ON THE KUNEN-SHELAH PROPERTIES IN BANACH SPACES

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ABSTRACT. We introduce and study the Kunen-Shelah properties KS_i, $i = 0, 1, ..., 7$. Let us highlight for a Banach space X some of our results: **(1)** X^* has a w^*-nonseparable equivalent dual ball **iff** X has an ω_1-*polyhedron* (i.e., a bounded family $\{x_i\}_{i<\omega_1}$ such that $x_j \notin \overline{\text{co}}(\{x_i : i \in \omega_1 \setminus \{j\}\})$ for every $j \in \omega_1$) **iff** X has an *uncountable bounded almost biorthonal system* (UBABS) of type η, for some $\eta \in [0, 1)$, (i.e., a bounded family $\{(x_\alpha, f_\alpha)\}_{1\leq\alpha<\omega_1} \subset X \times X^*$ such that $f_\alpha(x_\alpha) = 1$ and $|f_\alpha(x_\beta)| \leq \eta$, if $\alpha \neq \beta$); **(2)** if X has an uncountable ω-independent system then X has an UBABS of type η for every $\eta \in (0, 1)$; **(3)** if X has not the property (C) of Corson, then X has an ω_1-polyhedron; **(4)** X has not an ω_1-polyhedron **iff** X has not a *convex right-separated* ω_1*-family* (i.e., a bounded family $\{x_i\}_{i<\omega_1}$ such that $x_j \notin \overline{\text{co}}(\{x_i : j < i < \omega_1\})$ for every $j \in \omega_1$) **iff** every w^*-closed convex subset of X^* is w^*-separable **iff** every convex subset of X^* is w^*-separable **iff** $\mu(X) = 1$, $\mu(X)$ being the Finet-Godefroy index of X (see [1]).

1. **Introduction.** If X is a Banach space and θ an ordinal, a family $\{x_\alpha : \alpha < \theta\} \subset X$ is said to be a θ-*basic sequence* if there exists $1 \leq K < \infty$ such that for every $n < m$ in $\mathbb{N}$, every families $\lambda_i \in \mathbb{R}$, $i = 1, ..., m$, and $\alpha_1 < ... < \alpha_n < ... < \alpha_m < \theta$ we have $\| \sum_{i=1}^{i=n} \lambda_i x_{\alpha_i}\| \leq K \| \sum_{i=1}^{i=m} \lambda_i x_{\alpha_i}\|$. A family $\{x_i\}_{i\in I} \subset X$ is a basic sequence if it is a θ-basic sequence for some ordinal θ. If $K = 1$ the basic sequence is said to be *monotone.* A *biorthogonal system* in X is a family $\{(x_i, x_i^*) : i \in I\} \subset X \times X^*$ such that

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$x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0,\ i, j \in I,\ i \neq j$. A *Markuschevich system* (in short, a M-system) in X is a biorthogonal system $\{(x_i, x_i^*) : i \in I\}$ in X such that $\{x_i^* : i \in I\}$ is total on $\overline{[\{x_i : i \in I\}]}$ (see [14]).

It is well known (see [14, pg. 599]) that if the density of a Banach space X satisfies Dens$(X) \geq \aleph_1$, then X has a monotone ω_1-basic sequence. Also if Dens$(X) > \mathfrak{c}$, X has a monotone ω_1-basic sequence, because in this case an easy calculation shows that w^*-Dens$(X^*) \geq \aleph_1$. However, if $\aleph_1 \leq$ Dens$(X) \leq \mathfrak{c}$ and w^*-Dens$(X^*) \leq \aleph_0$, X can fail to have an uncountable basic sequence, even an uncountable biorthogonal system. Indeed, Shelah [13] constructed under the axiom $\diamondsuit_{\aleph_1}$ -an axiom which implies the continuum hypothesis (CH)- a nonseparable Banach space S that fails to have an uncountable biorthogonal system. Later Kunen [8, p. 1123] constructed under (CH) a Hausdorff compact space K such that $C(K)$ is nonseparable and has not an uncountable biorthogonal system, among other pathological interesting properties.

A Banach space X is said to have the *Kunen-Shelah property* KS_0 (resp. KS_1) if X has not an uncountable basic sequence (resp. an uncountable Markuschevich system). A Banach space X is said to have the *Kunen-Shelah property* KS_2 if X has not an uncountable biorthogonal system. Clearly, $KS_2 \Rightarrow KS_1 \Rightarrow KS_0$.

The first example of a Banach space X such that $X \in KS_0$ but $X \notin KS_2$ was given in [9] and is the space of Johnson-Lindentrauss JL_2 (see [4]). The properties KS_2 and KS_1 were separated in [2] (see also [1]), where it was proved that if a Banach space X has the property (C) of Corson and w^*-Dens $(X^*) \leq \aleph_0$, then $X \in KS_1$.

Question 1. There exists a Banach space X such that $X \in KS_0$ but $X \notin KS_1$?

In this paper we study some structures similar to uncountable biorthogonal systems, namely: uncountable ω-independent families, ω_1-polyhedrons, uncountable bounded almost biorthogonal systems (UBABS), etc. The lack of these structures allows us to define the Kunen-Shelah properties KS_3, KS_4, etc.

In Section 2 we prove that a Banach space X has an ω_1-polyhedron iff X has an UBABS iff X^* has a w^*-nonseparable dual equivalent ball. Section 3 deals with uncountable ω-independent families. In Section 4 it is proved that X has not an ω_1-polyhedron iff every w^*-closed convex subset of X^* is w^*-separable. In Section 5 we answer some questions posed by Finet and Godefroy [1] concerning the index $\mu(X)$. In Section 6 we prove that a space X has not a convex right-separated ω_1-family iff every w^*-closed convex subset of X^* is w^*-separable. Finally, in Section 7 we show that X has an ω_1-polyhedron iff X has a convex right-separated ω_1-family, whence every w^*-closed convex subset of X^* is w^*-separable iff every convex subset of X^* is so.

Let us introduce some notation. ω_1 is the first uncountable ordinal, $|A|$ the cardinal of the set A and $\mathfrak{c} = |\mathbb{R}|$. If X is a Banach space, X^* denotes its dual, $B(X)$ and $S(X)$ the closed unit ball and sphere of X, resp., and $B(x,r)$ the closed ball with radius r and center x. If $A \subset X$ we denote by $[A]$ the linear subspace spanned by A. Recall that a Banach space X is said to have the *property (C)* of Corson (in short, $X \in (C)$) if $\cap_{i\in I} C_i \neq \emptyset$ whenever $\{C_i : i \in I\}$ is a family of closed bounded convex subsets of X with the countable intersection property, i.e., $\emptyset \neq \cap_{i\in J} C_i$ for every countable subset $J \subset I$.

2. **UBABS and ω_1-polyhedrons.** If X is a Banach space, a bounded family $\{(x_\alpha, f_\alpha)\}_{1\leq\alpha<\omega_1} \subset X \times X^*$ is said to be an *uncountable bounded almost biorthogonal system* (in short, an UBABS), if there exist a real number $0 \leq \eta < 1$ such that $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \leq \eta$, if $\alpha \neq \beta$. If in addition we have $|f_\alpha(x_\beta)| \leq \eta$ for $\alpha \neq \beta$, then the UBABS $\{(x_\alpha, f_\alpha)\}_{1\leq\alpha<\omega_1} \subset X \times X^*$ is said to be of type η. Define the index $\tau(X)$ as follows:

$$\tau(X) = \inf\{0 \leq \eta < 1 : X \text{ has an UBABS of type } \eta\},$$

where $\inf\{\emptyset\} = 1$. Clearly, $\tau(X)$ is invariant by isomorphisms and: (1) if X has an uncountable biorthogonal system, then $\tau(X) = 0$; (2) $\tau(X) < 1$ iff X has an UBABS.

If τ is a cardinal, a bounded family $\{x_i\}_{i\in\tau}$ in a Banach space X is said to be a *τ-polyhedron* iff $x_j \notin \overline{\text{co}}(\{x_i\}_{i\in\tau\setminus\{j\}})$ for every $j \in \tau$. In a dual Banach space X^* one can define a w^*-τ-polyhedron in a analogous way, using the w^*-topology instead of the w-topology.

Proposition 2.1. *A Banach space X has an ω_1-polyhedron iff X^* has an w^*-ω_1-polyhedron.*

Proof. Let $\{x_\alpha\}_{\alpha<\omega_1} \subset B(X)$ be an ω_1-polyhedron. By the Hahn-Banach Theorem there exists $f_\alpha \in S(X^*)$ such that:

$$f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : i \in \omega_1 \setminus \{\alpha\}\} =: e_\alpha.$$

By passing to a subsequence, we can suppose that there exist $0 < \epsilon < \infty$ and $r \in \mathbb{R}$ such that $f_\alpha(x_\alpha) - e_\alpha \geq \epsilon > 0$ and $|r - f_\alpha(x_\alpha)| \leq \frac{\epsilon}{4}$, $\forall \alpha < \omega_1$. Hence, if $\alpha, \beta < \omega_1$ with $\alpha \neq \beta$, we have:

$$f_\alpha(x_\alpha) \geq r - \frac{\epsilon}{4} > r - \frac{3\epsilon}{4} \geq f_\beta(x_\beta) - \epsilon \geq \epsilon_\beta \geq f_\beta(x_\alpha),$$

which implies that $\{f_\alpha\}_{\alpha<\omega_1}$ is an w^*-ω_1-polyhedron in X^*.

The converse implication is analogous. □

Let us see in the following Proposition the relation between ω_1-polyhedrons and UBABS.

Proposition 2.2. *For a Banach space X the following are equivalent:*

(1) X has an ω_1-polyhedron; (2) X has an UBABS of type η for some $0 \leq \eta < 1$; (3) X has an UBABS.

Proof. (1) $\Rightarrow$ (2). If X has an uncountable biorthogonal system, then clearly X has an UBABS of type 0.

In the contrary case, w^*-Dens$(X^*) \leq \aleph_0$. Let $\{x_\alpha\}_{1\leq\alpha<\omega_1} \subset X$ be an ω_1-polyhedron. Assume that $x_1 = 0$ and that $\|x_\alpha\| \leq 1$. For each $1 \leq \alpha < \omega_1$

consider $f_\alpha \in S(X^*)$ such that:

$$1 \geq f_\alpha(x_\alpha) > \sup\{f_\alpha(x_i) : 1 \leq i < \omega_1,\ i \neq \alpha\} =: \rho_\alpha.$$

Observe that $\rho_\alpha \geq 0$ if $\alpha \neq 1$. By passing to an uncountable subsequence, it can be assumed that there are real numbers $0 < \epsilon, r \leq 1$ such that $f_\alpha(x_\alpha) - \rho_\alpha \geq \epsilon$ and $|r - f_\alpha(x_\alpha)| < \frac{\epsilon}{8}$ for every $2 \leq \alpha < \omega_1$. Since w^*-Dens$(X^*) \leq \aleph_0$, by passing again to a subsequence, we assume that there exists $z \in X^*$ such that $z(x_\alpha) > 0$ and $|\frac{z(x_\beta)}{z(x_\alpha)} - 1| < \frac{\epsilon}{8}$ for every $2 \leq \alpha, \beta < \omega_1$. Then, if $g_\alpha = f_\alpha + \frac{z}{z(x_\alpha)}$, $2 \leq \alpha < \omega_1$, we have:

$$g_\alpha(x_\alpha) = f_\alpha(x_\alpha) + 1 \geq r - \frac{\epsilon}{8} + 1 > r - \frac{6\epsilon}{8} + 1 \geq f_\alpha(x_\alpha) - \frac{7\epsilon}{8} + 1 \geq$$

$$\geq \sup\{g_\alpha(x_\beta) : 2 \leq \beta < \omega_1, \beta \neq \alpha\} \geq \inf\{g_\alpha(x_\beta) : 2 \leq \beta < \omega_1, \beta \neq \alpha\} \geq -\frac{\epsilon}{8}.$$

Denote $h_\alpha = \frac{g_\alpha}{g_\alpha(x_\alpha)}$. Then, for $2 \leq \alpha, \beta < \omega_1, \alpha \neq \beta$, we have $h_\alpha(x_\alpha) = 1$ and:

$$-\frac{\frac{\epsilon}{8}}{r - \frac{\epsilon}{8} + 1} \leq -\frac{\frac{\epsilon}{8}}{g_\alpha(x_\alpha)} \leq h_\alpha(x_\beta) = \frac{g_\alpha(x_\beta)}{g_\alpha(x_\alpha)} \leq \frac{r + 1 - \frac{6\epsilon}{8}}{r + 1 - \frac{\epsilon}{8}}.$$

So, $\{(x_\alpha, h_\alpha) : 2 \leq \alpha < \omega_1\} \subset X \times X^*$ is an UBABS of type η such that:

$$0 \leq \eta = \max\{\frac{\frac{\epsilon}{8}}{r - \frac{\epsilon}{8} + 1}, \frac{r + 1 - \frac{6\epsilon}{8}}{r + 1 - \frac{\epsilon}{8}}\} < 1.$$

(2) $\Rightarrow$ (3) is obvious and (3) $\Rightarrow$ (1) is clear because, if $\{(x_\alpha, f_\alpha)\}_{1 \leq \alpha < \omega_1} \subset X \times X^*$ is an UBABS, then $\{x_\alpha\}_{\alpha < \omega_1}$ is an ω_1-polyhedron. $\square$

Let us consider some results on representation in polyhedrons, that we need later. If $\{x_i\}_{i \in I}$ is a w^*-τ-polyhedron in a dual Banach space X^* with $\tau = \mathrm{card}(I)$ and $K = \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i \in I})$, the *core* of K is the set:

$$K_0 = \mathrm{core}(K) = \cap\{\overline{\mathrm{co}}^{w^*}(\{x_i\}_{i \in I \setminus A}) : A \subset I, A \text{ finite}\}.$$

Define the function $\lambda : K \to [0, 1]$ as follows:

$$\forall k \in K,\ \lambda(k) = \sup\{\lambda \in [0, 1] : \exists u \in K, \exists i \in I \text{ such that } k = \lambda x_i + (1 - \lambda)u\}.$$

Let $H = \{x \in K : \lambda(x) = 0\}$. Since for every finite subset $A \subset I$, each $x \in K$ has the expression $x = \sum_{i \in A} \lambda_i x_i + (1 - \mu)u$ with $u \in \overline{\mathrm{co}}^{w^*}(\{x_i\}_{i \in I \setminus A})$, $\lambda_i \in [0, 1], i \in A, \mu = \sum_{i \in A} \lambda_i \leq 1$, it can be seen easily that $H \subset K_0$.

Lemma 2.3. *Let $\{x_i\}_{i\in I}$ be a w^*-τ-polyhedron in the dual Banach space X^*, $\tau = card(I)$, $K = \overline{co}^{w^*}(\{x_i\}_{i\in I\}$, K_0 =core(K) and $H = \{x \in K : \lambda(x) = 0\}$. If $x \in K$, there exist a sequence of positive numbers $\{\mu_n\}_{n\geq 1}$ with $0 \leq \sum_{n\geq 1} \mu_n = \mu \leq 1$, a sequence of subindices (maybe no distinct) $\{i_n\}_{n\geq 1} \subset I$ and $u \in H$ such that $x = \sum_{n\geq 1} \mu_n \cdot x_{i_n} + (1-\mu)u$.*

Proof. Clearly the statement is true if $x \in H$. Assume that $x \notin H$, i.e., $\lambda(x) > 0$. Choose $0 < \frac{1}{2}\lambda(x) \leq \lambda_1 \leq 1$, $i_1 \in I$, and $u_1 \in \overline{\text{co}}^{w^*}(\{x_i\}_{i\in I\setminus\{i_1\}})$ such that $x = \lambda_1 x_{i_1} + (1-\lambda_1)u_1$. If $u_1 \in H$, we end. In the contrary case, $\lambda(u_1) > 0$ and we choose $0 < \frac{1}{2}\lambda(u_1) \leq \lambda_2 \leq 1$, $i_2 \in I$, and $u_2 \in \overline{\text{co}}^{w^*}(\{x_i\}_{i\in I\setminus\{i_2\}}$ such that $u_1 = \lambda_2 x_{i_2} + (1-\lambda_2)u_2$. By reiteration, there are two possibilities:

(A) $u_m \in H$ for some $m \in \mathbb{N}$. Then we obtain the representation:

$$x = \sum_{k=1}^{m} \lambda_k \cdot P_{k-1} \cdot x_{i_k} + P_m u_m,\ P_k = \prod_{k=1}^{n}(1-\lambda_k),\ P_0 = 1. \tag{1}$$

(B) Always $u_m \notin H$. As P_m decreases in (1), there exists $\lim_{m\geq 1} P_m = P \in [0,1]$. We have two cases:

(1) :$\underline{P > 0}$. Observe that $P > 0$ iff the series $\sum_{k\geq 1} \lambda_k < +\infty$. In consequence the series $\sum_{k\geq 1} \lambda_k P_{k-1} x_{i_k}$ converges and $u_m \to u \in K$ as $m \to \infty$. So from (1) we obtain $x = \sum_{k\geq 1} \lambda_k \cdot P_{k-1} \cdot x_{i_k} + Pu$. We claim that $\lambda(u) = 0$. Indeed, suppose that $\mu := \lambda(u) > 0$ and pick $q \in \mathbb{N}$ such that $P/P_q > 1/2$, $\lambda_{q+1} < \mu/8$. Then:

$$u_q = \frac{1}{P_q}\left(\sum_{j\geq 1} \lambda_{q+j} P_{q+j-1} x_{q+j} + Pu\right),$$

which implies that $\lambda(u_q) \geq \frac{P}{P_q}\lambda(u) = \frac{P}{P_q}\mu > \frac{\mu}{2}$. Since $0 < \frac{1}{2}\lambda(u_q) \leq \lambda_{q+1} \leq 1$, we obtain $\frac{\mu}{8} > \lambda_{q+1} \geq \frac{\mu}{4}$, a contradiction.

(2) :$\underline{P = 0}$. In this case $P_m u_m \to 0$ as $m \to \infty$ and we obtain the representation $x = \sum_{k\geq 1} \lambda_k P_{k-1} x_{i_k}$ with $\sum_{k\geq 1} \lambda_k P_{k-1} = 1$.

□

In order to connect the existence of an UBABS in a Banach space X with the w^*-nonseparability of dual equivalent unit balls of X^*, we introduce the index $\sigma(X)$. If $K \subset X^*$ is a disc (i.e., a convex symmetric subset of X^*), define the index $\sigma(K)$ as:

$$\sigma(K) = \max\{0 \leq t \leq 1 : \exists A \subset K, |A| \leq \aleph_0,\ tK \subseteq \overline{\text{co}}^{w^*}(A \cup (-A))\}$$

Observe that $0 \leq \sigma(K) < 1$ iff K is w^*-nonseparable and that there exists a countable subset $A \subset K$ such that $\sigma(K) \cdot K \subset \overline{\text{co}}^{w^*}(A \cup (-A))$.

Lemma 2.4. *Let X be a Banach space, $K \subset X^*$ a w^*-nonseparable disc and $\sigma(K) < \rho \leq 1$. Then there exists $\epsilon = \epsilon(\rho) > 0$ (depending on ρ) such that for every countable subset $A \subset K$ there exists $k \in K$ satisfying $dist(\rho k, \overline{co}^{w^*}(A \cup (-A))) \geq \epsilon$.*

Proof. In the contrary case, there exist a sequence of real numbers $\epsilon_n \downarrow 0$ and a sequence of countable subsets $A_n \subset K,\ n \geq 1$, such that every $k \in K$ satisfies $\text{dist}(\rho k, \overline{\text{co}}^{w^*}(A_n \cup (-A_n))) < \epsilon_n$. So, if $A = \cup_{n\geq 1} A_n$ we have $\rho K \subset \overline{\text{co}}^{w^*}(A \cup (-A))$, a contradiction. □

Define the index $\sigma(X)$, X a Banach space, as follows:

$$\sigma(X) = \inf\{\sigma(K) : K \subset X^* \text{ a dual equivalent ball of } X^*\}.$$

It is clear that $\sigma(X)$ is invariant by isomorphisms.

Proposition 2.5. *For a Banach space X we have that:*

$$\sigma(X) = \inf\{\sigma(K) : K \subset X^* \ \textit{a } w^*\textit{-compact disc}\ \}.$$

Proof. Obviously $\sigma(X) \geq \inf\{\sigma(K) : K \subset X^* \text{ a } w^*\text{-compact disc }\}$. In order to prove the contrary inequality, it is enough to see that $\sigma(X) \leq \sigma(K)$ for any w^*-compact disc $K \subset X^*$. So, fix a w^*-compact disc $K \subset X^*$. Assume that K is w^*-nonseparable, pick $\sigma(K) < \rho < 1$ and let $\epsilon = \epsilon(\rho) > 0$ be given by Lemma 2.4. For $0 < \delta < \epsilon$ such that $\rho + \delta < 1$ consider $H_\delta = K + \delta B(X^*)$, which is an equivalent dual ball of X^*. We claim that $\sigma(H_\delta) \leq \rho + \delta$. Indeed, let $\rho + \delta < t \leq 1$ and $A \subset H_\delta$ a countable subset.

Then $A \subset A_1 + A_2$, where $A_1 \subset K$, $A_2 \subset \delta B(X^*)$ are countable subsets. Assume that $tH_\delta \subset \overline{\mathrm{co}}^{w^*}(A \cup (-A))$. As $\overline{\mathrm{co}}^{w^*}(A \cup (-A)) \subset \overline{\mathrm{co}}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*)$, we get:

$$tK \subset tH_\delta \subset \overline{\mathrm{co}}^{w^*}(A_1 \cup (-A_1)) + \delta B(X^*),$$

which implies that , $\forall k \in K$, $\mathrm{dist}(tk, \overline{\mathrm{co}}^{w^*}(A_1 \cup (-A_1)) \leq \delta$. But by Lemma 2.4 there exists $k \in K$ such that $\mathrm{dist}(\rho k, \overline{\mathrm{co}}^{w^*}(A_1 \cup (-A_1))) \geq \epsilon$. Thus $\mathrm{dist}(tk, \overline{\mathrm{co}}^{w^*}(A_1 \cup (-A_1)) > \delta$, a contradiction. So, $tH_\delta \not\subseteq \overline{\mathrm{co}}^{w^*}(A \cup (-A))$ and $\sigma(H_\delta) \leq \rho + \delta$, $\forall 0 < \delta < \epsilon$. Hence, $\sigma(X) \leq \rho$, for every $\sigma(K) < \rho < 1$, and from this fact we conclude that $\sigma(X) \leq \sigma(K)$. □

Proposition 2.6. *If X is a Banach space then $\sigma(X) \leq \tau(X)$.*

Proof. Assume that $\tau(X) < \eta < 1$ and choose an UBABS $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset X \times X^*$ of type η such that $\|f_\alpha\| = 1$ and $\|x_\alpha\| \leq M$, $\forall \alpha < \omega_1$, for some $0 < M < \infty$. Clearly, $\{\pm f_\alpha\}_{\alpha<\omega_1}$ is an w^*-ω_1-polyhedron. Denote $K = \overline{\mathrm{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha<\omega_1})$, $K_0 =$ core (K) and $H = \{z \in K : \lambda(z) = 0\}$. It is easy to see that $|z(x_\alpha)| \leq \eta$ for every $z \in K_0$ and $\alpha < \omega_1$. We claim that $\sigma(K) \leq \eta$. Indeed, let $A \subset K$ be countable. By Lemma 2.3 there exists $\gamma < \omega_1$ such that:

$$A \subset \overline{\mathrm{co}}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H) \subset \overline{\mathrm{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H).$$

Clearly, $\overline{\mathrm{co}}^{w^*}(A \cup (-A)) \subset \overline{\mathrm{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H)$ and for every $\gamma < \rho < \omega_1$ and every $z \in \overline{\mathrm{co}}^{w^*}(\{\pm f_\alpha\}_{\alpha\leq\gamma} \cup H)$ we have $|z(x_\rho)| \leq \eta$.

Hence, for every $\gamma < \rho < \omega_1$ and $\eta < t \leq 1$ we have that $tf_\rho \notin \overline{\mathrm{co}}^{w^*}(A \cup (-A))$. So $\sigma(K) \leq \eta$ and from this fact we conclude that $\sigma(X) \leq \tau(X)$. □

Now we prove for a Banach space X that $\sigma(X) = 1$ iff $\tau(X) = 1$.

Proposition 2.7. *A Banach space X has an UBABS of type η, for some $\eta \in [0, 1)$, iff X^* has a w^*-nonseparable equivalent dual unit ball. So, $\sigma(X) = 1$ iff $\tau(X) = 1$.*

Proof. Firstly, if X has an UBABS of type η, for some $\eta \in [0,1)$ (i.e., $\tau(X) < 1$), then by Prop. 2.6 we have $\sigma(x) < 1$ (i.e., X^* has a w^*-nonseparable equivalent dual unit ball).

Assume now that X is a Banach space with $\sigma(X) < 1$ equipped with an equivalent norm such that $\sigma(B(X^*)) < 1$. Fix $\rho > 0$ with $\sigma(B(X^*)) < \rho < 1$. If $A \subset S(X)$ and $\epsilon \geq 0$ we put:

$$(A,\epsilon)^{\perp} = \{z \in X^* : |z(x)| \leq \epsilon,\ \forall x \in A\} \text{ and } S((A,\epsilon)^{\perp}) = S(X^*) \cap (A,\epsilon)^{\perp}.$$

Clearly $\epsilon B(X^*) + A^{\perp} \subset (A,\epsilon)^{\perp}$.

Claim 0. If $A \subset S(X)$ and $A^{\perp} \neq \{0\}$, then $\epsilon S(X^*) \subset \mathbf{co}(S((A,\epsilon)^{\perp}))$ for $0 \leq \epsilon < 1$.

Indeed, let $u \in \epsilon S(X^*)$ be arbitrary and pick some $v \in A^{\perp} \setminus \{0\}$. We can find $\lambda, \mu > 0$ such that $u + \lambda v, u - \mu v \in S(X^*)$. Thus, $u + \lambda v, u - \mu v \in S((A,\epsilon)^{\perp})$. Let $t \in (0,1)$ be such that $t\lambda + (1-t)(-\mu) = 0$. Then $u = t(u+\lambda v) + (1-t)(u - \mu v) \in \operatorname{co}(S((A,\epsilon)^{\perp}))$.

Claim 1. For every countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exists $f \in S((A,\sqrt{\rho})^{\perp})$ such that $\sqrt{\rho} f \notin \overline{\mathbf{co}}^{w^*}(F \cup (-F))$.

The opposite means that $\sqrt{\rho} S((A,\sqrt{\rho})^{\perp}) \subset \overline{\operatorname{co}}^{w^*}(F \cup (-F))$. By Claim 0 we have $\sqrt{\rho} S(X^*) \subset \operatorname{co}(S((A,\sqrt{\rho})^{\perp}))$. So:

$$\rho B(X^*) \subset \overline{\operatorname{co}}^{w^*}(\rho S(X^*)) \subset \overline{\operatorname{co}}^{w^*}(\sqrt{\rho} S((A,\sqrt{\rho})^{\perp})) \subset \overline{\operatorname{co}}^{w^*}(F \cup (-F)),$$

a contradiction because $\sigma(B(X^*)) < \rho$. So, Claim 1 holds.

Claim 2. There exist $0 \leq \delta < \epsilon \leq 1 - \sqrt{\rho}$ such that for every countable subsets $A \subset S(X)$ and $F \subset S(X^*)$ there exist $f_0 \in S((A,\sqrt{\rho})^{\perp})$ and $x_0 \in S(X)$ such that $f_0(x_0) \geq 1 - \delta$ and $f(x_0) \leq 1 - \epsilon,\ \forall f \in F$.

Denote by $\mathcal{R} = \{r = (r_1, r_2) \in \mathbb{Q} \times \mathbb{Q} : 0 < r_1 < r_2 \leq 1 - \sqrt{\rho}\}$. As $\mathcal{R}$ is countable, we can put $\mathcal{R} = \{r_n\}_{n \geq 1}$. If Claim 2 is false, for every pair $r_n = (r_{n1}, r_{n2}) \in \mathcal{R}$ we can choose countable subsets $A_n \subset S(X)$, $F_n \subset S(X^*)$, $n \geq 1$, such that for every $g \in S((A_n,\sqrt{\rho})^{\perp})$ and every $x \in S(X)$ either $g(x) < 1 - r_{n1}$ or there exists $f \in F_n$ with $f(x) > 1 - r_{n2}$. Let

$A=\cup_{n\geq 1}A_n,\ F=\cup_{n\geq 1}F_n$. By Claim 1 there exists $f_0\in S((A,\sqrt{\rho})^{\perp})$ such that $\sqrt{\rho}f_0\notin \overline{\mathrm{co}}^{w^*}(F\cup(-F))$. By the Hahn-Banach Theorem there exists $y\in S(X)$ such that:

$$\sqrt{\rho}f_0(y)>\sup\{|f(y)|:f\in F\}=:\gamma_0\geq 0.$$

Choose a sequence $\{z_n\}_{n\geq 1}\subset S(X)$ such that:

$$\lim_{n\to\infty} f_0(z_n)=\|f_0\|=1 \text{ and } 1-f_0(z_n)<\tfrac{1}{n}(f_0(y)-\gamma_0),\ n\geq 1. \tag{2}$$

Then $f_0(\frac{z_n+\frac{1}{n}y}{\|z_n+\frac{1}{n}y\|})=1-\delta_n$ with:

$$0\leq\delta_n=\frac{\|z_n+\frac{1}{n}y\|-f_0(z_n)-\frac{1}{n}f_0(y)}{\|z_n+\frac{1}{n}y\|}\leq\frac{1-f_0(z_n)+\frac{1}{n}(1-f_0(y))}{\|z_n+\frac{1}{n}y\|}.$$

Hence, $\lim_{n\to\infty}\delta_n=0$. On the other hand, for every $f\in F$:

$$f(\frac{z_n+\frac{1}{n}y}{\|z_n+\frac{1}{n}y\|})\leq\frac{1+\frac{1}{n}\gamma_0}{\|z_n+\frac{1}{n}y\|}=1-\epsilon_n,$$

where:

$$\epsilon_n=\frac{\|z_n+\frac{1}{n}y\|-1-\frac{1}{n}\gamma_0}{\|z_n+\frac{1}{n}y\|}\leq\frac{1+\frac{1}{n}-1-\frac{1}{n}\gamma_0}{\|z_n+\frac{1}{n}y\|}=\frac{\frac{1}{n}(1-\gamma_0)}{\|z_n+\frac{1}{n}y\|}$$

and

$$\epsilon_n=\frac{\|z_n+\frac{1}{n}y\|-1-\frac{1}{n}\gamma_0}{\|z_n+\frac{1}{n}y\|}>\frac{\|z_n+\frac{1}{n}y\|-f_0(z_n)-\frac{1}{n}f_0(y)}{\|z_n+\frac{1}{n}y\|}=\delta_n\geq 0$$

by (2). Pick any $n\in\mathbb{N}$ such that $\frac{\frac{1}{n}(1-\gamma_0)}{\|z_n+\frac{1}{n}y\|}\leq 1-\sqrt{\rho}$. Then $0\leq\delta_n<\epsilon_n\leq 1-\sqrt{\rho}$ and there is some $m\in\mathbb{N}$ such that $\delta_n\leq r_{m1}<r_{m2}\leq\epsilon_n$. Let $x_0=\frac{z_n+\frac{1}{n}y}{\|z_n+\frac{1}{n}y\|}\in S(X)$ and observe that $f_0\in S((A_m,\sqrt{\rho})^{\perp}),\ f_0(x_0)\geq 1-\delta_n$ and $f(x_0)\leq 1-\epsilon_n,\ \forall f\in F$. Then $f_0\in S((A_m,\sqrt{\rho})^{\perp}),\ f_0(x_0)\geq 1-r_{m1}$ and $f(x_0)\leq 1-\epsilon_n\leq 1-r_{m2},\ \forall f\in F_m$, a contradiction. So, Claim 2 holds.

Let $0\leq\delta<\varepsilon\leq 1-\sqrt{\rho}$ be from Claim 2. We will construct a transfinite sequence $\{(x_\alpha,f_\alpha)\}_{\alpha<\omega_1}\subset S(X)\times S(X^*)$ so that for every $\alpha<\omega_1$:

$$f_\alpha(x_\alpha)\geq 1-\delta \tag{3}$$

and

$$f_\alpha(x_\beta) \leq 1 - \varepsilon \ \text{ if } \ \alpha \neq \beta. \tag{4}$$

On the first step, we take $x_1 \in S(X)$ and $f_1 \in S(X^*)$ such that $f_1(x_1) = 1$. Let $1 < \alpha_0 < \omega_1$ and suppose constructed the family $\{(x_\alpha, f_\alpha) : \alpha < \alpha_0\}$ fulfilling the conditions (3) and (4). Let us apply the Claim 2, putting there $F = \{f_\alpha : \alpha < \alpha_0\}$ and $A = \{x_\alpha : \alpha < \alpha_0\}$. Denote the received elements x_0 and f_0 by x_{α_0} and f_{α_0}. The inequality (3) for $\alpha = \alpha_0$ is satisfied by construction. The inequality (4) for $\alpha = \alpha_0$ and $\beta < \alpha_0$ holds because $f_0 \in S((A, \sqrt{\rho})^\perp)$ and $\varepsilon \leq 1 - \sqrt{\rho}$. For $\beta = \alpha_0$ and $\alpha < \alpha_0$ it follows because $\sup\{f(x_0) : f \in F\} \leq 1 - \varepsilon$. Now the set $\{(\overline{x}_\alpha, \overline{f}_\alpha)\}_{\alpha<\omega_1}$, where $\overline{x}_\alpha = x_\alpha$, $\overline{f}_\alpha = f_\alpha / f_\alpha(x_\alpha)$, $1 \leq \alpha < \omega_1$, is an uncountable bounded (by $(1-\delta)^{-1}$) almost biorthogonal system. □

Proposition 2.8. *Let X be a Banach space such that $\sigma(X) < \frac{1}{3}$. Then $\tau(X) \leq \frac{2\sigma(X)}{1-\sigma(X)}$. So, for every Banach space we have that: (1) $\sigma(X) = 0$ iff $\tau(X) = 0$; (2) $\sigma(X) = 0$ whenever X has an uncountable biorthogonal system.*

Proof. (A) Let $\|\cdot\|$ be an equivalent norm on X such that the corresponding dual unit ball $B(X^*)$ satisfies $\sigma(B(X^*)) < \frac{1}{3}$. It is enough to prove that there exists in X an UBABS of type $\eta \leq \frac{2a}{1-a}$, for every $\sigma(B(X^*)) < a < \frac{1}{3}$. So, fix some $\sigma(B(X^*)) < a < \frac{1}{3}$. By induction we choose a family $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset S(X) \times S(X^*)$ such that:

$$f_\alpha(x_\alpha) > \frac{1-a}{2} \ \text{ but } |f_\alpha(x_\beta)| < a, \ \text{ if } \alpha \neq \beta. \tag{5}$$

Pick $(x_1, f_1) \in S(X) \times S(X^*)$ satisfying $f_1(x_1) = 1$. Let $\alpha < \omega_1$ and assume that we have chosen $\{(x_\beta, f_\beta)\}_{\beta<\alpha} \subset S(X) \times S(X^*)$ fulfilling (5). Denote:

$$A_\alpha = \overline{[\{x_\beta : \beta < \alpha\}]} \ \text{ and } F_\alpha = \overline{\text{co}}^{w^*}(\{\pm f_\beta : \beta < \alpha\} \cup G_0),$$

where $G_0 \subset B(X^*)$ is a countable symmetric subset 1-norming on A_α. By [15, Lemma 4.3] there exists $x_\alpha \in S(X)$ such that $\sup\{|f(x_\alpha)| : f \in F_\alpha\} <$

a. We claim that $\text{dist}(x_\alpha, A_\alpha) > \frac{1-a}{2}$. Indeed, pick $z \in A_\alpha$ and observe that, if $\|z\| < \frac{1+a}{2}$, then clearly $\|z - x_\alpha\| > \frac{1-a}{2}$, and if $\|z\| \geq \frac{1+a}{2}$, then:

$$\|z - x_\alpha\| \geq \sup\{f(z - x_\alpha) : f \in F_\alpha\} \geq$$
$$\geq \|z\| - \sup\{f(x_\alpha) : f \in F_\alpha\} > \frac{1+a}{2} - a = \frac{1-a}{2}.$$

This fact means that, if $Q : X \to \frac{X}{A_\alpha}$ is the canonical quotient mapping, then $\|Q(x_\alpha)\| > \frac{1-a}{2}$. So, as $(\frac{X}{A_\alpha})^* = A_\alpha^\perp$ there exists $f_\alpha \in S(X^*) \cap A_\alpha^\perp$ such that $f_\alpha(x_\alpha) > \frac{1-a}{2}$. Thus we have chosen the pair (x_α, f_α) and this completes the induction.

Now put $\tilde{f}_\alpha = \frac{f_\alpha}{f_\alpha(x_\alpha)}$ and consider the family $\mathfrak{F} = \{(x_\alpha, \tilde{f}_\alpha)\}_{\alpha<\omega_1}$ and observe that:

(a) $\mathfrak{F}$ is bounded because $\|x_\alpha\| = 1$ and:

$$\|\tilde{f}_\alpha\| = \frac{\|f_\alpha\|}{|f_\alpha(x_\alpha)|} < \frac{1}{\frac{1-a}{2}} = \frac{2}{1-a} < \frac{2}{1-\frac{1}{3}} = 3.$$

(b) $\tilde{f}_\alpha(x_\alpha) = 1$ and $|\tilde{f}_\alpha(x_\beta)| = \frac{|f_\alpha(x_\beta)|}{f_\alpha(x_\alpha)} < \frac{a}{\frac{1-a}{2}} = \frac{2a}{1-a} < 1$ if $\alpha \neq \beta$.

So, $\mathfrak{F}$ is an UBABS of type $\eta \leq \frac{2a}{1-a}$.

(B) (1) follows from (A) and Prop. 2.6. (2) follows from the definition of $\tau(X)$ and (1). □

3. **On ω-independence. The Kunen-Shelah property KS_3.** A family $\{x_i\}_{i\in I}$ in a Banach space X is said to be *ω-independent* if for every sequence $(i_n)_{n\geq 1} \subset I$ of distinct indices, and every sequence $(\lambda_n)_{n\geq 1} \subset \mathbb{R}$, the series $\sum_{n=1}^\infty \lambda_n x_{i_n}$ converges (in norm) to 0 iff $\lambda_n = 0$ for every $n \geq 1$ (see [6],[12]). A Banach space X is said to have the *Kunen-Shelah property* KS_3 if X has not an uncountable ω-independent family. Of course, every biorthogonal family is ω-independent (i.e., $KS_3 \Rightarrow KS_2$), but there are ω-independent families which are not merely biorthogonal systems. Here is one example: $X = C([0,1]^{\omega_1})$ and $\{f_\alpha^n\}_{\alpha<\omega_1, n\geq 1}$ defined as

$$f_\alpha^n\left((t_\gamma)_{\gamma<\omega_1}\right) = t_\alpha^n$$

for every $x = (t_\gamma)_{\gamma<\omega_1} \in [0,1]^{\omega_1}$. This family is ω-independent but not a biorthogonal system by the Theorem of Müntz-Szasz (see [11, 15.26 Th.]).

Question 2. Does a Banach space have an uncountable biorthogonal system whenever it has an uncountable ω-independent family?

Unfortunately, the indices $\sigma(X), \tau(X)$ do not separate the properties KS_2 and KS_3, because as we prove in the following, if $X \in KS_3$, then $\sigma(X) = 0$.

Lemma 3.1. *Let X be a Banach space, $\{x_i\}_{1\leq i<\omega_1} \subset X$ an uncountable bounded ω-independent family, $H \subset X$ a closed separable subspace and $N \in \mathbb{N}$. Then there exist ordinal numbers $\rho < \gamma < \omega_1$ such that $x_\rho \notin \overline{co}\,(H \cup \{\pm N x_i\}_{\gamma\leq i<\omega_1})$.*

Proof. Without loss of generality suppose that $\|x_i\| \leq 1,\ \forall i < \omega_1$. Assume that for every pair of ordinal numbers ρ, γ such that $\rho < \gamma < \omega_1$ we have $x_\rho \in \overline{\text{co}}\,(H \cup \{\pm N x_i\}_{\gamma\leq i<\omega_1})$. For $n \in \mathbb{N}$ and $\rho < \gamma < \omega_1$, denote $D_\gamma = \text{co}(\{\pm N x_i\}_{\gamma\leq i<\omega_1})$ and

$$H(\rho,\gamma,n) = \left\{(u,\lambda) \in H \times (0,1] : \exists v \in D_\gamma \text{ with } \|\lambda u + (1-\lambda)v - x_\rho\| < \frac{1}{2n}\right\}.$$

If $\rho < \gamma < \gamma' < \omega_1$ and $n \geq 1$, by the hypothesis and the definition of $H(\rho,\gamma,n)$, we have $H(\rho,\gamma,n) \neq \emptyset, H(\rho,\gamma,n+1) \subset H(\rho,\gamma,n) \supset H(\rho,\gamma',n)$.

For $\beta < \omega_1$ and $n \geq 1$ define:

$$H(\beta,n) = \text{cl}\,(\cup\{H(\rho,\gamma,n) : \beta \leq \rho < \gamma < \omega_1\})\,.$$

where "cl" means the closure in $H \times (0,1]$. Clearly, for $\beta < \beta'$ and $n \geq 1$ we have:

$$\emptyset \neq H(\beta',n) \subset H(\beta,n) \supset H(\beta,n+1).$$

Since $H \times (0,1]$ is hereditarily Lindelöff , for each $n \geq 1$ there exists $\beta_n < \omega_1$ such that for every $\beta_n \leq \beta < \omega_1$ we have $H(\beta,n) = H(\beta_n,n)$. So, for every $(u,\lambda) \in H(\beta_n,n)$ and every $\beta_n \leq \beta < \omega_1$ we have $(u,\lambda) \in H(\beta,n)$, which implies that there exist $\beta \leq \rho < \gamma < \omega_1$ and $v \in D_\gamma$ such that:

$$\|x_\rho - (\lambda u + (1-\lambda)v)\| < \tfrac{1}{n}.$$

Let $\beta_0 = \sup_{n\geq 1}\beta_n$ and fix $\beta_0 \leq \rho < \gamma < \omega_1$ and $n \geq 1$. Pick $(u,\mu) \in H(\rho,\gamma,n)$ and $w \in D_\gamma$ such that $\|x_\rho - (\mu u + (1-\mu)w)\| < \frac{1}{2n}$. Since $(u,\mu) \in H(\beta_0, n) = H(\gamma, n)$, there exist $\gamma \leq \sigma < \theta < \omega_1$ and $v \in D_\theta$ such that $\|x_\sigma - (\mu u + (1-\mu)v)\| < \frac{1}{n}$.

Denote $T = x_\sigma - (\mu u + (1-\mu)v)$. Then we have $\mu u = x_\sigma - T - (1-\mu)v$ and:

$$\|x_\rho - (x_\sigma - T - (1-\mu)v + (1-\mu)w)\| < \frac{1}{2n}.$$

Since $\|T\| < \frac{1}{n}$, we obtain:

$$\|x_\rho - (x_\sigma - (1-\mu)v + (1-\mu)w)\| =$$
$$= \|x_\rho - (x_\sigma - T - (1-\mu)v + (1-\mu)w) - T\| \leq$$
$$\leq \|x_\rho - (x_\sigma - T - (1-\mu)v + (1-\mu)w)\| + \|T\| < \frac{1}{2n} + \frac{1}{n} = \frac{3}{2n}.$$

Since $x_\sigma, v, w \in E_\gamma := \overline{[\{x_i\}_{\gamma\leq i<\omega_1}]}$, if $n \to \infty$ (with ρ, γ fixed), we obtain that $x_\rho \in E_\gamma$ (in particular, this implies that $E_{\beta_0} = E_\beta$, $\forall \beta_0 \leq \beta < \omega_1$). Denote $S = x_\rho - (x_\sigma - (1-\mu)v + (1-\mu)w)$. Then:

$$x_\rho = S + \mu v + (1-\mu)w + x_\sigma - v.$$

Taking into account that $\mu v + (1-\mu)w, -v \in D_\gamma$, $x_\sigma \in \frac{1}{N}D_\gamma$ and that $\|S\| < \frac{3}{2n}$, we finally get $x_\rho \in \mathrm{cl}((1+\frac{1}{N})D_\gamma + D_\gamma) = \mathrm{cl}((2+\frac{1}{N})D_\gamma)$. So, x_ρ is an accumulation point of $F_\gamma := (2+\frac{1}{N})D_\gamma$ (because $x_\rho \in \overline{F_\gamma} \setminus F_\gamma$).

In consequence, we can conclude that every x_i, $\beta_0 \leq i < \omega_1$, is an accumulation point of every F_γ for $\gamma < \omega_1$.

Let $(a_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} a_n = 0$, $\sum_{n\geq 1} a_n = \infty$, and let $b_n = \sup_{m>n} a_m$. Fix $\beta_0 < \tau < \omega_1$. Using the proof of [6, Th. 3], like in [12], we can construct inductively a sequence of signs $\{\epsilon_n\}_{n\geq 1}$, a sequence of real numbers $\{\lambda_r^n\}_{n\geq 1, 1\leq r\leq k(n)}$ and a sequence of ordinals $\{\gamma_r^n\}_{n\geq 1, 1\leq r\leq k(n)}$ such that:

(1) $\sum_{r=1}^{k(n)} |\lambda_r^n| \leq 2N+1$, for every $n \geq 1$.

(2) $\tau < \gamma_1^n < \gamma_2^n < \cdots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \cdots < \omega_1$, for every $n \geq 1$.

(3) $x_\tau + \sum_{n\geq 1} a_n\epsilon_n y_n = 0$, where $y_n = \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}$.

Let us see the two first steps of this argument. Denote $K = \{x_i\}_{\tau<i<\omega_1}$.

Step 1. By the proof of [6, Th. 3] we can find $p_1 \in \mathbb{N}$, a finite sequence of (not necessarily distinct) elements $\{h_n\}_{1\leq n\leq p_1} \subset K$ and a finite sequence of signs $\{\epsilon_n\}_{1\leq n\leq p_1}$ such that:

$$\|x_\tau + \sum_{n=1}^{p_1} a_n\epsilon_n h_n\| < 2^{-1},$$

$$\|x_\tau + \sum_{n=1}^{j} a_n\epsilon_n h_n\| < b_1 + 1 + 2^{-1}, \text{ for } 1 \leq j \leq p_1.$$

Since $h_n \in \mathrm{cl}(F_\beta)$, $\forall\beta_0 \leq \beta < \omega_1$, we can find, for $1 \leq n \leq p_1$, real numbers $\{\lambda_r^n\}_{1\leq r\leq k(n)}$ with $\sum_{r=1}^{k(n)} |\lambda_r^n| \leq 2N+1$, and ordinals $\{\gamma_r^n\}_{r=1}^{k(n)}$ such that:

(a) $\tau < \gamma_1^n < \gamma_2^n < \cdots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \cdots < \omega_1$.
(b) $\|x_\tau + \sum_{n=1}^{p_1} a_n\epsilon_n \cdot \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}\| < 2^{-1}$.
(c) $\|x_\tau + \sum_{n=1}^{j} a_n\epsilon_n \cdot \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}\| < b_1 + 1 + 2^{-1}$, for $1 \leq j \leq p_1$.

Step 2. Let $u_1 = x_\tau + \sum_{n=1}^{p_1} a_n\epsilon_n \cdot \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}$. By the proof of [6, Th. 3] we can find $p_1 < p_2 \in \mathbb{N}$, a finite sequence of (not necessarily distinct) elements $\{h_n\}_{p_1+1\leq n\leq p_2} \subset K$ and a finite sequence of signs $\{\epsilon_n\}_{p_1+1\leq n\leq p_2}$ such that:

$$\|u_1 + \sum_{n=p_1+1}^{p_2} a_n\epsilon_n h_n\| < 2^{-2},$$

$$\|u_1 + \sum_{n=p_1+1}^{j} a_n\epsilon_n h_n\| < b_{p_1} + 2^{-1} + 2^{-2}, \text{ for } p_1 + 1 \leq j \leq p_2.$$

Since $h_n \in \mathrm{cl}(F_\beta)$, $\forall\beta_0 \leq \beta < \omega_1$, we can find, for $p_1 < n \leq p_2$, real numbers $\{\lambda_r^n\}_{1\leq r\leq k(n)}$ with $\sum_{r=1}^{k(n)} |\lambda_r^n| \leq 2N+1$, and ordinals $\{\gamma_r^n\}_{r=1}^{k(n)}$ such that:

(a) $\gamma_{k(p_1)}^{p_1} < \gamma_1^n < \gamma_2^n < \cdots < \gamma_{k(n)}^n < \gamma_1^{n+1} < \cdots < \omega_1$.
(b) $\|u_1 + \sum_{n=p_1+1}^{p_2} a_n\epsilon_n \cdot \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}\| < 2^{-2}$.
(c) $\|u_1 + \sum_{n=p_1+1}^{j} a_n\epsilon_n \cdot \sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}\| < b_{p_1} + 2^{-1} + 2^{-2}$, for $p_1 < j \leq p_2$.

Now by reiteration we obtain the complete construction. It is easy to see that the series $x_\tau + \sum_{n\geq 1} a_n \epsilon_n \left(\sum_{r=1}^{k(n)} \lambda_r^n x_{\gamma_r^n}\right)$ converges to zero. This proves that $\{x_i\}_{i<\omega_1}$ is not ω-independent, a contradiction. So, we can choose $\rho < \gamma < \omega_1$ such that $x_\rho \notin \overline{\text{co}}(H \cup \{\pm N x_i\}_{\gamma \leq i < \omega_1})$. □

Proposition 3.2. *Let a Banach space X have an uncountable ω-independent family $\{x_\alpha\}_{1\leq\alpha<\omega_1}$. Then for every $0 < \eta < 1$, there exist an uncountable subsequence $\{\alpha_i\}_{i<\omega_1} \subset \omega_1$ and an UBABS $\{(z_i, f_i)\}_{i<\omega_1} \subset X \times X^*$ of type η such that $z_i = x_{\alpha_i}$ and $f_i(z_j) = 0$ for $j < i < \omega_1$. So, $\tau(X) = 0$ and X has an ω_1-polyhedron.*

Proof. Let $\{x_i\}_{1\leq i<\omega_1} \subset X$ be an uncountable ω-independent family and suppose, without loss of generality, that $\|x_i\| \leq 1$ for every $i < \omega_1$. Let $N \in \mathbb{N}$ be such that $1/N \leq \eta$. In the following we choose by induction two subsequences of ordinal numbers $\{i_\alpha, j_\alpha\}_{\alpha<\omega_1}, i_\alpha < j_\alpha \leq i_\beta < j_\beta < \omega_1$, for $\alpha < \beta < \omega_1$, such that:

$$x_{i_\alpha} \notin \overline{\text{co}}\left(\overline{[\{x_{i_\beta} : \beta < \alpha\}]} \cup \{\pm N x_j\}_{j_\alpha \leq j < \omega_1}\right). \tag{6}$$

Indeed, let $\alpha < \omega_1$ and assume that we have chosen $\{i_\beta, j_\beta\}_{\beta<\alpha}$ satisfying (6). Put $H = \overline{[\{x_{i_\beta}\}_{\beta<\alpha}]}$ and $\nu = \sup_{\beta<\alpha}\{j_\beta\}$ (if $\alpha = 1$, put $H = \{0\}$ and $\nu = 1$). By Lemma 3.1 there exist $\nu \leq \rho < \gamma < \omega_1$ such that $x_\rho \notin \overline{\text{co}}\,(H \cup \{\pm N x_i\}_{\gamma\leq i<\omega_1})$. So, we put $i_\alpha = \rho$, $j_\alpha = \gamma$ and this completes the induction. Let $z_\alpha = x_{i_\alpha}, \alpha < \omega_1$. By (6) we have $z_\alpha \notin \overline{\text{co}}\left(\overline{[\{z_\beta : \beta < \alpha\}]} \cup \{\pm N z_j\}_{\alpha<j<\omega_1}\right)$. So, by the Hahn-Banach Theorem there exists $f_\alpha \in X^*$ such that:

$$1 = f_\alpha(z_\alpha) > \sup\{f_\alpha(x) : x \in \overline{\text{co}}\left(\overline{[\{z_\beta : \beta < \alpha\}]} \cup \{\pm N z_j\}_{\alpha<j<\omega_1}\right)\}.$$

Clearly, $f_\alpha(z_\beta) = 0$, if $\beta < \alpha$, and $|f_\alpha(N z_\beta)| < 1$, i.e., $|f_\alpha(z_\beta)| < 1/N$, if $\alpha < \beta < \omega_1$. Finally, choosing an uncountable subsequence $A \subset \omega_1$ with $\{\|f_\alpha\| : \alpha \in A\}$ bounded, then $\{(z_\alpha, f_\alpha) : \alpha \in A\}$ is the UBABS of type η we are looking for. □

4. **The Kunen-Shelah property KS_4.** A Banach space X is said to have the *Kunen-Shelah property* KS_4 if X has not an ω_1-polyhedron. The implication $KS_4 \Rightarrow KS_3$ was proved in [3]. It also follows from Prop. 3.2 and from Prop. 7.3 and a result of Sersouri [12].

Proposition 4.1. *Let Z be a Banach space and $X \subset Z$ a closed subspace such that Z/X is separable. Then the following are equivalent: (a) $Z \in KS_4$; (b) $X \in KS_4$.*

Proof. (a)⇒(b). This is obvious.

(b)⇒(a). Assume that $Z \notin KS_4$ and prove that $X \notin KS_4$. By Prop. 2.2 there exists in Z an UBABS $\{(z_\alpha, f_\alpha) : \alpha < \omega_1\}$ of type $\eta \in [0, 1)$ with $\|f_\alpha\| \le M$, $\forall \alpha < \omega_1$, for some $0 < M < \omega_1$. Denote $\epsilon := 1 - \eta$. Since Z/X is separable, there exists an uncountable subset $I \subset \omega_1$ such that, if $Q : Z \to Z/X$ is the canonical quotient mapping, then $\|Qz_\alpha - Qz_\beta\| < \frac{\epsilon}{4M}$ for every $\alpha, \beta \in I$. Fix $\tau \in I$ and denote $y_\alpha = z_\alpha - z_\tau$, $\forall \alpha \in I$. Since $\|Qy_\alpha\| < \frac{\epsilon}{4M}$, there exists $x_\alpha \in X$ such that $\|x_\alpha - y_\alpha\| < \frac{\epsilon}{4M}$, $\forall \alpha \in I$. Then for each $\alpha, \beta \in I$, $\alpha \ne \beta$, we have:

$$f_\alpha(x_\alpha) = f_\alpha(y_\alpha) + f_\alpha(x_\alpha - y_\alpha) \ge f_\alpha(y_\alpha) - M\frac{\epsilon}{4M} = f_\alpha(z_\alpha) - f_\alpha(z_\tau) - \frac{\epsilon}{4} =$$
$$= 1 - f_\alpha(z_\tau) - \frac{\epsilon}{4} > \eta - f_\alpha(z_\tau) + \frac{\epsilon}{4} \ge f_\alpha(z_\beta) - f_\alpha(z_\tau) + \frac{\epsilon}{4} =$$
$$= f_\alpha(y_\beta) + \frac{\epsilon}{4} = f_\alpha(y_\beta) + M\frac{\epsilon}{4M} \ge f_\alpha(x_\beta),$$

which implies that $\{x_\alpha : \alpha \in I\}$ is an uncountable polyhedron in X, i.e., $X \in KS_4$. □

In the following we obtain some characterizations of the property KS_4. Let us see some previous lemmas.

Lemma 4.2. *Let X be a locally convex topological space, $\tau = \sigma(X, X^*)$, $f \in X^* \setminus \{0\}$, $C \subset f^{-1}(1)$ a bounded convex subset and $B = co(C \cup (-C))$. Then C is τ-separable iff B is τ-separable.*

Proof. Clearly, B is τ-separable whenever C is τ-separable. For the converse implication suppose that B is τ-separable and choose a countable subset

$A \subset C$ such that $D := \{tx - (1-t)y : x, y \in A, t \in [0,1]\}$ is τ-dense in B. Now it is an easy exercise to prove that $C \subset \tau\text{-cl}(A)$, i.e., C is τ-separable. □

Lemma 4.3. *Let X be a locally convex topological space, $\tau = \sigma(X, X^*)$, $C \subset X$ a convex subset such that for some $f \in X^*$ there exists a countable subset $\mathcal{R} \subset \mathbb{R}$ satisfying:*

(1) $\emptyset \neq (\inf\{f(x) : x \in C\}, \sup\{f(x) : x \in C\}) \subset \overline{\mathcal{R}}$.
(2) $C_r := \{x \in C : f(x) = r\}$ *is τ-separable, for each* $r \in \mathcal{R}$.

Then C is τ-separable.

Proof. By hypothesis $\inf\{f(x) : x \in C\} < \sup\{f(x) : x \in C\}$. For each $r \in \mathcal{R}$, choose a countable subset $A_r \subset C_r$ such that $C_r \subset \tau\text{-cl}(A_r)$. Let $A = \cup_{r \in \mathcal{R}} A_r$ be a countable subset of C. We claim that A is τ-dense in C. Indeed, pick $z_0 \in C$ arbitrarily and let U be a τ-neighborhood of z_0 in C. By hypothesis, there exists some $r \in \mathcal{R}$ such that $C_r \cap U \neq \emptyset$. So, $A_r \cap U \neq \emptyset$, whence $A \cap U \neq \emptyset$. □

Proposition 4.4. *Let X be a Banach space. The following are equivalent:*

(1) $X \in KS_4$.
(2) *$K \subset X^*$ is w^*-separable whenever K is a w^*-compact convex symmetric subset such that $\|\cdot\|$-int$(K) \neq \emptyset$.*
(3) *$K \subset X^*$ is w^*-separable whenever K is a w^*-compact convex symmetric subset, i.e., $\sigma(X) = 1 = \tau(X)$.*
(4) *$K \subset X^*$ is w^*-separable whenever K is a w^*-closed convex symmetric subset.*
(5) *$K \subset X^*$ is w^*-separable whenever K is a w^*-closed convex subset.*

Proof. (1) $\Rightarrow$ (2). This follows from Prop. 2.7 and Prop. 2.2, because if $K \subset X^*$ is a w^*-compact convex symmetric subset such that $\|\cdot\|\text{-int}(K) \neq \emptyset$, then K is the dual unit ball of X^* when X is equipped with the equivalent norm $|\cdot|$ such that $|x| = \sup\{x^*(x) : x \in K\}$ for every $x \in X$.

(2) $\Rightarrow$ (3). Let $K \subset X^*$ be a w^*-compact convex symmetric subset and denote $K_n = K + \frac{1}{n}B(X^*)$, which is w^*-compact convex symmetric subset of X^* with nonempty interior. By (2) there is a countable family $\{x_{n,m}\}_{m\geq 1} \subset K_n$ such that $K_n = \overline{\{x_{n,m} : m \geq 1\}}^{w^*}$ for every $n \geq 1$. Pick $k_{n,m} \in K$ such that $\|k_{n,m} - x_{n,m}\| \leq \frac{1}{n}$. Then it is easy to see that $K = \overline{\{k_{n,m} : n, m \geq 1\}}^{w^*}$.

(3) $\Rightarrow$ (4). Let $K \subset X^*$ be a w^*-closed convex symmetric subset and denote $K_n = K \cap nB(X^*)$. By (3) K_n is w^*-separable and so K, because $K = \cup_{n\geq 1} K_n$.

(4) $\Rightarrow$ (5). It is enough to prove that if $K \subset X^*$ is a w^*-compact convex subset, then K is w^*-separable. Without loss of generality, assume that $0 \notin K$. Let $f \in X$ be such that $0 < \min\{f(k) : k \in K\} \leq \max\{f(k) : k \in K\} < \infty$. If $t \in [\min\{f(k) : k \in K\}, \max\{f(k) : k \in K\}]$, denote $K_t = \{k \in K : f(k) = t\}$ and $C_t = \overline{\text{co}}^{w^*}(K_t \cup (-K_t))$. By (4) and Lemma 4.2 each C_t is w^*-separable. So, from Lemma 4.3 we get that K is w^*-separable.

(5) $\Rightarrow$ (1). Suppose that there exists in X a bounded ω_1-polyhedron $\{x_i\}_{i<\omega_1}$. By Prop. 2.2, there exists in X an UBABS $\{(x_\alpha, f_\alpha)\}_{\alpha<\omega_1} \subset X \times X^*$ such that $\|f_\alpha\| = 1$, $\|x_\alpha\| \leq M$, $f_\alpha(x_\alpha) = 1$ and $f_\alpha(x_\beta) \leq 1 - \epsilon$, for every $\alpha, \beta < \omega_1, \alpha \neq \beta$, and some $1 \geq \epsilon > 0$, $1 \leq M < +\infty$. Let $K = \overline{\text{co}}^{w^*}(\{f_\alpha : \alpha < \omega_1\})$. Consider the w^*-open slices $U_\alpha = \{k \in K : k(x_\alpha) > 1 - \frac{\epsilon}{3}\}$ for all $\alpha < \omega_1$. Then U_α is a w^*-open neighborhood of f_α in K and we can easily realize that $U_\alpha \cap U_\beta = \emptyset$, whenever $\alpha \neq \beta$. Thus K is w^*-nonseparable, a contradiction to (5). So, $X \in KS_4$. □

Question 3. Let X be a Banach space. If $\tau(X) < 1$, is $\tau(X) = 0$? If $\tau(X) = 0$, does X have an uncountable ω-independent family?

5. **The Finet-Godefroy indices.** If X is a Banach space, the Finet-Godefroy indices $d_\infty(X)$ and $\mu(X)$ were introduced in [1] and defined as follows:

$$d_\infty(X) = \inf\{d(X, Y) : Y \text{ subspace of } \ell_\infty(\mathbb{N})\}$$

where $d(X,Y)$ is the Banach-Mazur distance. Clearly, $d_\infty(X)$ depends upon the norm $\|\cdot\|$ of X and we see easily that: (i) $d_\infty(X) \in [1,\infty]$; (ii) $d_\infty(X) < \infty$ iff X is isomorphic to a subspace of $\ell_\infty(\mathbb{N})$; (iii) $d_\infty(X, \|\cdot\|) = 1$ iff $(X, \|\cdot\|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$ iff the dual unit ball $B(X^*)$ is w^*-separable. The corresponding isomorphic invariant index is:

$$\mu(X) = \sup\{d_\infty(X, |\cdot|)\}$$

where the supremum is computed over the set of equivalent norms on X.

Proposition 5.1. *Let X be a Banach space. Then:*

(1) $\mu(X) = \sigma(X)^{-1}$ $(0^{-1} = \infty)$.

(2) *If X has an uncountable ω-independent system, then $\mu(X) = \infty$.*

Proof. (1) This follows from [1, Lemma III.1] and a simple calculation.

(2) By Prop. 3.2 and Prop. 2.8 we get that $\sigma(X) = 0$. Now apply (1). ☐

The following questions are proposed in [1] :

(1) It is clear that $\mu(X) = 1$ if X is separable. Is the converse true?

(2) Does there exist a nonseparable Banach space X such that every quotient of X is isometric to a subspace of $\ell_\infty(\mathbb{N})$?

In the following we answer these questions.

Proposition 5.2. *Let X be a Banach space. The following are equivalent:*

(1) $X \in KS_4$.

(2) *Every quotient of $(X, |\cdot|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$, for every equivalent norm $|\cdot|$ of X.*

(3) $\mu(X) = 1$.

(4) *Every quotient of X satisfies the property KS_4.*

Proof. (1) $\Rightarrow$ (2). Let $|\cdot|$ be an equivalent norm on X, $Y \subset X$ a closed subspace and $Z = (X/Y, |\cdot|)$ the corresponding quotient space. Clearly, we have $(B(Z^*), w^*) = (B(Y^\perp), w^*)$. But $(B(Y^\perp), w^*)$ is w^*-separable by Prop. 4.4. So, Z is isometric to a subspace of $\ell_\infty(\mathbb{N})$.

(2) $\Rightarrow$ (3). By (2) $d_\infty(X, |\cdot|) = 1$ for every equivalent norm $|\cdot|$ on X. So, $\mu(X) = 1$.

(3) $\Rightarrow$ (4). Since $\mu(X/Y) \le \mu(X)$ for every quotient X/Y (see [1, Th. III-2]), (3) implies that $\mu(X/Y) = 1$, i.e., $\sigma(X/Y) = 1$. So, by Prop. 4.4 we get that $X/Y \in KS_4$.

(4) $\Rightarrow$ (1). This is obvious. □

Corollary 5.3. *If X is either the space $C(K)$, under CH and K being the Kunen compact space, or the space S of Shelah, under $\diamondsuit_{\aleph_1}$, then X is nonseparable, $\mu(X) = 1$ and every quotient of $(X, |\cdot|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$, for every equivalent norm $|\cdot|$ of X.*

Proof. This follows from Prop. 5.2 since in both cases $X \in KS_4$ (see Section 6). □

Remarks. (1) The fact that every quotient of $(X, |\cdot|)$ is isometric to a subspace of $\ell_\infty(\mathbb{N})$ for every equivalent norm $|\cdot|$ of X, when $X = C(K)$, K being the Kunen compact, was shown in [5, Cor. 4.5].

(2) In [1] is asked if $\mu(X) = \infty$ whenever a Banach space X satisfies $\mu(X) > 1$. In fact, it is not known a Banach space X such that $1 < \mu(X) < \infty$. Observe that $1 < \mu(X) < \infty$ implies that $X \in KS_3$ but $X \notin KS_4$, because: (i) $1 < \mu(X) < \infty$ iff $1 > \sigma(X) > 0$ by Prop. 5.1; (ii) $1 > \sigma(X)$ iff $X \notin KS_4$ by Prop. 4.4; (iii) and $\sigma(X) > 0$ implies $X \in KS_3$ by Prop. 3.2 and Prop. 2.8.

6. **The Kunen-Shelah property KS_5.** Let θ be an ordinal. A *convex right-separated θ-family* in a Banach space X is a bounded family $\{x_i\}_{i<\theta} \subset X$ such that $x_j \notin \overline{\text{co}}(\{x_i : j < i < \theta\})$ for every $j \in \theta$. A family of convex closed bounded subsets $\{C_\alpha\}_{\alpha<\theta}$ in the Banach space X is said to be a *contractive* (resp. *expansive*) *θ-onion* iff $C_\alpha \subsetneqq C_\beta$ (resp. $C_\beta \subsetneqq C_\alpha$) whenever $\beta < \alpha < \theta$. It is easy to prove that X has a contractive θ-onion iff X has a convex right-separated θ-family. In the dual Banach space X^* one can define a contractive (resp. expansive) w^*-θ-onion in a analogous way, using the w^*-topology instead of the w-topology.

A Banach space X is said to have the *Kunen-Shelah property* KS_5 if X has not a contractive uncountable onion. If X has a τ-polyhedron $\{x_\alpha : \alpha < \tau\}$, it is clear that $\{C_\alpha : \alpha < \tau\},\ C_\alpha = \overline{\text{co}}(\{x_\beta : \alpha < \beta < \tau\})$, is a contractive τ-onion. So, the property KS_5 implies KS_4, whence by Prop. 3.2 we get $KS_5 \Rightarrow KS_3$, a result proved by Sersouri in [12].

Proposition 6.1. *Let X be a Banach space. Then:*

(1) X has a contractive ω_1-onion iff X^* has an expansive w^*-ω_1-onion.

(2) X has an expansive ω_1-onion iff X^* has a contractive w^*-ω_1-onion.

(3) X is nonseparable iff X^* has a contractive w^*-ω_1-onion.

Proof. (1) Assume that X has a contractive ω_1-onion, i.e., there exists a sequence $\{x_\alpha\}_{\alpha<\omega_1} \subset B(X)$ such that $x_\alpha \notin \overline{\text{co}}(\{x_\beta\}_{\alpha<\beta<\omega_1})$. By the Hahn-Banach Theorem there exists $f_\alpha \in X^*$ such that:

$$f_\alpha(x_\alpha) > \sup\{f_\alpha(x_\beta) : \alpha < \beta < \omega_1\} =: e_\alpha.$$

By passing to a subsequence, we can suppose that there exist $0 < \epsilon, M < \infty$ and $r \in \mathbb{R}$ such that $\|f_\alpha\| \leq M,\ f_\alpha(x_\alpha) - e_\alpha \geq \epsilon > 0$ and $|r - f_\alpha(x_\alpha)| \leq \frac{\epsilon}{4},\ \forall \alpha < \omega_1$. Hence, if $\beta < \alpha < \omega_1$, we have:

$$f_\alpha(x_\alpha) \geq r - \frac{\epsilon}{4} > r - \frac{3\epsilon}{4} \geq f_\beta(x_\beta) - \epsilon \geq e_\beta \geq f_\beta(x_\alpha),$$

which implies that $f_\alpha \notin \overline{\text{co}}^{w^*}(\{f_\beta : \beta < \alpha\}) =: K_\alpha$, i.e., $\{K_\alpha : \alpha < \omega_1\}$ is an expansive w^*-ω_1-onion in X^*.

The converse implication is analogous.

(2) Use the same argument that in (1).

(3) Apply (2) and that X has an expansive ω_1-onion iff X is nonseparable. $\square$

A Banach space has the property $HL(1)$ (in short, $X \in HL(1)$) whenever in every family of open semi-spaces $\{U_i\}_{i\in I}$ of X there exists a countable subset $\{i_n\}_{n\geq 1} \subset I$ such that $\cup_{n\geq 1} U_{i_n} = \cup_{i\in I} U_i$, i.e., every closed convex subset of X is the intersection of a countable family of closed semi-spaces of X.

Proposition 6.2. *Let X be a Banach space. Then the following are equivalent: (1) $X \in KS_5$; (2) Every convex subset of X^* is w^*-separable; (3) $X \in HL(1)$.*

Proof. (1) $\Leftrightarrow$ (2). By Prop. 6.1, X has not a contractive uncountable onion iff X^* has not an expansive uncountable w^*-onion and it is trivial to prove that this occurs iff every convex subset of X^* is w^*-separable.

(2) $\Rightarrow$ (3). Suppose that $X \notin HL(1)$ and let $\mathfrak{F} = \{U_i\}_{i<\omega_1}$ be an uncountable family of open semi-spaces of X such that $\mathfrak{F}$ has not a countable subcover. Assume that $U_i = \{x \in X : x_i^*(x) < a_i\}$, with $a_i \neq 0$, for all $i < \omega_1$ (if $a_i = 0$, for some $i < \omega_1$, we put the family $U_{in} = \{x \in X : x_i^*(x) < -\frac{1}{n}\}, n \geq 1$, instead of U_i). Dividing by $|a_i|$, we can suppose that each U_i has the expression $U_i = \{x \in X : y_i^*(x) < \epsilon_i\}$ with $\epsilon_i = \pm 1$ and $y_i^* = x_i^*/|a_i|$. Putting $\mathfrak{F}_1 = \{U_i \in \mathfrak{F} : \epsilon_i = +1\}$ and $\mathfrak{F}_2 = \{U_i \in \mathfrak{F} : \epsilon_i = -1\}$, it is clear that either $\mathfrak{F}_1$ or $\mathfrak{F}_2$ has not countable subcover.

Assume that $\mathfrak{F}_1$ doesn't admit a countable subcover (the argument for $\mathfrak{F}_2$ is similar). So, there exists an uncountable family $\{V_\alpha : \alpha < \omega_1\} \subset \mathfrak{F}_1$, $V_\alpha = \{x \in X : z_\alpha^*(x) < 1\}$, such that there exists $x_\alpha \in V_\alpha \setminus \cup_{\beta<\alpha} V_\beta$, $\forall \alpha < \omega_1$. Put $A = \text{co}\{z_i^*\}_{i<\omega_1}$, which is w^*-separable, by hypothesis. Thus, we can find $\rho < \omega_1$ such that $A \subset \overline{\text{co}}^{w^*}(\{z_i^*\}_{i\leq\rho})$. Pick $\rho < \alpha < \omega_1$. As $x_\alpha \in V_\alpha \setminus \cup_{\beta<\alpha} V_\beta$, we get that $z_\alpha^*(x_\alpha) < 1$ and $z_\beta^*(x_\alpha) \geq 1$ for every $\beta < \alpha$. Let $C = \{x^* \in X^* : x^*(x_\alpha) \geq 1\}$, which is a convex w^*-closed subset of X^*. Since $z_i^* \in C$ for all $i \leq \rho$, it follows that $A \subset C$. So, $z_\alpha^* \notin C$ and $z_\alpha^* \in A$, a contradiction which proves (3).

(3) $\Rightarrow$ (1). Suppose that X has a contractive ω_1-onion $\{C_\alpha\}_{\alpha<\omega_1}$. We choose vectors $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and a sequence of open semi-spaces $\{U_\alpha\}_{\alpha<\omega_1}$ such that $x_\alpha \in U_\alpha$ and $U_\alpha \cap C_{\alpha+1} = \emptyset$. Clearly, no countable subfamily of $\{U_\alpha\}_{\alpha<\omega_1}$ covers $\{x_\alpha\}_{\alpha<\omega_1}$, which contradicts to (3). $\square$

Remark. If X is a Banach space, we put $X \in L(1)$ if from every cover of X by open semi-spaces we can choose a countable subcover. Clearly, X has the property (C) of Corson iff $X \in L(1)$. Since $X \in HL(1) \Rightarrow X \in L(1)$, we have that $X \in KS_5$ implies $X \in$(C).

Proposition 6.3. *If X is either the space $C(K)$, under CH and K being the Kunen compact space, or the space S of Shelah, under $\diamondsuit_{\aleph_1}$, then $X \in KS_5$*

Proof. The space $C(K)$, K being the Kunen compact space, satisfies $C(K) \in KS_5$ because for every uncountable family $\{x_i : i \in I\} \subset C(K)$, there exists $j \in I$ such that $x_j \in \text{ wcl}(\{x_i : i \in I \setminus \{j\}\})$. It is clear that a space with this property cannot have an ω_1-onion.

The space S of Shelah satisfies (see [13, Lemma 5.2]) that if $\{y_i\}_{i<\omega_1} \subset S$ is an uncountable sequence, then $\forall \epsilon > 0, \forall n \geq 1$, there exist $i_0 < i_1 < ... < i_n < \omega_1$ such that:

$$\|y_{i_0} - \frac{1}{n}(y_{i_1} + ... + y_{i_n})\| \leq \frac{1}{n}\|y_{i_0}\| + \epsilon. \tag{7}$$

Assume that S has an ω_1-onion $\{C_\alpha : 1 \leq \alpha < \omega_1\}$ with $C_1 \subset B(S)$. Choose $x_\alpha \in C_\alpha \setminus C_{\alpha+1}$ and let $\eta_\alpha := \text{ dist}(x_\alpha, C_{\alpha+1})$ which satisfies $\eta_\alpha > 0$. By passing to a subsequence, it can be assumed that $\eta_\alpha \geq \eta > 0, \ \forall \alpha < \omega_1$. Let $m \in \mathbb{N}$ be such that $\frac{1}{m} < \frac{\eta}{2}$. By (7) there exists $i_0 < i_1 < ... < i_m < \omega_1$ such that:

$$\|x_{i_0} - \frac{1}{m}(x_{i_1} + ... + x_{i_m})\| \leq \frac{1}{m}\|x_{i_0}\| + \frac{\eta}{2} < \eta.$$

Since $\frac{1}{m}(x_{i_1} + ... + x_{i_m}) \in C_{i_0+1}$ and $\text{ dist}(x_{i_0}, C_{i_0+1}) \geq \eta$, we get a contradiction which proves that $S \in KS_5$. □

7. KS_4 **and** KS_5 **are equivalent.** If X is Asplund or has the property (C) of Corson, it is easy to prove that $X \in KS_4 \Leftrightarrow X \in KS_5$. In the following we prove the equivalence $KS_5 \Leftrightarrow KS_4$ in general. A sequence $\{C_\alpha : \alpha < \omega_1\}$ of convex closed bounded subset of a Banach space X is said to be a *generalized* ω_1*-onion* iff $\emptyset \neq C_\alpha \subset C_\beta$, for $\beta < \alpha$, and there exists a subsequence $\{\alpha_\beta\}_{\beta<\omega_1} \subset \omega_1$, with $\alpha_{\beta_1} < \alpha_{\beta_2}$ if $\beta_1 < \beta_2$, such that $C_{\alpha_{\beta_1}} \neq C_{\alpha_{\beta_2}}$, i.e., $\{C_{\alpha_\beta} : \beta < \omega_1\}$ is an ω_1-onion. If $C \subset X$ is a subset,

denote by cone(C) the closed convex cone generated by C. Observe that, if C is a convex subset, then cone$(C) =$ cl$(\cup_{\lambda\geq 0}\lambda C)$.

Lemma 7.1. *Let X be a Banach space, $C \subset X$ a convex closed separable subset of X and $\{C_\alpha : 1 \leq \alpha < \omega_1\}$ a generalized ω_1-onion of X.*

(1) *If dist$(C, C_\alpha) = 0$ for every $\alpha < \omega_1$, then for every $\epsilon > 0$ there exists $c_\epsilon \in C$ such that dist$(c_\epsilon, C_\alpha) \leq \epsilon$ for every $\alpha < \omega_1$.*

(2) *There are two disjoint alternatives, namely:*

  (A) *either there exist two ordinals $\beta < \alpha < \omega_1$ and $z \in C_\beta$ such that $z \notin \overline{co}([C] \cup \ cone(C_\alpha))$,*

  (B) *or for every pair of ordinals $\beta < \alpha < \omega_1$ we have $C_\beta \subset \overline{co}([C] \cup$ cone$(C_\alpha))$. In this case, we have:*

$$\overline{co}([C] \cup \ cone(C_\alpha)) = \overline{co}([C] \cup \ cone(C_\beta)), \ \forall \alpha, \beta < \omega_1,$$

  and for every $\epsilon > 0$ there exists $c_\epsilon \in X$ such that dist$(c_\epsilon, C_\alpha) \leq \epsilon$ for every $\alpha < \omega_1$.

Proof. (1) For every $\alpha < \omega_1$ and $n \geq 1$ let $C(\alpha, n) = \{x \in C : \text{ dist}(x, C_\alpha) \leq 1/n\}$. Then $\{C(\alpha, n) : \alpha < \omega_1\}$ is a family of nonempty closed convex subset such that $C(\alpha, n) \supset C(\beta, n)$, if $\alpha < \beta$, with the countable intersection property. Since C is separable, we conclude that $\cap_{\alpha<\omega_1} C(\alpha, n) \neq \emptyset$ for every $n \geq 1$. So, if for every $n \geq 1$ we pick $c_n \in \cap_{\alpha<\omega_1} C(\alpha, n)$, then dist$(c_n, C_\alpha) \leq 1/n$ for every $\alpha < \omega_1$.

(2) Clearly, the alternatives (A) and (B) are disjoint. Suppose that (B) holds. Since $[C]$ is separable there exist two ordinals $\beta_0 < \alpha_0 < \omega_1$ and $z_0 \in C_{\beta_0} \setminus C_{\alpha_0}$ such that $z_0 \notin \overline{[C]}$ but $z_0 \in \overline{\text{co}}([C] \cup \text{ cone}(C_\alpha))$ for every $\alpha < \omega_1$.

Claim. If $H = \overline{[C \cup \{z_0\}]}$, then dist$(H, C_\alpha) = 0$ for every $\alpha < \omega_1$.

Indeed, let $\epsilon_0 = \text{ dist}(z_0, \overline{[C]})$ and $n_0 \geq 1$ such that $\frac{2}{n_0} < \epsilon_0$. Observe that for every $\alpha < \omega_1$ and $\epsilon > 0$ we can choose $\lambda \in [0, 1), \ \mu > 0, \ w \in [C]$ and

$v \in C_\alpha$ such that:

$$\|\lambda w + (1-\lambda)\mu v - z_0\| \leq \epsilon. \tag{8}$$

Let $M > 0$ be such that $C_1 \subset B(0; M)$. We claim that if we pick $\alpha < \omega_1$, $n \geq n_0$, $\lambda \in [0,1)$, $\mu > 0$, $w \in [C]$ and $v \in C_\alpha$ fulfilling (8) with $\epsilon = 1/n$, then $(1-\lambda)\mu \geq \frac{1}{n_0 M}$. Indeed, in the contrary case we would have:

$$\epsilon_0 \leq \|\lambda w - z_0\| = \|\lambda w + (1-\lambda)\mu v - z_0 - (1-\lambda)\mu v\| \leq$$

$$\leq \|\lambda w + (1-\lambda)\mu v - z_0\| + \|(1-\lambda)\mu v\| \leq \frac{1}{n_0} + \frac{1}{n_0} < \epsilon_0,$$

which is a contradiction. So, for every $\alpha < \omega_1$, $n \geq n_0$, $\lambda \in [0,1)$, $\mu > 0$, $w \in C$ and $v \in C_\alpha$ fulfilling (8) with $\epsilon = 1/n$ we have:

$$\|\frac{z_0}{(1-\lambda)\mu} - \frac{\lambda}{(1-\lambda)\mu} w - v\| \leq \frac{1}{(1-\lambda)\mu n} \leq \frac{n_0 M}{n}$$

and this proves that $\mathrm{dist}(H, C_\alpha) = 0$ for every $\alpha < \omega_1$.

As H is separable, given $\epsilon > 0$, applying (1) we can choose a vector $c_\epsilon \in X$ such that $\mathrm{dist}(c_\epsilon, C_\alpha) \leq \epsilon$ for every $\alpha < \omega_1$, and this completes the proof. □

Proposition 7.2. *Let X be a Banach space without the property (C) of Corson. Then there exists a sequence $\{(y_\alpha, y_\alpha^*) : \alpha < \omega_1\} \subset X \times X^*$ such that $y_\alpha^*(y_\alpha) = 1$ for all $\alpha < \omega_1$ but $y_\alpha^*(y_\beta) = 0$, if $\beta < \alpha$, and $y_\alpha^*(y_\beta) \leq 0$, if $\beta > \alpha$. So, X has a ω_1-polyhedron and $X \notin KS_4$.*

Proof. Since X doesn't satisfy the property (C) of Corson, it is easy to see that there exists in X a ω_1-onion $\{C_\alpha : \alpha < \omega_1\}$ such that $\cap_{\alpha<\omega_1} C_\alpha = \emptyset$. Using a transfinite inductive process with ω_1 steps we construct:

(1) A sequence of numbers $\{n_\alpha : \alpha < \omega_1\}$ with $n_\alpha \in \{0,1\}$ such that if $p(\alpha) = |\{\beta \leq \alpha : n_\beta = 1\}|$ then $p(\alpha) < \aleph_0$.
(2) Two sequences of ordinals $\{\rho_\gamma, \tau_\gamma : \gamma < \omega_1\}$ such that $1 \leq \rho_\gamma < \tau_\gamma \leq \rho_\beta < \omega_1$ if $\gamma < \beta < \omega_1$.
(3) For each $\alpha < \omega_1$ a generalized ω_1-onion $\{C_\beta^{(\alpha)} : \rho_\alpha \leq \beta < \omega_1\}$ such that $C_\gamma \supset C_\gamma^{(\alpha)} \supset C_\gamma^{(\beta)} \neq \emptyset$ if $\alpha \leq \beta < \omega_1$ and $\rho_\beta \leq \gamma < \omega_1$.

(4) If $n_\alpha = 0$ we choose an element $y_\alpha \in C^{(\alpha)}_{\rho_\alpha}$ such that if $H_\alpha = \overline{[\{y_\beta : \beta < \alpha, n_\beta = 0\}]}$ then $y_\alpha \notin \overline{\text{co}}(H_\alpha \cup \text{ cone}(C^{(\alpha)}_{\tau_\alpha}))$. Also, in this case we demand that $C^{(\alpha)}_\gamma = \cap_{\beta<\alpha} C^{(\beta)}_\gamma$ for every $\rho_\alpha \leq \gamma < \omega_1$.

(5) If $n_\alpha = 1$ we do not choose the element y_α. Instead of we pick a vector $a_{p(\alpha)} \in X$ such that $C^{(\alpha)}_\beta \subset B(a_{p(\alpha)}, 2^{-p(\alpha)})$ for every $\tau_\alpha \leq \beta < \omega_1$, which will imply that:

$$\text{diam}(C^{(\alpha)}_\beta) \leq 2^{-p(\alpha)+1} \text{ and } \text{dist}(a_{p(\alpha)}, C^{(\alpha)}_\beta) \leq 2^{-p(\alpha)}, \ \forall \tau_\alpha \leq \beta < \omega_1.$$

Begin the construction.

Step 1. In this step we choose $n_1 = 0,\ \rho_1 = 1,\ \tau_1 = 2,\ C^{(1)}_\beta = C_\beta$, for every $1 \leq \beta < \omega_1$, $y_1 \in C_1 \setminus C_2$ arbitrary and $H_1 = \{0\}$.

Step $\alpha + 1 < \omega_1$. Suppose constructed all the steps $\beta \leq \alpha$ satisfying the above requirements and construct the step $\alpha+1$. By hypothesis $\{C^{(\alpha)}_\beta : \tau_\alpha \leq \beta < \omega_1\}$ is a generalized ω_1-onion. By Lemma 7.1 there are two disjoint alternatives:

(A) There exist two ordinals $\tau_\alpha \leq \beta_0 < \alpha_0 < \omega_1$ and a vector $z_0 \in C^{(\alpha)}_{\beta_0}$ such that $z_0 \notin \overline{\text{co}}(H_\alpha \cup \text{ cone}(C^{(\alpha)}_{\alpha_0}))$. In this case we do $\rho_{\alpha+1} = \beta_0,\ \tau_{\alpha+1} = \alpha_0,\ n_{\alpha+1} = 0,\ y_{\alpha+1} = z_0$ and $C^{(\alpha+1)}_\beta = C^{(\alpha)}_\beta$ for every $\rho_{\alpha+1} \leq \beta < \omega_1$.

(B) If (A) doesn't hold, there exists $c \in X$ such that $\text{dist}(c, C^{(\alpha)}_\beta) \leq 2^{-(p(\alpha)+2)}$ for every $\tau_\alpha \leq \beta < \omega_1$. In this case we do $n_{\alpha+1} = 1,\ p(\alpha+1) = p(\alpha) + 1,\ \rho_{\alpha+1} = \tau_\alpha,\ \tau_{\alpha+1} = \tau_\alpha + 1,\ a_{p(\alpha+1)} = c$ and $C^{(\alpha+1)}_\beta = B(a_{p(\alpha+1)}, 2^{-p(\alpha+1)}) \cap C^{(\alpha)}_\beta$ for every $\rho_{\alpha+1} \leq \beta < \omega_1$. Since $n_{\alpha+1} = 1$ we do not choose $y_{\alpha+1}$.

Step $\alpha < \omega_1$, α a limit ordinal. Let $\alpha < \omega_1$ be a limit ordinal, suppose constructed all the steps $\beta < \alpha$ satisfying the above requirements and construct the step α.

Claim : $|\{\beta < \alpha : n_\beta = 1\}| < \aleph_0$**.**

Indeed, in the contrary case we would have a sequence of ordinals $\{\beta_m\}_{m\geq 1} \uparrow \alpha,\ \beta_m < \beta_{m+1} < \alpha$, such that $n_{\beta_m} = 1$ for every $m \geq 1$.

Obviously $p(\beta_m) \uparrow +\infty$ when $m \to \infty$. The sequence $\{a_{p(\beta_m)}\}_{m\geq 1}$ is a Cauchy sequence. Indeed, if $r < s$ are two integers, for every $\tau_{\beta_s} \leq \beta < \omega_1$, since $C_\beta^{(\beta_s)} \subset C_\beta^{(\beta_r)}$, we have:

$$\operatorname{dist}(a_{p(\beta_r)}, a_{p(\beta_s)}) \leq \operatorname{dist}(a_{p(\beta_r)}, C_\beta^{(\beta_r)}) + \operatorname{diam}(C_\beta^{(\beta_r)}) + \operatorname{dist}(a_{p(\beta_s)}, C_\beta^{(\beta_r)}) \leq$$
$$\leq 2^{-p(\beta_r)} + 2^{-p(\beta_r)+1} + 2^{-p(\beta_s)} \underset{r,s\to\infty}{\longrightarrow} 0.$$

Let $a_0 := \lim_{m\to\infty} a_{p(\beta_m)}$ and $\gamma_0 = \sup\{\tau_\beta : \beta < \alpha\}$. Then $a_0 \in C_\gamma$ for every $\gamma_0 \leq \gamma < \omega_1$ because:

$$\operatorname{dist}(a_0, C_\gamma) \leq \operatorname{dist}(a_0, a_{p(\beta_m)}) + \operatorname{dist}(a_{p(\beta_m)}, C_\gamma^{(\beta_m)}) \underset{m\to\infty}{\longrightarrow} 0.$$

Hence $\cap_{\alpha<\omega_1} C_\alpha \neq \emptyset$, a contradiction which proves the Claim.

Denote as above $\gamma_0 = \sup\{\tau_\beta : \beta < \alpha\}$ and let $D_\gamma := \cap_{\beta<\alpha} C_\gamma^{(\beta)}$ for every $\gamma_0 \leq \gamma < \omega_1$. By the Claim and the construction of the previous steps we have that:

(a) There exists an ordinal $\delta_0 < \alpha$ such that $n_\delta = 0$ for every $\delta_0 \leq \delta < \alpha$. So, $p(\delta) = p(\delta_0)$ for every $\delta \in [\delta_0, \alpha)$.

(b) For every $\gamma_0 \leq \gamma < \omega_1$ we have $D_\gamma = C_\gamma^{(\delta_0)}$, which by induction hypothesis implies that $\{D_\gamma : \gamma_0 \leq \gamma < \omega_1\}$ is a generalized ω_1-onion.

If $H_\alpha := \overline{[\{y_\beta : \beta < \alpha, n_\beta = 0\}]}$, by Lemma 7.1 we have the following disjoint alternatives:

(A) There are two ordinals $\gamma_0 \leq \beta_0 < \alpha_0 < \omega_1$ and a vector $z_0 \in D_{\beta_0}$ such that $z_0 \notin \overline{\operatorname{co}}(H_\alpha \cup \operatorname{cone}(D_{\alpha_0}))$. In this case we do $\rho_\alpha = \beta_0,\ \tau_\alpha = \alpha_0,\ n_\alpha = 0,\ y_\alpha = z_0$ and $C_\beta^{(\alpha)} = D_\beta$ for every $\rho_\alpha \leq \beta < \omega_1$.

(B) If (A) doesn't hold, there exists $c \in X$ such that $\operatorname{dist}(c, D_\gamma) \leq 2^{-p(\delta_0)+2}$ for every $\gamma_0 \leq \gamma < \omega_1$. In this case we do $n_\alpha = 1,\ p(\alpha) = p(\delta_0) + 1,\ \rho_\alpha = \gamma_0,\ \tau_\alpha = \rho_\alpha + 1,\ a_{p(\alpha)} = c$ and $C_\gamma^{(\alpha)} = B(a_{p(\alpha)}, 2^{-p(\alpha)}) \cap D_\gamma$ for $\gamma_0 \leq \gamma < \omega_1$. Since $n_\alpha = 1$ we do not choose y_α.

And this completes the induction.

Obviously, there exists $\rho < \omega_1$ such that $n_\alpha = 0$ for every $\rho \leq \alpha < \omega_1$, which gives us the sequence $\{y_\alpha : \rho \leq \alpha < \omega_1\}$ fulfilling that $y_\alpha \notin \overline{\text{co}}([\{y_\beta : \rho \leq \beta < \alpha\}] \cup \text{cone}(\{y_\beta : \alpha < \beta < \omega_1\})) =: K_\alpha$ for every $\rho \leq \alpha < \omega_1$. So, by the Hahn-Banach theorem there exists $y^*_\alpha \in X^*$ such that $y^*_\alpha(y_\alpha) = 1$ but $\sup\{y^*_\alpha(y) : y \in K_\alpha\} < 1$. In particular, $y^*_\alpha(y_\beta) = 0$, if $\rho \leq \beta < \alpha$, and $y^*_\alpha(y_\beta) \leq 0$ if $\alpha < \beta < \omega_1$. □

Proposition 7.3. *Let X be a Banach space. We have:*

(1) If $X \in KS_4$, then $X \in (C)$; (2) $X \in KS_4$ iff $X \in KS_5$.

Proof. (1) This follows from Prop. 7.2 where it is proved that if $X \notin (C)$ then X has an ω_1-polyhedron.

(2) Clearly, $X \in KS_5$ implies $X \in KS_4$. Assume that $X \in KS_4$. By (1) we have that $X \in (C)$. In order to prove that $X \in KS_5$, by Prop. 6.2 it is enough to prove that every convex subset $C \subset X^*$ is w^*-separable. Since $X \in KS_4$, $\overline{C}^{w^*}$ is w^*-separable by Prop. 4.4. So, there exists a countable family $\{z_n : n \geq 1\} \subset \overline{C}^{w^*}$ w^*-dense in $\overline{C}^{w^*}$. Since $X \in (C)$, by [10, pg. 147] there exists a countable family $\{z_{nm} : n, m \geq 1\} \subset C$ such that $z_n \in \overline{\text{co}}^{w^*}(\{z_{nm} : m \geq 1\})$ for every $n \geq 1$. So, C is w^*-separable. □

Remarks. A nonseparable Banach space X has the *Kunen-Shelah property* KS_6 if for every uncountable family $\{x_i\}_{i \in I} \subset X$ there exists $j \in I$ such that $x_j \in wcl(\{x_i\}_{i \in I \setminus \{j\}})$ (wcl=weak closure). Clearly, $KS_6 \Rightarrow KS_5$. It seems that the unique known example of a Banach space X such that $X \in KS_6$ is the space $X = C(K)$, K being the Kunen compact space ([8, p. 1123]) constructed by Kunen under CH. This space $C(K)$ of Kunen has more interesting pathological properties. For example, $(C(K))^n, w^n)$ is hereditarily Lindelöf for every $n \in \mathbb{N}$. In view of this situation, we can introduce the property KS_7. A Banach space X is said to have the Kunen-Shelah property KS_7 iff (X^n, w^n) is for every $n \in \mathbb{N}$. It can be easily proved that $KS_7 \Rightarrow KS_6$. We do not know either if the Shelah space S has the property KS_6 or if the properties KS_5, KS_6 and KS_7 can be separated.

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