformly bumpable and are modeled on k-admissible Banach spaces. Then M and N are weakly C^k equivalent as Finsler manifolds if and only if $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^k Finsler isometry. In particular, h is a C^{k-1} Finsler isometry whenever $k \geq 2$.*

In order to prove Theorem 2.3, we shall follow the ideas of the Riemmanian case [12]. Let us divide the proof into several propositions.

Proposition 2.4. *Let M and N be C^k Finsler manifolds such that N is modeled on a k-admissible Banach space Y. Let $h : M \to N$ be a map such that $T : C_b^k(N) \to C_b^k(M)$ given by $T(f) = f \circ h$ is continuous. Then h is $||T||$-Lispchitz for the Finsler metrics.*

Proof. For every $y \in N$, let us take a chart $\psi_y : V_y \to Y$ with $\psi_y(y) = 0$. Let us consider the equivalent norm on Y, $|||\cdot|||_y := ||d\psi_y^{-1}(0)(\cdot)||_y$ and fix $\varepsilon > 0$. Let us define the ball $B_{|||\cdot|||_y}(z,t) := \{w \in Y : |||w - z|||_y < t\}$.

Fact. For every $r > 0$ such that $B_{|||\cdot|||_y}(0,r) \subset \psi_y(V_y)$ and every $\widetilde{\varepsilon} > 0$, there exists a C^k smooth and Lipschitz function $f_y : Y \to \mathbb{R}$ such that

(1) $f_y(0) = r$,
(2) $||f_y||_\infty := \sup\{|f_y(z)| : z \in Y\} = r$,
(3) $\mathrm{Lip}(f_y) \leq (1+\varepsilon)^2$ for the norm $|||\cdot|||_y$,
(4) $f_y(z) = 0$ for every $z \in Y$ with $|||z|||_y \geq r$, and
(5) $|||z|||_y \leq r - f_y(z) + \widetilde{\varepsilon}$ for every $|||z|||_y \leq r$.

Let us prove the Fact. First of all, let us take $r > 0$, $\widetilde{\varepsilon} > 0$ and $0 < \alpha < \min\{1, \frac{\varepsilon}{4}, \frac{2\widetilde{\varepsilon}}{5r}\}$. Since N is a C^k Finsler manifold modeled on a k-admissible Banach space Y, there are an open subset $B \supset \{x \in Y : |||x|||_y \leq 1\}$ of Y and a C^k smooth function $g : B \to \mathbb{R}$ such that

(i) $|g(x) - |||x|||_y| < \alpha/2$ on B, and
(ii) $\mathrm{Lip}(g) \leq (1 + \alpha/2)$ for the norm $|||\cdot|||_y$.

Now, let us take a C^∞ smooth and Lipschitz function $\theta : \mathbb{R} \to [0,1]$ such that

(i) $\theta(t) = 0$ whenever $t \leq \alpha$,
(ii) $\theta(t) = 1$ whenever $t \geq 1 - \alpha$,
(iii) $\mathrm{Lip}(\theta) \leq (1 + \varepsilon)$, and
(iv) $|\theta(t) - t| \leq 2\alpha$ for every $t \in [0, 1 + \alpha]$.

Let us define
$$f(x) = \begin{cases} \theta(g(x)) & \text{if } x \in B, \\ 1 & \text{if } x \in Y \setminus B. \end{cases}$$
It is straightforward to verify that f is well-defined, C^k smooth, $f(x) = 1$ whenever $|||x|||_y \geq 1$ and $f(x) = 0$ whenever $|||x|||_y \leq \alpha/2$. Let us now consider $f_y : Y \to [0, r]$ as $f_y(z) = r(1 - f(\frac{z}{r}))$, which is C^k smooth, Lipschitz and satisfies:

(i) $f_y(0) = r$,
(ii) $||f_y||_\infty = r$,
(iii) $|f_y(z) - f_y(x)| \leq (1+\varepsilon)(1+\alpha/2)|||z - x|||_y \leq (1+\varepsilon)^2 |||z - x|||_y$,
(iv) $f_y(z) = 0$ for every $z \in Y$ with $|||z|||_y \geq r$,
(v) $|||\frac{z}{r}|||_y \leq \frac{\alpha}{2} + g(\frac{z}{r}) \leq \frac{\alpha}{2} + 2\alpha + f(\frac{z}{r})$ for every $|||z|||_y \leq r$. Thus, $|||z|||_y \leq r(\frac{\alpha}{2} + 2\alpha) + r - f_y(z) \leq \widetilde{\varepsilon} + r - f_y(z)$ for every $|||z|||_y \leq r$.

Let us now prove Proposition 2.4. Let us fix $p_1, p_2 \in M$ and $\varepsilon > 0$. Let us consider $\sigma : [0,1] \to M$ a piecewise C^1 smooth path in M joining p_1 and p_2, with $\ell(\sigma) \leq d_M(p_1, p_2) + \varepsilon$. Since $h : M \to N$ is continuous, the path $\widehat{\sigma} := h \circ \sigma : [0,1] \to N$, joining $h(p_1)$ and $h(p_2)$, is continuous as well. For every $q \in \widehat{\sigma}([0,1])$, there is $0 < r_q < 1$ and a chart $\psi_q : V_q \to Y$ such that $\psi_q(q) = 0$, $B_N(q, r_q) \subset V_q$ and the bijection $\psi_q : V_q \to \psi_q(V_q)$ is $(1+\varepsilon)$-bi-Lipschitz for the norm $||d\psi_q^{-1}(0)(\cdot)||_q$ in Y (see Lemma 1.3). Since $\widehat{\sigma}([0,1])$ is a compact set of N, there is a finite family of points $0 = t_1 < t_2 < \cdots < t_m = 1$ and a family of open intervals $\{I_k\}_{k=1}^m$ covering the interval $[0,1]$ so that, if we define $q_k := \widehat{\sigma}(t_k)$ and $r_k := r_{q_k}$, for every $k = 1, \ldots, m$, we have

(a) $\widehat{\sigma}(I_k) \subset B_N(q_k, r_k/(1+\varepsilon))$,
(b) $I_j \cap I_k \neq \emptyset$ if, and only if, $|j - k| \leq 1$.

It is clear that $\widehat{\sigma}([0,1]) \subset \bigcup_{k=1}^m B_N(q_k, \frac{r_k}{1+\varepsilon})$. Now, let us select a point $s_k \in I_k \cap I_{k+1}$ such that $t_k < s_k < t_{k+1}$, for every $k = 1, \ldots, m-1$. Let us write $a_k := \widehat{\sigma}(s_k)$, for every $k = 1, \ldots, m-1$, $\psi_k := \psi_{q_k}$, $V_k := V_{q_k}$ and $|||\cdot|||_k := ||d\psi_k^{-1}(0)(\cdot)||_{q_k}$, for every $k = 1, \ldots, m$. Notice that $a_k \in B_N(q_k, \frac{r_k}{1+\varepsilon}) \cap B_N(q_{k+1}, \frac{r_{k+1}}{1+\varepsilon})$, for every $k = 1, \ldots, m-1$. Since $\psi_k : V_k \to \psi_k(V_k)$ is $(1+\varepsilon)$-bi-Lipschitz for the norm $|||\cdot|||_k$ in Y, we deduce that $\psi_k(a_k) \in B_{|||\cdot|||_k}(0, r_k)$, for every $k = 1, \ldots, m-1$.

Now, let us we apply the above Fact to r_k, ε and $\widetilde{\varepsilon} = \varepsilon/2m$ to obtain functions $f_k : Y \to [0, r_k]$ satisfying properties (1)–(5), $k = 1, \ldots, m$. Let us define the C^k smooth and Lipschitz functions $g_k : N \to [0, r_k]$ as $g_k(z) = f_k(\psi_k(z))$ for every $z \in V_k$ and $g_k(z) = 0$ for $z \notin V_k$, $k = 1, \ldots, m$. Then,

(i) $g_k \in C_b^k(N)$;
(ii) $g_k(q_k) = r_k$;
(iii) $|g_k(z) - g_k(x)| \leq (1+\varepsilon)^3 d_N(z, x)$ for all $z, x \in N$;
(iv) If $z \in \psi_k^{-1}(B_{|||\cdot|||_k}(0, r_k))$, then $|||\psi_k(z)|||_k \leq r_k$ and from condition (5) on the Fact, we obtain

$$d_N(z, q_k) \leq (1+\varepsilon)|||\psi_k(z) - \psi_k(q_k)|||_k = (1+\varepsilon)|||\psi_k(z)|||_k \leq (1+\varepsilon)(r_k - g_k(z) + \varepsilon/2m).$$

The Lipschitz constant of $g_k \circ h$, for $k = 1, \ldots, m$, is the following
$$\begin{aligned} \operatorname{Lip}(g_k \circ h) &\leq ||g_k \circ h||_{C_b^1(M)} = ||T(g_k)||_{C_b^1(M)} \leq ||T|| ||g_k||_{C_b^1(N)} = \\ &= ||T|| \max\{||g_k||_\infty, ||dg_k||_\infty\} \leq ||T||(1+\varepsilon)^3. \end{aligned}$$

Now, since $r_k = g_k(q_k) = g_k(h(\sigma(t_k)))$ and $\psi_k(h(\sigma(s_k))) \in B_{|||\cdot|||_k}(0, r_k)$, we have

$$
\begin{aligned}
d_N(h(p_1),h(p_2)) \leq \sum_{k=1}^{m-1} [d_N(h(\sigma(t_k)), h(\sigma(s_k))) + d_N(h(\sigma(s_k)), h(\sigma(t_{k+1})))] \leq \\
\leq \sum_{k=1}^{m-1} (1+\varepsilon)[g_k(q_k) - g_k(h(\sigma(s_k))) + \\
+ g_{k+1}(q_{k+1}) - g_{k+1}(h(\sigma(s_k))) + \varepsilon/m] \leq \\
\leq \sum_{k=1}^{m-1} (1+\varepsilon)[\mathrm{Lip}(g_k \circ h) d_M(\sigma(t_k), \sigma(s_k)) + \\
+ \mathrm{Lip}(g_{k+1} \circ h) d_M(\sigma(t_{k+1}), \sigma(s_k)) + \varepsilon/m] \leq \\
\leq \sum_{k=1}^{m-1} ||T||(1+\varepsilon)^4 [d_M(\sigma(t_k), \sigma(s_k)) + d_M(\sigma(t_{k+1}), \sigma(s_k))] + \varepsilon(1+\varepsilon) \leq \\
\leq \sum_{k=1}^{m-1} ||T||(1+\varepsilon)^4 \ell(\sigma_{|_{[t_k, t_{k+1}]}}) + \varepsilon(1+\varepsilon) = ||T||(1+\varepsilon)^4 \ell(\sigma) + \varepsilon(1+\varepsilon) \leq \\
\leq ||T||(1+\varepsilon)^4 (d_M(p_1, p_2) + \varepsilon) + \varepsilon(1+\varepsilon)
\end{aligned}
$$

for every $\varepsilon > 0$. Thus, h is $||T||$-Lipschitz. □

Lemma 2.5. *Let M and N be C^k Finsler manifolds such that N is modeled on a Banach space with a Lipschitz C^k smooth bump function. Let $h : M \to N$ be a homeomorphism such that $f \circ h \in C^k_b(M)$ for every $f \in C^k_b(N)$. Then, h is a weakly C^k smooth function on M.*

Proof. Let us fix $x \in M$ and $\varepsilon = 1$. There are charts $\varphi : U \to X$ of M at x and $\psi : V \to Y$ of N at $h(x)$ satisfying inequalities (1.1) and (1.2) on U and V, respectively. We can assume that $h(U) \subset V$. Since Y admits a Lipschitz and C^k smooth bump function and $\psi(h(U))$ is an open neighborhood of $\psi(h(x))$ in Y, there are real numbers $0 < s < r$ such that $B(\psi(h(x)), s) \subset B(\psi(h(x)), r) \subset \psi(h(U))$ and a Lipschitz and C^k smooth function $\alpha : Y \to \mathbb{R}$ such that $\alpha(y) = 1$ for $y \in B(\psi(h(x)), s)$ and $\alpha(y) = 0$ for $y \notin B(\psi(h(x)), r)$. Let us define $U_0 := h^{-1}(\psi^{-1}(B(\psi(h(x)), s))) \subset U$, which is an open neighborhood of x in M.

Let us check that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is C^k smooth on $\varphi(U_0) \subset X$ for all $y^* \in Y^*$. Following the proof of [9, Theorem 4], we define $g : N \to \mathbb{R}$ as $g(y) = 0$ whenever $y \notin V$ and $g(y) = \alpha(\psi(y)) \cdot y^*(\psi(y))$ whenever $y \in V$. It is clear that $g \in C^k_b(N)$ and, by assumption, $g \circ h \in C^k_b(M)$. Now, it follows that $\psi(h(\varphi^{-1}(z))) \in B(\psi(h(x)), s)$ for every $z \in \varphi(U_0)$. Thus

$$
\begin{aligned}
y^* \circ (\psi \circ h \circ \varphi^{-1})(z) = y^*(\psi(h(\varphi^{-1}(z)))) = \alpha(\psi(h(\varphi^{-1}(z))))y^*(\psi(h(\varphi^{-1}(z)))) = \\
= g(h(\varphi^{-1}(z))) = g \circ h \circ \varphi^{-1}(z),
\end{aligned}
$$

for every $z \in \varphi(U_0)$. Since $(g \circ h) \circ \varphi^{-1}$ is C^k smooth on $\varphi(U_0)$, we have that $y^* \circ (\psi \circ h \circ \varphi^{-1})$ is C^k smooth on $\varphi(U_0)$. Thus $\psi \circ h \circ \varphi^{-1}$ is weakly C^k smooth on $\varphi(U_0)$ and h is weakly C^k smooth on M. □

Proof of Theorem 2.3. If $h : M \to N$ is a weak C^k Finsler isometry, we can define the operator $T : C_b^k(N) \to C_b^k(M)$ by $T(f) = f \circ h$. Let us check that T is well defined. For every $x \in M$, there are charts $\varphi : U \to X$ of M at x and $\psi : V \to Y$ of N at $h(x)$, such that $h(U) \subset V$ and $\psi \circ h \circ \varphi^{-1}$ is weakly C^k smooth on $\varphi(U) \subset X$. Also, $f \circ \psi^{-1}$ is C^k smooth on $\psi(V) \subset Y$. Thus, by [15, Proposition 4.2], $(f \circ \psi^{-1}) \circ (\psi \circ h \circ \varphi^{-1}) = f \circ h \circ \varphi^{-1}$ is C^k smooth on $\varphi(U)$. Therefore, $f \circ h$ is C^k smooth on U. Since this holds for every $x \in M$, we deduce that $f \circ h$ is C^k smooth on M. Moreover, T is an algebra isomorphism with $||T(f)||_{C_b^1(M)} = ||f \circ h||_{C_b^1(M)} = ||f||_{C_b^1(N)}$ for every $f \in C_b^k(N)$.

Conversely, let $T : C_b^k(N) \to C_b^k(M)$ be a normed algebra isometry. Then, we can define the function $h : H(C_b^k(M)) \to H(C_b^k(N))$ by $h(\varphi) = \varphi \circ T$ for every $\varphi \in H(C_b^k(M))$. The function h is a bijection. Moreover, h is an homeomorphism. Recall that we identify $x \in M$ with $\delta_x \in H(C_b^k(M))$. Thus, $h(x) = h(\delta_x) = \delta_x \circ T$. Since h is an homeomorphism, by Proposition 1.9, we obtain for every $p \in N$ a unique point $x \in M$ such that $h(\delta_x) = \delta_p$. Let us check that $T(f) = f \circ h$ for all $f \in C_b^k(N)$. Indeed, for every $x \in M$ and every $f \in C_b^k(N)$,

$$T(f)(x) = \delta_x(T(f)) = (\delta_x \circ T)(f) = h(\delta_x)(f) = \delta_{h(x)}(f) = f(h(x)) = f \circ h(x).$$

Now, from Proposition 2.4 and Lemma 2.5 we deduce that h is a weak C^k Finsler isometry. □

Remark 2.6. *It is worth mentioning that, for Riemannian manifolds, every metric isometry is a C^1 Finsler isometry. This result was proved by S. Myers and N. Steenrod [21] in the finite-dimensional case and by I. Garrido, J.A. Jaramillo and Y.C. Rangel [12] in the general case. Also, S. Deng and Z. Hou [5] obtained a version for finite-dimensional Riemannian-Finsler manifolds. Nevertheless, there is no a generalization, up to our knowledge, of the Myers-Steenrod theorem for all Finsler manifolds. Thus, for $k = 1$ we can only assure that the metric isometry obtained in Theorem 2.3 is weakly C^1 smooth.*

Let us finish this note with some interesting corollaries of Theorem 2.3. First, recall that every separable Banach space with a Lipschitz C^k smooth bump function satisfies condition (A.2) and every WCG Banach space with a C^1 smooth bump function satisfies condition (A.1) for $k = 1$.

Corollary 2.7. *Let M and N be complete, C^1 Finsler manifolds that are C^1 uniformly bumpable and are modeled on WCG Banach spaces. Then M and N are weakly C^1 equivalent as Finsler manifolds if, and only if, $C_b^1(M)$ and $C_b^1(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^1(N) \to C_b^1(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^1 Finsler isometry.*

Notice that the assumptions of Corollary 2.7 hold if M and N are modeled on Banach spaces with separable dual.

Corollary 2.8. *Let M and N be complete, separable C^k Finsler manifolds that are modeled on Banach spaces with a Lipschitz and C^k smooth bump function. Then M and N are weakly C^k equivalent as Finsler manifolds if and only if $C_b^k(M)$ and*

$C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a weak C^k Finsler isometry. In particular, h is a C^{k-1} Finsler isometry whenever $k \geq 2$.

Since every weakly C^k smooth function with values in a finite-dimensional normed space is C^k smooth and every finite-dimensional C^k Finsler manifold is C^k uniformly bumpable [18], we obtain the following Myers-Nakai result for finite-dimensional C^k Finsler manifolds.

Corollary 2.9. *Let M and N be complete and finite dimensional C^k Finsler manifolds. Then M and N are C^k equivalent as Finsler manifolds if, and only if, $C_b^k(M)$ and $C_b^k(N)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^k(N) \to C_b^k(M)$ is of the form $T(f) = f \circ h$ where $h : M \to N$ is a C^k Finsler isometry.*

We obtain an interesting application of Finsler manifolds to Banach spaces. Recall the well-known Mazur-Ulam Theorem establishing that every surjective isometry between two Banach spaces is affine.

Corollary 2.10. *Let X and Y be WCG Banach spaces with C^1 smooth bump functions. Then X and Y are isometric if, and only if, $C_b^1(X)$ and $C_b^1(Y)$ are equivalent as normed algebras. Moreover, every normed algebra isomorphism $T : C_b^1(Y) \to C_b^1(X)$ is of the form $T(f) = f \circ h$ where $h : X \to Y$ is a surjective isometry. In particular, h and h^{-1} are affine isometries.*

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